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A NOTE ON THE DIOPHANTINE EQUATION $|a^x - b^y| = c$

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Abstract

Let a, b, and c be positive integers. We show that if $(a, b) = (N^k - 1, N)$, where $N, k \geq 2$, then there is at most one positive integer solution (x, y) to the exponential Diophantine equation $|a^x - b^y| = c$, unless $(N, k) = (2, 2)$. Combining this with results of Bennett [3] and the first author [6], we stated all cases for which the equation $|(N^k \pm 1)^x - N^y| = c$ has more than one positive integer solutions (x, y).

1. Introduction

Let a, b, x, and y be positive integers and c an integer. The Diophantine equation

$$a^x - b^y = c \tag{1}$$

has a very rich history. It has been studied by many authors (see for examples [2], [3], [5], [6], [7], [9], [10], [13], [14], [15], [16], [17], [19], [20]). This Diophantine equation has some connections with Group Theory [1] and with Hugh Edgar's problem (i.e., the number of solutions (m, n) of $p^m - q^n = 2^h$) [4]. In 1936, Herschfeld [7] proved that equation (1) has at most one solution in positive integers x, y if $(a, b) = (3, 2)$ and c is sufficiently large. The same year, Pillai [13], (see also [14]) extended Herschfeld's result to any a, b with $\gcd(a, b) = 1$, $a > b \geq 2$, and $|c| > c_0(a, b)$, where $c_0(a, b)$ is a computational constant depending on a and b. Moreover, Pillai has conjectured that if $a = 3$ and $b = 2$ then $c_0(3, 2) = 13$. In 1982, this conjecture was proved by Stroeker and Tijdeman [19]. For more information about the history of this Diophantine equation, one can see for example [2], [15], [16], [19].

In this paper, we consider the exponential Diophantine equation

$$|a^x - b^y| = c. \tag{2}$$

There are infinitely many pairs (a, b) such that the equation (2) has at least two solutions. For example, let r and s be positive integers with $r \neq s$ and

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$\max\{r, s\} > 1$. If $a = (b^r + b^s)/2$ and $c = |a - b^r|$, then $(x, y) = (1, r)$ and $(1, s)$ both satisfy equation (2).

In 2003, Bennett [3] proved the following result.

THEOREM 1.1. *If N and c are positive integers with $N \geq 2$, then the Diophantine equation*

$$|(N+1)^x - N^y| = c$$

has at most one solution in positive integers x and y, unless

$$(N, c) \in \{(2, 1), (2, 5), (2, 7), (2, 13), (2, 23), (3, 13)\}.$$

In the first two of these cases, there are precisely 3 solutions, while the last four cases have 2 solutions apiece.

Very recently, the first author [6] extended Theorem 1.1 to obtain:

THEOREM 1.2. *If $(a, b) = (N^k + 1, N)$ with $\min\{N, k\} \geq 2$, then equation (2) has at most one solution, except $(N, k, c) \in \{(2, 2, 3), (2, 2, 123), (2, t, 2^t - 1)\}$ $(t \geq 3)$. In the first case, there are precisely 3 solutions, while the last two cases have 2 solutions.*

The aim of this paper is to study the number of solution of the equation

$$|(N^k - 1)^x - N^y| = c \tag{3}$$

and to prove the following result:

THEOREM 1.3. *If $(a, b) = (N^k - 1, N)$ with $\min\{N, k\} \geq 2$ and $(N, k) \neq (2, 2)$, then equation (2) has at most one positive integer solution (x, y).*

Naturally, from Theorems 1.1–1.3 we state

COROLLARY 1.4. *If $(a, b) = (N^k \pm 1, N)$ with $N \geq 2$, then equation (2) has at most one solution, unless*

$$(a, b, c) \text{ or } (b, a, c) \in \{(2, 3, 1), (2, 3, 5), (2, 3, 7), (2, 3, 13), (2, 3, 23), (3, 4, 13), (2, 5, 3), (2, 5, 123), (2, 2^t + 1, 2^t - 1)\ (t \geq 3)\}.$$

These cases having more than one solution are listed here:

$$3 - 2 = 2^2 - 3 = 3^2 - 2^3 = 1$$
$$2^3 - 3 = 3^2 - 2^2 = 2^5 - 3^3 = 5$$
$$5 - 2 = 2^3 - 5 = 2^7 - 5^3 = 3$$

and

$$3^2 - 2 = 2^4 - 3^2 = 7$$
$$2^4 - 3 = 2^8 - 3^5 = 13$$
$$3^3 - 2^2 = 2^5 - 3^2 = 23$$
$$5^3 - 2 = 2^7 - 5 = 123$$
$$(2^t + 1) - 2 = 2^{t+1} - (2^t + 1) = 2^t - 1.$$

The organization of this paper is as follows. In Section 2, we prove some useful results and recall a result due to Mignotte [11]. The proof of Theorem 1.3 will be given in Section 3 by the means of lower bounds for linear forms in two logarithms.

2. Preliminary work

Let p be a prime and let $\operatorname{ord}_p(n)$ denote the highest exponent of p in the prime factorization of an integer n. We define the number $\nu_p(n)$ by $\nu_p(n) = p^{-\operatorname{ord}_p(n)}$. (This corresponds to $|n|_p$ defined on pages 200–201 of [12]). Moreover $\log_p(1+n)$ denotes the p-adic logarithm of n. The p-adic logarithm satisfies the identity $\log_p(1+n) = \sum_{r=1}^{\infty}(-1)^{r+1}\frac{n^r}{r}$.

We have the following result.

LEMMA 2.1. *Let x, y, N, and k be positive integers with $N \geq 2$. If*

$$N^y \equiv 1 \pmod{(N^k - 1)^x}, \tag{4}$$

then $d \mid y$, where

(5)
$$d = \begin{cases} k(N^k - 1)^{x-1}, & \text{if } 2 \mid N \text{ or } x = 1 \text{ or } N^k \equiv 1 \pmod 4, \\ 2^{1-\operatorname{ord}_2(N^k+1)}k(N^k-1)^{x-1}, & \text{otherwise.} \end{cases}$$

PROOF. As $N^k - 1 \mid N^y - 1$, we get $k \mid y$. So there exists a positive integer z such that $y = kz$. The congruence (4) gives $N^{2y} \equiv 1 \pmod{2(N^k-1)^x}$. Let p be a divisor of $N^k - 1$. If $p = 2$, then we have $N^{2kz} \equiv 1 \pmod 4$ and if p is odd, then we have $N^{2kz} \equiv 1 \pmod p$. Also we know that $\nu_p(N^{2kz} - 1) = \nu_p(\log_p(N^{2kz}))$, see Mordell [12]. Then we obtain

$$\begin{aligned} \nu_p(2(N^k - 1)^x) \geq \nu_p(N^{2kz} - 1) &= \nu_p(z \log_p(N^{2k})) \\ &= \nu_p(z)\nu_p(\log_p(N^{2k})) = \nu_p(z)\nu_p(N^{2k} - 1) \\ &= \nu_p(z)\nu_p(N^k - 1)\nu_p(N^k + 1). \end{aligned}$$

Thus we have

$$\operatorname{ord}_p(2(N^k-1)^{x-1}) \le \operatorname{ord}_p(z) + \operatorname{ord}_p(N^k+1). \tag{6}$$

When p is odd, as $p \mid N^k - 1$ we get $p \nmid N^k + 1$, i.e., $\operatorname{ord}_p(N^k+1) = 0$. If $2 \mid N$, then p and any divisor of N^k+1 are both odd. By inequality (6), we get the first case. If $2 \nmid N$, then we need to consider $p = 2$, then from (6) we have

$$1 + \operatorname{ord}_2((N^k-1)^{x-1}) \le \operatorname{ord}_2(z) + \operatorname{ord}_2(N^k+1). \tag{7}$$

We put $z = 2^\alpha z'$, $2 \nmid z'$ and $N^k - 1 = 2^\beta \mu$, $2 \nmid \mu$, then applying inequality (6) we get $\mu^{x-1} \mid z'$. Similarly applying inequality (7) with 2^α and 2^β, we have $1+\beta(x-1) < \alpha + \operatorname{ord}_2(N^k+1)$. Thus we obtain $2^{1-\operatorname{ord}_2(N^k+1)}(N^k-1)^{x-1} \mid z$, so the remaining cases are proved.

We can prove the following lemma using a similar argument.

LEMMA 2.2. *Let x, y, N, and k be positive integers with $N \ge 2$, $y \ge k \ge 2$. If*

$$(N^k-1)^x \equiv 1 \pmod{N^y}, \tag{8}$$

then $\tau N^{y-k} \mid x$, *where* $\tau = \begin{cases} 1, & \text{if } N \text{ is even,} \\ 2, & \text{if } N \text{ is odd.} \end{cases}$

PROOF. Let p be a divisor of N. It is easy to see that $2 \mid x$. If $p = 2$, then we have $(N^k-1)^x \equiv 1 \pmod 4$. Otherwise, if p is odd, thus $(N^k-1)^x \equiv 1 \pmod p$. We know $\nu_p((N^k-1)^x - 1) = \nu_p(\log_p((N^k-1)^x))$. This and condition (8) imply

$$\nu_p(N^y) \ge \nu_p(x/2)\nu_p(N^k-2)\nu_p(N^k).$$

Thus we obtain

$$\operatorname{ord}_p(N^{y-k}) \le \operatorname{ord}_p(x/2) + \operatorname{ord}_p(N^k-2).$$

In the case $2 \nmid N$, we don't need to consider $p = 2$. We immediately get the result. If $2 \mid N$, since $k \ge 2$, this implies $N^k \equiv 0 \pmod 4$. Then we have $\operatorname{ord}_p(N^k-2) = 1$. So we obtain $\operatorname{ord}_p(N^{y-k}) \le \operatorname{ord}_p(x)$.

Now we recall the following result on linear forms in two logarithms due to Mignotte (see [11], Corollary of Theorem 2, page 110). For any non-zero algebraic number γ of degree d over Q, whose minimal polynomial over Z is $a\prod_{j=1}^{d}(X-\gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d}\log\max(1, |\gamma^{(j)}|)\right)$$

its absolute logarithmic height.

LEMMA 2.3. *Consider the linear form*

$$\Lambda = b_1 \log\alpha_1 - b_2\log\alpha_2,$$

where b_1 and b_2 are positive integers. Suppose that α_1, α_2 are multiplicatively independent. Put

$$D = [\mathsf{Q}(\alpha_1, \alpha_2) : \mathsf{Q}]/[\mathsf{R}(\alpha_1, \alpha_2) : \mathsf{R}]$$

and let ρ, λ, a_1 and a_2 be positive real numbers with $\rho \geq 4$, $\lambda = \log\rho$,

$$a_i \geq \max\{1, (\rho-1)\log|\alpha_i| + 2Dh(\alpha_i)\}, \qquad (i = 1, 2)$$

and

$$a_1 a_2 \geq \max\{20, 4\lambda^2\}.$$

Further suppose h is a real number with

$$h \geq \max\left\{3.5,\ 1.5\lambda,\ D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+1.377\right)+0.023\right\},$$

$\chi = h/\lambda$, $\upsilon = 4\chi + 4 + 1/\chi$. Then we have the lower bound

$$\log|\Lambda| \geq -(C_0 + 0.06)(\lambda + h)^2 a_1 a_2, \tag{9}$$

where

$$C_0 = \frac{1}{\lambda^3}\left\{\left(2+\frac{1}{2\chi(\chi+1)}\right) \cdot \left(\frac{1}{3}+\sqrt{\frac{1}{9}+\frac{4\lambda}{3\upsilon}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{32\sqrt{2}(1+\chi)^{3/2}}{3\upsilon^2\sqrt{a_1a_2}}}\right)\right\}^2.$$

3. Proof of Theorem 1.3

Suppose that the equation

$$|(N^k-1)^x - N^y| = c > 0$$

has two solutions (x_i, y_i) $(i = 1, 2)$ with $1 \le x_1 \le x_2$ satisfying the condition

$$N \ge 2, \quad k \ge 2 \quad \text{and} \quad (N,k) \ne (2,2). \tag{10}$$

PROPOSITION 3.1. *The equation*

$$(N^k-1)^{x_1} + (N^k-1)^{x_2} = N^{y_1} + N^{y_2} \tag{11}$$

has no solution (x_1, x_2, y_1, y_2) *with the condition* (10).

PROOF. We rewrite equation (11) into the form

$$(N^k-1)^{x_1}((N^k-1)^{x_2-x_1}+1) = N^{\min\{y_1,y_2\}}(N^{|y_2-y_1|}+1).$$

Since $\gcd(N^k-1, N) = 1$, we have $N^{|y_2-y_1|}+1 \equiv 0 \pmod{N^k-1}$. Therefore, there exist positive integers p, q such that $|y_2-y_1| = pk+q$, for $0 \le q < k$. Then we obtain

$$-1 \equiv N^{|y_2-y_1|} \equiv N^{pk+q} = (N^k)^p N^q \equiv N^q \pmod{N^k-1}.$$

Thus we get $N^q+1 \equiv 0 \pmod{N^k-1}$. This implies $N^k-1 \le N^q+1$. But as $q < k$, we get $N^k-1 \le N^{k-1}+1$. It follows that $N^{k-1}(N-1) \le 2$. This is impossible when $(N,k) \ne (2,2)$. So Proposition 3.1 is proved.

Let us consider the equation

$$(N^k-1)^{x_1} - N^{y_1} = (N^k-1)^{x_2} - N^{y_2} = \pm c, \qquad c > 0, \tag{12}$$

with $x_1 < x_2$ and $y_1 < y_2$. Taking equation (12) modulo N, we have

$$(-1)^{x_1} \equiv (-1)^{x_2} \pmod{N}.$$

If $N > 2$, it follows that

$$x_1 \equiv x_2 \pmod{2}. \tag{13}$$

We rewrite equation (12) into the form

$$(N^k-1)^{x_1}((N^k-1)^{x_2-x_1}-1) = N^{y_1}(N^{y_2-y_1}-1). \tag{14}$$

Since $x_2 - x_1$ is even, so $N^k \mid (N^k-1)^{x_2-x_1}-1$. Thus N^k divides the right side of equation (14). As $\gcd(N^{y_2-y_1}-1, N) = 1$, we have $y_1 \ge k$.

From Lemma 2.1, we have $k \mid y_1 \Leftrightarrow k \mid y_2$. It is easy to show that the special case $k \mid y_1$ or $k \mid y_2$ can be solved by Theorem 1.1. In fact, if $k \mid y_i$ $(i = 1, 2)$ then there exist positive integers t_1 and t_2 such that $y_1 = t_1 k$ and $y_2 = t_2 k$. Let us put $M = N^k - 1$, thus the equation

$$|(M+1)^X - M^Y| = c$$

have the solutions $(X, Y) = (x_1, t_1)$ and (x_2, t_2). From Theorem 1.1, we have $M \leq 3$. Thus we get $N^k - 1 \leq 3$ which contradicts the condition (10). Therefore, using equation (14), we will consider

$$y_1 > k \qquad \text{and} \qquad k \nmid y_i \quad (i = 1, 2). \tag{15}$$

Assume $N = 2$. Considering equation (12) modulo 2^k gives

$$(-1)^{x_1} - 2 \equiv (-1)^{x_2} \pmod{2^k}.$$

Using condition (10), we get $k \geq 3$. This leads to $2 \mid x_1$ and $2 \nmid x_2$.

PROPOSITION 3.2. *If the equation*

$$(N^k - 1)^{x_1} - N^{y_1} = (N^k - 1)^{x_2} - N^{y_2} = c > 0 \tag{16}$$

has solutions (x_1, x_2, y_1, y_2) *with the condition* (10)*, then* $N^k - 1 < 24379$.

PROOF. Either $y_1 > k$ or $2 \mid x_1$ implies $x_1 \geq 2$. We set

$$\Lambda = x_2 \log(N^k - 1) - y_2 \log(N).$$

Then we have

$$0 < \Lambda < e^{\Lambda} - 1 = \frac{c}{N^{y_2}} < \frac{(N^k - 1)^{x_1}}{N^{y_2}}.$$

On the other hand, using equation (14) we get $N^{y_2 - y_1} \equiv 1 \pmod{(N^k - 1)^{x_1}}$. Then from Lemma 2.1 with $x_1 \geq 2$ and $N^k - 1 \geq 2^3 - 1 > 2^{2.8}$, we have

$$y_2 - y_1 \geq k \left(\frac{N^k - 1}{2} \right)^{x_1 - 1} \geq k \left(\frac{N^k - 1}{2} \right)^{0.5 x_1} > k (N^k - 1)^{0.32 x_1}.$$

Thus we obtain

$$\Lambda < \frac{((y_2 - y_1)/k)^{3.125}}{N^{y_2}} < \frac{{y_2}^{3.125}}{N^{y_2}}.$$

We know that $\Lambda < ((y_2 - y_1)/k)^{3.125}/N^{y_2} \leq (y_2/2)^{3.125}/2^{y_2}$. The function $(y/2)^{3.125}/2^y$ is a maximum when y is between 4 and 5, so $\Lambda < 0.548$. Now we apply Lemma 2.3 to Λ. We take

$$D = 1, \;\; \alpha_1 = N^k - 1, \;\; \alpha_2 = N, \;\; b_1 = x_2, \;\; b_2 = y_2 \tag{17}$$

and

$$a_1 = (\rho + 1)\log(N^k - 1), \qquad a_2 = (\rho + 1)\log N. \tag{18}$$

Since $N \geq 4$ with $k = 2$ or $N \geq 2$ with $k \geq 3$, we choose $\rho = 4.8$. It satisfies $a_1 a_2 \geq \max\{20, 4\lambda^2\}$. The fact $\Lambda > 0$ implies

$$\frac{x_2}{\log N} > \frac{y_2}{\log(N^k - 1)}.$$

We take

$$h = \max\left\{8.56, \log\left(\frac{x_2}{\log N}\right) + 0.82\right\}.$$

First we suppose

$$h = \log\left(\frac{x_2}{\log N}\right) + 0.82,$$

then

$$\frac{x_2}{\log N} \geq 2299.$$

We obtain $C_0 < 0.627$, then we have

$$\log|\Lambda| > -23.12\left(\log\left(\frac{x_2}{\log N}\right) + 2.389\right)^2 \log(N^k - 1)\log N.$$

We have

$$\frac{x_2}{\log N} = \frac{y_2}{\log(N^k - 1)} + \frac{\Lambda}{\log(N^k - 1)\log N} < \frac{y_2}{\log(N^k - 1)} + 0.407.$$

Combining this and bounds of Λ, we have

$$\begin{aligned}\frac{x_2}{\log N} &< 0.407 + \frac{3.125\log y_2}{\log(N^k - 1)\log N} + 23.12\left(\log\left(\frac{x_2}{\log N}\right) + 2.389\right)^2 \\ &< 1.698 + 2.317\log\left(\frac{x_2}{\log N}\right) + 23.12\left(\log\left(\frac{x_2}{\log N}\right) + 2.389\right)^2.\end{aligned}$$

We get

$$\frac{x_2}{\log N} < 2415.$$

Next we suppose $h = 8.56$, then we have also

$$\frac{x_2}{\log N} < e^{8.56 - 0.82} \leq 2299 < 2415.$$

Since $y_2/\log(N^k - 1) < x_2/\log N$, thus

$$y_2 < 2415\log(N^k - 1). \tag{19}$$

Using (15), (19), and Lemma 2.1, we obtain

$$N^k - 1 < k\left(\frac{N^k - 1}{2}\right)^{x_1 - 1} + y_1 < y_2 < 2415\log(N^k - 1). \tag{20}$$

This implies $N^k - 1 < 24397$.

PROPOSITION 3.3. *If the equation*

$$N^{y_1} - (N^k - 1)^{x_1} = N^{y_2} - (N^k - 1)^{x_2} = c > 0 \tag{21}$$

has solutions (x_1, x_2, y_1, y_2) *with the condition* (10)*, then* $N^k - 1 < 42455$.

PROOF. We will use a similar method to that of Proposition 3.2. We set again

$$\Lambda = x_2\log(N^k - 1) - y_2\log(N).$$

Then we obtain

$$0 < -\Lambda < e^{-\Lambda} - 1 = \frac{c}{(N^k - 1)^{x_2}} < \frac{N^{y_1}}{(N^k - 1)^{x_2}}. \tag{22}$$

The fact that the left side of equation (21) is positive implies $y_1 > k$. From equation (14), we get $(N^k - 1)^{x_2 - x_1} \equiv 1 \pmod{N^{y_1}}$. So Lemma 2.2 gives $x_2 - x_1 \geq N^{y_1 - k}$. Therefore, as $N^k \geq 8$, then we obtain

$$-\Lambda < \frac{N^k}{N^k - 1}\cdot\frac{N^{y_1 - k}}{(N^k - 1)^{x_2 - 1}} < \frac{1.15(x_2 - x_1)}{(N^k - 1)^{x_2 - 1}} \leq \frac{1.15(x_2 - 1)}{(N^k - 1)^{x_2 - 1}}.$$

From congruence (13), we have $x_2 - 1 \geq x_2 - x_1 \geq 2$ and $x_2 - 1 \geq 2x_2/3$. Then we obtain

$$-\Lambda < \frac{0.77x_2}{(N^k - 1)^{2x_2/3}}. \tag{23}$$

Again, by $x_2 \geq 3$ and $N^k \geq 8$ we get $-\Lambda < 0.05$. Thus we have

$$\frac{x_2}{\log N} < \frac{y_2}{\log(N^k - 1)} < \frac{x_2}{\log N} + \frac{0.05}{\log(N)\log(N^k - 1)} < \frac{x_2}{\log N} + 0.038. \tag{24}$$

Now we apply Lemma 2.3 to $-\Lambda$. We take the same parameters as those in (17), (18) and we choose $\rho = 4.1$. Here we have

$$h = \max\left\{9.10, \log\left(\frac{y_2}{\log(N^k - 1)}\right) + 0.81\right\}.$$

First we suppose

$$h = \log\left(\frac{y_2}{\log(N^k - 1)}\right) + 0.81,$$

then

$$\frac{y_2}{\log(N^k - 1)} > 3983. \tag{25}$$

We have $C_0 < 0.859$ and thus

$$\log|-\Lambda| > -23.91\left(\log\left(\frac{y_2}{\log(N^k - 1)}\right) + 2.22\right)^2 \log(N^k - 1)\log N.$$

On the other hand, by inequality (23) we get

$$\log|-\Lambda| < -0.27 + \log x_2 - \frac{2}{3}x_2\log(N^k - 1).$$

The upper and lower bounds imply

$$\frac{x_2}{\log N} < \frac{1.5\log x_2 - 0.405}{\log(N^k - 1)\log N} + 35.87\left(\log\left(\frac{y_2}{\log(N^k - 1)}\right) + 2.22\right)^2.$$

Using this and the middle terms of (24), we get

$$\frac{y_2}{\log(N^k - 1)} < 1.12\log\left(\frac{y_2}{\log(N^k - 1)}\right) + 35.87\left(\log\left(\frac{y_2}{\log(N^k - 1)}\right) + 2.22\right)^2.$$

It results

$$\frac{y_2}{\log(N^k - 1)} < 3969.$$

This contradicts inequality (25).

Next, we suppose $h = 9.10$. Then we have

$$\frac{y_2}{\log(N^k - 1)} < e^{9.10-0.81} < 3984.$$

Since $x_2/\log N < y_2/\log(N^k - 1)$, thus

$$x_2 < 3984\log N. \tag{26}$$

By (15), (26) and Lemma 2.2, we get

$$N^{y_1-k} \le x_2 - x_1 < x_2 < 3984\log N \le 3984\log(N^{y_1-k}). \tag{27}$$

This implies $N^{y_1-k} < 42455$. If $y_1 - k \geq k$, we have $N^k < 42455$.

Otherwise, suppose that $y_1 - k \leq k - 1$. From equation (21) we have $(N^k - 1)^{x_1} < N^{y_1}$. Then we obtain

$$(N^k - 1)^{x_1} < N^{2k-1}.$$

If $x_1 \geq 2$, then we have $N^{2k} - 2N^k < N^{2k-1}$. This implies that $N^{k-1}(N-1) < 2$, which is impossible. It remains $x_1 = 1$. Now from (22) and $y_1 \leq 2k - 1$, we have

$$\left|\frac{\log(N^k - 1)}{\log N} - \frac{y_2}{x_2}\right| < \frac{1}{x_2(N^k - 1)^{x_2-2}\log N}. \tag{28}$$

Using $x_2 \geq 3$ and $N^k = 8$, we get $(N^k - 1)^{x_2-2}\log N > 2x_2$. Thus we obtain

$$\left|\frac{\log(N^k - 1)}{\log N} - \frac{y_2}{x_2}\right| < \frac{1}{2x_2^2}.$$

Thus y_2/x_2 is a convergent in the simple continued fraction expansion to $\log(N^k - 1)/\log N$. It is known that (see [8]), if p_r/q_r is the r'th such convergent, then

$$\left|\frac{\log(N^k - 1)}{\log N} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1} + 2)q_r^2},$$

where a_{r+1} is the $(r + 1)$st partial quotient to $\log(N^k - 1)/\log N$. In the continued fraction expansion

$$\frac{\log(N^k - 1)}{\log N} = [k - 1, 1, a_2, \ldots],$$

by direct computation, one gets $q_2 = a_2 + 1$ and

$$(N^k - 1)\log N - 1 < a_2 < N^k \log N - 1.$$

Let $y_2/x_2 = p_r/q_r$ for some nonnegative integer r. From inequality (26) we have $q_r \leq x_2 < 3984\log N$. If $N^k - 1 > 3984$, then $q_2 - 1 = a_2 > (N^k - 1)\log N - 1 \geq 3894\log N - 1 > q_r - 1$. This implies $r < 2$. But $q_0 = q_1 = 1$ such that $x_2 = 1$, which is impossible. Then we have $N^k - 1 \leq 3984$. This completes the proof of Proposition 3.3.

Finally, running a MAPLE scripts by Scott and Styer [18], we found all solutions of the equation

$$a^x - b^y = c$$

in the range $1 < a, b < 53000$, which are listed in [17]. This helps us to check the remaining cases stated in Propositions 3.2 and 3.3. We found no solution

(x, y) satisfying $(a, b) = (N^k - 1, N)$ with condition (10). Combining this with Proposition 3.1 completes the proof of Theorem 1.3.

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