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DECOMPOSABLE PROJECTIONS RELATED TO THE FOURIER AND FLIP AUTOMORPHISMS

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Abstract

In this paper we classify Fourier invariant projections g in the irrational rotation C*-algebra that can be decomposed in the form

$$g = f + \sigma(f) + \sigma^2(f) + \sigma^3(f)$$

for some Fourier orthogonal projection f, where σ is the Fourier transform automorphism. The analogous result is shown for the flip automorphism as well as the existence of flip-orthogonal projections. Both classifications are achieved by means of topological invariants (given by unbounded traces) and the canonical trace. We also show (in both the flip and Fourier cases) that invariant projections h are subprojections of orthogonal decompositions g for some projection f such that $\tau(f) = \tau(h)$ (where τ is the canonical trace).

1. Introduction and Main Results

Our setting is the irrational rotation C*-algebra A_θ generated by unitaries U, V enjoying the commutation relation $VU = \lambda UV$ where $\lambda = e(\theta) := e^{2\pi i\theta}$. The *flip* automorphism on A_θ is the canonical automorphism ϕ given by $\phi(U) = U^{-1}$, $\phi(V) = V^{-1}$ and the *Fourier transform* is the canonical automorphism defined by $\sigma(U) = V$, $\sigma(V) = U^{-1}$. The flip automorphism was extensively studied in [1], [2], and [3]; it prompted research interest in the other three canonical automorphisms of the rotation C*-algebra.

Definition 1.1. A projection f is *flip-orthogonal* if $f\phi(f) = 0$. A projection f is *σ^*-orthogonal* if $f, \sigma(f), \sigma^2(f), \sigma^3(f)$ are mutually orthogonal projections (so that their sum is a Fourier invariant projection).

A quick rundown of the results of this paper are as follows:

(1) A_θ contains flip-orthogonal projections (Theorem 1.2);

(2) topological classification of flip invariant projections in A_θ that can be decomposed as $f + \phi(f)$ for some flip orthogonal projection f (Theorem 1.3);

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(3) topological classification of Fourier invariant projections that can be decomposed as

$$f + \sigma(f) + \sigma^2(f) + \sigma^3(f) \tag{1.1}$$

for some σ^*-orthogonal projection f (Theorem 1.7);

(4) we generalize the result in [18] that the Fourier transform on a corner algebra of A_θ is conjugate to the tensor product of respective Fourier transforms on M_q and another rotation algebra (Theorem 1.5), which in [18] was proved for a dense G_δ set of θ's and specialized rationals p/q – strict conditions that we remove. (Theorem 1.5 is used to prove (3).)

(5) We show in both the flip and Fourier cases that invariant projections are subprojections of an orthogonal decomposition such as (1.1) for some f. (See Theorems 1.4 and 1.8 for more precise statements.)

Phillips' notion of tracial Rokhlin property [9] involves projections that are approximately orthogonal, a property he showed to hold for the canonical finite order automorphisms (including the flip and Fourier) of the irrational rotation algebra.

For any positive integer q, define

$$\widetilde{q} = \tfrac{1}{2}(q - r)$$

where $r \in \{1, 2\}$ is such that $q \equiv r \mod 2$.

STANDING HYPOTHESIS. In the statements of Theorems 1.2–1.8 below, let $\theta > 0$ be irrational and p/q be a positive rational (in reduced form, $q > 0$) such that $0 < q|q\theta - p| < 1$. For example, any convergent of θ satisfies this inequality.

THEOREM 1.2. *For each $k \le \widetilde{q}$ there exists a smooth flip-orthogonal projection in A_θ of trace $k|q\theta - p|$.*

It would seem that in view of this, and the fact that this is the case also for the Fourier transform (see Theorem 1.6 below or [16]), that projections orthogonal with respect to the hexic and cubic transforms exist also. One of the advantages of getting such projections lies in their use in constructing basic building blocks (of type I) in A_θ that are invariant and approximate A_θ – such as is shown in [17] where two orthogonal matrix algebras over the unit circle are mapped onto each other by an order 4 automorphism quite similar to the Fourier transform (which is in fact approximately unitarily equivalent to the Fourier transform). In fact, since the latter inductive limit automorphism is equal to the Fourier transform on the K-theory of A_θ, by Elliott's Classification Theorem [6] they are approximately unitarily equivalent.

Next, we determine when a flip invariant projection g in A_θ can be decomposed as $g = e + \phi(e)$ for some flip orthogonal projection e. This is not always possible, even if g has "even" trace, because a necessary condition for such a decomposition to hold is that the topological invariants of g, given by the unbounded traces ϕ_{ij}, all vanish. (These maps are defined in the Preliminaries.) Henceforth by "topological invariants" we will mean the quantized numbers given by the unbounded traces on projections (as they are indeed quantized). For indeed, if $e\phi(e) = 0$ then one has $\phi_{ij}(e) = \phi_{ij}(ee) = \phi_{ij}(\phi(e)e) = 0$, and by the ϕ-invariance of ϕ_{ij} one necessarily has $\phi_{ij}(g) = 0$. To see that this latter equality is not always the case for flip invariant projections one calculates the ϕ_{ij} on the flip invariant Rieffel projections in [14] (Lemma 2.1), and finds that they are not all zero.

The following theorem shows that the topological invariants given by the unbounded traces ϕ_{ij} (along with the canonical trace) are the only (topological) obstructions for such decomposition.

THEOREM 1.3. *For each flip invariant projection g in A_θ with vanishing topological invariants and of "even" trace $\tau(g) = 2k|q\theta - p|$, where $k \le \widetilde{q}$, there is a flip orthogonal projection f in A_θ such that*

$$g = f + \phi(f). \tag{1.2}$$

By consequence of these results we obtain the following result that shows that invariant projections are related to flip orthogonal projections in a natural way.

THEOREM 1.4. *Let g be a flip invariant projection in A_θ of trace $\tau(g) = k|q\theta - p|$ where $k \le \widetilde{q}$. Then there exists a flip orthogonal projection f in A_θ such that*

$$\tau(f) = \tau(g) \qquad \text{and} \qquad g \le f + \phi(f).$$

We obtain the analogous results for the Fourier transform (Theorems 1.6, 1.7, 1.8), but in order to accomplish this we will need to prove the following result (which will occupy our attention).

THEOREM 1.5. *Let σ be the Fourier transform on A_θ. Then there exists a Fourier invariant smooth projection e in A_θ of trace $\tau(e) = q|q\theta - p|$ and a smooth *-isomorphism*

$$\eta : eA_\theta e \to M_q \otimes A_{\theta'} \qquad \text{such that} \quad \eta\sigma = (\Sigma \otimes \sigma')\eta \tag{1.3}$$

where Σ and σ' are Fourier transform automorphisms on M_q and $A_{\theta'}$, respectively, given by

$$\Sigma(u) = v, \qquad \Sigma(v) = u^*, \qquad \sigma'(a) = b^*, \qquad \sigma'(b) = a, \tag{1.4}$$

where $M_q = C^*(u, v)$ *and* u, v *are order* q *unitary matrices with* $vu = e\left(\frac{p}{q}\right)uv$, *and* $A_{\theta'}$ *is generated by unitaries* a, b *with* $ba = e(\theta')ab$, *where* θ' *is some irrational in the* $\mathrm{GL}(2, \mathsf{Z})$ *orbit of* θ. *The isomorphism* η *is "smooth" in that it induces an isomorphism of the canonical smooth algebras* $eA_\theta^\infty e$ *and* $M_q \otimes A_{\theta'}^\infty$.

In [18] we constructed the projection e of Theorem 1.5 but obtained the commutative equality (1.3) with the automorphism $\Sigma \otimes \sigma'$ only under the very specialized case that θ belongs to a dense G_δ set of irrationals and has an infinite sequence of rational approximants p/q such that p is a perfect square and $q \equiv 0 \mod 4$. Theorem 1.5 does away with all these restrictions, and allows us to obtain the following more general results.

By analogy with $\widetilde{q}$, for any positive integer q, we define

$$\widehat{q} = \tfrac{1}{4}(q - r) \in \mathsf{Z} \tag{1.5}$$

where $r \in \{1, 2, 3, 4\}$ is such that $q \equiv r \mod 4$.

THEOREM 1.6. *For each* $k \leq \widehat{q}$ *there exists a* σ^**-orthogonal projection in* A_θ *of trace* $k|q\theta - p|$.

These results are used in proving the following two theorems.

THEOREM 1.7. *Let* g *be a Fourier invariant projection in* A_θ *with vanishing topological invariants and of "quartic" trace* $\tau(g) = 4k|q\theta - p|$, *where* $k \leq \widehat{q}$. *Then there exists a* σ^**-orthogonal projection* f *in* A_θ *such that* g *has the orthogonal decomposition*

$$g = f + \sigma(f) + \sigma^2(f) + \sigma^3(f). \tag{1.6}$$

THEOREM 1.8. *Let* g *be a Fourier invariant projection in* A_θ *of trace* $\tau(g) = k|q\theta - p|$ *where* $k \leq \widehat{q}$. *Then there exists a* σ^**-orthogonal projection* f *in* A_θ *and Fourier invariant projections* g_1, g_2, g_3, *mutually orthogonal and with* g, *such that*

$$\tau(f) = \tau(g) = \tau(g_j), \qquad g + g_1 + g_2 + g_3 = f + \sigma(f) + \sigma^2(f) + \sigma^3(f)$$

and $\mathbf{T}_4[g_r] = \Omega^{-r}\mathbf{T}_4[g]$, *where* $\Omega = (1; i, i; -1, -1, -1)$.

The Connes-Chern character invariant $\mathbf{T}_4$ is defined near the end of the Preliminaries (in terms of canonical and unbounded traces).

We conjecture that if the projection g in Theorem 1.7 is approximately central, then the projection f could be chosen to be so as well.

Presumably, the same decomposition results hold for the cubic and hexic transforms of A_θ studied in [4]. (The converse of Theorem 1.7 is trivial.)

REMARK 1.9. In all that follows we assume that θ is a fixed irrational number, with $0 < \theta < 1$, that p/q is any rational number such that $0 < q(q\theta - p) < 1$ (and $q > 0$). The reason it suffices to assume that $q(q\theta - p) > 0$ in the proofs even though in the statements of theorems we assume $q|q\theta - p| < 1$, is that if $q\theta - p < 0$, then $p - q\theta = q(1-\theta) - (q-p) > 0$ and the results can be applied to $1-\theta$ and the rational $(q-p)/q$. Consequently, our results apply to any convergent of θ since they all enjoy the inequality $q|q\theta - p| < 1$.

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2. Preliminaries

CONVENTION 2.1. It will be convenient to introduce projective equality "$X \doteq Y$," for matrices or operators X, Y, to mean that $X = cY$ for some complex number c with $|c| = 1$. Unless the contrary is stated, we write $m \equiv n$ for congruence mod q (q being fixed in the process). We write δ_d^k for the divisor δ-function given to be 1 when $d|k$ and 0 otherwise. We clearly have $\sum_{j=0}^{n-1} e(mj/n) = n\delta_n^m$.

Let p/q be a positive rational number in lowest terms and set $\lambda_0 = e(p/q)$. Let u, v be any two unitaries in $M_q = M_q(\mathbb{C})$ of order q such that $vu = \lambda_0 uv$. Such unitaries generate M_q (as $u^j v^k$ form a basis, $j, k = 0, 1, \ldots, q-1$). If u', v' is any other pair of order q unitaries satisfying $v'u' = \lambda_0 u'v'$, then there is a unique automorphism of M_q such that $u \mapsto u'$, $v \mapsto v'$.

It is easy to check the equation

$$(v^m u^n)^k = \lambda_0^{-mnk(k-1)/2} v^{mk} u^{nk}. \tag{2.1}$$

There is a natural homomorphism $\alpha : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{Aut}(M_q)$ sending the matrix $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to the automorphism α_X given by

$$\alpha_X(v) = \lambda_0^{ac(q-1)/2} v^a u^c, \qquad \alpha_X(u) = \lambda_0^{bd(q-1)/2} v^b u^d, \tag{2.2}$$

where we write $\lambda_0^t := e(pt/q)$ for any real t.

Using (2.1) it is easy to check that these unitaries have order q and satisfy the same relation $vu = \lambda_0 uv$, so that the automorphism α_X exists, and that $\alpha_X \alpha_Y = \alpha_{XY}$.

If an automorphism μ of M_q has the generic form

$$\mu(v) \doteq v^a u^c, \qquad \mu(u) \doteq v^b u^d \tag{2.3}$$

we will refer to $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ as "its matrix" (which is unique mod q); and if this matrix is chosen in $\mathrm{SL}(2, \mathsf{Z})$ then $\mu(v) \doteq \alpha_X(v)$, $\mu(u) \doteq \alpha_X(u)$.

We denote by $\mathrm{SL}(2, \mathsf{Z}_q)$ the group of 2 by 2 matrices with entries in the commutative ring $\mathsf{Z}_q = \mathsf{Z}/q\mathsf{Z}$ whose determinant is 1 mod q. There is a canonical map $\mathrm{SL}(2, \mathsf{Z}) \to \mathrm{SL}(2, \mathsf{Z}_q)$.

LEMMA 2.2. *The canonical group homomorphism* $\mathrm{SL}(2, \mathsf{Z}) \to \mathrm{SL}(2, \mathsf{Z}_q)$ *is surjective.*

PROOF. Fix a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that $ad - bc = 1 + xq$ for some integer x. Without loss of generality one may assume that b, d are relatively prime. (This is because (b, d) has order q in $\mathsf{Z}_q \oplus \mathsf{Z}_q$ and any such element has integer representatives that are relatively prime.) Pick integers k, ℓ such that $kb + \ell d = 1$. Then one has $(a - x\ell q)d - b(c + xkq) = 1$ and the matrix $\left[\begin{smallmatrix} a-x\ell q & b \\ c+xkq & d \end{smallmatrix}\right]$ is a preimage in $\mathrm{SL}(2, \mathsf{Z})$.

In this paper we will use the Rieffel framework that we have set up in [18] – which was based on Rieffel's equivalence bimodule construction in [12] – the essentials of which we recollect here for our calculations below.

Let M be a locally compact Abelian group and dm the Haar measure on M. In the case of finite discrete M, each point has Haar measure $1/\sqrt{|M|}$. When $M = \mathsf{R}$, dm is the usual Lebesgue measure. (Recall that these Haar measure normalizations ensure that two iterations of the Fourier transform of $f(t)$ is equal to $f(-t)$.) We will be interested in the case when M is self-dual: $\widehat{M} \cong M$. In this case there is a canonical pairing $\langle m, m' \rangle : M \times M \to \mathsf{T}$ (where T is the unit circle). For $m, m' \in \mathsf{Z}_q$ and $s, t \in \mathsf{R}$ the pairings are $\langle m, m' \rangle = e(mm'/q)$ and $\langle s, t \rangle = e(st)$.

If f is a continuous complex function of compact support on M, the Fourier transform is given by $\widehat{f}(m') = \int_M f(m)\overline{\langle m, m' \rangle}\, dm$. Letting $G = M \times \widehat{M}$, the Heisenberg projective unitary representation is given by

$$\pi : G \to \mathcal{U}(L^2(M)), \qquad [\pi_{(m,s)} f](n) = \langle n, s \rangle f(n + m), \tag{2.4}$$

for $m, n \in M$, $s \in \widehat{M}$. The canonical Heisenberg cocycle on G is given by the "first-and-last" pairing

$$\mathfrak{h}((m, s), (m', s')) = \langle m, s' \rangle. \tag{2.5}$$

One has

$$\pi_x \pi_y = \mathfrak{h}(x, y)\pi_{x+y}, \qquad \pi_x^* = \mathfrak{h}(x, x)\pi_{-x} \tag{2.6}$$

for $x, y \in G$. If D is a lattice in G, then its dual lattice is

$$D^\perp = \{y \in G : \mathfrak{h}(x, y)\overline{\mathfrak{h}(y, x)} = 1, \ \forall x \in D\}.$$

Rieffel's Theorem 2.15 in [12] states that the Schwartz space $\mathscr{S}(M)$ on M can be completed into is an equivalence bimodule with the twisted convolution C*-algebras $C^*(D, \mathfrak{h})$ and $C^*(D^\perp, \overline{\mathfrak{h}})$ acting on the left and the right, respectively. The C*-algebra $C^*(D, \mathfrak{h})$ is generated by the unitaries π_x, for $x \in D$, while $C^*(D^\perp, \overline{\mathfrak{h}})$ is the *opposite* algebra of the C*-algebra generated by the unitaries π_y^*, for $y \in D^\perp$. Thus, $C^*(D, \mathfrak{h}) = C^*(\pi_x : x \in D)$ and $C^*(D^\perp, \overline{\mathfrak{h}}) = C^*(\pi_y^* : y \in D^\perp)^{\mathrm{opp}}$. (To make matters clearer, we may use "$\bullet$" to denote the opposite multiplication of $C^*(D^\perp, \overline{\mathfrak{h}})$ – thus $x \bullet y = yx$.)

On $\mathscr{S}(M)$ there are C*-valued inner products given by

$$\begin{aligned}\langle f_1, f_2\rangle_D &= |G/D| \sum_{w\in D} \langle f_1, f_2\rangle_D(w)\pi_w \\ &= |G/D| \sum_{w\in D} \langle f_1, \pi_w f_2\rangle_{L^2(M)}\pi_w\end{aligned} \tag{2.7}$$

$$\begin{aligned}\langle f_1, f_2\rangle_{D^\perp} &= \sum_{z\in D^\perp} \langle f_1, f_2\rangle_{D^\perp}(z)\pi_z^* \\ &= \sum_{z\in D^\perp} \langle \pi_z f_2, f_1\rangle_{L^2(M)}\pi_z^*\end{aligned} \tag{2.8}$$

having the module properties

$$\langle f_1, f_2\rangle_D^* = \langle f_2, f_1\rangle_D, \qquad \langle f_1, f_2\rangle_{D^\perp}^* = \langle f_2, f_1\rangle_{D^\perp} \tag{2.9}$$

$$a \bullet \langle f_1, f_2\rangle_{D^\perp} \bullet b = \langle f_1 a^*, f_2 b\rangle_{D^\perp} \tag{2.10}$$

for $a, b \in C^*(D^\perp, \overline{\mathfrak{h}})$ and $f_1, f_2 \in \mathscr{S}(M)$.

Using Rieffel's equivalence bimodule construction, applied to a suitably chosen lattice, in [18] we constructed a Fourier invariant smooth projection e in A_θ of trace $q(q\theta - p)$ and an isomorphism

$$\mu : eA_\theta e \to M_q \otimes A_{\theta'} \qquad \text{such that} \quad \mu\sigma = (\sigma_1 \otimes \sigma_2)\mu \tag{2.11}$$

where σ_1 and σ_2 are automorphisms of M_q and $A_{\theta'}$ (respectively) given by

$$\sigma_1(x) = W_0^* \sigma'(x) W_0 \qquad (x \in M_q) \tag{2.12}$$

where W_0 is some unitary in M_q, where $\sigma_2 = \sigma'$ on $A_{\theta'}$, and σ' is the Fourier automorphism of $M_q \otimes A_{\theta'}$ given by

$$\sigma'(V_1) = V_2, \quad \sigma'(V_2) = V_1^*, \quad \sigma'(V_3) = V_4, \quad \sigma'(V_4) = V_3^* \tag{2.13}$$

where $M_q = C^*(V_3, V_4)$, $A_{\theta'} = C^*(V_1, V_2)$, and the unitaries here satisfy the relations

(2.14)
$$V_1V_2 = e(\theta')V_2V_1, \quad V_3V_4 = e\big(\tfrac{p}{q}\big)V_4V_3, \quad V_jV_k = V_kV_j, \quad V_k^q = I,$$

(using the usual multiplication of operators on the underlying Hilbert space) for $j = 1, 2$ and $k = 3, 4$, where θ' is in the standard $\mathrm{GL}(2, \mathsf{Z})$ orbit of θ. In the notation of the beginning of this section, $v = V_3, u = V_4$ and $\sigma' \doteq \alpha_J$ where $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

With τ and τ' being the canonical traces on A_θ and $M_q \otimes A_{\theta'}$, respectively, from [18] (Sections 2 and 4) their Morita equivalence yields the equation[1]

$$\tau(\mu^{-1}(y)) = q(q\theta - p)\tau'(y) \tag{2.15}$$

for all y in $M_q \otimes A_{\theta'}$. (The trace on M_q being the usual q^{-1} times the sum of the diagonal entries.) Note that $y = 1$ gives $\tau(e) = q(q\theta - p)$.

Further, the unitary $W_0 \in M_q$ has the Hilbert module inner product form $W_0 = \langle \varphi, \widehat{\varphi} \rangle_{D_0^\perp}$ where

$$\varphi(n, m) = \frac{1}{\sqrt{q}} e\big(\tfrac{1}{q}[an^2 + bnm + \gamma m^2]\big), \tag{2.16}$$

for some integer constants a, b, γ (to be shortly specified), and $\widehat{\varphi}$ is its (discrete) Fourier transform. In our construction [18] we realized the matrix algebra M_q as twisted group C*-algebras $M_q \cong C^*(D_0, \mathfrak{h})$ and $M_q \cong C^*(D_0^\perp, \overline{\mathfrak{h}})$ (as in Rieffel [12]) with respect to the groups $M_0 = \mathsf{Z}_q \times \mathsf{Z}_q$, $G_0 = M_0 \times \widehat{M}_0 = M_0 \times M_0$. We took the lattice D_0 in G_0, and its complement $D_0^\perp$, with bases:

$$\begin{aligned} D_0 :\ & \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ -p_3 & -p_4 & p_1 & p_2 \end{bmatrix}, \\ D_0^\perp :\ & \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} p_2 & -p_1 & -p_4 & p_3 \\ p_4 & -p_3 & p_2 & -p_1 \end{bmatrix} \end{aligned} \tag{2.17}$$

where, by Lagrange's theorem, one can write $p = p_1^2 + p_2^2 + p_3^2 + p_4^2$ as a sum of four integer squares. Here, V_3, V_4 are realized as order q unitaries such that $V_k \doteq \pi_{-\delta_k}$ (in terms of the Heisenberg unitaries).

[1] In [18] we had the trace relation $\tau(\langle f, g \rangle_D) = q(q\theta - p)\,\tau'(\langle g, f \rangle_{D^\perp})$ for a suitable lattice subgroup D and it's complement $D^\perp$, and upon using the isomorphism μ, which is given by $\mu^{-1}(y) = \langle \xi y, \xi \rangle_D$ for $y \in M_q \otimes A_{\theta'}$, where $\langle \xi, \xi \rangle_{D^\perp} = 1$, one obtains (2.15).

In [18] we had the following notation

$$
\begin{aligned}
r_1 &= -p_1 - 2\gamma p_3 + bp_4, & s_1 &= p_1 - 2ap_3 - bp_4,\\
r_2 &= p_2 + 2ap_4 - bp_3, & s_2 &= p_2 - 2\gamma p_4 - bp_3,\\
r_3 &= p_3 - 2\gamma p_1 + bp_2, & s_3 &= p_3 + 2ap_1 + bp_2,\\
r_4 &= -p_4 + 2ap_2 - bp_1, & s_4 &= p_4 + 2\gamma p_2 + bp_1.
\end{aligned} \tag{2.18}
$$

Then

$$\Delta := r_1r_4 - r_2r_3 = s_1s_4 - s_2s_3. \tag{2.19}$$

We showed that there exist integers a, b, γ such that Δ is relatively prime to q (Proposition 3.1 of [18]), that $\langle\varphi, \varphi\rangle_{D_0} = I$, and $\langle\varphi, \varphi\rangle_{D_0^\perp} = I$ (both being the identity matrix of M_q). (We note that the covolume here is $|G_0/D_0| = 1$.)

The algebra $C^*(D_0^\perp, \bar{\mathfrak{h}}) \cong M_q$ is generated by the unitaries V_3, V_4 as above. On each of these algebras we have respective Fourier transforms $\sigma_0(\pi_x) = \overline{\mathfrak{h}(x, x)}\pi_{R_0x}$ for $x \in D_0$, and $\sigma_0'(\pi_y) = \overline{\mathfrak{h}(y, y)}\pi_{R_0y}$ for $y \in D_0^\perp$, where $R_0(u; v) = (-v; u)$ for $u, v \in M_0$. They satisfy the properties

$$\sigma_0(\langle\phi_1, \phi_2\rangle_{D_0}) = \langle\widehat{\phi_1}, \widehat{\phi_2}\rangle_{D_0}, \qquad \sigma_0'(\langle\phi_1, \phi_2\rangle_{D_0^\perp}) = \langle\widehat{\phi_1}, \widehat{\phi_2}\rangle_{D_0^\perp}. \tag{2.20}$$

If τ_0, τ_0' are the canonical normalized traces on $C^*(D_0, \mathfrak{h})$ and $C^*(D_0^\perp, \bar{\mathfrak{h}})$, respectively, then $\tau_0(\langle\phi_1, \phi_2\rangle_{D_0}) = \tau_0'(\langle\phi_2, \phi_1\rangle_{D_0^\perp})$, for $\phi_j \in \mathscr{S}(M_0)$ (all complex functions on M_0, which in the general case is Schwartz space), since we have $|G_0/D_0| = 1$.

We have the following formulas that we will take the liberty of using below.

$$\widehat{bf} = \sigma_0(b)\widehat{f}, \qquad \widehat{fa} = \widehat{f}\sigma_0'(a), \tag{2.21}$$

$$\tau_0(\langle f_1, f_2\rangle_{D_0}\langle f_3, f_4\rangle_{D_0}) = \tau_0'(\langle f_4, f_1\rangle_{D_0^\perp}\langle f_2, f_3\rangle_{D_0^\perp}) \tag{2.22}$$

for $f, f_j \in \mathscr{S}(M_0)$, $a \in C^*(D_0, \mathfrak{h})$, $b \in C^*(D_0^\perp, \bar{\mathfrak{h}})$.

We also have

$$\sigma'(W_0) = \sigma_0'(W_0) = W_0^* \tag{2.23}$$

which easily follows from (2.20), the definition of W_0, $\widehat{\widehat{\varphi}} = \varphi$, and (2.9). Further, note that we have $\sigma' \doteq \sigma_0' \doteq \alpha_J$ on M_q.

In the later sections of [18] we confined ourselves to the special case that θ is in a dense G_δ set of irrationals with the property that there are infinitely many rationals p/q such that $|q(q\theta - p)| < 1$ where $p = p_1^2$ is a perfect square and q is divisible by 4. This helped to simplify the calculation of W_0 and its relation with V_3, V_4, since it enabled us to take $a = \gamma = 0$ and $b = 1$ so that we could take $\varphi(n, m) = q^{-1/2}e\big(\frac{1}{q}nm\big)$, making the automorphism σ_1 equal to the inverse of σ'. However, without making such restrictions, and allowing for general constants a, b, γ, and for arbitrary p, one cannot in general expect exact equality of σ_1 with the inverse of σ'. The best one can say is that σ_1 is conjugate to the inverse σ' (which we prove in this paper), and this will meet our purposes.

Finally, we recollect the unbounded trace functionals that give us the topological invariants in both the flip and Fourier cases.

For the flip automorphism case the unbounded traces are given on generic vectors $U^m V^n$ by (see [14] or [13])

$$\phi_{ij}(U^m V^n) = \lambda^{-mn/2}\delta_2^{i-m}\delta_2^{j-n} \tag{2.24}$$

for $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1)$, $m, n \in \mathsf{Z}$, and δ_d^k is the divisor function (where $VU = \lambda UV$). These define linear functionals on the canonical smooth *-subalgebra A_θ^∞. They are ϕ-invariant and satisfy the ϕ-trace property $\phi_{ij}(xy) = \phi_{ij}(\phi(y)x)$ for all x, y in A_θ^∞. Clearly, on the fixed point subalgebra of A_θ^∞ under the flip they are traces. On the crossed product C*-algebra $A_\theta \rtimes_\phi \mathsf{Z}_2$ they induce densely defined trace functionals given by

$$T_{ij}(a + bW) = \phi_{ij}(b) \tag{2.25}$$

where $a, b \in A_\theta^\infty$ and W is the (order 2) canonical unitary of the crossed product. It was shown in [13] that these traces and the canonical trace induce an injective homomorphism on K_0:

$$K_0(A_\theta \rtimes_\phi \mathsf{Z}_2) \to \mathsf{R}^5, \quad x \mapsto (\tau(x); T_{00}(x), T_{01}(x), T_{10}(x), T_{11}(x)) \tag{2.26}$$

(see Proposition 3.2 of [13]). By transforming the situation to the fixed point algebra (using the canonical Morita equivalence between the fixed point algebra and the crossed product) one easily sees that we have the injective homomorphism on $K_0(A_\theta^\phi)$:

$$\begin{aligned} &\mathbf{T}_2 : K_0(A_\theta^\phi) \to \mathsf{R}^5, \\ &\qquad \mathbf{T}_2(x) = (\tau(x); \phi_{00}(x), \phi_{01}(x), \phi_{10}(x), \phi_{11}(x)). \end{aligned} \tag{2.27}$$

For the Fourier case we have the twisted trace maps

$$\psi_{20}(U^m V^n) = \lambda^{-mn/2}\delta_2^m\delta_2^n, \qquad \psi_{10}(U^m V^n) = \lambda^{(m-n)^2/4}\delta_2^{m-n}, \tag{2.28}$$

$$\psi_{21}(U^m V^n) = \lambda^{-mn/2}\delta_2^{m-1}\delta_2^{n-1}, \quad \psi_{11}(U^m V^n) = \lambda^{(m-n)^2/4}\delta_2^{m-n-1}, \tag{2.29}$$

$$\psi_{22}(U^m V^n) = \lambda^{-mn/2}\,\delta_2^{m-n-1}, \tag{2.30}$$

where ψ_{1j} are σ-invariant σ-traces and ψ_{2j} are σ-invariant σ^2-traces. By the same token as the flip case one has the associated injective group homomorphism

$$\mathbf{T}_4 : K_0(A_\theta^\sigma) \to \mathsf{C}^6,$$
$$\mathbf{T}_4(x) = (\tau(x); \psi_{10}(x), \psi_{11}(x); \psi_{20}(x), \psi_{21}(x), \psi_{22}(x))$$

This map was shown to be injective in [15] for a dense G_δ set of θ's (that includes the rationals), but since by [5] or [10] one has $K_0(A_\theta^\sigma) \cong \mathsf{Z}^9$ for all θ the map is injective for all irrational θ. By the cancellation theorem for A_θ^σ (see Proposition 5.2 of [8]) one knows that two projections a and b in A_θ^σ are unitarily equivalent by a unitary in A_θ^σ if and only if $\mathbf{T}_4(a) = \mathbf{T}_4(b)$.

3. The Flip Case

From the Preliminaries we have a Fourier invariant projection e in A_θ of trace $q(q\theta - p)$ and an isomorphism

$$\mu : eA_\theta e \to M_q \otimes A_{\theta'}$$

satisfying $\mu\sigma = (\sigma_1 \otimes \sigma_2)\mu$ where σ_1 and σ_2 are automorphisms of order 4 on M_q and $A_{\theta'}$, respectively, given by (2.12) and (2.13) (note $\sigma_2 = \sigma'$ on $A_{\theta'}$). From this intertwining relation one has $\mu\sigma^2 = (\sigma_1^2 \otimes \sigma_2^2)\mu$ or

$$\mu\phi = (\phi_1 \otimes \phi_2)\mu \tag{3.1}$$

where ϕ_1 is the flip $u \mapsto u^{-1}$, $v \mapsto v^{-1}$ on M_q and ϕ_2 the flip on $A_{\theta'}$. Note that $\sigma_1^2 = \phi_1$ follows from (2.23). Indeed, from (2.12) and (2.23) one has

$$\begin{aligned}\sigma_1^2(x) &= W_0^*\sigma'(W_0^*\sigma'(x)W_0)W_0 = W_0^*\sigma'(W_0^*)\sigma'^2(x)\sigma'(W_0)W_0 \\ &= \sigma'^2(x) = \phi_1(x).\end{aligned} \tag{3.2}$$

From (3.1) one can obtain flip-orthogonal projections in $eA_\theta e$ (and hence in A_θ, since e is flip invariant) from ϕ_1-orthogonal projections in M_q.

We can faithfully represent the unitaries u, v by the standard matrices

$$u = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_0^{q-1} \end{bmatrix}, \qquad v = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.3}$$

($vu = \lambda_0 uv$ where $\lambda_0 = e^{2\pi i p/q}$) and the flip automorphism ϕ_1 can be represented by $\mathrm{Ad}_w = w(\ )w^*$ where w is the order 2 unitary

$$w = \left[\delta_q^{i+j}\right]_{i,j=0}^{q-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.4}$$

REMARK 3.1. We can now quickly see that flip-orthogonal projections exist in A_θ. Pick any column unit vectors a and b (in $\mathbb{C}^q$) such that $wa = a$ and $wb = -b$ (as the linear transformation $v \to wv$ has eigenvalues ± 1). Let $y = \frac{1}{\sqrt{2}}(a+b)$. Then it is clear that $y^*wy = 0$, so that $yy^*wyy^* = 0$, and the projection $f' = yy^*$ in M_q is such that $f'\phi_1(f') = 0$. Thus $f = \mu^{-1}(f' \otimes 1)$ is a non-zero projection in A_θ such that $f\phi(f) = 0$.

We now turn to the proof of Theorem 1.2.

It is easy to obtain the dimensions of the eigenspaces of eigenvalue 1 and -1 for w. Suppose $a = [a_0, a_1, \ldots, a_{q-1}]^T$ (transpose of a row vector) is an eigenvector of w of eigenvalue 1: $wa = a$. Then since

$$wa = [a_0, a_{q-1}, a_{q-2}, \ldots, a_2, a_1]^T \tag{3.5}$$

one has $a_{q-j} = a_j$ for $j = 1, 2, \ldots, q-1$ and a_0 is arbitrary. The eigendimensions will depend on the parity of q.

If q is even, then $a_{q/2}$ is the midpoint of the sequence $a_1, \ldots, a_{q-1}$. Hence

$$a = [a_0, a_1, \ldots, a_{q/2-1}, a_{q/2}, a_{q/2-1}, \ldots, a_1]^T \tag{3.6}$$

so that the $(+1)$-eigenspace is $1 + \frac{q}{2}$ dimensional. Consequently, the (-1)-eigenspace will be $\frac{q}{2} - 1$ dimensional.

If q is odd, then the sequence $a_1, \ldots, a_{q-1}$ has no midpoint but becomes

$$a_1, \ldots, a_{(q-1)/2}, a_{(q-1)/2}, \ldots a_1,$$

so it contains $\frac{1}{2}(q-1)$ independent parameters. Thus the $(+1)$-eigenspace has dimension $1 + \frac{1}{2}(q-1) = \frac{1}{2}(q+1)$, and the (-1)-eigenspace is $\frac{1}{2}(q-1)$ dimensional.

Using our notation for $\widetilde{q}$, in either parity case the (-1)-eigenspace of w has dimension $\widetilde{q}$ and the $(+1)$-eigenspace has dimension $q - \widetilde{q}$ which is at least $\widetilde{q}$. (Note $2\widetilde{q} < q$.)

Now fix $k \le \widetilde{q}$. One picks an orthonormal set of k (column) vectors $a^1, \ldots, a^k$ in the $(+1)$-eigenspace of w and an orthonormal set of k vectors

$b^1, \dots, b^k$ in the (-1)-eigenspace of w. Let $y_j = \frac{1}{\sqrt{2}}(a^j + b^j)$. Then $y_j^* y_\ell = \delta_{j\ell}$, so that $y_j y_j^*$ are mutually orthogonal projections of rank 1, and $y_j^* w y_k = 0$ for all j, k, so that $y_j y_j^* w y_k y_k^* = 0$. Letting $f' = y_1 y_1^* + \dots + y_k y_k^* \in M_q$ one has a projection of rank k such that $f' \phi_1(f') = 0$. This yields the projection $f = \mu^{-1}(f' \otimes 1)$ in A_θ has trace $k(q\theta - p)$ and is flip-orthogonal. This proves Theorem 1.2.

We now turn to the proof of Theorem 1.3, and begin with a flip-invariant projection g in A_θ whose topological invariants vanish and has "even" trace $\tau(g) = 2k(q\theta - p)$, where $k \le \widetilde{q}$. Thus,

$$\mathbf{T}_2(g) = (2k(q\theta - p); 0, 0, 0, 0).$$

Since $k \le \widetilde{q}$, Theorem 1.2 gives us a flip-orthogonal projection f whose trace is $k(q\theta - p)$. As we noted above, since f is flip-orthogonal one has $\phi_{jk}(f) = 0$ for all j, k, so that the projection $h = f + \phi(f)$ is flip-invariant, has vanishing topological invariants, and trace $2k(q\theta - p)$. That is, $\mathbf{T}_2(h) = \mathbf{T}_2(g)$. Since the fixed point C*-algebra A_θ^ϕ has cancellation by Corollary 5.6 of [13], it follows that g is unitarily equivalent to $h = f + \phi(f)$ by a unitary that is flip invariant. Therefore, $g = z + \phi(z)$ where z is a flip-orthogonal projection. This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.5

In this section we prove the following Proposition which proves Theorem 1.5 in view of the setup of the Preliminaries.

PROPOSITION 4.1. *The automorphism $\sigma_1(x) = W_0^* \sigma'(x) W_0$ of $M_q = C^*(V_3, V_4)$ is conjugate to σ'^{-1} in* $\mathrm{Aut}(M_q)$.

PROOF. We consider instead the automorphism $\eta(x) = W_0 \sigma'(x) W_0^*$, and observe that η is conjugate to σ_1. Indeed, this follows from the equality $\sigma' \eta = \sigma_1 \sigma'$. Therefore, the Proposition follows once we show that η is conjugate to σ'^{-1}.

Fix $j = 3, 4$ and for simplicity, let $x = V_4^{-\ell} V_3^{-k}$ where k, ℓ are arbitrary. Since $W_0 = \langle \varphi, \widehat{\varphi} \rangle_{D_0^\perp}$, and using (2.10) and (2.20), we have

$$\begin{aligned}
\tau'(W_0 V_j W_0^* \cdot V_4^{-\ell} V_3^{-k}) &= \tau'(\langle \varphi, \widehat{\varphi} \rangle_{D_0^\perp} V_j \langle \widehat{\varphi}, \varphi \rangle_{D_0^\perp} x) && (4.1)\\
&= \tau'(x \bullet \langle \widehat{\varphi}, \varphi \rangle_{D_0^\perp} \bullet V_j \bullet \langle \varphi, \widehat{\varphi} \rangle_{D_0^\perp}) \\
&= \tau'(\langle \widehat{\varphi} x^*, \varphi V_j \rangle_{D_0^\perp} \langle \varphi, \widehat{\varphi} \rangle_{D_0^\perp}) \\
&= \tau(\langle \varphi V_j, \varphi \rangle_{D_0} \langle \widehat{\varphi}, \widehat{\varphi} x^* \rangle_{D_0}) \\
&= \tau(\langle \varphi V_j, \varphi \rangle_{D_0} \sigma_0(\langle \varphi, \varphi y \rangle_{D_0}) && (4.2)
\end{aligned}$$

where y is such that $\sigma_0'(y) = x^*$, i.e. $y = V_4^{-k} V_3^{\ell} \simeq \pi_{-\ell\delta_3+k\delta_4}$.

Since V_j and y are generic unitaries, this calculation shows that we need to compute inner products of the form $\langle \varphi, \varphi\pi_\delta \rangle_{D_0} = \langle \varphi, \pi_\delta(\varphi) \rangle_{D_0}$ for general $\delta \in D_0^\perp$. This we do now, where, as in [18] (Lemma 4.1), φ is given by (2.16) (and the integer constants a, b, γ satisfy the constraint that Δ is relatively prime to q).

We have

$$\begin{aligned}
&\langle \varphi, \pi_\delta(\varphi) \rangle_{D_0}(m\varepsilon_1 + n\varepsilon_2) \\
&\quad = \langle \varphi, \pi_\delta(\varphi) \rangle_{D_0}(mp_1 - np_3, mp_2 - np_4; mp_3 + np_1, mp_4 + np_2) \\
&\quad = \frac{1}{q} \sum_{r,s=0}^{q-1} \varphi(r, s) \overline{\pi_\delta(\varphi)(r + mp_1 - np_3, s + mp_2 - np_4)} \\
&\qquad\qquad \cdot e\big(-\tfrac{1}{q}[r(mp_3 + np_1) + s(mp_4 + np_2)]\big).
\end{aligned}$$

Writing $\delta = (c_1, c_2; c_3, c_4) \in D_0^\perp$ one has

$$\begin{aligned}
&\varphi(r, s)\overline{\pi_\delta(\varphi)(r + \alpha, s + \beta)} \\
&\quad = \frac{1}{q} e\big(\tfrac{1}{q}[ar^2 + brs + \gamma s^2]\big) e\big(-\tfrac{1}{q}[c_3(r + \alpha) + c_4(s + \beta)]\big) \\
&\qquad \cdot e\big(-\tfrac{1}{q}[a(r + \alpha + c_1)^2 + b(r + \alpha + c_1)(s + \beta + c_2) + \gamma(s + \beta + c_2)^2]\big) \\
&\quad = \frac{1}{q} e\big(-\tfrac{1}{q}[Rr + Ss + C]\big)
\end{aligned}$$

where

$$R = c_3 + 2a(\alpha + c_1) + b(\beta + c_2) \tag{4.3}$$

$$S = c_4 + 2\gamma(\beta + c_2) + b(\alpha + c_1) \tag{4.4}$$

$$C = c_3\alpha + c_4\beta + a(\alpha + c_1)^2 + b(\alpha + c_1)(\beta + c_2) + \gamma(\beta + c_2)^2 \tag{4.5}$$

are independent of r, s and

$$\alpha = mp_1 - np_3, \qquad \beta = mp_2 - np_4. \tag{4.6}$$

Thus,

$$\begin{aligned}
&\langle \varphi, \pi_\delta(\varphi) \rangle_{D_0}(m\varepsilon_1 + n\varepsilon_2) \\
&\quad = \frac{1}{q^2} \sum_{r,s=0}^{q-1} e\big(-\tfrac{1}{q}[Rr + Ss + C]\big) e\big(-\tfrac{1}{q}[r(mp_3 + np_1) + s(mp_4 + np_2)]\big) \\
&\quad = \delta_q^{R+mp_3+np_1} \delta_q^{S+mp_4+np_2} e\big(-\tfrac{C}{q}\big).
\end{aligned}$$

Here, note that

$$\begin{aligned} R + mp_3 + np_1 &= s_3 m + s_1 n + c_3 + 2ac_1 + bc_2 \\ S + mp_4 + np_2 &= s_4 m + s_2 n + c_4 + 2\gamma c_2 + bc_1 \end{aligned}$$

where the s_j are given by (2.18).

As the coefficent matrix of the system in m, n

$$\begin{bmatrix} s_4 & s_2 \\ s_3 & s_1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \equiv \begin{bmatrix} -c_4 - 2\gamma c_2 - bc_1 \\ -c_3 - 2ac_1 - bc_2 \end{bmatrix} \tag{4.7}$$

has determinant $\Delta = s_1 s_4 - s_2 s_3$ relatively prime to q, it has a unique solution for m, n mod q. This means that the inner product $\langle \varphi, \pi_\delta(\varphi) \rangle_{D_0}$ is a generic unitary, i.e.

$$\langle \varphi, \pi_\delta(\varphi) \rangle_{D_0} \doteq \pi_{m\varepsilon_1 + n\varepsilon_2} \tag{4.8}$$

where m, n is the unique solution of (4.7) modulo q. From this one gets

$$\langle \varphi V_j, \varphi \rangle_{D_0} = \langle \varphi, \varphi V_j^* \rangle_{D_0} \doteq \langle \varphi, \pi_{\delta_j}(\varphi) \rangle_{D_0} \doteq \pi_{m_j \varepsilon_1 + n_j \varepsilon_2} \tag{4.9}$$

where m_j, n_j satisfy (4.7) for δ_j, $j = 3, 4$. For $j = 3$ one has $\delta_3 = (p_2, -p_1; -p_4, p_3)$ (mod q class) so that the system (4.7) becomes

$$\begin{bmatrix} s_4 & s_2 \\ s_3 & s_1 \end{bmatrix} \begin{bmatrix} m_3 \\ n_3 \end{bmatrix} \equiv \begin{bmatrix} -r_3 \\ -r_4 \end{bmatrix} \tag{4.10}$$

for some integers m_3, n_3 (where the r_j are given in (2.18)). Similarly, for $\delta_4 = (p_4, -p_3; p_2, -p_1)$ there are integers m_4, n_4 such that

$$\begin{bmatrix} s_4 & s_2 \\ s_3 & s_1 \end{bmatrix} \begin{bmatrix} m_4 \\ n_4 \end{bmatrix} \equiv \begin{bmatrix} -r_1 \\ -r_2 \end{bmatrix}. \tag{4.11}$$

Now in the above we had $y \doteq \pi_{-\ell\delta_3 + k\delta_4} = \pi_\delta$ where

$$\delta = -\ell\delta_3 + k\delta_4 = (kp_4 - \ell p_2, -kp_3 + \ell p_1; kp_2 + \ell p_4, -kp_1 - \ell p_3) \tag{4.12}$$

and where k, ℓ are arbitrary at this point. Its associated system is

$$s_4 m + s_2 n \equiv (kp_1 + \ell p_3) - 2\gamma(-kp_3 + \ell p_1) - b(kp_4 - \ell p_2) \tag{4.13}$$

$$s_3 m + s_1 n \equiv -(kp_2 + \ell p_4) - 2a(kp_4 - \ell p_2) - b(-kp_3 + \ell p_1) \tag{4.14}$$

or

$$s_4 m + s_2 n \equiv -(-p_1 - 2\gamma p_3 + b p_4)k + (p_3 - 2\gamma p_1 + b p_2)\ell \equiv -r_1 k + r_3 \ell \tag{4.15}$$

$$s_3 m + s_1 n \equiv -(p_2 + 2a p_4 - b p_3)k + (-p_4 + 2a p_2 - b p_1)\ell \equiv -r_2 k + r_4 \ell \tag{4.16}$$

or

$$\begin{bmatrix} s_4 & s_2 \\ s_3 & s_1 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} \equiv \begin{bmatrix} -r_1 & r_3 \\ -r_2 & r_4 \end{bmatrix}\begin{bmatrix} k \\ \ell \end{bmatrix} \tag{4.17}$$

the m, n solution of which gives

$$\langle \varphi, \varphi y\rangle_{D_0} \doteqdot \langle \varphi, \pi_{-\ell\delta_3 + k\delta_4}(\varphi)\rangle_{D_0} \doteqdot \pi_{m\varepsilon_1 + n\varepsilon_2} \tag{4.18}$$

hence

$$\sigma_0(\langle \varphi, \varphi y\rangle_{D_0}) \doteqdot \sigma_0(\pi_{m\varepsilon_1 + n\varepsilon_2}) \doteqdot \pi_{-n\varepsilon_1 + m\varepsilon_2}. \tag{4.19}$$

Combining these with (4.2) one gets

$$\tau'(W_0 V_j W_0^* V_4^{-\ell} V_3^{-k}) = \tau(\langle \varphi V_j, \varphi\rangle_{D_0} \sigma_0(\langle \varphi, \varphi y\rangle_{D_0})) \doteqdot \tau(\pi_{m_j\varepsilon_1 + n_j\varepsilon_2}\pi_{-n\varepsilon_1 + m\varepsilon_2}) \tag{4.20}$$

$$\doteqdot \tau(\pi_{(m_j - n)\varepsilon_1 + (n_j + m)\varepsilon_2}) \tag{4.21}$$

$$\doteqdot \delta_q^{m_j - n}\delta_q^{n_j + m} \tag{4.22}$$

which is non-zero if and only if

$$\begin{bmatrix} m \\ n \end{bmatrix} \equiv \begin{bmatrix} -n_j \\ m_j \end{bmatrix}. \tag{4.23}$$

These show that

$$W_0 V_j W_0^* \doteqdot V_3^k V_4^\ell \tag{4.24}$$

for $j = 3, 4$ where (k, ℓ) is the unique solution of (4.17) associated with $(m, n) \equiv (-n_j, m_j)$. For $j = 3$,

$$\begin{bmatrix} m \\ n \end{bmatrix} \equiv \begin{bmatrix} -n_3 \\ m_3 \end{bmatrix} \tag{4.25}$$

so that (4.17) becomes

$$\begin{bmatrix} s_4 & s_2 \\ s_3 & s_1 \end{bmatrix} \begin{bmatrix} -n_3 \\ m_3 \end{bmatrix} \equiv \begin{bmatrix} -r_1 & r_3 \\ -r_2 & r_4 \end{bmatrix} \begin{bmatrix} k_3 \\ \ell_3 \end{bmatrix} \tag{4.26}$$

for some unique k_3, ℓ_3. Similarly, for $j = 4$ one has integers k_4, ℓ_4 such that

$$\begin{bmatrix} s_4 & s_2 \\ s_3 & s_1 \end{bmatrix} \begin{bmatrix} -n_4 \\ m_4 \end{bmatrix} \equiv \begin{bmatrix} -r_1 & r_3 \\ -r_2 & r_4 \end{bmatrix} \begin{bmatrix} k_4 \\ \ell_4 \end{bmatrix}. \tag{4.27}$$

We thus have

$$W_0 V_3 W_0^* \doteq V_3^{k_3} V_4^{\ell_3}, \qquad W_0 V_4 W_0^* \doteq V_3^{k_4} V_4^{\ell_4}. \tag{4.28}$$

This gives

$$\eta(V_3) \doteq V_3^{k_4} V_4^{\ell_4}, \qquad \eta(V_4) \doteq V_3^{-k_3} V_4^{-\ell_3} \tag{4.29}$$

so that η has matrix $\begin{bmatrix} k_4 & -k_3 \\ \ell_4 & -\ell_3 \end{bmatrix} = KJ$ where K, J are defined below in (4.31).

The congruences (4.26) and (4.27) can be combined together by writing

$$SJM \equiv RK \tag{4.30}$$

where

$$\begin{aligned} S &= \begin{bmatrix} s_4 & s_2 \\ s_3 & s_1 \end{bmatrix}, \quad R = \begin{bmatrix} -r_1 & r_3 \\ -r_2 & r_4 \end{bmatrix}, \quad M = \begin{bmatrix} m_3 & m_4 \\ n_3 & n_4 \end{bmatrix}, \\ K &= \begin{bmatrix} k_3 & k_4 \\ \ell_3 & \ell_4 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{4.31}$$

Also, (4.10) and (4.11) can be expressed by the congruence

$$SM \equiv -\begin{bmatrix} r_3 & r_1 \\ r_4 & r_2 \end{bmatrix} = -RJ. \tag{4.32}$$

We now eliminate M by substituting it from (4.32) into (4.30). To do this note that S is in $\mathrm{GL}(2, \mathsf{Z}_q)$ since its determinant Δ is relatively prime to q. (It's inverse S^{-1} in $\mathrm{GL}(2, \mathsf{Z}_q)$ has determinant congruent to Δ', where $\Delta'\Delta \equiv 1 \bmod q$.) Thus we have $M \equiv -S^{-1}RJ$, and (4.30) gives

$$K \equiv -R^{-1}SJS^{-1}RJ. \tag{4.33}$$

The matrix of η is therefore (since $J^2 = -I$)

(4.34)
$$\Sigma_1 := \begin{bmatrix} k_4 & -k_3 \\ \ell_4 & -\ell_3 \end{bmatrix} = KJ \equiv R^{-1}SJS^{-1}R = (R^{-1}ST)\cdot J^{-1}\cdot(R^{-1}ST)^{-1}$$

where $T = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and the matrix $R^{-1}ST$ is in $\mathrm{SL}(2, \mathsf{Z}_q)$. (Note: $TJ^{-1}T = J$.) From this we see that Σ_1 is in $\mathrm{SL}(2, \mathsf{Z}_q)$ and is conjugate to J^{-1} in $\mathrm{SL}(2, \mathsf{Z}_q)$.

By Lemma 2.2 we may lift the matrix $R^{-1}ST$ to a matrix Q in $\mathrm{SL}(2, \mathsf{Z})$, so that $\Sigma := QJ^{-1}Q^{-1}$ is a preimage in $\mathrm{SL}(2, \mathsf{Z})$ for Σ_1.

From $Q^{-1}\Sigma Q = J^{-1}$, and in view of the canonical homomorphism $\alpha : \mathrm{SL}(2, \mathsf{Z}) \to \mathrm{SL}(2, \mathsf{Z}_q)$ mentioned in the Preliminaries, we see that η is conjugate to the automorphism $\alpha_Q^{-1}\eta\alpha_Q =: \eta'$ which has the form

$$\eta'(V_3) = \mu V_4^{-1}, \qquad \eta'(V_4) = \nu V_3 \tag{4.35}$$

for some phase constants μ, ν. Now it is easy to see that η' is conjugate to $\alpha_{J^{-1}} = \sigma'^{-1}$, the (inverse) Fourier transform on M_q (with respect to the pair V_3, V_4). Since η^2 is just the flip ($V_j \mapsto V_j^{-1}$, $j = 3, 4$) the same is the case for $(\eta')^2$, which means that $\mu = \nu = \pm 1$. If $\mu = \nu = 1$ we are done. If $\mu = \nu = -1$ the automorphism $\zeta(V_3) = -V_4$, $\zeta(V_4) = V_3^{-1}$ satisfies $\zeta\eta' = \alpha_{J^{-1}}\zeta$ so that η' is still conjugate to $\alpha_{J^{-1}} = \sigma'^{-1}$.

This proves that η, and hence σ_1, is conjugate to σ'^{-1} (the inverse Fourier transform on M_q) in $\mathrm{Aut}(M_q)$.

The proof of Theorem 1.5 now follows:

From the Preliminaries we have the isomorphism $\mu : eA_\theta e \to M_q \otimes A_{\theta'}$ satisfying $\mu\sigma = (\sigma_1 \otimes \sigma')\mu$. By Proposition 4.1, $\sigma_1 = \beta\sigma'^{-1}\beta^{-1}$ for some automorphism β of M_q. Letting $\eta = (\beta^{-1} \otimes id)\mu$ and Σ the Fourier transform of M_q given by $\Sigma(u) = v$, $\Sigma(v) = u^*$, we obtain $\eta\sigma = (\Sigma \otimes \sigma')\eta$, which is (1.3), and proves Theorem 1.5. (The smoothness of η follows directly from that of μ, which is smooth by its construction in [18].)

5. The Fourier Case

We are now ready to prove Theorem 1.7 (as well as Theorem 1.6), and so we begin with a Fourier invariant projection g in A_θ whose (Fourier) topological invariants vanish, and has "quartic" trace $\tau(g) = 4k(q\theta - p)$, where $k \le \widehat{q}$.

Recall: $\widehat{q} = \frac{1}{4}(q - r) \in \mathsf{Z}$ where $r \in \{1, 2, 3, 4\}$ is such that $q \equiv r \bmod 4$.

By hypothesis, we have $\mathbf{T}_4(g) = (4k(q\theta - p); 0, 0; 0, 0, 0)$. The proof will be complete once we have established the existence of a σ^*-orthogonal projection

f such that $\tau(f) = k(q\theta - p)$. For then, one has the Fourier invariant projection $\varepsilon = f + \sigma(f) + \sigma^2(f) + \sigma^3(f)$ such that $\mathbf{T}_4(\varepsilon) = \mathbf{T}_4(g)$, so that ε and g are unitarily equivalent by a Fourier invariant unitary (by the last paragraph of the Preliminaries). Hence g has the form asserted by Theorem 1.7.

We now claim, thanks to Theorem 1.5, that the existence of such σ^*-orthogonal projection f arises from σ'^*-orthogonal projections in M_q.

To show this, we use the isomorphism $\eta : eA_\theta e \to M_q \otimes A_{\theta'}$ satisfying the intertwining relation (1.3). From the latter, if f_1 is a projection in M_q that is Σ^*-orthogonal, then $\eta^{-1}(f_1 \otimes 1)$ is a σ^*-orthogonal projection in A_θ. Thus we need to construct a Σ^*-orthogonal projection f_1 in M_q of rank k (where $k \le \widehat{q}$). To do this, we use the matrix representation (3.3) for u, v with respect to which Σ is given by Ad_L where L is the order 4 unitary

$$L = \frac{1}{\sqrt{q}}\big[\lambda_0^{ij}\big]_{i,j=0}^{q-1} = \frac{1}{\sqrt{q}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^{q-1} \\ 1 & \lambda_0^2 & \lambda_0^4 & \cdots & \lambda_0^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_0^{q-1} & \lambda_0^{2(q-1)} & \cdots & \lambda_0^{(q-1)^2} \end{bmatrix}. \tag{5.1}$$

One has the commutation relations $Lv = u^{-1}L$, $Lu = vL$, $L^2 = w$ (and w is the symmetry matrix in (3.4)). Therefore, the action of Σ on u, v is the same as the action of $\mathrm{Ad}_L = L(\ )L^*$ on u, v, respectively.

From [7] (Section 5) the dimension of each of the eigenspaces of L (of eigenvalues $\pm 1, \pm i$) is at least $\widehat{q}$. In fact, $\widehat{q}$ is the smallest of their dimensions and corresponds to eigenvalue $-i$ or i.

Since $k \le \widehat{q}$, there exists an orthonormal set of k vectors $v_1(t), \dots, v_k(t)$ (viewed as column vectors in $\mathbb{C}^q$ of our matrix representation) in the eigenspace of L of eigenvalue $t \in \{\pm 1, \pm i\}$. Let

$$y_j = \tfrac{1}{2}\big[v_j(-i) + v_j(i) + v_j(-1) + v_j(1)\big]$$

for $j = 1, \dots, k$. Then $y_j^* y_\ell = \delta_{j\ell}$ so that $\{y_j y_j^*\}_{j=1}^k$ are mutually orthogonal projections of rank 1 and their sum

$$f_1 := y_1 y_1^* + \cdots + y_k y_k^*$$

is a rank k projection such that $f_1 L f_1 = 0$ and $f_1 L^2 f_1 = 0$, since $y_j^* L y_\ell = y_j^* L^2 y_\ell = 0$ for all j, ℓ. This means that f_1 is orthogonal to $\Sigma(f_1)$ and $\Sigma^2(f_1)$.

Using (2.15) the trace is

$$\tau(f) = \tau(\eta^{-1}(f_1 \otimes 1)) = q(q\theta - p)\tau'(f_1) \tag{5.2}$$
$$= q(q\theta - p) \cdot \frac{k}{q} = k(q\theta - p),$$

as required. This completes the proof of Theorem 1.7, and the existence of the projection f proves Theorem 1.6.

It now remains to prove Theorems 1.4 and 1.8 of the Introduction. But first we need a lemma.

LEMMA 5.1. *If $\omega^4 = 1$ and θ is irrational, there are smooth elements $x_1, \ldots, x_n$ in A_θ such that*

$$\sigma(x_j) = \omega x_j \qquad \textit{and} \qquad \sum_{j=1}^{n} x_j^* x_j = 1 \tag{5.3}$$

The analogous result holds for the flip automorphism ϕ and $\omega = \pm 1$.

PROOF. The proof is essentially Rieffel's normalization trick in Proposition 2.1 of [11], except that we have in addition the σ-covariance condition in (5.3), so we detail it out for our situation carefully. Let $E = \{x \in A_\theta : \sigma(x) = \omega x\}$.

First, let us show that if there are elements $x_1, \ldots, x_n$ in A_θ such that (5.3) holds (here n is fixed), then by approximating each x_j by a smooth element we obtain (5.3) with smooth x_j. Given $x \in E$ and $\epsilon > 0$ pick a smooth element $x' \in A_\theta$ such that $\|x - x'\| < \epsilon$, so that also $\|\omega^j x - \sigma^j(x')\| < \epsilon$. Letting $y = x' + \omega^{-1}\sigma(x') + \omega^{-2}\sigma^2(x') + \omega^{-3}\sigma^3(x')$ one checks that $y \in E$, $\|y - 4x\| < 4\epsilon$, and clearly y is smooth. Hence $y/4$ is within ϵ of x. Therefore, given $x_1, \ldots, x_n$ in A_θ enjoying (5.3), we can approximate x_j by a smooth element $x_j' \in E$ such that the corresponding sum $\sum_{j=1}^n (x_j')^* x_j' = a$ is a smooth positive element in A_θ^σ close to 1 and is therefore invertible. This gives $\sum_{j=1}^n (x_j' a^{-1/2})^* (x_j' a^{-1/2}) = 1$ where $x_j' a^{-1/2}$ is a smooth element in E, so that (5.3) holds with smooth elements.

Therefore, it suffices to show (5.3) without requiring that the elements x_j be smooth. Let J be the set of finite sums $\sum_j x_j^* y_j$ where $x_j, y_j \in E$. Clearly $J \subseteq A_\theta^\sigma$ and is easily seen to be a two-sided ideal in A_θ^σ which, since θ is irrational and A_θ^σ is simple, is therefore a dense ideal in A_θ^σ. Hence there are elements $x_j, y_j \in E$ ($i, j = 1, \ldots, n$ for some n) such that $\sum_j x_j^* y_j = d$ is close to 1, and hence is an invertible element in A_θ^σ. Replacing y_j by $y_j d^{-1} \in E$ we may assume $\sum_j x_j^* y_j = 1$ where $x_j, y_j \in E$. Writing the elements of $E^n = E \oplus \cdots \oplus E$ as column $n \times 1$ matrices, and setting $X = (x_1, \ldots, x_n)^T$

and $Y = (y_1, \ldots, y_n)^T$ (transposes), we can express the preceding sum as $X^*Y = 1$, the identity of A_θ, where $X, Y \in E^n$. The matrix $P = YX^*$ is clearly an idempotent in $M_n(A_\theta^\sigma)$. We can with no loss of generality assume P is a projection. (This is because P is similar to a projection Q, say $P = SQS^{-1}$ for some invertible S in $M_n(A_\theta^\sigma)$, and upon making the substitutions $X_1 = S^*X$ and $Y_1 = S^{-1}Y$ one gets $X_1^*Y_1 = 1$, $Y_1X_1^* = Q$, and $X_1, Y_1 \in E^n$.) Now as X^*X and Y^*Y are positive elements in A_θ^σ we can write $X^*X = a^*a$ and $Y^*Y = bb^*$ for some $a, b \in A_\theta^\sigma$. Letting $Z = Xb \in E^n$ one has $ZZ^* = P$ and $Z^*Z = c^*c$ where $c = ab \in A_\theta^\sigma$. To check the first of these, note that $P = P^* = XY^*$ so that we have

$$ZZ^* = Xbb^*X^* = (XY^*)(YX^*) = PP = P.$$

For the second, $Z^*Z = b^*X^*Xb = b^*a^*ab = c^*c$. The condition $X^*Y = 1$ gives $1 = X^*YX^*Y = X^*PY = X^*ZZ^*Y = X^*Xbb^*X^*Y = a^*abb^* = a^*cb^*$. As the C*-algebra A_θ is *finite* (where $xy = 1 \Rightarrow yx = 1$) this implies $c^*c = b^*(a^*c) = 1$. Therefore, $Z^*Z = 1$ and Z is in E^n so that we obtain (5.3).

Proof of Theorem 1.4. By Lemma 5.1, there are smooth elements $x_1, \ldots, x_n$ in A_θ such that

$$\phi(x_j) = -x_j \qquad \text{and} \qquad \sum_{j=1}^n x_j^* x_j = 1.$$

Consider the matrix projection $Q = [x_i g x_j^*]_{i,j=1}^n$ in $M_n(A_\theta^\phi)$ (and which is smooth if g is smooth). Clearly this projection has the same (canonical) trace as that of g. Its topological invariants given by a ϕ-trace $\phi_{k\ell}$ (defined on A_θ^∞ where $k, \ell = 0, 1$) are

$$\phi_{k\ell}(Q) = \sum_i \phi_{k\ell}(x_i g x_i^*) = -\sum_i \phi_{k\ell}(x_i^* x_i g) = -\phi_{k\ell}(g).$$

This means that $\phi_{k\ell}(g \oplus Q) = 0$ for all ϕ-traces so that its Connes-Chern character is $\mathbf{T}_2(g \oplus Q) = (2\tau(g); 0, 0, 0, 0)$.

Since by hypothesis $k \le \tilde{q}$, Theorem 1.2 gives us a smooth flip orthogonal projection z of trace $\tau(z) = k(q\theta - p) = \tau(g)$. Hence $\mathbf{T}_2(g \oplus Q) = \mathbf{T}_2(z + \phi(z))$. Since $\mathbf{T}_2$ is injective on $K_0(A_\theta^\phi)$ and A_θ^ϕ has the cancellation property, it follows that there exists a unitary w in $M_{n+1}(A_\theta^\phi)$ such that

$$w(g \oplus Q)w^* = (z + \phi(z)) \oplus \mathbf{O}_n$$

(where $\mathbf{O}_n$ is the zero n by n matrix). Since any subprojection of the right hand side is of the form $e' \oplus \mathbf{O}_n$ for some subprojection e' of $z + \phi(z)$, there are orthogonal subprojections e_1, e_2 of $z + \phi(z)$ such that

$$w(g \oplus \mathbf{O}_n)w^* = e_1 \oplus \mathbf{O}_n, \quad w(0 \oplus Q)w^* = e_2 \oplus \mathbf{O}_n, \quad e_1 + e_2 = z + \phi(z).$$

The first of these says that g and e_1 give the same class in $K_0(A_\theta^\phi)$, so that again by cancellation there is a unitary v in A_θ^ϕ such that

$$g = ve_1v^* \leq v(z + \phi(z))v^* = f + \phi(f)$$

where $f = vzv^*$ is a flip orthogonal projection satisfying the stated properties.

PROOF OF THEOREM 1.8. By Lemma 5.1, for $r = 0, 1, 2, 3$, there are smooth elements $x_{r1}, \dots, x_{rn_r}$ in A_θ such that

$$\sigma(x_{rj}) = i^r x_{rj} \qquad \text{and} \qquad \sum_{j=1}^{n_r} x_{rj}^* x_{rj} = 1 \tag{5.4}$$

(where $i = \sqrt{-1}$). (Of course, for $r = 0$ we take $n_0 = 1$ and $x_{01} = 1$.)

Consider the matrix projection $Q_r = [x_{rj} g x_{rk}^*]_{j,k=1}^{n_r}$ in $M_{n_r}(A_\theta^\sigma)$ (and which is smooth if g is smooth). By (5.4) this projection has the same (canonical) trace as that of g. Its topological invariants, given by the σ^s-trace $\psi_{s\ell}$ (defined on A_θ^∞ where $s = 1, 2$), are

$$\begin{aligned}\psi_{s\ell}(Q_r) &= \sum_j \psi_{s\ell}(x_{rj} g x_{rj}^*) = \sum_j \psi_{s\ell}(\sigma^s(x_{rj}^*) x_{rj} g) \\ &= i^{-rs} \sum_j \psi_{s\ell}(x_{rj}^* x_{rj} g) = i^{-rs} \psi_{s\ell}(g)\end{aligned}$$

which gives

$$\begin{aligned}&\mathbf{T}_4[Q_r] \\ &= (\tau(g);\, \psi_{10}(Q_r), \psi_{11}(Q_r);\, \psi_{20}(Q_r), \psi_{21}(Q_r), \psi_{22}(Q_r)) \\ &= (\tau(g);\; i^{-r}\psi_{10}(g), i^{-r}\psi_{11}(g);\, (-1)^r\psi_{20}(g), (-1)^r\psi_{21}(g), (-1)^r\psi_{22}(g)) \\ &= \Omega^{-r}\mathbf{T}_4[g]\end{aligned}$$

where $\Omega = (1; i, i; -1, -1, -1)$ (and we used coordinatewise multiplication). Adding over $r = 1, 2, 3$, gives

$$\mathbf{T}_4([g \oplus Q_1 \oplus Q_2 \oplus Q_3]) = (4\tau(g); 0, 0; 0, 0, 0).$$

Now since $\tau(g) = k(q\theta - p)$ and $k \leq \widehat{q}$, by Theorem 1.6 there exists a σ^*-orthogonal projection h such that $\tau(h) = \tau(g)$, so that

$$\begin{aligned}\mathbf{T}_4([h + \sigma(h) + \sigma^2(h) + \sigma^3(h)]) &= (4\tau(g); 0, 0; 0, 0, 0)\\ &= \mathbf{T}_4([g \oplus Q_1 \oplus Q_2 \oplus Q_3]).\end{aligned}$$

By injectivity of $\mathbf{T}_4$ on $K_0(A_\theta^\sigma)$, and the cancellation property of A_θ^σ, there is a unitary W in $M_{m+1}(A_\theta^\sigma)$ such that

$$W(g \oplus Q_1 \oplus Q_2 \oplus Q_3)W^* = (h + \sigma(h) + \sigma^2(h) + \sigma^3(h)) \oplus \mathbf{O}_m$$

where $m = n_1 + n_2 + n_3$. Since any subprojection of the right hand side is of the form $e' \oplus \mathbf{O}_m$ for some subprojection e' of $h + \sigma(h) + \sigma^2(h) + \sigma^3(h)$ in A_θ^σ, there are mutually orthogonal Fourier invariant subprojections e_0, e_1, e_2, e_3 of $h + \sigma(h) + \sigma^2(h) + \sigma^3(h)$ such that

$$\begin{aligned}W(g \oplus \mathbf{O} \oplus \mathbf{O} \oplus \mathbf{O})W^* &= e_0 \oplus \mathbf{O}_m\\ W(0 \oplus Q_1 \oplus \mathbf{O} \oplus \mathbf{O})W^* &= e_1 \oplus \mathbf{O}_m\\ W(0 \oplus \mathbf{O} \oplus Q_2 \oplus \mathbf{O})W^* &= e_2 \oplus \mathbf{O}_m\\ W(0 \oplus \mathbf{O} \oplus \mathbf{O} \oplus Q_3)W^* &= e_3 \oplus \mathbf{O}_m.\end{aligned}$$

The first of these says that the projections g and e_0 of A_θ^σ are stably equivalent so that again by cancellation they are unitarily equivalent, $g = w^* e_0 w$, by a unitary w in A_θ^σ. Thus we have

$$wgw^* + e_1 + e_2 + e_3 = h + \sigma(h) + \sigma^2(h) + \sigma^3(h)$$

or

$$g + g_1 + g_2 + g_3 = f + \sigma(f) + \sigma^2(f) + \sigma^3(f)$$

where $g_j = w^* e_j w$ are Fourier invariant and $f = w^* h w$ is σ^*-orthogonal. Clearly, $\mathbf{T}_4[g_r] = \Omega^{-r}\mathbf{T}_4[g]$.

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