MATH. SCAND. 107 (2010), 198–208

# ON THE WEAK DIFFERENTIABILITY OF $u \circ f^{-1}$

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## Abstract

Let $p \geq n-1$ and suppose that $f : \Omega \to \mathsf{R}^n$ is a homeomorphism in the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$. Further let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ where $q = \frac{p}{p-(n-1)}$ and for $q > n$ we also assume that $u$ is continuous. Then $u \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}(f(\Omega))$ and if we moreover assume that $f$ is a mapping of finite distortion, then $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega))$.

## 1. Introduction

In this paper we address the following issue. Suppose that $\Omega \subset \mathsf{R}^n$ is a domain, suppose that $f : \Omega \to \mathsf{R}^n$ is a homeomorphism in the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ (see Preliminaries for the definition) and $u$ is a function in $W^{1,q}_{\mathrm{loc}}(\Omega)$ for some properly chosen $q = q(p,n)$. Under which conditions can we then conclude that the composition satisfies $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega))$ or that $u \circ f^{-1}$ is weakly differentiable in some weaker sense?

Let us consider the following elementary example. Let us denote $h(x) = C(x) + x$, where $C(x)$ denotes the usual Cantor ternary function, and set $f(x_1, x_2, \dots, x_n) = [h^{-1}(x_1), x_2, \dots, x_n]$. It is easy to check that $f$ is a homeomorphism and $f$ is Lipschitz, but $f^{-1}$ fails the ACL condition and therefore $f^{-1} \notin W^{1,1}_{\mathrm{loc}}$. This shows that even the composition of the identity and an inverse of a Lipschitz mapping may not be weakly differentiable. However one may check that the inverse of this particular $f$ is weakly differentiable in the weaker sense, namely it has bounded variation $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$ (see Preliminaries for the definition). Surprisingly we show that the composition always satisfies $u \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$ if $f$ and $u$ are regular enough.

THEOREM 1.1. (i) *Let $n = 2$ and suppose that $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathsf{R}^2)$ is a homeomorphism and let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be continuous. Then $u \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}(f(\Omega))$.*

(ii) *Let $p \geq n-1$, $f : \Omega \to \mathsf{R}^n$ be a homeomorphism and let $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$. Further assume that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ where $q = \frac{p}{p-(n-1)}$ and for $q > n$ we moreover assume that $u$ is continuous. Then $u \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}(f(\Omega))$.*

* The author was supported in part by GAČR 201/06/P100 and in part by MSM 0021620839.
Received 14 May 2009.

Let us remind the reader that for $q > n$ there is always a continuous representative of $u \in W^{1,q}_{\mathrm{loc}}$. A picking of the correct representative of $u$ is needed otherwise $u \circ f^{-1}$ is not even necessarily measurable. If $p \geq n$ and thus $q \leq n$ then $f$ satisfies the Lusin $(N)$ condition (see [10]) and therefore the validity of the result for one representative implies the result for any representative.

Moreover it is possible to show that the composition satisfies $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}$ if we add the requirement that $f$ has finite distortion. We say the homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ has finite distortion if $J_f(x) \geq 0$ a.e. and $J_f(x) = 0 \Rightarrow |Df(x)| = 0$ a.e. (for basic properties and examples see e.g. [8]).

THEOREM 1.2. *Let $p \geq n-1$, $f : \Omega \to \mathsf{R}^n$ be a homeomorphism of finite distortion and let $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$. Further assume that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ where $q = \frac{p}{p-(n-1)}$ and for $q > n$ we moreover assume that $u$ is continuous. Then $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega))$.*

*Moreover for every $n \geq 2$, $p > n-1$ and $0 < \varepsilon < q-1$ there exists homeomorphism of finite distortion $f \in W^{1,p}((-1,1)^n, (-1,1)^n)$ and continuous function $u \in W^{1,q-\varepsilon}((-1,1)^n)$, $q = \frac{p}{p-(n-1)}$, such that $u \circ f^{-1} \notin W^{1,1}_{\mathrm{loc}}((-1,1)^n)$.*

Let us note that the positive part of our Theorem 1.2 was known before [14] under additional technical assumptions while Theorem 1.1 about BV regularity of the composition is entirely new. Moreover counterexamples in Theorem 1.2 showing sharpness are also new.

The additional requirements in [14] were that $p > n-1$ and that $f$ satisfies the Lusin $(N)$ and $(N^{-1})$ condition. In comparison with [14] we address the limiting case $p = n-1$ and $q = \infty$, we remove the unnecessary condition $(N)$ and we replace the $(N^{-1})$ condition by a weaker condition that $f$ has finite distortion. Let us recall that any homeomorphism $f \in W^{1,1}_{\mathrm{loc}}$ which satisfies $J_f(x) \geq 0$ a.e. and $(N^{-1})$ condition has finite distortion, but there is homeomorphism of finite distortion which fails the $(N^{-1})$ condition (see [9]).

From some recent results ([6], [12], [7], [2]) we already know that $f^{-1} \in W^{1,1}_{\mathrm{loc}}$ (or $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$ if we do not know that $f$ has finite distortion) and hence we can simplify and shorten the proof from [14]. We obtain that $u_i \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}$ (or $u_i \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$), where $u_i$ are the usual smooth convolution approximation of $u$. This fact comes with the uniform key estimate of the derivative of the composition and thus we can pass to the limit in the standard way to obtain our result.

Let us recall that for any $n \geq 3$ and $0 < \varepsilon < 1$ there exists a homeomorphism $f \in W^{1,n-1-\varepsilon}$ such that $f^{-1} \notin \mathrm{BV}_{\mathrm{loc}}$ (see [7, Example 3.1] or [4, Example 1.3]). Therefore we cannot expect any regularity of $u \circ f^{-1}$ for any $p$ below the natural exponent $n-1$. Moreover for every $n \geq 2$, $\delta > 0$ and

$p \in [1, \infty)$ there exists a homeomorphism of finite distortion $f \in W^{1,p}$ such that $f^{-1} \notin W^{1,1+\delta}_{\mathrm{loc}}$ (see [6, Example 6.1]). Therefore we cannot expect any higher Sobolev regularity of $u \circ f^{-1}$ than $W^{1,1}$.

There are a lot of known results about the Sobolev regularity of the composition $u \circ g$ (see e.g. [3], [13], [5] and references given there) but our results are different. We impose the condition on $g^{-1}$ rather than on the function $g$. In this way the assumption are easier, because we only assume some integrability of the derivative and not the integrability of some ratio of derivative and the jacobian. We are not aware of any result for BV regularity of $u \circ g$ similar to Theorem 1.1 (ii).

## 2. Preliminaries

### *2.1. Notation*

The Lebesgue measure of a set $A \subset \mathsf{R}^n$ is denoted by $\mathscr{L}_n(A)$. A mapping $f : \Omega \to \mathsf{R}^n$ is said to satisfy the Lusin condition $(N)$ if $\mathscr{L}_n(f(A)) = 0$ for every $A \subset \Omega$ such that $\mathscr{L}_n(A) = 0$. Analogously, $f$ is said to satisfy the Lusin condition $(N^{-1})$ if $\mathscr{L}_n(f^{-1}(A)) = 0$ for every $A \subset \mathsf{R}^n$ such that $\mathscr{L}_n(A) = 0$.

By $|\mu|$ we denote the total variation of the signed measure $\mu$.

### *2.2. Functions of bounded variation*

Let $\Omega \subset \mathsf{R}^n$ be open and $m \in \mathsf{N}$. A function $h \in L^1(\Omega)$ is of bounded variation, $h \in \mathrm{BV}(\Omega)$, if the distributional partial derivatives of $h$ are measures with finite total variation in $\Omega$: there are Radon (signed) measures $\mu_1, \dots, \mu_n$ defined in $\Omega$ so that for $i = 1, \dots, n$, $|\mu_i|(\Omega) < \infty$ and

$$\int_\Omega h(x) D_i\varphi(x)\, dx = -\int_\Omega \varphi(x)\, d\mu_i(x)$$

for all $\varphi \in C_0^\infty(\Omega)$. We say that $f \in L^1(\Omega, \mathsf{R}^m)$ belongs to $\mathrm{BV}(\Omega, \mathsf{R}^m)$ if the coordinate functions of $f$ belong to $\mathrm{BV}(\Omega)$. Analogously we define the Sobolev space: $f \in W^{1,p}(\Omega, \mathsf{R}^m)$ if $f \in L^p(\Omega, \mathsf{R}^m)$ and the distributional derivatives of the coordinate functions are in $L^p(\Omega, \mathsf{R}^n)$. Further, $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathsf{R}^m)$ (or $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^m)$) requires that $f \in \mathrm{BV}(\Omega', \mathsf{R}^m)$ (or $f \in W^{1,p}(\Omega', \mathsf{R}^m)$) for each open $\Omega' \subset\subset \Omega$. For an introduction to the theory of BV and $W^{1,p}$ spaces see [1], [15]. The function $h : \Omega \to \mathsf{R}^m$ is said to be a representative of $g : \Omega \to \mathsf{R}^m$ if $h = g$ almost everywhere with respect to Lebesgue measure.

Let $h \in \mathrm{BV}((0,1), \mathsf{R}^n)$ be a continuous one to one map. Variation coincides with the pointwise variation (see [1, Theorem 3.27]) and thus we can estimate the measure of each interval $(a, b) \subset (0, 1)$ by

$$|Dh|\big((a,b)\big) \le C\mathscr{H}_1\big(h((a,b))\big)$$

and therefore

$$\int_0^1 U(h(s))\, d|Dh|(s) \le \int_{h((0,1))} U(t)\, d\mathscr{H}_1(t) \tag{2.1}$$

for each nonnegative continuous function $U : h((0,1)) \to \mathsf{R}$.

*2.3. BVL and ACL condition*

It is a well-known fact (see e.g. [1, Section 3.11]) that a function $g \in L^1_{\mathrm{loc}}(\Omega)$ is in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ (or in $W^{1,1}_{\mathrm{loc}}(\Omega)$) if and only if there is a representative which has bounded variation (or is an absolutely continuous function) on almost all lines parallel to coordinate axes and the variation on these lines is integrable.

More precisely, let $i \in \{1, 2, \dots, n\}$, $Q_0 = (0,1)^n$ and by $\pi_i$ we denote the projection to the hyperplane perpendicular to $i$-th coordinate axis. For $y \in \pi_i(Q_0)$ we denote $g_{i,y}(t) = g(y + t\mathbf{e}_i)$. Let $g \in L^1_{\mathrm{loc}}(Q_0)$. Then $g \in \mathrm{BV}_{\mathrm{loc}}(Q_0)$ if and only if for every $i \in \{1, \dots, n\}$ the function $g_{i,y}(t) \in \mathrm{BV}\big((0,1)\big)$ for $\mathscr{L}_{n-1}$ almost every $y \in \pi_i(Q_0)$ and moreover

$$\int_{\pi_i(Q_0)} |Dg_{i,y}|\big((0,1)\big)\, d\mathscr{L}_{n-1}(y) < \infty.$$

In this case we can estimate the total variation of $Dg$ by

$$|Dg|(Q_0) \le C \sum_{i=1}^{n} \int_{\pi_i(Q_0)} |Dg_{i,y}|\big((0,1)\big)\, d\mathscr{L}_{n-1}(y). \tag{2.2}$$

## 3. Regularity of the composition

We will need the following version of the coarea formula (see [14, Lemma 3.2]).

LEMMA 3.1. *Let $p > n-1$ and let $f \in W^{1,p}(\Omega, \mathsf{R}^n)$ be a homeomorphism such that $f(\Omega) = (0,1)^n$. Set $\pi(x_1, \dots, x_n) = [x_2, \dots, x_n]$ and $S(x) = \pi \circ f(x)$. Then for every non-negative measurable function $h$ we have*

$$\int_{(0,1)^{n-1}} \int_{S^{-1}(y)} h(s)\, d\mathscr{H}_1(s)\, d\mathscr{L}_{n-1}(y) = \int_\Omega h(x) J_S(x)\, dx,$$

*where $J_S$ is the square root of the sums of the squares of the determinants of the $n-1$ by $n-1$ minors of the differential matrix of $S$.*

PROOF OF THEOREM 1.1. Let us first assume that $p = n-1$ and $q = \infty$. From [7, Theorem 1.1] and [2, Theorem 1] we know that any homeomorphism in $\mathrm{BV}_{\mathrm{loc}}$ for $n = 2$ or in $W^{1,n-1}_{\mathrm{loc}}$ for $n \ge 3$ satisfies $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$. Now the composition of the Lipschitz function $g$ and a $\mathrm{BV}_{\mathrm{loc}}$ mapping $f^{-1}$ lies in $\mathrm{BV}_{\mathrm{loc}}$ (see [1, Theorem 3.16]).

Further assume that $p > n - 1$. As before we can use [2, Theorem 1] to conclude that $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$. Suppose first that $u$ is smooth. Then the function $u \circ f^{-1}$ is continuous and belongs to $\mathrm{BV}_{\mathrm{loc}}(f^{-1}(\Omega))$. We are aiming for a local estimate of the total variation of the measure $D(u \circ f^{-1})$.

Without loss of generality we will suppose that $f(\Omega) = (0, 1)^n$, $\Omega$ is bounded and $f \in W^{1,p}(\Omega, \mathsf{R}^n)$. Let $\pi : \mathsf{R}^n \to \mathsf{R}^{n-1}$ be given by $\pi(x_1, \dots, x_n) = [x_2, \dots, x_n]$ and for each $y \in \mathsf{R}^{n-1}$ let us denote $\lambda_y = \{[x, y] : x \in (0, 1)\}$.

Our mapping $f^{-1}$ has bounded variation on $\lambda_y$ for $\mathscr{L}_{n-1}$ almost every $y \in (0, 1)^{n-1}$. Fix such a $y$, and denote the natural parametrization of $\lambda_y$ by $p_y$. Then $f^{-1} \circ p_y$ has bounded variation and we have the equality of the following two measures on the interval $(0, 1)$ (see [1, Theorem 3.96])

$$\frac{d}{dt}(u \circ f^{-1} \circ p_y) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(f^{-1}(y, t))(\mu_y)_j, \tag{3.1}$$

here $(\mu_y)_j$ denotes the distributional derivative of the $j$-th coordinate function of $f^{-1} \circ p_y$. We estimate the right hand side of (3.1), we integrate with respect to $y$, use (2.1), Lemma 3.1 and Hölder's inequality to obtain

$$\begin{aligned}
&\int_{(0,1)^{n-1}} \int_0^1 |\nabla u(f^{-1}(t, y))| \, d|\mu_y|(t) \, d\mathscr{L}_{n-1}(y) \\
&\quad \le C \int_{(0,1)^{n-1}} \int_{f^{-1}(\lambda_y)} |\nabla u| \, d\mathscr{H}_1 \, d\mathscr{L}_{n-1}(y) \\
&\quad \le C \int_\Omega |\nabla u(x)| \, |Df(x)|^{n-1} \, dx \\
&\quad \le C \Big( \int_\Omega |\nabla u|^q \Big)^{\frac{1}{q}} \Big( \int_\Omega |Df^{-1}|^p \Big)^{\frac{n-1}{p}}.
\end{aligned} \tag{3.2}$$

Similarly to (3.1) and (3.2) we can estimate the derivatives of $u \circ f^{-1}$ on lines parallel to other coordinate axes. Using (2.2) we finally obtain

$$|D(u \circ f^{-1})|(Q_0) \le C \Big( \int_\Omega |\nabla u|^q \Big)^{\frac{1}{q}} \Big( \int_\Omega |Df^{-1}|^p \Big)^{\frac{n-1}{p}}. \tag{3.3}$$

Now let us return to the case when $u$ is not smooth. Denote by $u_i$ the usual convolution approximations to $u$. Consider two indices $i$ and $j$ and let $v = u_i - u_j$. Applying (3.3) to $v$ we easily see that $D(u_i \circ f^{-1})$ forms a

Cauchy sequence in Radon measures. It follows that $u_i \circ f^{-1}$ forms also a Cauchy sequence in $L^1_{\mathrm{loc}}$ (see [1, Chapter 3.4]).

If $n-1 < p < n$ and thus $q > n$ then $u$ is continuous. Now $u_i$ converge (locally) uniformly to $u$ and thus $u_i \circ f^{-1}$ converge uniformly to $u \circ f^{-1}$. Since $u_i \circ f^{-1}$ is Cauchy in BV it follows that $u \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$ (see [15] or [1]).

If $p \geq n$ then $f$ satisfies the Lusin $(N)$ condition [10]. Now $u_i$ converges to $u$ almost everywhere and thus $u_i \circ f^{-1}$ converges to $u \circ f^{-1}$ almost everywhere. Since $u_i \circ f^{-1}$ is Cauchy in $L^1_{\mathrm{loc}}$ we obtain that $u_i \circ f^{-1}$ converges to $u \circ f^{-1}$ in $L^1_{\mathrm{loc}}$. This and $u_i \circ f^{-1}$ being Cauchy in BV implies that $u \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$ (see [15] or [1]).

Proof of the first part of Theorem 1.2. Under the additional assumption of $f$ being of finite distortion we know [2, Theorem 1.2] that $f^{-1} \in W^{1,1}_{\mathrm{loc}}$. From [1, Theorem 3.16 and Corollary 3.19] we know that the composition of Lipschitz function and $W^{1,1}$ mapping is in $W^{1,1}$. The derivative of $f^{-1}$ is now an $L^1$ function and not the measure and therefore similarly to the previous proof we obtain that $u_i \circ f^{-1}$ forms a Cauchy sequence in $W^{1,1}$ and therefore analogously to the previous proof we obtain $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}$.

## 4. Construction of examples

The following general construction of examples of mappings of finite distortion was introduced in [6] (see also [5]). Here we give only the brief overview of the construction, for details see [6, Section 5].

### *4.1. Canonical transformation*

If $c \in \mathsf{R}^n$, $a, b > 0$, we use the notation

$$Q(c,a,b) := [c_1 - a, c_1 + a] \times \cdots \times [c_{n-1} - a, c_{n-1} + a] \times [c_n - b, c_n + b].$$

for the interval with center at $c$ and halfedges $a$ in the first $n-1$ coordinates and $b$ in the last coordinate. For $Q = Q(c,a,b)$ we set

$$\varphi_Q(y) = (c_1 + ay_1, \ldots, c_{n-1} + ay_{n-1}, c_n + by_n).$$

Let $P$, $P'$ be concentric intervals, $P = Q(c,a,b)$, $P' = Q(c,a',b')$, where $0 < a < a'$ and $0 < b < b'$. We set

$$\varphi_{P,P'}(t,y) = (1-t)\varphi_P(y) + t\varphi_{P'}(y), \qquad t \in [0,1],\ y \in \partial Q_0.$$

Now, we consider two rectangular annuli, $P' \setminus P^\circ$, and $\tilde{P}' \setminus \tilde{P}^\circ$, where $P = Q(c,a,b)$, $P' = Q(c,a',b')$, $\tilde{P} = Q(\tilde{c},\tilde{a},\tilde{b})$ and $\tilde{P}' = Q(\tilde{c},\tilde{a}',\tilde{b}')$, The mapping

$$h = \varphi_{\tilde{P},\tilde{P}'} \circ (\varphi_{P,P'})^{-1}$$

is called the *canonical transformation* of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$.

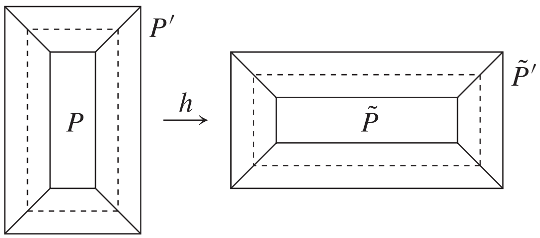


FIGURE 1. The canonical transformation of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$ for $n = 2$.

This homeomorphism maps $\partial P'$ onto $\partial \tilde{P}'$ and $\partial P$ onto $\partial \tilde{P}$ linearly and the boundaries of rectangles between $P'$ and $P$ are mapped to the corresponding boundaries of rectangles between $\tilde{P}'$ and $\tilde{P}$ also linearly.

*4.2. Construction of a Cantor set and a mapping*

By $\mathsf{V}$ we denote the set of $2^n$ vertices of the cube $[-1, 1]^n =: Q_0$. The sets $\mathsf{V}^k = \mathsf{V} \times \cdots \times \mathsf{V}$, $k \in \mathsf{N}$, will serve as the sets of indexes for our construction. If $\boldsymbol{w} \in \mathsf{V}^k$ and $v \in \mathsf{V}$, then the concatenation of $\boldsymbol{w}$ and $v$ is denoted by $\boldsymbol{w}^\wedge v$. The following two results are proven in [6].

LEMMA 4.1. *Let $n \geq 2$. Suppose that we are given two sequences of positive real numbers $\{a_k\}_{k\in\mathsf{N}_0}$, $\{b_k\}_{k\in\mathsf{N}_0}$,*

$$a_0 = b_0 = 1; \tag{4.1}$$

$$a_k < a_{k-1}, \quad b_k < b_{k-1}, \quad \textit{for } k \in \mathsf{N}. \tag{4.2}$$

*Then there exist unique systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathsf{N}}\mathsf{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathsf{N}}\mathsf{V}^k}$ of intervals*

$$Q_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, 2^{-k}a_k, 2^{-k}b_k), \qquad Q'_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, 2^{-k}a_{k-1}, 2^{-k}b_{k-1}) \tag{4.3}$$

*such that*

$$Q'_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathsf{V}^k,\ \textit{are nonoverlaping for fixed } k \in \mathsf{N}, \tag{4.4}$$

$$Q_{\boldsymbol{w}} = \bigcup_{v\in\mathsf{V}} Q'_{\boldsymbol{w}^\wedge v} \ \textit{for each } \boldsymbol{w} \in \mathsf{V}^k,\ k \in \mathsf{N}, \tag{4.5}$$

$$c_v = \frac{1}{2}v, \qquad v \in \mathsf{V}, \tag{4.6}$$

$$c_{\boldsymbol{w}^\wedge v} = c_{\boldsymbol{w}} + \sum_{i=1}^{n-1} 2^{-k}a_k v_i \mathbf{e}_i + 2^{-k}b_k v_n \mathbf{e}_n, \tag{4.7}$$

$$\boldsymbol{w} \in \mathsf{V}^k,\ k \in \mathsf{N},\ v = (v_1, \ldots, v_n) \in \mathsf{V}.$$

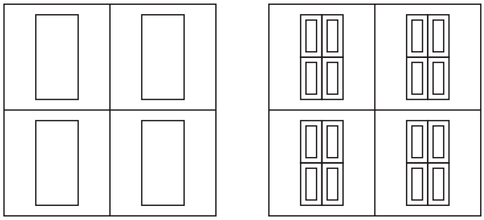

FIGURE 2. Intervals $Q_{\boldsymbol{v}}$ and $Q'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathsf{V}^1$ and $\boldsymbol{v} \in \mathsf{V}^2$ for $n = 2$.

In this way we construct a Cantor type set $E$ in $\mathsf{R}^n$ which is a product of one-dimensional Cantor sets $E_a$ and $E_b$, that is

$$E_a \times \cdots \times E_a \times E_b = E = \bigcap_{k=1}^{\infty} \bigcup_{v \in \mathsf{V}^k} Q_v.$$

Our next aim is to construct a homeomorphism which maps this Cantor type set onto a similar Cantor type set in a natural canonical way. Then we choose a suitable parameters of this construction to obtain a counterexample in the second part of Theorem 1.2.

THEOREM 4.2. *Let $n \geq 2$. Suppose that we are given four sequences of positive real numbers $\{a_k\}_{k \in \mathsf{N}_0}$, $\{b_k\}_{k \in \mathsf{N}_0}$, $\{\tilde{a}_k\}_{k \in \mathsf{N}_0}$, $\{\tilde{b}_k\}_{k \in \mathsf{N}_0}$,*

$$a_0 = b_0 = \tilde{a}_0 = \tilde{b}_0 = 1; \tag{4.8}$$

$$a_k < a_{k-1}, \quad b_k < b_{k-1}, \quad \tilde{a}_k < \tilde{a}_{k-1}, \quad \tilde{b}_k < \tilde{b}_{k-1}, \quad \textit{for } k \in \mathsf{N}. \tag{4.9}$$

*Let the systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \bigcup_{k \in \mathsf{N}} \mathsf{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \bigcup_{k \in \mathsf{N}} \mathsf{V}^k}$ of intervals be as in Lemma 4.1, and similarly systems $\{\tilde{Q}_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \bigcup_{k \in \mathsf{N}} \mathsf{V}^k}$, $\{\tilde{Q}'_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \bigcup_{k \in \mathsf{N}} \mathsf{V}^k}$ of intervals be associated with the sequences $\{\tilde{a}_k\}$ and $\{\tilde{b}_k\}$. Then there exists a unique sequence $\{f^k\}$ of bilipschitz homeomorphisms of $Q_0$ onto itself such that*

(a) *$f^k$ maps each $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathsf{V}^m$, $m = 1, \ldots, k$, onto $\tilde{Q}'_{\boldsymbol{v}} \setminus \tilde{Q}_{\boldsymbol{v}}$ canonically,*

(b) *$f^k$ maps each $Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathsf{V}^k$, onto $\tilde{Q}_{\boldsymbol{v}}$ affinely.*

*Moreover,*

$$|f^k - f^{k+1}| \lesssim 2^{-k}, \qquad |(f^k)^{-1} - (f^{k+1})^{-1}| \lesssim 2^{-k}. \tag{4.10}$$

*The sequence $f^k$ converges uniformly to a homeomorphism $f$ of $Q_0$ onto $Q_0$.*

*4.3. Completion of the proof of Theorem 1.2*

PROOF OF THE SECOND PART OF THEOREM 1.2. Set $\alpha = n-1$ and $\gamma = q-1-\frac{\varepsilon}{2}$. It is easy to check that

$$(n-1)\gamma + \alpha - \gamma p > 0 \quad \text{and} \quad (n-1)\gamma + \alpha - \alpha(q-\varepsilon) > 0. \tag{4.11}$$

Set

$$a_k = \frac{1}{(k+1)^\gamma}, \quad b_k = \frac{1}{(k+1)^\alpha}, \quad \tilde{a}_k = \frac{1}{2} + \frac{1}{(k+1)^\alpha}, \quad \tilde{b}_k = \frac{1}{(k+1)^\alpha},$$

and use Theorem 4.2 to obtain our mapping $f$. Recall that due to the symmetry of the construction (see [6, Remark 5.5]) we obtain that $f^{-1}$ is given by the same theorem applied to sequences $\tilde{a}_k$, $\tilde{b}_k$, $a_k$ and $b_k$.

Similarly as in [6, Section 6] or [5, Section 5] we obtain that $f \in W^{1,p}(Q_0, \mathsf{R}^n)$ and that

$$\int_{Q_0} |Df|^p \sim C \sum_{k\in\mathsf{N}} \frac{1}{k^{1+(n-1)\gamma+\alpha}} k^{\gamma p} < \infty, \tag{4.12}$$

where the finiteness of the sum follows from (4.11). As a brief hint (see [6] or [5] for details) for this let us point out that

$$\mathscr{L}_n(Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}) = \frac{1}{2^{(k-1)n}}\left(a_{k-1}^{n-1} b_{k-1} - a_k^{n-1} b_k\right) \sim \frac{1}{2^{kn} k^{1+(n-1)\gamma+\alpha}}$$

for every $\boldsymbol{v} \in \mathsf{V}^k$, we have $2^{kn}$ rectangular annuli like that in each step and the derivative of $f_k$ and thus also of $f$ on this annuli is at most

$$\max\left\{\frac{\tilde{a}_{k-1}}{a_{k-1}}, \frac{\tilde{a}_{k-1}-\tilde{a}_k}{a_{k-1}-a_k}, \frac{\tilde{b}_{k-1}}{b_{k-1}}, \frac{\tilde{b}_{k-1}-\tilde{b}_k}{b_{k-1}-b_k}\right\} \sim k^\gamma.$$

In fact we need to show that the sequence $f_k$ is Cauchy in $W^{1,p}$ but this easily follows from the convergence of the sum in (4.12). Note that $f$ is clearly a mapping of finite distortion since each $f_k$ is a mapping of finite distortion and a measure of a set where $f$ is not equal to some $f_k$ has measure zero.

Set

$$\tilde{\tilde{a}}_k = \frac{1}{(k+1)^\gamma}, \qquad \tilde{\tilde{b}}_k = \frac{1}{2} + \frac{1}{2(k+1)^\gamma}$$

and use Theorem 4.2 for sequences $a_k$, $b_k$, $\tilde{\tilde{a}}_k$ and $\tilde{\tilde{b}}_k$ to obtain a mapping $g$. Similarly as above we use (4.11) to obtain $g \in W^{1,q-\varepsilon}(Q_0, \mathsf{R}^n)$ since

$$\int_{Q_0} |Dg|^{q-\varepsilon} \sim C \sum_{k\in\mathsf{N}} \frac{1}{k^{1+(n-1)\gamma+\alpha}} k^{\alpha(q-\varepsilon)} < \infty.$$

Recall (see [6, Remark 5.6]) that $g \circ f^{-1}$ can be obtained by Theorem 4.2 applied to sequences $\tilde{a}_k$, $\tilde{b}_k$, $\tilde{\tilde{a}}_k$ and $\tilde{\tilde{b}}_k$. We claim that $g \circ f^{-1}$ does not satisfy the ACL-condition and therefore $g \circ f^{-1} \notin W^{1,1}_{\mathrm{loc}}(Q_0, Q_0)$. It is clear from the construction in Theorem 4.2 that there are Cantor sets $E_a$, $E_b$, $E_{\tilde{a}}$ and $E_{\tilde{b}}$ such that $g \circ f^{-1}$ maps the Cantor set $E_a \times E_a \times \cdots \times E_a \times E_b$ onto the Cantor set $\tilde{E}_a \times \tilde{E}_a \times \cdots \times \tilde{E}_a \times \tilde{E}_b$. Clearly $\mathscr{L}_1(\tilde{E}_a) = \mathscr{L}_1(E_b) = 0$, $\mathscr{L}_1(\tilde{E}_b) > 0$, $\mathscr{L}_1(E_a) > 0$ and it is not difficult to check that for every $y \in [-1, 1]^{n-1}$ such that $y \in E_a \times \cdots \times E_a$ there exists $\tilde{y} \in \tilde{E}_a \times \cdots \times \tilde{E}_a$ such that $g \circ f^{-1}(\{y\} \times E_b) = \{\tilde{y}\} \times \tilde{E}_b$. It follows that $g \circ f^{-1}(y, \cdot)$ does not satisfy the one-dimensional Luzin condition $(N)$ and therefore cannot be absolutely continuous there. Since $\mathscr{L}_{n-1}(E_a \times \cdots \times E_a) > 0$ we obtain that $g \circ f^{-1}$ does not satisfy the ACL-condition. Another possibility how to show $g \circ f^{-1} \notin W^{1,1}$ is to compute

$$\int_{Q_0} |\nabla g \circ f^{-1}| \sim C \sum_{k \in \mathsf{N}} \big(\tilde{a}_{k-1}^{n-1} \tilde{b}_{k-1} - \tilde{a}_k^{n-1} \tilde{b}_k\big) \frac{\tilde{\tilde{b}}_{k-1}}{\tilde{b}_{k-1}} \sim \sum_{k \in \mathsf{N}} \frac{1}{k^{1+\alpha}} k^{\alpha} = \infty.$$

For the statement of Theorem 1.1 simply pick $f$ as above and set $u = g_n$, i.e., the $n$-th coordinate function of $g$.

Acknowledgement. The author would like to thank Jan Malý and Pekka Koskela for drawing his attention to the problem and for many stimulating conversations.

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