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ASYMPTOTICS OF EIGENVALUES OF REGULAR STURM-LIOUVILLE PROBLEMS WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITION FOR INTEGRABLE POTENTIAL

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Abstract

In this paper we obtain asymptotic estimates of eigenvalues for regular Sturm-Liouville problems having the eigenparameter in the boundary condition without smoothness conditions on q.

1. Introduction

In this paper we consider the boundary value problem

$$\tau y := -y'' + qy = \lambda, \qquad y, t \in [a, b], \tag{1}$$

$$a_1 y(a) - a_2 y'(a) = \lambda(a_1' y(a) - a_2' y'(a)), \qquad a_1, a_2, a_1', a_2' \in \mathsf{R}, \tag{2}$$

$$y(b)\cos\beta + y'(b)\sin\beta = 0, \qquad \beta \in [0, \pi], \tag{3}$$

where λ is a real parameter, $q(t)$ is a real-valued function. We assume that $q(t)$ is integrable on $[a, b]$. This problem differs from the usual regular Sturm-Liouville problem in the sense that the eigenvalue parameter λ is contained in the boundary condition at a. Problems of this type arise from the method of separation of variables applied to mathematical models for certain physical problems including that of heat conduction and wave propagation, etc. [7]. It is shown by Walter [14] that this problem is a self-adjoint problem if the relation

$$\delta := \begin{vmatrix} a_1' & a_1 \\ a_2' & a_2 \end{vmatrix} > 0 \tag{4}$$

holds.

The purpose of this paper is to obtain asymptotic approximations for the eigenvalues λ_n of (1)–(3) when the condition (4) is satisfied under the sole condition that $q(t)$ is integrable on $[a, b]$. Approximations of this type have been derived before. We mention in particular [6], [7] and [1]. Fulton's approach in

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[6] is based on an iteration of the usual Volterra integral equation, producing an asymptotic expansion of the solution in higher powers of $1/\lambda^{\frac{1}{2}}$ as $\lambda \to \infty$ and in [7] is based on the analysis of [13] for regular Sturm-Liouville problems on a finite closed interval and involves some operator-theoretical results of [14]. The approach used in [1] is based on an iterative procedure solving the associated Riccati equation and producing an asymptotic expansion of the solution in the higher powers of $1/\lambda^{\frac{1}{2}}$ as $\lambda \to \infty$ for smooth $q(t)$. There is also a vast amount of literature dealing with asymptotic estimates of eigenvalues for standard Sturm-Liouville problems with regular endpoints [2], [3], [4], [5], [8], [9], [10], [12], [13].

Here we follow the similar approach in [3], [9], [11]. In this paper we introduce a method of obtaining the asymptotic form of λ_n for the problem (1)–(3) when the condition (4) is satisfied under the sole condition that $q(t)$ is integrable on $[a, b]$ as $n \to \infty$.

We assume without loss of generality, that $q(t)$ has mean value zero. That is

$$\int_a^b q(t)\,dt = 0. \tag{5}$$

2. The results

Our results include the following four distinct cases concerning a_2' and β as pointed out in [7]. These are $a_2' \neq 0$, $\beta \neq 0$; $a_2' \neq 0$, $\beta = 0$; $a_2' = 0$, $\beta \neq 0$; and $a_2' = 0$, $\beta = 0$.

THEOREM 1. *The eigenvalues λ_n of* (1)–(3) *satisfy as $n \to \infty$*

(i) $a_2' \neq 0$, $\beta \neq 0$

$$\begin{aligned} \lambda_n^{\frac{1}{2}} = \frac{(n+1)\pi}{(b-a)} + \frac{1}{2(n+1)\pi}\Bigg\{ & 2\frac{a_1'}{a_2'} + 2\cot\beta \\ & + \sin\left(\frac{2(n+1)\pi a}{b-a}\right)\int_a^b q(t)\sin\left(\frac{2(n+1)\pi t}{b-a}\right)dt \\ & + \cos\left(\frac{2(n+1)\pi a}{b-a}\right)\int_a^b q(t)\cos\left(\frac{2(n+1)\pi t}{b-a}\right)dt\Bigg\} \\ & + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)), \end{aligned} \tag{6}$$

(ii) $a_2' \neq 0,\ \beta = 0$

$$\lambda_n^{\frac{1}{2}} = \frac{(2n+3)\pi}{2(b-a)} + \frac{1}{(2n+3)\pi}\Bigg\{2\frac{a_1'}{a_2'} + \sin\left(\frac{2(n+3)\pi a}{b-a}\right)\int_a^b q(t)\sin\left(\frac{(2n+3)\pi t}{b-a}\right)dt + \cos\left(\frac{(2n+3)\pi a}{b-a}\right)\int_a^b q(t)\cos\left(\frac{(2n+3)\pi t}{b-a}\right)dt\Bigg\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)). \tag{7}$$

THEOREM 2. *The eigenvalues λ_n of* (1)–(3) *satisfy as $n \to \infty$*
(i) $a_2' = 0,\ \beta \neq 0$

$$\lambda_n^{\frac{1}{2}} = \frac{(2n+3)\pi}{2(b-a)} + \frac{1}{(2n+3)\pi}\Bigg\{2\frac{a_2}{a_1'} + 2\cot\beta - \sin\left(\frac{2(n+3)\pi a}{b-a}\right)\int_a^b q(t)\sin\left(\frac{(2n+3)\pi t}{b-a}\right)dt - \cos\left(\frac{(2n+3)\pi a}{b-a}\right)\int_a^b q(t)\cos\left(\frac{(2n+3)\pi t}{b-a}\right)dt\Bigg\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)), \tag{8}$$

(ii) $a_2' = 0,\ \beta = 0$

$$\lambda_n^{\frac{1}{2}} = \frac{(n+2)\pi}{(b-a)} + \frac{1}{2(n+2)\pi}\Bigg\{2\frac{a_2}{a_1'} - \sin\left(\frac{2(n+2)\pi a}{b-a}\right)\int_a^b q(t)\sin\left(\frac{2(n+2)\pi t}{b-a}\right)dt - \cos\left(\frac{2(n+2)\pi a}{b-a}\right)\int_a^b q(t)\cos\left(\frac{2(n+2)\pi t}{b-a}\right)dt\Bigg\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)), \tag{9}$$

where $\eta(n)$ is defined by (22).

For $a = 0,\ b = \pi$ we get the following corollaries:

COROLLARY 3. *The eigenvalues* λ_n *of* (1)–(3) *satisfy as* $n \to \infty$

(i) $a_2' \neq 0,\ \beta \neq 0$

(10)
$$\lambda_n^{\frac{1}{2}} = (n+1) + \frac{1}{2(n+1)\pi}\left\{2\frac{a_1'}{a_2'} + 2\cot\beta + \int_0^\pi q(t)\cos(2(n+1)t)\,dt\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)),$$

(ii) $a_2' \neq 0,\ \beta = 0$

(11)
$$\lambda_n^{\frac{1}{2}} = \frac{(2n+3)}{2} + \frac{1}{(2n+3)\pi}\left\{2\frac{a_1'}{a_2'} + \int_0^\pi q(t)\cos((2n+3)t)\,dt\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)).$$

COROLLARY 4. *The eigenvalues* λ_n *of* (1)–(3) *satisfy as* $n \to \infty$

(i) $a_2' = 0,\ \beta \neq 0$

(12)
$$\lambda_n^{\frac{1}{2}} = \frac{(2n+3)}{2} + \frac{1}{(2n+3)\pi}\left\{2\frac{a_2}{a_1'} + 2\cot\beta - \int_0^\pi q(t)\cos((2n+3)t)\,dt\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)),$$

(ii) $a_2' = 0,\ \beta = 0$

(13)
$$\lambda_n^{\frac{1}{2}} = (n+2) + \frac{1}{2(n+2)\pi}\left\{2\frac{a_2}{a_1'} - \int_0^\pi q(t)\cos(2(n+2)t)\,dt\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)).$$

As an illustration of our results we give the following example:

Let $q(t) = t^{-\frac{1}{2}} - 2\pi^{-\frac{1}{2}}$. Using (10)–(13) we get the following estimates on λ.

$a_2' \neq 0,\ \beta \neq 0$

$$\lambda_n^{\frac{1}{2}} = (n+1) + \frac{1}{2(n+1)\pi}\left\{2\frac{a_1'}{a_2'} + 2\cot\beta + \frac{\pi C\big(2\sqrt{n+1}\big)}{\sqrt{n+1}}\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n));$$

$a_2' \neq 0,\ \beta = 0$

$$\lambda_n^{\frac{1}{2}} = \frac{(2n+3)}{2} + \frac{1}{(2n+3)\pi}\left\{2\frac{a_1'}{a_2'} + \frac{\sqrt{2}\pi C\big(\sqrt{2(2n+3)}\big)}{\sqrt{2n+3}}\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n));$$

$a_2' = 0, \beta \neq 0$

$$\lambda_n^{\frac{1}{2}} = \frac{(2n+3)}{2} + \frac{1}{(2n+3)\pi}\left\{2\frac{a_2}{a_1'} + 2\cot\beta - \frac{\sqrt{2}\pi C\big(\sqrt{2(2n+3)}\big)}{\sqrt{2n+3}}\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n));$$

$a_2' = 0, \beta = 0$

$$\lambda_n^{\frac{1}{2}} = (n+2) + \frac{1}{2(n+2)\pi}\left\{2\frac{a_2}{a_1'} - \frac{\pi C\big(2\sqrt{n+2}\big)}{\sqrt{n+2}}\right\} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)),$$

where $C(x)$ is the Fresnel integral defined as $C(x) = \int_0^x \cos(t^2)\,dt$.

3. The Method

We associate with (1) the Riccati equation

$$v' = -\lambda + q - v^2. \tag{14}$$

We define

$$S(t,\lambda) := \operatorname{Re}\{v(t,\lambda)\}, \tag{15}$$

$$T(t,\lambda) := \operatorname{Im}\{v(t,\lambda)\}. \tag{16}$$

It is shown in [2] that any real-valued solution of (1) is in the form

$$y(t,\lambda) = R(t,\lambda)\cos(\theta(t,\lambda)) \tag{17}$$

with

$$\frac{R'(t,\lambda)}{R(t,\lambda)} = S(t,\lambda), \tag{18}$$

$$\theta'(t,\lambda) = T(t,\lambda). \tag{19}$$

Our approach to calculating λ_n is to approximate those λ which are such that

$$\theta(b,\lambda) - \theta(a,\lambda) = \int_a^b T(t,\lambda)\,dt. \tag{20}$$

We suppose that there exist functions $A(t)$ and $\eta(\lambda)$ so that

$$\left|\int_t^b e^{2i\lambda^{\frac{1}{2}}x} q(x)\,dx\right| \le A(t)\eta(\lambda) \qquad \text{for all} \quad t \in [a,b], \tag{21}$$

where

(i) $A(t)$ is a decreasing function of t,

(ii) $\eta(\lambda) \to 0$ as $\lambda \to \infty$,

(iii) $A(.) \in L^1[a,b]$.

For $q \in L^1[a,b]$ the existence of the A and η functions may be established for λ positive as follows.

We note that, avoiding the trivial case $\int_t^b |q(x)|\,dx = 0$,

$$\left|\int_t^b e^{2i\lambda^{\frac{1}{2}}x} q(x)\,dx\right| \le \int_t^b |q(x)|\,dx < \infty,$$

so, if we define

$$F(t,\lambda) := \begin{cases} \left|\int_t^b e^{2i\lambda^{\frac{1}{2}}x} q(x)\,dx\right| \Big/ \int_t^b |q(x)|\,dx & \text{if } \int_t^b |q(x)| \neq 0, \\ 0 & \text{if } \int_t^b |q(x)| = 0, \end{cases} \tag{22}$$

then $0 \le F(t,\lambda) \le 1$ and we set $\eta(\lambda) := \sup_{a \le t \le b} F(t,\lambda)$. Note that $\eta(\lambda)$ is well defined by (22) and $\eta(\lambda) \to 0$ as $\lambda \to \infty$ [11].

We set $A(t) := \int_t^b |q(x)|\,dx$. Then $A(t)$ is clearly a decreasing function of t and

$$\begin{aligned} \int_a^b A(t)\,dt &= \int_a^b \int_t^b |q(x)|\,dx\,dt = \int_a^b |q(x)| \int_a^x dt\,dx \\ &= \int_a^b |q(x)|(x-a)\,dx \le a \int_a^b |q(x)|\,dx. \end{aligned} \tag{23}$$

Our method of approximating a solution of (14) is similar to that of [11]. We consider (14) on $[a,b]$ and set

$$v(t,\lambda) := i\lambda^{\frac{1}{2}} + \sum_{n=1}^{\infty} v_n(t,\lambda). \tag{24}$$

Substitution of (24) into (14) and rearrangement then gives

$$v_1' + 2i\lambda^{\frac{1}{2}} v_1 + v_2' + 2i\lambda^{\frac{1}{2}} v_2 + \sum_{n=3}^{\infty}\big(v_n' + 2i\lambda^{\frac{1}{2}} v_n\big) = q - v_1^2 - \sum_{n=3}^{\infty}\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right).$$

We choose the v_n so that

$$(25)\qquad \begin{cases} v_1' + 2i\lambda^{\frac{1}{2}} v_1 = q, \\ v_2' + 2i\lambda^{\frac{1}{2}} v_2 = -v_1^2, \\ v_n' + 2i\lambda^{\frac{1}{2}} v_n = -\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right) \quad \text{for } n = 3, 4, \ldots \end{cases}$$

and

$$(26)\qquad \begin{cases} v_1(t,\lambda) = -e^{-2i\lambda^{\frac{1}{2}} t}\displaystyle\int_t^b e^{2i\lambda^{\frac{1}{2}} x} q(x)\,dx, \\ v_2(t,\lambda) = e^{-2i\lambda^{\frac{1}{2}} t}\displaystyle\int_t^b e^{2i\lambda^{\frac{1}{2}} x} v_1^2(x,\lambda)\,dx, \\ v_n(t,\lambda) = e^{-2i\lambda^{\frac{1}{2}} t}\displaystyle\int_t^b e^{2i\lambda^{\frac{1}{2}} x}\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right)dx. \end{cases}$$

It is proven in [2] that $\sum_{n=1}^{\infty} v_n'(t,\lambda)$ is uniformly absolutely convergent and the series $i\lambda^{\frac{1}{2}} + \sum_{n=1}^{\infty} v_n'(t,\lambda)$ is thus a solution of (14) and

$$(27)\qquad T(t,\lambda) = \lambda^{\frac{1}{2}} + \operatorname{Im}\sum_{n=1}^{\infty} v_n(t,\lambda).$$

It is also proven in [2] that there exist a sequence $\{k_n\}$ of real numbers with

$$(28)\qquad |v_n(t,\lambda)| \le k_n \eta(\lambda)^n,$$

(Lemma 2.2, [2]).

It may easily be obtained, by a change in the order of integration and (5) that

$$\int_a^b v_1(t,\lambda)\,dt = \frac{-i}{2\lambda^{\frac{1}{2}}}\int_a^b q(x)\big(1 - e^{2i\lambda^{\frac{1}{2}}(x-a)}\big)\,dx$$

$$= \frac{i}{2\lambda^{\frac{1}{2}}} \int_a^b e^{2i\lambda^{\frac{1}{2}}(x-a)} q(x)\,dx, \tag{29}$$

$$\int_a^b v_2(t,\lambda)\,dt = \frac{i}{2\lambda^{\frac{1}{2}}} \int_a^b v_1^2(x,\lambda)\big(1 - e^{2i\lambda^{\frac{1}{2}}(x-a)}\big)\,dx \tag{30}$$

and

$$\begin{aligned} &\int_a^b v_n(t,\lambda)\,dt \\ &= \frac{i}{2\lambda^{\frac{1}{2}}} \int_a^b \Big(v_{n-1}^2(x,\lambda) + 2v_{n-1}(x,\lambda)\sum_{m=1}^{n-2} v_m(x,\lambda)\Big)\big(1 - e^{2i\lambda^{\frac{1}{2}}(x-a)}\big)\,dx. \end{aligned} \tag{31}$$

It is shown in [11] that any real valued solution $y(t,\lambda)$ of (1) is of the form

$$y(t,\lambda) = R(t,\lambda)\cos\theta(t,\lambda), \tag{32}$$

hence

$$y'(t,\lambda) = R'(t,\lambda)\cos\theta(t,\lambda) - R(t,\lambda)\theta'(t,\lambda)\sin\theta(t,\lambda). \tag{33}$$

We now determine the conditions under which the first boundary condition (2) and the second boundary condition (3) are satisfied. Considering (32) and (33) one observes that equation (2) holds if

$$\begin{aligned} R(a,\lambda)\Big\{\cos\theta(a,\lambda)\Big[a_1 - a_2\frac{R'(a,\lambda)}{R(a,\lambda)} \\ - \lambda\Big(a_1' - a_2'\frac{R'(a,\lambda)}{R(a,\lambda)}\Big)\Big] + \sin\theta(a,\lambda)(a_2 - \lambda a_2')\theta'(a,\lambda)\Big\} = 0. \end{aligned} \tag{34}$$

We can write (34) as

$$R(a,\lambda)\sin(\gamma_1 + \theta(a,\lambda)) = 0, \tag{35}$$

where

$$\sin\gamma_1 = a_1 - a_2\frac{R'(a,\lambda)}{R(a,\lambda)} - \lambda\Big(a_1' - a_2'\frac{R'(a,\lambda)}{R(a,\lambda)}\Big),$$

$$\cos\gamma_1 = (a_2 - \lambda a_2')\theta'(a,\lambda).$$

From (18)–(19)

$$\sin\gamma_1 = a_1 - a_2 S(a,\lambda) - \lambda(a_1' - a_2' S(a,\lambda)), \tag{36}$$

$$\cos\gamma_1 = (a_2 - \lambda a_2')T(a,\lambda). \tag{37}$$

We define $\Omega := \tan \gamma_1$. From (35) the first boundary condition (2) is satisfied

(i) for $a_2' \neq 0$

$$\theta(a, \lambda) = -\gamma_1 = -\tan^{-1}(\Omega). \tag{38}$$

For $a_2' = 0$, (34) reduces to

$$R(a, \lambda)\left\{\cos\theta(a, \lambda)\left[a_1 - a_2\frac{R'(a, \lambda)}{R(a, \lambda)} - \lambda a_1'\right] + a_2 \sin\theta(a, \lambda)\theta'(a, \lambda)\right\}$$

$$= 0$$

$$= R(a, \lambda)\sin(\delta_1 + \theta(a, \lambda)), \tag{39}$$

where

$$\sin\delta_1 = a_1 - a_2\frac{R'(a, \lambda)}{R(a, \lambda)} - \lambda a_1',$$
$$\cos\delta_1 = a_2\theta'(a, \lambda).$$

Again from (18)–(19)

$$\sin\delta_1 = a_1 - a_2 S(a, \lambda) - \lambda a_1', \tag{40}$$

$$\cos\delta_1 = a_2 T(a, \lambda). \tag{41}$$

We define $\Gamma := \cot\delta_1$. From (39) the first boundary condition (2) is satisfied

(ii) for $a_2' = 0$

$$\theta(a, \lambda) = -\delta_1 = -\cot^{-1}(\Gamma). \tag{42}$$

Similarly the second boundary condition (3) holds if

$$R(b, \lambda)\left\{\cos\theta(b, \lambda)\left[\cos\beta + \frac{R'(b, \lambda)}{R(b, \lambda)}\sin\beta\right] - \sin\theta(b, \lambda)\theta'(b, \lambda)\sin\beta\right\} = 0.$$

One can write (42) as

$$R(b, \lambda)\sin(\gamma_2 - \theta(b, \lambda)) = 0, \tag{43}$$

where

$$\sin\gamma_2 = \cos\beta + S(b, \lambda)\sin\beta, \tag{44}$$

$$\cos\gamma_2 = T(b, \lambda)\sin\beta. \tag{45}$$

We define $\eta := \tan\gamma_2$. From (43) the second boundary condition (3) is satisfied

(iii) for $\beta \neq 0$

$$\theta(b,\lambda) = \gamma_2 + (n+1)\pi = \tan^{-1}(\eta) + (n+1)\pi. \tag{46}$$

For $\beta = 0$, (43) reduces to

$$R(b,\lambda)\cos\theta(b,\lambda) = 0. \tag{47}$$

From (47), the second boundary condition (3) is satisfied

(iv) for $\beta = 0$

$$\theta(b,\lambda) = \frac{\pi}{2} + (n+1)\pi. \tag{48}$$

4. Proof of the results

We first give the following theorems:

THEOREM 5. *Let $q(t)$ be a real-valued integrable function on $[a,b]$. Then the eigenvalues of* (1)–(3) *satisfy as $\lambda \to \infty$*

(i) *for $a_2' \neq 0$ and $\beta \neq 0$*

$$(n+1)\pi = \int_a^b T(t,\lambda)dt - \tan^{-1}(\Omega) - \tan^{-1}(\eta), \tag{49}$$

(ii) *for $a_2' \neq 0$ and $\beta = 0$*

$$\frac{(2n+3)\pi}{2} = \int_a^b T(t,\lambda)dt - \tan^{-1}(\Omega). \tag{50}$$

THEOREM 6. *Let $q(t)$ be a real-valued integrable function on $[a,b]$. Then the eigenvalues of* (1)–(3) *satisfy as $\lambda \to \infty$*

(i) *for $a_2' = 0$ and $\beta \neq 0$*

$$(n+1)\pi = \int_a^b T(t,\lambda)dt - \cot^{-1}(\Gamma) - \tan^{-1}(\eta), \tag{51}$$

(ii) *for $a_2' = 0$ and $\beta = 0$*

$$\frac{(2n+3)\pi}{2} = \int_a^b T(t,\lambda)dt - \cot^{-1}(\Gamma), \tag{52}$$

where $S(t,\lambda)$, $T(t,\lambda)$ *are defined by* (15)–(16) *and*

$$\Omega = \frac{a_1 - a_2 S(a,\lambda) - \lambda[a_1' - a_2' S(a,\lambda)]}{(a_2 - \lambda a_2')T(a,\lambda)}, \tag{53}$$

$$\eta = \frac{\cos\beta + S(b,\lambda)\sin\beta}{\sin\beta T(b,\lambda)}, \tag{54}$$

$$\Gamma = \frac{a_2 T(a,\lambda)}{a_1 - a_2 S(a,\lambda) - \lambda a_1'}. \tag{55}$$

PROOF OF THEOREM 5 AND THEOREM 6. Theorem 5(i) follows from (20), (38) and (46). Theorem 5(ii) follows from (20), (38) and (48). Similarly Theorem 6(i) follows from (20), (42) and (46). Theorem 6(ii) follows from (20), (42) and (48).

In the following lemma, we evaluate $\tan^{-1}(\Omega)$, $\tan^{-1}(\eta)$ and $\cot^{-1}(\Gamma)$.

LEMMA 7. *As* $\lambda \to \infty$
(i)

$$\tan^{-1}(\Omega) = \lambda^{-\frac{1}{2}}\left[\frac{a_1'}{a_2'} + \sin(2\lambda^{\frac{1}{2}}a + \zeta_a)\right] + O(\lambda^{-\frac{1}{2}}\eta(\lambda)^2), \tag{56}$$

(ii)

$$\tan^{-1}(\eta) = \lambda^{-\frac{1}{2}}\cot\beta + O(\lambda^{-\frac{1}{2}}\eta(\lambda)^2), \tag{57}$$

(iii)

$$\cot^{-1}(\Gamma) = \frac{\pi}{2} + \frac{a_2\lambda^{\frac{-1}{2}}}{a_1'} + O(\lambda^{-1}\eta(\lambda)^2), \tag{58}$$

where a_1', a_2' *and* β *are as in* (2), (3) *and*

$$\sin\zeta_t = \int_t^b q(x)\cos(2\lambda^{\frac{1}{2}}x)\,dx, \tag{59}$$

$$\cos\zeta_t = \int_t^b q(x)\sin(2\lambda^{\frac{1}{2}}x)\,dx. \tag{60}$$

PROOF. From (15), (24) and (28)

$$S(t,\lambda) = -\sin(2\lambda^{\frac{1}{2}}t + \zeta_t) + O(\eta(\lambda)^2).$$

From (16), (24) and (28)

$$T(t,\lambda) = \lambda^{\frac{1}{2}} - \cos(2\lambda^{\frac{1}{2}}t + \zeta_t) + O(\eta(\lambda)^2).$$

Hence

$$S(a,\lambda) = -\sin(2\lambda^{\frac{1}{2}}a + \zeta_a) + O(\eta(\lambda)^2), \tag{61}$$

$$T(a,\lambda) = \lambda^{\frac{1}{2}} - \cos(2\lambda^{\frac{1}{2}}a + \zeta_a) + O(\eta(\lambda)^2), \tag{62}$$

$$S(b,\lambda) = -\sin(2\lambda^{\frac{1}{2}}b + \zeta_b) + O(\eta(\lambda)^2), \tag{63}$$

$$T(b,\lambda) = \lambda^{\frac{1}{2}} - \cos(2\lambda^{\frac{1}{2}}b + \zeta_b) + O(\eta(\lambda)^2). \tag{64}$$

From (36) and (37)

$$\tan\gamma_1 = \frac{a_1 - a_2 S(a,\lambda) - \lambda(a_1' - a_2' S(a,\lambda))}{(a_2 - \lambda a_2')T(a,\lambda)}. \tag{65}$$

Substituting the values of $S(a,\lambda)$ and $T(a,\lambda)$ given by (61) and (62) into (65), one obtains

$$\tan\gamma_1 = \lambda^{-\frac{1}{2}}\left[\frac{a_1'}{a_2'} + \sin(2\lambda^{\frac{1}{2}}a + \zeta_a)\right] + O(\lambda^{-\frac{1}{2}}\eta(\lambda)^2). \tag{66}$$

Similarly from (44) and (45)

$$\tan\gamma_2 = \frac{\cos\beta + S(b,\lambda)\sin\beta}{\sin\beta T(b,\lambda)}. \tag{67}$$

Substituting the values of $S(b,\lambda)$ and $T(b,\lambda)$ given by (63) and (64) into (67), one obtains

$$\tan\gamma_2 = \lambda^{-\frac{1}{2}}\cot\beta + O(\lambda^{-\frac{1}{2}}\eta(\lambda)^2). \tag{68}$$

From (40) and (41)

$$\cot\delta_1 = \frac{a_2 T(a,\lambda)}{a_1 - a_2 S(a,\lambda) - \lambda a_1'}. \tag{69}$$

Substituting the values of $S(a,\lambda)$ and $T(a,\lambda)$ given by (61) and (62) into (69), one obtains

$$\cot\delta_1 = -\frac{a_2}{a_1'}\lambda^{-\frac{1}{2}} + O(\lambda^{-\frac{1}{2}}\eta(\lambda)^2). \tag{70}$$

Proof is done by using (66), (68) and (70) together with inverse trigonometric series.

USE OF REVERSION TO PROVE THEOREM 1 AND THEOREM 2. By (5), (28), (49) and (29)–(31)

$$\lambda_n^{\frac{1}{2}} = \frac{(n+1)\pi}{(b-a)} + O(n^{-1}\eta(n)), \tag{71}$$

where we have written $\eta(n)$ for $\eta\left(\frac{(n+1)\pi}{(b-a)}\right)$. Also

$$\lambda_n^{-\frac{1}{2}} = \frac{(b-a)}{(n+1)\pi} + O(n^{-3}\eta(n)) \tag{72}$$

and

$$\begin{aligned} e^{2i\lambda_n^{\frac{1}{2}}(x-a)} &= \exp\left\{2i\left(\frac{n+1}{b-a}\right)\pi(x-a) + O(n^{-1}\eta(n))\right\} \\ &= \exp\left\{2i\left(\frac{n+1}{b-a}\right)\pi(x-a)\right\} + O(n^{-1}\eta(n)). \end{aligned} \tag{73}$$

Thus, by (28), (49), (56), (57) and (71)–(73)

$$\begin{aligned} &\lambda_n^{\frac{1}{2}}(b-a) \\ &\quad = (n+1)\pi + \frac{(b-a)}{(n+1)\pi}\left\{\frac{a_1'}{a_2'} + \sin\left(2\left(\frac{(n+1)\pi}{(b-a)}\right)a + \zeta_a\right) + \cot\beta\right\} \\ &\qquad - \operatorname{Im}\left\{\frac{i(b-a)}{2(n+1)\pi}\int_a^b e^{2i\left(\frac{(n+1)\pi}{(b-a)}\right)(x-a)} q(x)\,dx\right\} \\ &\qquad + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)). \end{aligned}$$

Finally, Theorem 1(i) is proved by substituting the values of $\sin\zeta_a$ and $\cos\zeta_a$ given by (59), (60) respectively.

Similarly by (5), (28), (50) and (29)–(31)

$$\lambda_n^{\frac{1}{2}} = \frac{(2n+3)\pi}{2(b-a)} + O(n^{-1}\eta(n)), \tag{74}$$

$$\lambda_n^{-\frac{1}{2}} = \frac{2(b-a)}{(2n+3)\pi} + O(n^{-3}\eta(n)) \tag{75}$$

and

$$\begin{aligned} e^{2i\lambda_n^{\frac{1}{2}}(x-a)} &= \exp\left\{i\left(\frac{(2n+3)}{b-a}\right)\pi(x-a) + O(n^{-1}\eta(n))\right\} \\ &= \exp\left\{i\left(\frac{(2n+3)}{b-a}\right)\pi(x-a)\right\} + O(n^{-1}\eta(n)). \end{aligned} \tag{76}$$

Again, by (28), (50), (56) and (74)–(76)

$$\lambda_n^{\frac{1}{2}}(b-a) = \frac{(2n+3)}{2}\pi + \frac{2(b-a)}{(2n+3)\pi}\left\{\frac{a_1'}{a_2'} + \sin\left(\frac{(2n+3)\pi}{(b-a)}\right)a + \zeta_a\right\}$$
$$- \operatorname{Im}\left\{\frac{i(b-a)}{(2n+3)\pi}\int_a^b e^{i\left(\frac{(2n+3)\pi}{(b-a)}\right)(x-a)}q(x)\,dx\right\}$$
$$+ O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)).$$

Theorem 1(ii) is proved by substituting the values of $\sin\zeta_a$ and $\cos\zeta_a$ given by (59), (60) respectively.

Similar to the proof of Theorem 1, Theorem 2 is proved by using Theorem 6 and Lemma 7.

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