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LATTICE GAUGE FIELD THEORY AND PRISMATIC SETS

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Abstract

We study prismatic sets analogously to simplicial sets except that realization involves prisms, i.e., products of simplices rather than just simplices. Particular examples are the prismatic subdivision of a simplicial set S and the prismatic star of S. Both have the same homotopy type as S and in particular the latter we use to study lattice gauge theory in the sense of Phillips and Stone. Thus for a Lie group G and a set of parallel transport functions defining the transition over faces of the simplices, we define a classifying map from the prismatic star to a prismatic version of the classifying space of G. In turn this defines a G-bundle over the prismatic star.

1. Introduction

In the study of global properties of locally trivial fibre bundles it is a fundamental difficulty that the usual combinatorial methods of algebraic topology depends on the use of simplicial complexes which structure behaves badly with respect to local trivializations. By a theorem of Johnson [11], the base and total space of a locally trivial smooth fibre bundle with projection $\pi : E \to B$ can be triangulated in such a way that π is a simplicial map. But obviously even in this case a general fibre is not a simplicial complex in any natural way. However such a fibre has a natural decomposition into prisms, i.e., products of simplices, and the whole triangulated bundle gives the basic example of a prismatic set, analogous to the notion of a simplicial set derived from a simplicial complex. Prismatic sets were introduced and used by the second author and R. Ljungmann in [7] (see also Ljungmann's thesis [12]) in order to construct an explicit fibre integration map in smooth Deligne cohomology, see also Dupont-Kamber [6]. But the important special case of the prismatic subdivision of a simplicial set was used in Akyar [1] in connection with "Lattice Gauge Theory" in the sense of Phillips-Stone [19], [21]. Similar constructions have been used in other connections, see e.g. McClure-Smith [16] or Brasselet-Teissier [2]. One can see Lüscher [13] for further information about Lattice Gauge Fields.

In this paper we shall give a more systematic treatment of prismatic sets and their properties but we shall concentrate on the applications to lattice gauge

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theory extending the work of Phillips and Stone to arbitrary simplicial sets and all dimensions. For an arbitrary simplicial set S and a given Lie group G together with a set of parallel transport functions in their sense, we construct a prismatic set $\bar{P}_{\cdot}S$ of the same homotopy type as S and a classifying map from $\bar{P}_{\cdot}S$ to a prismatic version of the standard model for BG, for a reference see Segal [22]. This is one of our main results (Theorem 8.1). Geometrically, for S a simplicial complex, $\bar{P}_{\cdot}S$ is closely related to the nerve of the covering by stars of vertices (Theorem 5.1). In turn this gives a principal G-bundle with a connection and thus in principle gives rise via the usual Chern-Weil and Chern-Simons Theory to explicit formulas for characteristic classes (Corollary 8.2). We shall return to this elsewhere. One can see Cheeger-Simons [3], Chern-Simons [4], Dupont [5], Freed [9], Witten [23] for further information about Chern-Simons Theory.

The paper is organized as follows:

In Chapter 2, prismatic sets are defined and their various realizations are studied.

The third chapter introduces the prismatic triangulation of a simplicial map and in particular of a simplicial set. Furthermore, we comment on the calculation of the homology of the geometric realization of a prismatic set.

In Chapter 4 we study prismatic sets associated to stars of simplicial complexes. It turns out that the prismatic set $\bar{P}_{\cdot}S$ given in this chapter in the case of a simplicial complex is the nerve of the covering by stars of vertices.

In the fifth chapter, we compare the two star simplicial sets and prove that there is a natural surjective map $\bar{p} : \bar{P}_{\cdot}S \to P_{\cdot}\,\mathrm{St}\,S_{\cdot}$. It turns out that this map is an isomorphism for $S = K^s$, where K is a simplicial complex.

In Chapter 6, we introduce a prismatic version of the classifying space. This is done by replacing the Lie group G by the singular simplicial set of continuous maps $\mathrm{Map}(\Delta^q, G)$.

In Chapter 7, we introduce the notion of "compatible transition functions" similar to the "parallel transport functions" of Phillips-Stone [19] for a simplicial complex K. We show how a given bundle on the realization of a simplicial set and so-called "admissible trivializations" give rise to a set of compatible transition functions and vice versa. We end the chapter with a remark on the relation between the compatible transition functions and parallel transport along a piecewise linear path.

Finally in the last chapter we construct the classifying map for a given set of compatible transition functions. For this we construct a prismatic map from $\bar{P}_{\cdot}S$ to the prismatic model for the classifying space constructed in Chapter 6.

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2. Prismatic Sets

Prismatic sets are similar to simplicial sets but they are realized by using prisms instead of only simplices.

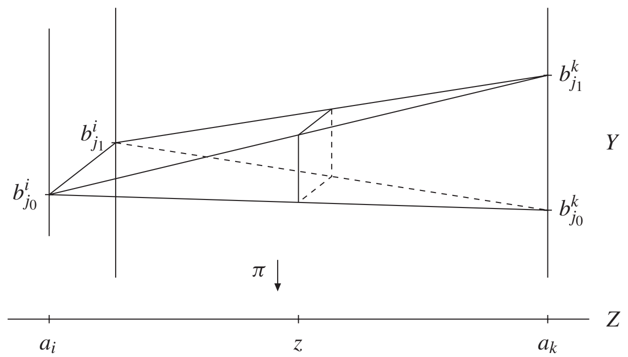
Let $\Delta^p = \left\{(t_0, \ldots, t_p) \in \mathsf{R}^{p+1} \;\middle|\; \sum_i t_i = 1, t_i \leq 1\right\}$ be a standard p-simplex given with barycentric coordinates. A prism is a product of simplices, that is, a set of the form $\Delta^{q_0 \cdots q_p} = \Delta^{q_0} \times \cdots \times \Delta^{q_p}$.

The motivating example is triangulated fibre bundles:

Example 2.1. Given a smooth fibre bundle $\pi : Y \to Z$ with $\dim Y = m+n$, $\dim Z = m$ and compact fibres possibly with boundary. By a theorem of Johnson [11], there are smooth triangulations K and L of Y and Z, respectively and a simplicial map $\pi' : K \to L$ in the following commutative diagram

$$\begin{tikzcd} {|K|} \arrow[r, "\approx"] \arrow[d, "{|\pi'|}"] & Y \arrow[d, "\pi"] \\ {|L|} \arrow[r, "\approx"] & Z, \end{tikzcd}$$

and the horizontal maps are homeomorphisms which are smooth on each simplex. Let K be an ordered simplicial complex as in Dwyer-Henn [8, Section 3] and let $|K| = \bigsqcup_{\tau \in K_k} \Delta^k \times \tau/\sim$, $k = 0, \ldots, \dim K$, be the geometric realization.



A simplex τ in K has vertices $\tau = (b^0_0, \ldots, b^0_{q_0} | \ldots | b^p_0, \ldots, b^p_{q_p})$ with $\sigma = (a_0, \ldots, a_p)$ in L such that $\pi'(b^i_j) = a_i$. So geometrically, for an open simplex

$\mathring{\sigma}$ in L, we have

$$\pi^{-1}(|\mathring{\sigma}|) \approx |\mathring{\sigma}| \times \bigsqcup_{\tau\in\pi^{-1}(\sigma)} \Delta^{q_0\cdots q_p} \times \tau.$$

We collect all these decompositions in the formal definition below using simplicial sets. For these we recall the notation but refer otherwise to Mac Lane [14], May [15].

DEFINITION 2.2. A simplicial set $S = \{S_q\}$ is a sequence of sets with face operators $d_i : S_q \to S_{q-1}$ and degeneracy operators $s_i : S_q \to S_{q+1}$, $i = 0, \ldots, q$, satisfying the *simplicial identities*:

$$d_i d_j = \begin{cases} d_{j-1}d_i & : \quad i < j \\ d_j d_{i+1} & : \quad i \geq j, \end{cases} \qquad s_i s_j = \begin{cases} s_{j+1}s_i & : \quad i \leq j \\ s_j s_{i-1} & : \quad i > j, \end{cases}$$

and

$$d_i s_j = \begin{cases} s_{j-1}d_i & : \quad i < j \\ \mathrm{id} & : \quad i = j, i = j+1 \\ s_j d_{i-1} & : \quad i > j+1. \end{cases}$$

EXAMPLE 2.3. A simplicial complex K gives a simplicial set where

$$K_p = \left\{ (a_{i_0}, \ldots, a_{i_p}) \;\middle|\; \begin{array}{l} \text{some non-decreasing sequences} \\ \text{for a given partial ordering of } K_0 \end{array} \right\}$$

is the set of p-simplices.

EXAMPLE 2.4. Given an open cover $\mathcal{U} = \{U_i\}$ of a smooth manifold Z we have the *nerve* $N\mathcal{U} = \{N\mathcal{U}(p)\}$ of the covering, where

$$N\mathcal{U}(p) = \bigsqcup_{i_0,\ldots,i_p} U_{i_0} \cap \cdots \cap U_{i_p},$$

and $(i_0, \ldots, i_p)$ is non-decreasing for a given partial order of the index set.

Let us denote $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0,\ldots,i_p}$. Then $N\mathcal{U}$ is a simplicial manifold, where the face and degeneracy maps come from the inclusions

$$d_j : U_{i_0,\ldots,i_p} \to U_{i_0,\ldots,\hat{i}_j,\ldots,i_p}$$
$$s_j : U_{i_0,\ldots,i_p} \to U_{i_0,\ldots,i_j,i_j,\ldots,i_p}.$$

That is, $N\mathcal{U}(p)$ is a smooth manifold for each p and the face and degeneracy maps are smooth. There is also a corresponding simplicial set $N_d\mathcal{U} = \{N_d\mathcal{U}(p)\}$ called the *discrete nerve* of the covering. Here $N_d\mathcal{U}(p)$ is simply the set consisting of an element for each non-empty intersection of $p+1$ open sets from $\mathcal{U}$. So there is a natural forgetful map $N\mathcal{U} \to N_d\mathcal{U}$.

NOTE. If S has only face operators, then it is called a Δ-set.

DEFINITION 2.5. Given $p \geq 0$, a $(p+1)$-multi-simplicial set S is a sequence $\{S_{q_0,\ldots,q_p}\}$ which is a simplicial set in each variable q_i, $i = 0, \ldots, p$ and such that the face and degeneracy operators

$$\begin{aligned} d_i^k &: S_{q_0,\ldots,q_p} \to S_{q_0,\ldots,q_k-1,\ldots,q_p} \\ s_i^k &: S_{q_0,\ldots,q_p} \to S_{q_0,\ldots,q_k+1,\ldots,q_p} \end{aligned}$$

commute with d_j^l, s_j^l for $k \neq l$ and $k, l = 0, \ldots, p$.

DEFINITION 2.6. i) A *prismatic set* P is a sequence $\{P_p\} = \{P_{p,q_0,\ldots,q_p}\}$ of $(p+1)$-multi-simplicial sets together with face operators

$$d_k : P_{p,q_0,\ldots,q_p} \to P_{p-1,q_0,\ldots,\hat{q}_k,\ldots,q_p}$$

commuting with d_j^l and s_j^l (interpreting $d_j^k = s_j^k = \mathrm{id}$ on the right) such that $\{P_p\}$ is a Δ-set.

ii) A prismatic set is called a *strong prismatic set* if similarly there are given degeneracy operators

$$s_k : P_{p,q_0,\ldots,q_p} \to P_{p+1,q_0,\ldots,q_k,q_k,\ldots,q_p}$$

making $\{P_p\}$ a simplicial set.

REMARK 1. We can also give another definition of a prismatic set in terms of functors of categories as follows:

Let $\mathbf{\Delta}$ be the simplicial category with objects $[n] = (0, \ldots, n)$ and non-decreasing functions as morphisms. Furthermore let $\mathbf{\Delta}_{\mathrm{in}} \subseteq \mathbf{\Delta}$ denote the subcategory allowing only *strictly increasing* functions as morphisms. In the category of small categories **Cat** consider for each $p = 0, 1, 2, \ldots$, the $(p+1)$-multi-simplicial category

$$\mathbf{\Pi}(p) = \mathbf{\Delta} \times \cdots \times \mathbf{\Delta} \qquad (p+1 \text{ factors}).$$

Now define a functor $\mathbf{\Pi}^{\mathrm{op}} : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}$ the category of small categories by $\mathbf{\Pi}^{\mathrm{op}}([p]) = \mathbf{\Pi}(p)^{\mathrm{op}}$. This gives a simplicial category with the k-th face map $d_k : \mathbf{\Pi}^{\mathrm{op}}(p) \to \mathbf{\Pi}^{\mathrm{op}}(p-1)$, $k = 0, \ldots, p$, given by deleting the k-th factor, and similarly the k-th degeneracy map given by a repetition. The Grothendieck construction for the *lax functor* $\mathbf{\Pi}_{\mathrm{in}}^{\mathrm{op}} = \mathbf{\Pi}^{\mathrm{op}}|\mathbf{\Delta}_{\mathrm{in}}^{\mathrm{op}}$ (cf. Goerss-Jardine [10, Chap. IX.3]) provides a small category $L(\mathbf{\Pi}_{\mathrm{in}}^{\mathrm{op}})$ together with a projection $\pi : L(\mathbf{\Pi}^{\text{op}}_{\text{in}}) \to \mathbf{\Delta}^{\text{op}}_{\text{in}}$. In fact objects of $L(\mathbf{\Pi}^{\text{op}}_{\text{in}})$ are just pairs (p, J) with $p \in \mathbf{\Delta}$ and $J \in \mathbf{\Pi}(p)$.

A *prismatic set* (respectively *strong prismatic set*) is now a functor: $L(\mathbf{\Pi}^{\text{op}}_{\text{in}}) \to$ **Sets** (respectively $L(\mathbf{\Pi}^{\text{op}}) \to$ **Sets**).

EXAMPLE 2.1 CONTINUED. Recall from [8] that an ordered simplicial complex K gives rise to a simplicial set K^s with the same realization. Here simplices are just non-decreasing tuples. That is,

$$K^s_n = \left\{ (a_{i_0}, \dots, a_{i_n}) \;\middle|\; \begin{array}{l} (a_{i_0}, \dots, a_{i_n}) \text{ a simplex of } K \\ \text{(with repetitions) } i_0 \le \cdots \le i_n. \end{array} \right\}$$

Similarly the situation in Example 2.1 gives a prismatic set as follows: $P_p(K/L)_{q_0 \dots q_p}$ consists of pairs (σ, τ), where σ is a $(p+1)$-tuple $\sigma = (a_0, \dots, a_p) \in L^s_p$ and $\tau = (b^0_0, \dots, b^0_{q_0} | \dots | b^p_0, \dots, b^p_{q_p})$ is a $(p+q+1)$-tuple satisfying the following:

1) The set of distinct vertices gives a simplex in K
2) Each group $|b^i_0, \dots, b^i_{q_i}|$ is non-decreasing.

Then we have face and degeneracy operators d^i_j, s^i_j deleting and repeating respectively each element in the groupings, whereas d_k, s_k deletes and repeates each grouping, respectively. It is now straight forward to check that this is a (strong) prismatic set.

EXAMPLE 2.7. For a given simplicial set S, consider the $(p+1)$-multi-simplicial set $E_p S = S \times \cdots \times S$, $(p+1)$-times. $d_i : E_p S \to E_{p-1} S$ is the projection which deletes the i-th factor. Similarly, the diagonal map $s_i : E_p S \to E_{p+1} S$ repeats the i-th factor. This is a strong prismatic set.

Prismatic sets have various realizations.

DEFINITION 2.8. First, we have for each p the thin (geometric) realization

$$|P_p| = \bigsqcup_{q_0, \dots, q_p} \Delta^{q_0 \dots q_p} \times P_{p, q_0, \dots, q_p} / \sim \tag{2.9}$$

with equivalence relation "$\sim$" generated by the face and degeneracy maps

$$\begin{aligned} \varepsilon^i_j &: \Delta^{q_0 \dots q_i \dots q_p} \to \Delta^{q_0 \dots q_i + 1 \dots q_p} \qquad \text{and} \\ \eta^i_j &: \Delta^{q_0 \dots q_i \dots q_p} \to \Delta^{q_0 \dots q_i - 1 \dots q_p}, \end{aligned}$$

respectively. Now $\{|P_p|\}$ is a Δ-space hence it gives a fat realization

$$\| |P_\cdot| \| = \bigsqcup_{p \ge 0} \Delta^p \times |P_p| / \sim \tag{2.10}$$

Here the face operators are $\pi_i \times d_i : \Delta^{q_0\cdots q_p} \times P_p \to \Delta^{q_0\cdots\hat{q}_i\cdots q_p} \times P_{p-1}$ where π_i is the projection $\pi_i : \Delta^{q_0\cdots q_p} \to \Delta^{q_0\cdots\hat{q}_i\cdots q_p}$ deleting the i-th factor. The further equivalence relation on $\||P_\cdot|\|$ given in (2.10) is thus generated by

$$(\varepsilon^i t, s, \sigma) \sim (t, \pi_i s, d_i\sigma), \quad t \in \Delta^{p-1}, \quad s \in \Delta^{q_0\cdots q_p}, \quad \sigma \in P_{p,q_0,\ldots,q_p}.$$

REMARK 2. For strong prismatic sets, the degeneracy operators s_i are determined by the diagonal map $\Delta_i : \Delta^{q_0\cdots q_p} \to \Delta^{q_0\cdots q_i q_i\cdots q_p}$ repeating the i-th factor. Hence for a strong prismatic set we have a thin realization

$$|P_\cdot| = ||P_\cdot||/\sim$$

given by the above and the further relation

$$(\eta^i t, s, \sigma) \sim (t, \Delta_i s, s_i\sigma), \quad t \in \Delta^{p+1}, \quad s \in \Delta^{q_0,\ldots,q_p}, \quad \sigma \in P_{p,q_0,\ldots,q_p}.$$

EXAMPLE 2.11. For a given simplicial set S and $E_p S$ as in Example 2.7 we have $\||E_\cdot S_\cdot|\|$ as the fat realization of the simplicial space whose p-th term is $|S_\cdot| \times \cdots \times |S_\cdot|$, $(p+1)$-times. This is a contractible space. In fact it is well-known that in general for any space X the simplicial space $E_p X = X \times \cdots \times X$, $(p+1)$-times, has a contractible fat realization.

3. Prismatic Triangulation

Let us return to the case of a triangulated fibre bundle $|K| \to |L|$. In this case the natural map

$$P_p(K/L)_{q_0,\ldots,q_p} \to K_{q_0+\cdots+q_p+p}$$

induces a homeomorphism

$$\begin{array}{ccc} |P_\cdot(K/L)| & \xrightarrow{\approx} & |K| \\ \big\downarrow{\scriptstyle |\pi'|} & & \big\downarrow{\scriptstyle \pi} \\ |L| & \xrightarrow[=]{} & |L| \end{array}$$

The top horizontal map in this diagram we shall call the prismatic triangulation homeomorphism

$$\lambda : |P_\cdot(K/L)| \xrightarrow{\approx} |K|.$$

It is induced by

(3.1) $$\lambda(t, s^0, \ldots, s^p, (\sigma, \tau)) = (t_0 s^0, \ldots, t_p s^p, \tau) \in \Delta^{p+q} \times K_{p+q},$$

where $(t, s, \sigma, \tau) \in \Delta^p \times \Delta^{q_0\cdots q_p} \times P_p(K/L)_{q_0\cdots q_p}$ and $q = q_0 + \cdots + q_p$.

NOTE. If $\mathring{\sigma}$ is an open p-simplex in L then λ provides a natural trivialization of $|K|_\sigma = \pi^{-1}(|\mathring{\sigma}|)$, that is, a homeomorphism

$$\lambda : |\mathring{\sigma}| \times |P_p(K/\sigma)| \xrightarrow{\approx} |K|_\sigma .$$

We can generalize this construction to any simplicial map:

EXAMPLE 3.2 (Prismatic triangulation of a simplicial map). Let $f : S \to \bar{S}$ be a simplicial map of simplicial sets and define $P_{\cdot}(f)$ by

$$P_p(f)_{q_0,\ldots,q_p} = \{(\bar{\sigma}, \sigma) \in \bar{S}_p \times S_{q_0+\cdots+q_p+p} \mid f(\sigma) = \mu_{q_0,\ldots,q_p}(\bar{\sigma})\}$$

where the corresponding map

$$\mu^{q_0,\ldots,q_p} : \Delta^{q_0+\cdots+q_p+p} \to \Delta^p$$

is given by

$$\{0, \ldots, q_0| \ldots |q_0 + \ldots + q_{p-1} + p, \ldots, q_0 + \cdots + q_p + p\} \to \{0, \ldots, p\}.$$

Explicitly

$$\mu_{q_0,\ldots,q_p} = \hat{s}_{q+p} \circ s_{(q_0+\ldots+q_p+p-1)\ldots(q_0+\cdots+q_{p-1}+p)} \circ \cdots \circ \hat{s}_{q_0} \circ s_{(q_0-1)\ldots(0)},$$

where the $\hat{s}_i$ are left out and

$$s_{(q_0+\ldots+q_i+i-1)\ldots(q_0+\cdots+q_{i-1}+i)} = s_{q_0+\cdots+q_i+i-1} \circ \cdots \circ s_{q_0+\cdots+q_{i-1}+i},$$

$i = 0, \ldots, p$. The boundary maps in the fibre direction

$$d^i_j : P_p(f)_{q_0,\ldots,q_p} \to P_p(f)_{q_0,\ldots,q_i-1,\ldots,q_p}$$

are inherited from the face operators defined on S_{q+p}. Thus

$$d^i_j(\bar{\sigma}, \sigma) = (\bar{\sigma}, d_{q_0+\cdots+q_{i-1}+i+j}\sigma).$$

Similarly the degeneracy maps s^i_j on $P_p(f)_{q_0,\ldots,q_p}$

$$s^i_j : P_p(f)_{q_0,\ldots,q_p} \to P_p(f)_{q_0,\ldots,q_i+1,\ldots,q_p}$$

are inherited from the ones on S_{q+p}. That is,

$$s^i_j(\bar{\sigma}, \sigma) = (\bar{\sigma}, s_{q_0+\cdots+q_{i-1}+i+j}\sigma).$$

The boundary maps

$$d_i : P_p(f)_{q_0,\ldots,q_p} \to P_{p-1}(f)_{q_0,\ldots,\hat{q}_i,\ldots,q_p}$$

are determined by the boundary maps defined on both S_{q+p} and $\bar{S}_p$. Thus

$$d_i(\bar{\sigma}, \sigma) = (d_i\bar{\sigma}, d_{q_0+\cdots+q_{i-1}+i} \circ \cdots \circ d_{q_0+\cdots+q_i+i}\sigma),$$

here the composition of the face operators can be shortly written as

$$d_{(q_0+\cdots+q_{i-1}+i)\ldots(q_0+\cdots+q_i+i)} = d_{q_0+\cdots+q_{i-1}+i} \circ \cdots \circ d_{q_0+\cdots+q_i+i}.$$

NOTE. $P_.(f)$ is a prismatic set, but in general not a strong one as we shall see in Remark 4 below.

THEOREM 3.3. *There is a pullback diagram*

$$\begin{tikzcd} \||P_.(f)|\| \arrow[r, "\lambda"] \arrow[d, "\|f\|"] & |S_.| \arrow[d, "|f|"] \\ \|\bar{S}_.\| \arrow[r, "\approx"] & |\bar{S}_.| \end{tikzcd}$$

In particular λ is a homotopy equivalence.

PROOF. The map $\lambda : \Delta^p \times \Delta^{q_0\cdots q_p} \times P_p(f)_{q_0\ldots q_p} \to \Delta^{q+p} \times S_{q+p}$ is given by $\lambda(t, s, \bar{\sigma}, \sigma) = (t_0 s^0, \ldots, t_p s^p, \sigma)$. For $t \in \mathring{\Delta}^p$, $u \in \mathring{\Delta}^{q+p}$ is uniquely of the form $u = (t_0 s^0, \ldots, t_p s^p)$, that is, $(u, \sigma) = \lambda(t, s, \bar{\sigma}, \sigma)$. In fact $\lambda : \mathring{\Delta}^p \times \mathring{\Delta}^{q_0\cdots q_p} \to \mathring{\Delta}^{p+q}$ is a diffeomorphism exhibiting $\mathring{\Delta}^{p+q}$ as the $(p+1)$-st join

$$\lambda : \mathring{\Delta}^{q_0} * \cdots * \mathring{\Delta}^{q_p} \xrightarrow{\approx} \mathring{\Delta}^{p+q}.$$

(Here the join is made using only the open interval (0,1).) The commutativity of the diagram follows from the definition of $P_.(f)$ hence λ factors over the pullback $\|\bar{S}_.\| \times_{|\bar{S}_.|} |S_.|$ in the diagram

$$\begin{tikzcd} \||P_.(f)|\| \arrow[r, "\Lambda"] \arrow[d, "\|f\|"] & \|\bar{S}_.\| \times_{|\bar{S}_.|} |S_.| \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & |S_.| \arrow[d, "|f|"] \\ \|\bar{S}_.\| \arrow[r, "\mathrm{id}"] & \|\bar{S}_.\| \arrow[r] & |\bar{S}_.| \end{tikzcd}$$

and we want to show that Λ is a homeomorphism.

Elements in the pullback $\|\bar{S}_.\| \times_{|\bar{S}_.|} |S_.|$ are represented by pairs $((t, \bar{\sigma}), (u, \sigma))$ such that $f(\sigma) = \mu_{q_0,\ldots,q_p}(\bar{\sigma})$ and $t = \mu^{q_0,\ldots,q_p}(u)$, where $\sigma \in S_{q+p}$, $\bar{\sigma} \in \bar{S}_q$. It follows from the above that over each open p-simplex $\|\mathring{\bar{\sigma}}\|$ in $|\bar{S}_.|$, Λ provides a homeomorphism onto its image

$$\Lambda : \|f\|^{-1}(\|\mathring{\bar{\sigma}}\|) \xrightarrow{\approx} (\mathrm{pr}_1)^{-1}(\|\mathring{\bar{\sigma}}\|).$$

Now Λ is shown to be a homeomorphism by induction over the skeleton of $\|\bar{S}_.\|$.

REMARK 3. For the case of a simplicial complex, notice the similarity of the above theorem with Example 2.1 cf. the note following (3.1).

EXAMPLE 3.4 (Prismatic triangulation of a simplicial set). Let S be a simplicial set and $\bar{S} = *$ the simplicial set with one element in each degree. Here $P_p(f) = P_p S$ is called the p-th *prismatic subdivision* of S and for each $t \in \mathring{\Delta}^p$ the map $\lambda_p(t, -) : |P_p S_.| \to |S_.|$ is a homeomorphism. In this case, Theorem 3.3 gives a homeomorphism $\Lambda : \||P_. S_.|\| \xrightarrow{\approx} \|*\| \times |S_.|$, where $\|*\| = \bigsqcup_n \Delta^n / \partial \Delta^n$. In particular $\lambda : \||P_. S_.|\| \to |S_.|$ is a homotopy equivalence. We shall call $P_. S$ the *prismatic triangulation* of S.

For later use, let us give the explicit construction of the $p+1$-prismatic set $P_. S$ and its realization:

$$P_p S_{q_0, \ldots, q_p} = S_{q_0 + \cdots + q_p + p}.$$

The face operators

$$d^i_j : P_p S_{q_0, \ldots, q_i, \ldots, q_p} = S_{q+p} \to P_p S_{q_0, \ldots, q_i - 1, \ldots, q_p} = S_{q+p-1}$$

are defined by

$$d^i_j := d_{q_0 + \cdots + q_{i-1} + i + j},$$

$j = 0, \ldots, q_i$. Similarly, the degeneracy operators

$$s^i_j : P_p S_{q_0, \ldots, q_i, \ldots, q_p} = S_{q+p} \to P_p S_{q_0, \ldots, q_i + 1, \ldots, q_i} = S_{q+p+1}$$

can be defined by

$$s^i_j := s_{q_0 + \cdots + q_{i-1} + i + j},$$

$j = 0, \ldots, q_i$. The face maps

$$d_{(i)} : P_p S_{q_0, \ldots, q_p} \to P_{p-1} S_{q_0, \ldots, \hat{q}_i, \ldots, q_p}$$

are the operators corresponding to the inclusions

$$\Delta^{q_0 + \cdots + \hat{q}_i + \cdots + q_p + p - 1} \to \Delta^{q_0 + \cdots + \cdots + q_p + p}$$

deleting the $q_i + 1$ basis vectors with indices $q_0 + \cdots + q_{i-1} + i, \ldots, q_0 + \cdots + q_i + i$.

REMARK 4. In terms of category theory, the prismatic triangulation can be considered as induced by the functor $\mathscr{P} : L(\mathbf{\Pi}^{\mathrm{op}}_{\mathrm{in}}) \to \mathbf{\Delta}^{\mathrm{op}}$ defined by

$$\begin{aligned}\mathscr{P}(p, ([q_0] \times \cdots \times [q_p])) = (0, \ldots, q_0 | q_0 + 1, \ldots, \\ q_0 + q_1 + 1 | \ldots | q_0 + \cdots + q_0 + \cdots + q_{p-1} + p, \ldots, q + p),\end{aligned}$$

that is, it takes the product of ordinals to an ordinal in the simplicial category $\mathbf{\Delta}$. Note that $P_{.}S$ is not a strong prismatic set since the morphism repeating one of the $[q_i]$'s above would not map to a non-decreasing sequence by the functor defined above.

Now we turn to the realizations. For the sequences of spaces $\{|P_{.}S_{.}|\}$, we obtain the fat realization:

$$\||P_{.}S_{.}|\| = \bigsqcup_{p\geq 0} \Delta^p \times |P_pS_{.}|/\sim,$$

where

$$|P_pS_{.}| = \bigsqcup \Delta^{q_0\cdots q_p} \times S_{q_0+\cdots+q_p+p}/\sim$$

and the face operators $|d_i| : |P_pS_{.}| \to |P_{p-1}S_{.}|$ are given by $|d_i| = \pi_i \times d_i$ with $\pi_i : \Delta^{q_0\cdots q_p} \to \Delta^{q_0\cdots\hat{q}_i\cdots q_p}$ beeing the natural projection.

Note that $\lambda_p : \Delta^p \times |P_pS_{.}| \to |S_{.}|$ satisfies

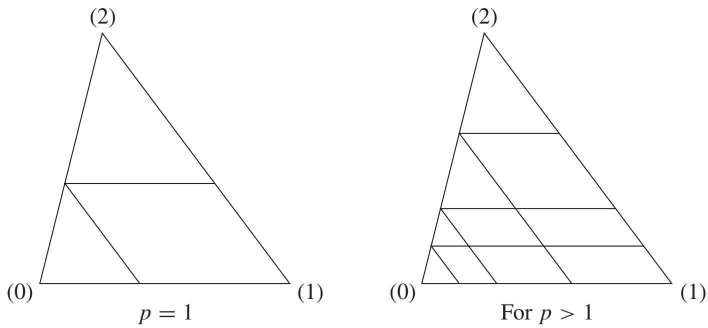
$$\lambda_p \circ (\varepsilon^i \times \mathrm{id}) = \lambda_{p-1} \circ (\mathrm{id} \times d_i).$$

Thus λ_p induces the map λ on the fat realization.

Let $\||P_{.}S_{.}|\|^p$ respectively $\||S_{.}|\|^p$ denote the sub-complexes generated by $\Delta^p \times |P_pS_{.}|$ respectively $\Delta^p \times |S_{.}|$. Then the restriction of Λ to $\||P_{.}S_{.}|\|^p$ is given by

$$\Lambda_p(t, s, \sigma) = (t, \lambda_p(t, s, \sigma)).$$

Example 3.5. For $S = \Delta^2$ via the map $\lambda_p : \||P_{.}S_{.}|\|^p \to |S_{.}|$, the image of the p-th prismatic subdivision $\{t\} \times |P_pS_{.}|$ is shown in the figures:

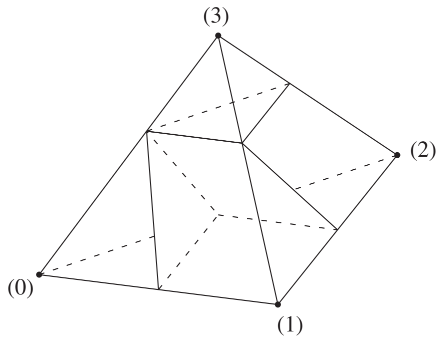


Here the "division points": $1 \geq x_1 \geq \cdots \geq x_p \geq 0$ are given by the interior coordinates

$$x_1 = 1 - t_0, x_2 = 1 - t_0 - t_1, \ldots, x_{p-1} = 1 - t_0 - \cdots t_{p-1}, x_p = t_p$$

for a point with barycentric coordinates $t = (t_0, \ldots, t_p)$.

For $p = 1$ and $S = \Delta^3$ the image is



COROLLARY 3.6. *The map Λ_p induce a homeomorphism*

$$\Lambda : \||P_{\cdot}S_{\cdot}|\| \to \||S_{\cdot}|\| \approx \|*\| \times |S_{\cdot}|.$$

COROLLARY 3.7. *The composite map* $\mathrm{proj}_2 \circ \Lambda = \lambda$

$$\||P_{\cdot}S_{\cdot}|\| \to \||S_{\cdot}|\| \to |S_{\cdot}|$$

is a homotopy equivalence.

REMARK 5. We can calculate the homology of the geometric realization of a prismatic set as follows:

A prismatic set P has a double complex $(C_{p,n}(P), \partial_V, \partial_H)$. Here

$$C_{p,n}(P) = \bigoplus_{q_0+\cdots+q_p=n} C_{p,q_0,\ldots,q_p}(P)$$

is the associated chain complex $C_p(P)$ generated by $P_{p,q_0,\ldots,q_p}$. The vertical boundary map is defined by using boundary maps in the fibre direction

$$\partial^i_F : C_{p,q_0,\ldots,q_p}(P) \to C_{p,q_0,\ldots,q_i-1,\ldots,q_p}(P)$$

given by $\partial^i_F = \sum(-1)^j d^i_j$, where, if $q_i = 0$ then $\partial^i_F = 0$. The total vertical boundary map is then

$$\partial_V = \partial^0{}_F + (-1)^{q_0+1}\partial^1{}_F + \cdots + (-1)^{q_0+\cdots+q_{p-1}+p}\partial^p{}_F.$$

There is also a horizontal boundary map

$$\partial_H = \partial_0 + (-1)^{q_0+1}\partial_1 + \cdots + (-1)^{q_0+\cdots+q_{p-1}+p}\partial_p,$$

where

$$\partial_k = \begin{cases} 0 & : \text{ if } q_k > 0 \\ d_k & : \text{ if } q_k = 0, \end{cases}$$

so that $\partial = \partial_V + \partial_H$ is a boundary map in the total complex $C_*(P)$ which is the cellular chain complex for the geometric realization. Hence it calculates the homology. In the case of $P_.(f)$ for $f : S \to \bar{S}$ a simplicial map, the double complex gives rise to a spectral sequence which for a triangulated fibre bundle is the usual Leray-Serre spectral sequence.

REMARK 6. For each p and each $t \in \mathring{\Delta}^p$, $\lambda_p(t)^{-1} : |S_.| \to \{t\} \times |P_p S_.|$ induces a map of cellular chain complexes

$$aw : C_*(S) \to C_{p,*}(P)$$

given by

$$aw(x) = \sum_{q_0+\cdots+q_p=n} s_{q_0+\cdots+q_{p-1}+p-1} \circ \cdots \circ s_{q_0}(x)_{(q_0,\ldots,q_p)},$$

where $x \in S_n$. This is related to the Alexander-Whitney map $C_*(S) \to C_*(S)^{\otimes(p+1)}$.

4. Prismatic Sets and Stars of Simplicial Complexes

For a simplicial set S and the prismatic triangulation $P_. S$ there is another closely related prismatic set $\bar{P}_. S$ which, as we shall see for a simplicial complex, is the nerve of the covering by stars of vertices considered as a prismatic set.

DEFINITION 4.1. For S a simplicial set let $\bar{P}_. S$ be the prismatic set given by

$$\bar{P}_p S_{q_0,\ldots,q_p} := S_{q_0+\cdots+q_p+2p+1},$$

where face and degeneracy operators on $\bar{P}_p S_{q_0,\ldots,q_p}$ are inherited from the ones of S_{q+2p+1} as follows:

Let $q = q_0 + \cdots + q_p$, the face operators

$$d^i_j : S_{q+2p+1} = \bar{P}_p S_{q_0,\ldots,q_p} \to S_{q+2p} = \bar{P}_p S_{q_0,\ldots,q_i-1,\ldots,q_p}$$

are defined by

$$d^i_j := d_{q_0+\cdots+q_{i-1}+2i+j}, \quad j = 0, \ldots, q_i \text{ but } j \neq q_i + 1,\ i = 0, \ldots, p.$$

So $\bar{P}_p S_{q_0,\ldots,q_p}$ has only $q + p + 1$ face operators, i.e., we skip the following $p + 1$ face operators

$$\{d_{q_0+1}, d_{q_0+q_1+3}, \ldots, d_{q+2p+1}\}.$$

Similarly the degeneracy operators

$$s^i_j : S_{q+2p+1} \to S_{q+2p+2}$$

can be defined by

$$s^i_j := s_{q_0+\ldots+q_{i-1}+2i+j}, \quad j = 0, \ldots, q_i, \text{ but } j \neq q_i + 1, \ i = 0, \ldots, p.$$

Furthermore the face operators

$$d_{(i)} : S_{q+2p+1} = \bar{P}_p S_{q_0,\ldots,q_p} \to S_{q+2p-q_i-1} = \bar{P}_{p-1} S_{q_0,\ldots,\hat{q}_i,\ldots,q_p}$$

corresponds to the inclusions

$$\Delta^{q+2p-q_i-1} \to \Delta^{q+2p+1}$$

deleting the $q_i + 2$ basis vectors with indices $q_0 + \cdots + q_{i-1} + 2i, \ldots, q_0 + \cdots + q_i + 2i + 1$. That is,

$$d_{(i)} = d_{q_0+\cdots+q_{i-1}+2i} \circ \cdots \circ d_{q_0+\cdots+q_i+2i+1}, \quad i = 0, \ldots, p.$$

Remark. As $P_{.}S$, $\bar{P}_{.}S$ is a prismatic set but in general not a strong prismatic set.

Realization of $\bar{P}_{.}S$. The equivalence relation on

$$\||\bar{P}_{.}S_{.}|\| = \bigsqcup_{p\geq 0} \Delta^p \times \Delta^{q_0\ldots q_p} \times \bar{P}_p S_{q_0,\ldots,q_p}/\sim$$

is given as described for (2.10).

The relations of $\bar{P}_{.}S$ to S and $P_{.}S$ are as follows:

Proposition 4.2. *Let $i : \|S_{.}\| \hookrightarrow \||\bar{P}_{.}S_{.}|\|$ be an inclusion defined for $(t, x) \in \Delta^p \times S_p$ by*

$$i(t, x) = (t, 1, s_0 \circ \cdots \circ s_p x) \in \Delta^p \times (\Delta^0)^{p+1} \times S_{2p+1} \subseteq \Delta^p \times |\bar{P}_p S_{.}|,$$

and let $r : \||\bar{P}_{.}S_{.}|\| \to \|S_{.}\|$ be the retraction defined for $(t, s, y) \in \Delta^p \times \Delta^{q_0\ldots q_p} \times S_{q+2p+1}$

$$r(t, s, y) = (t, d_{(0)\ldots(q_0)} \circ \hat{d}_{q_0+1} \circ \cdots \circ d_{(q_0+\cdots+q_{p-1}+2p)\ldots(q+2p)} \circ \hat{d}_{q+2p+1} y),$$

where the $\hat{d}_i$ are left out and $d_{(q_0+\cdots+q_{i-1}+2i)\ldots(q_0+\cdots+q_i+2i)} = d_{q_0+\cdots+q_{i-1}+2i} \circ \cdots \circ d_{q_0+\cdots+q_i+2i}$, $i = 0, \ldots, p$.

(1) *i is a deformation retract with the retraction r.*

(2) *There is a commutative diagram of homotopy equivalences*

$$\begin{tikzcd}
\|S_.\| \arrow[r, "i"] \arrow[dr] \arrow[ddr] & \||\bar{P}_.S_.|\| \arrow[d, "f"] \\
& \||P_.S_.|\| \arrow[d, "\Lambda"] \\
& \||S_.|\|
\end{tikzcd}$$

where $f : \Delta^p \times \Delta^{q_0 \cdots q_p} \times S_{q+2p+1} \to \Delta^p \times \Delta^{q_0 \cdots q_p} \times S_{q+p}$ *takes* $(t, s^0, \dots, s^p, x)$ *to* $(t, s^0, \dots, s^p, d_{q_0+1} \circ d_{q_0+q_1+3} \circ \cdots \circ d_{q+2p+1}x)$, $x \in S_{q+2p+1}$.

Here Λ is a homeomorphism by Corollary 3.6 and $\Lambda \circ f \circ i : \|S_.\| \to \||S_.|\| \approx \|*\| \times |S_.|$ is given by

$$\begin{aligned}
\Lambda \circ f \circ i(t, x) &= \Lambda \circ f(t, 1, s_0 \circ \cdots \circ s_p x) \\
&= \Lambda(t, 1, d_1 \circ d_3 \circ \cdots \circ d_{2p+1} \circ s_0 \circ \cdots \circ s_p x) \\
&= (t, \lambda(t, 1, d_1 \circ d_3 \circ \cdots \circ d_{2p+1} \circ s_0 \circ \cdots \circ s_p x)) \\
&= (t, t, d_1 \circ d_3 \circ \cdots \circ d_{2p+1} \circ s_0 \circ \cdots \circ s_p x) \\
&= (t, t, x)
\end{aligned}$$

which is clearly a homotopy equivalence. See [1] for further details of the proof.

For an ordered simplicial complex K, there is another prismatic complex defined using the stars of simplices. Classically, the open star $\mathrm{St}(\sigma)$ of a simplex σ in the realization $|K|$ is the union of all open simplices whose face is σ. The *star complex* is the union of closures

$$\bigcup_{\sigma \text{ simplex}} |\sigma| \times \overline{\mathrm{St}(\sigma)} = \bigcup_{(\sigma,\tau)} |\sigma| \times |\tau| \subseteq |K| \times |K|,$$

where (σ, τ) runs through pairs of simplices which are both faces of another simplex of K. This is a sub-complex of $|K| \times |K|$.

We describe the associated simplicial subset $\mathrm{St}(K^s) \subseteq (K^s \times K^s)_.$ as follows:

Let $(\sigma, \tau) \in K \times K$, where $\sigma = (a_{i_0}, \dots, a_{i_p})$, $\tau = (b_{j_0}, \dots, b_{j_r})$ such that there is another simplex $\sigma' = \sigma \cup \tau = (c_{k_0}, \dots, c_{k_n})$. By allowing repetitions and taking $\sigma' \in K^s$, we can assume $n = p + r + 1$ so that either $c_{k_s} = a_{i_t}$ or $c_{k_s} = b_{j_u}$, where $t = 0, \dots, p$, $u = 0, \dots, r$. Also we can assume $c_{k_n} = a_{i_p}$, and if $a_{i_t} = b_{j_u}$ then b_{j_u} comes before a_{i_t}. In other words (σ, τ) is of the form

$$\sigma = d_{(\nu_1)\dots(\nu_{r+1})}\sigma', \qquad \tau = d_{(\xi_1)\dots(\xi_{p+1})}\sigma',$$

where $0 \le \nu_1 < \cdots < \nu_{r+1} < n$ and $0 \le \xi_1 < \cdots < \xi_{p+1} \le n$ and $\xi_k \neq \nu_l$, $\forall k, l$. Therefore replacing (σ, τ) by $(s_{(\nu_{r+1})\dots(\nu_1)} \circ d_{(\nu_1)\dots(\nu_{r+1})}\sigma', s_{(\xi_{p+1})\dots(\xi_1)} \circ d_{(\xi_1)\dots(\xi_{p+1})}\sigma')$ in the product simplicial set $(K^s \times K^s)_.$, we arrive at the following definition for a general simplicial set S.

DEFINITION 4.3. Let $(S \times S)_.$ denote the product simplicial set with diagonal face and degeneracy operators. Let $\mathrm{St}(S)$ be the simplicial subset of $(S \times S)_.$ containing all simplices in degree $n = p + r + 1 = \deg(\sigma')$ of the form

$$(s_{(\nu_{r+1})\dots(\nu_1)} \circ d_{(\nu_1)\dots(\nu_{r+1})}\sigma', s_{(\xi_{p+1})\dots(\xi_1)} \circ d_{(\xi_1)\dots(\xi_{p+1})}\sigma'),$$

where $0 \le \nu_1 < \cdots < \nu_{r+1} < n$ and $0 \le \xi_1 < \cdots < \xi_{p+1} \le n$ with $\xi_k \neq \nu_l$, $\forall k, l$ as above. Here $s_{(\nu_{r+1})\dots(\nu_1)} = s_{\nu_{r+1}} \circ \cdots \circ s_{\nu_1}$, $d_{(\nu_1)\dots(\nu_{r+1})} = d_{\nu_1} \circ \cdots \circ d_{\nu_{r+1}}$, $s_{(\xi_{p+1})\dots(\xi_1)} = s_{\xi_{p+1}} \circ \cdots \circ s_{\xi_1}$ and $d_{(\xi_1)\dots(\xi_{p+1})} = d_{\xi_1} \circ \cdots \circ d_{\xi_{p+1}}$.

REMARK 7. The projection on the first factor $\pi_1 : S \times S \to S$ gives a simplicial map $\pi_1 : \mathrm{St}(S)_. \to S$. Hence, we obtain a prismatic set $P_. \mathrm{St}(S)_. = P_.(\pi_1)$ as in Example 3.2. Here for $r = q_0 + \cdots + q_p + p$ and $\sigma = s_{(\nu_{r+1})\dots(\nu_1)} \circ d_{(\nu_1)\dots(\nu_{r+1})}\sigma' = \mu_{q_0,\dots,q_p}\bar{\sigma}$, $\tau = s_{(\xi_{p+1})\dots(\xi_1)} \circ d_{(\xi_1)\dots(\xi_{p+1})}\sigma'$, we have

$$\begin{aligned} P_p \mathrm{St}(S)_{q_0,\dots,q_p} = \{(\bar{\sigma}, \sigma, \tau) \in S_p \times \mathrm{St}(S)_{q+p} \\ \subset S_p \times (S \times S)_{q+p} \mid \sigma, \tau \text{ given above}\}. \end{aligned}$$

That is, $\pi_1(\sigma, \tau) = \mu_{q_0,\dots,q_p}(\bar{\sigma})$, where $\bar{\sigma} = d_{(\nu_1)\dots(\nu_{r+1})}\sigma' \in S_p$. So the elements in $P_p \mathrm{St}(S)_{q_0,\dots,q_p}$ are of the form $(\bar{\sigma}, \mu_{q_0,\dots,q_p}\bar{\sigma}, \tau)$, where $\tau \in S_{q+p}$. Explicitly

$$\begin{aligned} \mu_{q_0,\dots,q_p} = \hat{s}_{q+p} \circ s_{(q+p-1)\dots(q_0+\cdots+q_{p-1}+p)} \\ \dots \hat{s}_{q_0+q_1+1} s_{(q_0+q_1)\dots(q_0+1)} \hat{s}_{q_0} s_{(q_0-1)\dots(0)}. \end{aligned}$$

5. Comparison of the two Star Prismatic Sets

We shall now prove that this is closely related to the prismatic set $\bar{P}_. S$ defined in the previous section.

THEOREM 5.1. (1) *There is a natural* (surjective) *map*

$$\bar{p} : \bar{P}_. S \to P_. \mathrm{St}(S)_.$$

(2) *If $S = K^s$, where K is a simplicial complex, then $\bar{p}$ is an isomorphism.*

Proof. (1) Take an element $\gamma \in \bar{P}_p S_{q_0,\dots,q_p} = S_{q_0+\cdots+q_p+2p+1}$. Then γ and $q_0, \dots, q_p$ determine an element $\bar{p}(\gamma)$ in $P_p\,\mathrm{St}(S)_{q_0,\dots,q_p}$ together with a $(p+1, q+p+1)$-partition $(\xi_1, \dots, \xi_p, \xi_{p+1}, \nu_1, \dots, \nu_{q+p+1})$ of $n = p+r+1$, where $r = q_0 + \dots + q_p + p$. Here

$$\begin{aligned}
\xi_1 &= q_0 + 1 \\
\xi_2 &= q_0 + q_1 + 3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
\xi_p &= q_0 + \cdots + q_{p-1} + 2p - 1 \\
\xi_{p+1} &= r + p + 1
\end{aligned}$$

and the ν's correspond to the complement, that is, $\nu_1, \dots, \nu_{q_0+1}, \nu_{q_0+2}, \dots, \nu_{q_0+q_1+2}, \dots, \nu_{q_0+\cdots+q_{p-1}+p+1}, \dots, \nu_{r+1}$, are $0, \dots, q_0, q_0+2, \dots, q_0+q_1+2, q_0+q_1+4, \dots, q_0+\dots+q_{p-2}+2p-2, \dots, q_0+\cdots+q_{p-1}+2p, q_0+\cdots+q_{p-1}+2p, \dots, r+p$, respectively. Then, in terms of Remark 7 at the end of Section 4, we define

$$\bar{p}(\gamma) = (\bar{\sigma}, \sigma, \tau) \in P_p\,\mathrm{St}(S)_{q_0,\dots,q_p} \subseteq S_p \times S_{q+p} \times S_{q+p}$$

where

$$\begin{aligned}
\bar{\sigma} &= d_{(0)\dots(q_0)} \circ \hat{d}_{q_0+1} \circ \cdots \circ d_{(q_0+\cdots+q_{p-1}+2p)\dots(q_0+\cdots+q_p+2p)} \circ \hat{d}_{q+2p+1}(\gamma) \\
&= d_{(\nu_1)\dots(\nu_{q+p+1})}(\gamma) = d_{(\nu_1)\dots(\nu_{r+1})}(\gamma) \\
\tau &= d_{q_0+1} \circ d_{q_0+q_1+3} \circ \cdots \circ d_{q_0+\cdots+q_p+2p+1}(\gamma) = d_{(\xi_1)\dots(\xi_{p+1})}(\gamma) \\
\sigma &= \hat{s}_{q_0+\cdots+q_p+p} \circ s_{(q_0+\cdots+q_p+p-1)\dots(q_0+\cdots+q_{p-1}+p)} \circ \hat{s}_{q_0+\cdots+q_{p-1}+p-1} \circ \\
&\qquad \cdots \circ \hat{s}_{q_0+q_1+1} \circ s_{(q_0+q_1)\dots(q_0+1)} \circ \hat{s}_{q_0} \circ s_{(q_0-1)\dots(0)}(\bar{\sigma}) \\
&= s_{(q+p-1)\dots(q+p-q_p)} \circ \cdots \circ s_{(q_0+q_1)\dots(q_0+1)} \circ s_{(q_0-1)\dots(0)}(\bar{\sigma}) \\
&= \mu_{q_0,\dots,q_p}(\bar{\sigma}).
\end{aligned}$$

Using the above expression for $\bar{\sigma}$ in terms of d's and γ, we get

$$\sigma = s_{(q+p-1)\dots(q+p-q_p)} \circ \cdots \circ s_{(q_0+q_1)\dots(q_0+1)} \circ s_{(q_0-1)\dots(0)} \circ d_{(\nu_1)\dots(\nu_{q+p+1})}(\gamma).$$

Now by using Definition 4.3 and Remark 7 we can choose γ as σ'. It follows that $(\sigma, \tau) \in \mathrm{St}(S)_{q+p}$ and hence $\bar{p}(\sigma') = (\bar{\sigma}, \sigma, \tau) \in P_p\,\mathrm{St}(S)_{q_0,\dots,q_p}$.

Now $\bar{p}$ is a surjective map: Suppose $(\bar{\sigma}, \sigma, \tau) \in P_p\,\mathrm{St}(S)_{q_0,\dots,q_p}$ and we shall find $\gamma \in \bar{P}_p S_{q_0,\dots,q_p}$ such that $\bar{p}(\gamma) = (\bar{\sigma}, \sigma, \tau)$. Here $(\bar{\sigma}, \sigma, \tau) \in P_p\,\mathrm{St}(S)_{q_0,\dots,q_p} \subset S_p \times (S \times S)_{q+p}$ is such that

$$\pi_1(\sigma, \tau) \in \mathrm{Im}\{\mu_{q_0,\dots,q_p} : S_p \to S_{q+p}\}$$

where $\bar{\sigma} \in S_p$. Again use the partition $(p+1, q+p+1)$ as above and again put $\gamma = \sigma'$ as in Remark 7. Hence for every $(\bar{\sigma}, \sigma, \tau) \in P_p \operatorname{St}(S)_{q_0,\ldots,q_p}$, there exist γ such that $\bar{p}(\gamma) = (\bar{\sigma}, \sigma, \tau) \in P_p \operatorname{St}(S)_{q_0,\ldots,q_p}$.

(2) If $S = K^s$, K simplicial complex then

$$P_p \operatorname{St}(K^s)_{q_0,\ldots,q_p} = \big\{(\bar{\sigma}, \sigma, \tau) \in K_p^s \times \operatorname{St}(K^s)_{q+p} \subset K_p^s \times (K^s \times K^s)_{q+p} \\ \mid \pi_1(\sigma, \tau) \in \operatorname{Im}\{\mu_{q_0,\ldots,q_p} : K_p^s \to K_{q+p}^s\}\big\}.$$

The map

$$\mu_{q_0,\ldots,q_p} : K_p^s \to K_{q+p}^s \text{ takes } (i_0, \ldots, i_p) \text{ to } (\underbrace{i_0, \ldots, i_0}_{(q_0+1)\text{-times}}, \ldots, \underbrace{i_p, \ldots, i_p}_{(q_p+1)\text{-times}}).$$

Then

$$\sigma = (a_{i_0}, \ldots, a_{i_0}, \ldots, a_{i_p}, \ldots, a_{i_p}) \in K_{q+p}^s,$$
$$\tau = (b_{j_0}, \ldots, b_{j_{q_0}}, \ldots, b_{j_{q_0+\cdots+q_{p-1}+p}}, \ldots, b_{j_{q+p}}) \in K_{q+p}^s.$$

By the definition $\bar{P}_p(K^s)_{q_0,\ldots,q_p} = P_p(K^s)_{q_0+1,\ldots,q_p+1}$. Then $\gamma \in K_{q+2p+1}^s$ is uniquely determined by σ and τ.

Explicitly the inverse map $\bar{p}^{-1} : P_p \operatorname{St}(K^s)_{q_0,\ldots,q_p} \to \bar{P}_p K_{q_0,\ldots,q_p}^s$ is defined by $\bar{p}^{-1}(\bar{\sigma}, \sigma, \tau) = \gamma$, where

$$\bar{\sigma} = (a_{i_0}, \ldots, a_{i_p}),$$
$$\tau = (b_{j_0}, \ldots, b_{j_{q+p}}) \qquad \text{and}$$
$$\gamma = (c_{k_0}, \ldots, c_{k_{q_0+1}} | \ldots | c_{k_{q_0+\cdots+q_{p-1}+2p}}, \ldots, c_{k_{q+2p+1}})$$

such that for $0 \le s \le q + 2p + 1$

$$c_{k_s} = \begin{cases} a_{i_{l-1}} & : \quad b_{j_s} \le a_{i_{l-1}} \\ b_{j_s} & : \quad a_{i_{l-1}} < b_{j_s} < a_{i_l} \\ a_{i_l} & : \quad a_{i_l} \le b_{j_s}, \end{cases}$$

$l = 1, \ldots, p$. Hence $\gamma \in \bar{P}_p K_{q_0,\ldots,q_p}^s$ exists and is uniquely determined by $(\sigma, \tau) \in \operatorname{St}(K^s)_{q+p}$.

Therefore $\bar{p} : \bar{P}_{.} K^s \to P_{.} \operatorname{St}(K^s)_{.}$ is an isomorphism.

Remark. Note that the above proof of injectivity does not work for general simplicial sets since in general σ' in Definition 4.3 is not uniquely defined by the two components in $(S \times S)$.

Remark. Notice that PK^s is different from the one given in Example 2.1 continued. It is not a strong prismatic set.

6. The Classifying Space and Lattice Gauge Theory

For the definition of a classifying map we need a prismatic version of the standard construction of the classifying space. We recall from Segal [22] the usual model of this.

Let G be a Lie group or more generally a topological group with 1 as the non-degenerate base point such that it has the same homotopy type as a CW complex and let NG be the nerve of G. Let $\bar{G}$ be the category with $\mathrm{Ob}(\bar{G}) = G$ and $\mathrm{Mor}(\bar{G}) = G \times G$, $\mathrm{source}(g_0, g_1) = g_1$, $\mathrm{target}(g_0, g_1) = g_0$ and the composition $(g_0, g_1) \circ (g_1, g_2) = (g_0, g_2)$ and let $N\bar{G}$ be the nerve of the category $\bar{G}$. Furthermore, the map $\gamma : N\bar{G} \to NG$ is the nerve of the functor $\gamma : \bar{G} \to G$ given by $\gamma(g_0, g_1) = g_0 {g_1}^{-1}$.

These two nerves are two simplicial spaces given by

$$\begin{aligned} NG(p) &= G \times \cdots \times G \qquad (p\text{-times}) \\ N\bar{G}(p) &= G \times \cdots \times G \qquad (p+1\text{-times}). \end{aligned}$$

By using the face and degeneracy operators on these simplicial spaces one has their realizations. The usual classifying space $BG = EG/G$ is constructed as a simplicial space $EG_p = G \times \cdots \times G$, $(p+1)$-times and $BG_p = (G \times \cdots \times G)/G$. We can identify $EG = \|N\bar{G}\| = \bigsqcup_{p \geq 0} \Delta^p \times G^{p+1}/\sim$ and $BG = \|NG\|$.

In order to make this simplicial space discrete we can replace G by the singular simplicial set of continuous maps $S_q G = \mathrm{Map}(\Delta^q, G)$ and $E_{.} S_{.} G$ as in Example 2.7 is a prismatic set. However we shall need another model constructed as follows:

Definition 6.1. A continuous map $a \in \mathrm{Map}(\Delta^p \times \Delta^{q_0 \cdots q_p}, G^{p+1})$ is called *restricted* if it has the form

$$a(t, s^0, \ldots, s^p) = (a_0(t, s^0), a_1(t, s^0, s^1), \ldots, a_p(t, s^0, \ldots, s^p)),$$

where $(t, s^0, \ldots, s^p) \in \Delta^p \times \Delta^{q_0 \cdots q_p}$ and if $a_j(\varepsilon^i t, s^0, \ldots, s^j)$ is independent of s^i for all $i < j$. Now we define

$$P_p EG_{q_0, \ldots, q_p} = \{a : \Delta^p \times \Delta^{q_0 \cdots q_p} \to G^{p+1} | a \text{ is restricted}\}.$$

$S_{.} G$ acts on this prismatic set diagonally (on the right). By the definition $P_{.} BG = P_{.} EG/S_{.} G$, that is,

$$P_p BG_{q_0, \ldots, q_p} = P_p EG_{q_0, \ldots, q_p}/S_p G.$$

PROPOSITION 6.2. *The evaluation maps give horizontal homotopy equivalences in the diagram*

$$\begin{tikzcd} {|||P_\cdot EG_\cdot|||} \arrow[r, "\mathrm{ev}"] \arrow[d, "|||\gamma|||"] & EG \arrow[d, "\gamma"] \\ {|||P_\cdot BG_\cdot|||} \arrow[r, "\mathrm{ev}"] & EG/G \end{tikzcd}$$

Furthermore the top map is equivariant with respect to the homomorphism $\mathrm{ev} : |S_\cdot G| \to G$.

PROOF. First notice that the evaluation map $\mathrm{ev} : |S_\cdot G| \to G$ is a homotopy equivalence. Also the equivariance is obvious by the commutative diagram

$$\begin{tikzcd} {|||P_\cdot EG_\cdot||| \times |S_\cdot G|} \arrow[r, "\mathrm{ev}\times\mathrm{ev}"] \arrow[d, "\cdot"] & EG \times G \arrow[d, "\cdot"] \\ {|||P_\cdot EG_\cdot|||} \arrow[r, "\mathrm{ev}\times\mathrm{ev}"] & EG \end{tikzcd}$$

where the vertical maps are given by quotients and the actions of $|S_\cdot G|$ and G are free. Since $|||P_\cdot EG_\cdot|||$ and EG are both contractible, the evaluation map induces a homotopy equivalence on the quotient. (See May [15, Chapter 3].)

7. Lattice Gauge Theory, Parallel Transport Function

In lattice gauge theory in the sense of Phillips and Stone [19] they construct for a given Lie group G and a simplicial complex K a G-bundle with connection on $|K|$ associated to a set of G-valued continuous functions defined over the faces of a simplex. These they call "parallel transport functions" since they are determined by parallel transport for the connection. In this section we shall introduce similar "compatible transition functions" for K replaced by a simplicial set S and in the following section we shall use these to construct a classifying map on the star complex $\bar{P}_\cdot S$. First we consider G-bundles over simplicial sets.

DEFINITION 7.1. A bundle over $|S_\cdot|$ is a sequence of bundles over $\Delta^p \times \sigma$ for all p, where $\sigma \in S_p$ and with commutative diagrams;

$$\begin{tikzcd} F_{d_j\sigma} \arrow[r, "\bar{\varepsilon}^j"] \arrow[d] & F_\sigma \arrow[d] \\ \Delta^{p-1} \times d_j\sigma \arrow[r, "\varepsilon^j"] & \Delta^p \times \sigma \end{tikzcd}$$

and

$$\begin{array}{ccc} F_{s_j\sigma} & \xrightarrow{\bar{\eta}^j} & F_\sigma \\ \downarrow & & \downarrow \\ \Delta^{p+1} \times s_j\sigma & \xrightarrow{\eta^j} & \Delta^p \times \sigma \end{array}$$

with the compatibility conditions:

$$\bar{\varepsilon}^j \bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i \bar{\varepsilon}^{j-1} & : \quad i < j \\ \bar{\varepsilon}^{i+1} \bar{\varepsilon}^j & : \quad i \geq j, \end{cases} \qquad \bar{\eta}^j \bar{\eta}^i = \begin{cases} \bar{\eta}^i \bar{\eta}^{j+1} & : \quad i \leq j \\ \bar{\eta}^{i-1} \bar{\eta}^j & : \quad i > j, \end{cases}$$

and

$$\bar{\eta}^j \bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i \bar{\eta}^{j-1} & : \quad i < j \\ 1 & : \quad i = j, i = j+1 \\ \bar{\varepsilon}^{i-1} \bar{\eta}^j & : \quad i > j+1. \end{cases}$$

Given a G-bundle $F \to |S_\cdot|$, G a Lie group, we can choose a trivialization $\varphi_\sigma : F_\sigma \to \Delta^p \times \sigma \times G$ for each non-degenerate $\sigma \in S_p$ since Δ^p is contractible. If σ is degenerate, that is, there exists τ such that $\sigma = s_i\tau$, then the trivialization of σ is defined as pullback of the trivialization of τ, that is, $\varphi_\sigma = {\eta^i}^*(\varphi_\tau)$.

Definition 7.2. A set of trivializations is called *admissible*, in case φ_σ for $\sigma = s_i\tau$ is given by $\varphi_\sigma = {\eta^i}^*(\varphi_\tau)$.

Lemma 7.3. *Admissible trivializations always exist.*

Now, let us define the transition functions for a simplex $\sigma \in S_p$:

Definition 7.4. Given a bundle $F \to |S_\cdot|$ and a set of trivializations, we get for each face τ of say $\dim \tau = q < p$ in σ, a *transition function* $\upsilon_{\sigma,\tau} : \Delta^q \to G$ as follows: The bundle map Θ given by the diagram

$$\begin{array}{ccc} \Delta^q \times (d_{(i_p)\ldots(i_0)}\sigma) \times G & \xrightarrow{\Theta} & \Delta^p \times (\sigma) \times G \\ \downarrow & & \downarrow \\ \Delta^q \times d_{(i_p)\ldots(i_0)}\sigma & \xrightarrow{\varepsilon^{i_0\ldots i_p}} & \Delta^p \times \sigma \end{array}$$

where $d_{(i_p)\ldots(i_0)}\sigma = \tau$, $\Theta = \varphi_\sigma \circ \bar{\varepsilon}^{i_0\ldots i_p} \circ \varphi^{-1}_{d_{(i_p)\ldots(i_0)}\sigma}$, $\bar{\varepsilon}^{i_0\ldots i_p} = \bar{\varepsilon}^{i_0} \circ \cdots \circ \bar{\varepsilon}^{i_p}$ and $\varepsilon^{i_0\ldots i_p} = \varepsilon^{i_0} \circ \cdots \circ \varepsilon^{i_p}$, determines $\upsilon_{\sigma,\tau}$ by the formula

$$\Theta(t, g) = (\varepsilon^{i_0} \circ \cdots \circ \varepsilon^{i_p}(t), \upsilon_{\sigma,\tau}(t)g), \qquad t \in \Delta^q, \quad g \in G.$$

This defines the set of *transition functions* $\{v_{\sigma,\tau} \mid \sigma \in S_p$ and τ is a face of $\sigma\}$ for the bundle over $|S_.|$.

REMARK. The transition functions are generalized lattice gauge fields. Classically lattice gauge fields are defined only on 1-skeletons but one can extend them to $p-1$ simplices for all p which gives rise to transition functions on Δ^p, as above.

We now list a number of propositions stating their properties. The proofs are straight forward. For details see Akyar [1].

PROPOSITION 7.5. *Given a bundle on a simplicial set and admissible trivializations, the transition function $v_{\sigma,\tau}$, where τ is a face of σ, satisfies;*

(i) *σ is non-degenerate: if $\gamma = d_j\sigma$ and $\tau = d_i\gamma$ then*

$$v_{\sigma,\tau} = (v_{\sigma,\gamma} \circ \varepsilon^i).v_{\gamma,\tau}.$$

This is called the cocycle condition.

(ii) *σ is degenerate: if $\sigma = s_j\sigma'$ and $\tau = d_i\sigma$ then when $i < j$ for $\tau = s_{j-1}\tau'$ one gets $\tau' = d_i\sigma'$ and when $i > j+1$ for $\tau = s_j\tau'$ one gets $\tau' = d_{i-1}\sigma'$. For the other cases, $i = j$ or $i = j+1$, $\tau = \sigma'$. Then the transition functions satisfy:*

$$v_{\sigma,\tau} = \begin{cases} v_{\sigma',\tau'} \circ \eta^{j-1} & : \quad i < j \\ 1 & : \quad i = j, i = j+1 \\ v_{\sigma',\tau'} \circ \eta^{j} & : \quad i > j+1. \end{cases}$$

(iii) *If τ is a composition of face operators of σ, e.g., $\tau = \tilde{d}^{p-(i-1)}\sigma$, $i = 1, \ldots, p$, where $\tilde{d}^{p-(i-1)} = d_i \circ \cdots \circ d_p$ then*

$$v_{\sigma,\tau} = (v_{\sigma,\tilde{d}^1\sigma} \circ (\varepsilon^i)^{p-i}).(v_{\tilde{d}^1\sigma,\tilde{d}^2\sigma} \circ (\varepsilon^i)^{p-i-1}) \\ \ldots (v_{\tilde{d}^{p-(i+1)}\sigma,\tilde{d}^{p-i}\sigma} \circ \varepsilon^i).v_{\tilde{d}^{p-i}\sigma,\tau}.$$

PROPOSITION 7.6. *Assume that we have a bundle over $|S_.|$. Then*

(1) *There exist admissible trivializations such that the transition functions are given by*

$$v_{\sigma,d_i\sigma} = 1 \qquad \text{if} \quad i < p.$$

(2) *For $\tau = \tilde{d}^{p-(i-1)}\sigma$, $i = 1, \ldots, p$, we get $v_{\sigma,\tau}$ as product of some transition functions:*

$$v_{\sigma,\tau} = (v_{\sigma} \circ (\varepsilon^i)^{p-i}).(v_{\tilde{d}^1\sigma} \circ (\varepsilon^i)^{p-i-1}).(v_{\tilde{d}^2\sigma} \circ (\varepsilon^i)^{p-i-2}) \\ \ldots (v_{\tilde{d}^{p-(i+1)}\sigma} \circ (\varepsilon^i)^1).(v_{\tilde{d}^{p-i}\sigma}).$$

(3) *The transition functions* $v_{\sigma,\tau}$ *satisfy the compatibility conditions:*

$$v_\sigma \circ \varepsilon^i = \begin{cases} v_{d_i\sigma} & : \ i < p-1 \\ v_{d_{p-1}\sigma}.v_{d_p\sigma}^{-1} & : \ i = p-1 \end{cases}$$

(4) *For a degenerate* σ, *we have*

$$v_{s_j\sigma} = \begin{cases} 1 & : \ i < j \\ 1 & : \ i = j, i = j+1 \\ v_\sigma \circ \eta^j & : \ i > j+1. \end{cases}$$

Proposition 7.7. *Given a bundle, one can find admissible trivializations such that the transition functions are determined by functions* $v_\sigma : \Delta^{p-1} \to G$ *for* $\sigma \in S_p$ *non-degenerate.*

Proposition 7.8. *Suppose given a set of transition functions*

$$v_\sigma : \Delta^{p-1} \to G$$

for $\sigma \in S_p$ *for all* p, *satisfying the compatibility conditions*

$$v_\sigma \circ \varepsilon^i = \begin{cases} v_{d_i\sigma} & : \ i < p-1 \\ v_{d_{p-1}\sigma}.v_{d_p\sigma}^{-1} & : \ i = p-1 \end{cases}$$

and

$$v_{s_j\sigma} = v_\sigma \circ \eta^j.$$

Then one can define for each $\sigma \in S_p$ *and each lower dimensional face* τ *of* σ, *a function* $v_{\sigma,\tau}$ *such that* (i) *and* (ii) *in Proposition 7.5 hold and such that*

$$v_{\sigma,\tau} = \begin{cases} v_\sigma & : \ i = p \\ 1 & : \ i < p. \end{cases}$$

Proposition 7.9. *Given a set of transition functions* $v_{\sigma,\tau}$ *satisfying* (i) *and* (ii) *in Proposition 7.5, there is a bundle* F *over* $|S_.|$ *and trivializations with transition functions* $v_{\sigma,\tau}$.

Corollary 7.10. *Given a set of functions* v_σ *satisfying the compatibility conditions in Proposition 7.6, one can construct a bundle* F *over* $|S_.|$ *and the trivializations with the transition functions* $v_{\sigma,d_p\sigma} = v_\sigma$ *and* $v_{\sigma,d_i\sigma} = 1$ *when* $i < p$ *and* $v_{s_i\sigma} = v_\sigma \circ \eta^i$ *for a degenerate* σ.

Definition 7.11. A set of functions $\{v_\sigma\}_{\sigma\in S}$ as in Proposition 7.8 are called a set of “compatible transition functions”.

We end this section by comparing these compatible transition functions with the "parallel transport functions" (p.t.f.) of Phillips and Stone [19]. For $S = K^s$ a set of such functions consist of a set of maps, $V_\sigma : c_\sigma \to G$ for each r-simplex σ of K, $r \geq 1$, c_σ is the $(r-1)$-cube given by $0 \leq s_{a_1} \leq 1, \dots, 0 \leq s_{a_{r-1}} \leq 1$, where $\sigma = \langle a_0, \dots, a_r \rangle \in K$ with the compatibility conditions

1. Cocycle condition

$$\begin{aligned} V_\sigma(s_{a_1}, \dots, s_{a_p} = 1, \dots, s_{a_{r-1}}) \\ = V_{\langle a_0,\dots,a_p\rangle}(s_{a_1}, \dots, s_{a_{p-1}}).V_{\langle a_p,\dots,a_r\rangle}(s_{a_{p+1}}, \dots, s_{a_{r-1}}). \end{aligned}$$

2. Compatibility condition

$$\begin{aligned} V_\sigma(s_{a_1}, \dots, s_{a_p} = 0, \dots, s_{a_{r-1}}) \\ = V_{\langle a_0,\dots,\hat{a}_p,\dots,a_{r-1}\rangle}(s_{a_1}, \dots, \hat{s}_{a_p}, \dots, s_{a_{r-1}}). \end{aligned}$$

Now, suppose we have compatible transition functions $\{v_\sigma\}$ for a principal G-bundle $E \to |K|$ with triangulated base. Then for $\sigma = \langle a_0, \dots, a_r \rangle$, the p.t.f. $V_\sigma : c_\sigma \to G$ is given by the parallel transport $E_{a_0} \to E_{a_r}$ along paths determined as follows:

Let $\sigma = \langle a_0, \dots, a_r \rangle \in K^s$ and $s = (s_{a_0}, \dots, s_{a_{r-1}}) \in c_\sigma$.

We pick $r-1$ points as $P_1, \dots, P_{r-1}$ so that P_1 is on the line segment from a_0 to a_1, that is,

$$P_1 = (1 - s_{a_1})a_0 + s_{a_1}a_1 = ((1 - s_{a_1}, s_{a_1}), \langle a_0, a_1 \rangle) \in |K|.$$

Similarly, P_2 is on the line segment from P_1 to a_2, $P_2 = (1 - s_{a_2})P_1 + s_{a_2}a_2$. Then

$$P_2 = ((1 - s_{a_2})(1 - s_{a_1}), (1 - s_{a_2})s_{a_1}, s_{a_2}, \langle a_0, a_1, a_2 \rangle).$$

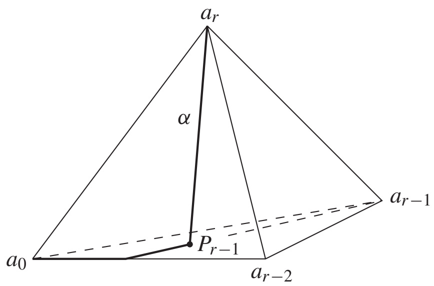
By continuing in the same way, we get

$$P_{r-1} = (1 - s_{a_{r-1}})P_{r-2} + s_{a_{r-1}}a_{r-1}.$$

Let α be the piecewise linear path from a_0 through $P_1, \dots, P_{r-1}$ to a_r. In other words, α is determined uniquely up to parametrization by $r-1$ numbers $s_{a_1}, \dots, s_{a_{r-1}}$. For $P_{r-1} = (t, d_r\sigma) \in \Delta^{r-1} \times K_{r-1}$, $d_r\sigma = \langle a_0, \dots, a_{r-1} \rangle$, the element

$$V_\sigma(s_1, \dots, s_{r-1}) = v_\sigma(t) \in G$$

is to be interpreted as the parallel transport along α.



8. The Classifying Map

THE CONSTRUCTION OF THE CLASSIFYING MAP. For a given set of compatible transition functions (c.t.f.) $\{v_\sigma\}$ satisfying Proposition 7.8 we have seen in Proposition 7.9 that there is an associated G-bundle F over $|S_\cdot|$. Recall that the composite map $\lambda \circ f : \|\,|\bar{P}_\cdot S_\cdot|\,\| \to \|\,|S_\cdot|\,\| \to |S_\cdot|$ is a homotopy equivalence. In this section, we construct a classifying map for the bundle $(\lambda \circ f)^* F$ over $\|\,|\bar{P}_\cdot S_\cdot|\,\|$.

THEOREM 8.1.

(1) *For given c.t.f.'s $\{v_\sigma\}$, there is a canonical prismatic map $m : \bar{P}_\cdot S \to P_\cdot BG$.*

(2) *The induced map of geometric realizations*

$$\mathrm{ev} \circ \|\,|m|\,\| = \bar{m} : \|\,|\bar{P}_\cdot S_\cdot|\,\| \xrightarrow{\|\,|m|\,\|} \|\,|P_\cdot BG_\cdot|\,\| \xrightarrow{\mathrm{ev}} BG$$

is a classifying map for the G-bundle $(\lambda \circ f)^* F$ over $\|\,|\bar{P}_\cdot S_\cdot|\,\|$.

PROOF. (1) The map $m : \bar{P}_\cdot S \to P_\cdot BG$ is defined as

$$m(\sigma) = [(a_0, a_1, \ldots, a_p)]$$

where $\sigma \in \bar{P}_p S_{q_0 \ldots q_p} = S_{q+2p+1}$, $q = q_0 + \cdots + q_p$ and $a_i : \Delta^p \times \Delta^{q_0 \ldots q_i} \to G$ are given in (8.2) below. First some notation. In the following, we use for convenience the interior coordinates $(x_1, \ldots, x_p)$ of the standard simplex with barycentric coordinates $(t_0, \ldots, t_p)$ as defined in Example 3.5. Similarly the interior coordinates of Δ^{q_i} are $s^i = (s^i_1, \ldots, s^i_{q_i})$. In these the map Λ from Section 3 is induced by the maps $\lambda_p : \Delta^p \times \Delta^{q_0+1 \ldots q_p+1} \to \Delta^{q+2p+1}$ given by

$$\begin{aligned}
&\lambda_p(x, s^0, 0, \ldots, 0, s^p, 0) \\
&\quad = \big(s^0_1(1 - x_1) + x_1, \ldots, s^0_{q_0}(1 - x_1) + x_1, x_1, x_1, s^1_1(x_1 - x_2) + x_2, \\
&\qquad \ldots, s^1_{q_1}(x_1 - x_2) + x_2, x_2, x_2, \ldots, s^{p-1}_1(x_{p-1} - x_p) + x_p, \\
&\qquad \ldots, s^{p-1}_{q_{p-1}}(x_{p-1} - x_p) + x_p, x_p, x_p, s^p_1 x_p, \ldots, s^p_{q_p} x_p, 0\big).
\end{aligned}$$

For convenience, we drop p in $\lambda_p(x,s)$ and write $\lambda(x,s)$. Next, let $\rho^{(i)}: \Delta^{q+2p+1} \to \Delta^{q_0+\cdots+q_{i-1}+2i-1}$ be the degeneracy map for $i = 1,\ldots,p$ defined by

$$\rho^{(i)} := \eta^{q_0+\cdots+q_{i-1}+2i-1} \circ \cdots \circ \eta^{q+2p}$$

deleting the last $q_i + \cdots + q_p + 2(p-i+1)$ coordinates. For example,

$$\begin{aligned}
&\rho^{(p)}\lambda(x,s)\\
&\quad= \big(s_1^0(1-x_1)+x_1,\ldots,s_{q_0}^0(1-x_1)+x_1,x_1,x_1,s_1^1(x_1-x_2)+x_2,\\
&\qquad \ldots,s_{q_1}^1(x_1-x_2)+x_2,x_2,x_2,\ldots,s_1^{p-1}(x_{p-1}-x_p)+x_p,\\
&\qquad \ldots,s_{q_{p-1}}^{p-1}(x_{p-1}-x_p)+x_p,x_p\big),
\end{aligned}$$

where $\rho^{(p)} := \eta^{q-q_p+2p-1} \circ \cdots \circ \eta^{q+2p}$ is deleting the last $q_p + 2$ coordinates. With this notation, the maps $a_i : \Delta^p \times \Delta^{q_0\cdots q_i} \to G$ defining the classifying map $m(\sigma)$ are given by

$$\begin{aligned}
a_p(x,s^0,0,\ldots,s^p,0) &= 1\\
a_{p-1}(x,s^0,0,\ldots,s^{p-1},0) &= v_{\sigma,d_{(p)}\sigma}(\rho^{(p)}(\lambda(x,s)))^{-1},\\
a_{p-2}(x,s^0,0,\ldots,s^{p-2},0) &= v_{\sigma,d_{(p-1)(p)}\sigma}(\rho^{(p-1)}(\lambda(x,s)))^{-1}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
a_1(x,s^0,0,s^1,0) &= v_{\sigma,d_{(2)\ldots(p)}\sigma}(\rho^{(2)}(\lambda(x,s)))^{-1}\\
a_0(x,s^0,0) &= v_{\sigma,d_{(1)\ldots(p)}\sigma}(\rho^{(1)}(\lambda(x,s)))^{-1}.
\end{aligned} \tag{8.2}$$

Then $m(\sigma)$ satisfy Definition 6.1 and it is straight forward that m is a prismatic map.

(2) For given c.t.f.'s v_σ, we now have the map of realizations $\||m|\| : \||\bar{P}_. S_.|\| \to \||P_. BG_.|\|$ given by

$$\||m|\|(x,s,\sigma) = (x,s,[(a_0,\ldots,a_p)]).$$

The classifying map $\bar{m} = \mathrm{ev} \circ \||m|\|$ and the associated bundle map are given as follows:

We have a bundle F on $|S_.|$ by Proposition 7.9 and for each p we have that $\lambda_p : \Delta^p \times |\bar{P}_p S_.| \to |S_.|$ is an epimorphism. Hence by pulling back we get bundles $\bar{F}_p \to \Delta^p \times |\bar{P}_p S_.|$ inducing a bundle map

$$\begin{array}{ccc}
\bar{F} & \longrightarrow & F\\
\downarrow & & \downarrow\\
\||\bar{P}_. S_.|\| & \longrightarrow & |S_.|
\end{array}$$

The transition functions are now used to define $\tilde{m}$ similarly to $\bar{m} = \mathrm{ev} \circ \| |m| \|$ in the commutative diagram

$$\begin{tikzcd}[column sep=large]
\bar{F} \arrow[rr, "\tilde{m}"] \arrow[d] & & EG \arrow[d, "\gamma"] \\
\| |\bar{P}_{\cdot} S_{\cdot}| \| \arrow[rr, "\bar{m}"] \arrow[dr, "\| |m| \|"'] & & BG \\
& \| |P_{\cdot} BG_{\cdot}| \| \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd} \tag{8.3}$$

For the construction of $\tilde{m}$ take (x, s, σ), $x \in \Delta^p$, $s \in \Delta^{q_0 \cdots q_p}$, $\sigma \in S_{q+2p+1}$. Here ev denotes the evaluation map as in Proposition 6.2. Then the fibre of $\bar{F}$ over $(x, s, \sigma) \in \Delta^p \times |\bar{P}_p S_{\cdot}|$ is the fibre of F at $(\lambda(x, s^0, 0, \ldots, s^p, 0), \sigma)$. Using the trivialization $\varphi_\sigma : F_\sigma \to \Delta^{q+2p+1} \times (\sigma) \times G$ and the projection on the last factor, we get an isomorphism $F_\sigma \to G$. Let $\bar{\varphi}_\sigma : \bar{F}_{(x,s,\sigma)} \to G$ be defined by

$$\bar{\varphi}_\sigma(\tilde{f}_{(x,s,\sigma)}) = \mathrm{proj}_2 \circ \varphi_\sigma(f_{(\lambda(x,s^0,0,\ldots,s^p,0),\sigma)}),$$

where $\tilde{f}_{(x,s,\sigma)} = ((x, s, \sigma), f_{(\lambda(x,s^0,0,\ldots,s^p,0),\sigma)}) \in \bar{F}_{(x,s,\sigma)}$, $f_{(\lambda(x,s^0,0,\ldots,s^p,0),\sigma)} \in F_\sigma$. On the other hand

$$\varphi_{d_{(p)}\sigma} : F_{d_{(p)}\sigma} \to \Delta^{q+2p-q_p-1} \times (d_{(p)}\sigma) \times G$$

gives us

$$\bar{\varphi}_{d_{(p)}\sigma} : F_{d_{(p)}\sigma} \to G.$$

By the definition,

$$\bar{\varphi}_\sigma(\bar{d}^{(p)} \tilde{f}) := v_{\sigma, d_{(p)}\sigma}(\rho^{(p)} \lambda(x, s^0, 0, \ldots, s^p, 0)).\bar{\varphi}_{d_{(p)}\sigma}(\tilde{f}),$$

where the transition function is

$$v_{\sigma, d_{(p)}\sigma} : \Delta^{q+2p-q_p-1} \to G.$$

The $(p+1)$-G-valued components of $\tilde{m}(\tilde{f}_{(x,s,\sigma)}) \in \Delta^p \times G^{p+1}$ are defined via the trivialization $\varphi_\sigma(\tilde{f})$ as follows: The $(p+1)$-st component is just $\bar{\varphi}_\sigma(\tilde{f})$ and by using the transition function $v_{\sigma, d_{(p)}\sigma}$ we find the p-th component as

$$v_{\sigma, d_{(p)}\sigma}(\rho^{(p)} \lambda(x, s^0, 0, \ldots, s^p, 0))^{-1}.\bar{\varphi}_\sigma(\tilde{f}).$$

We can apply the same method several times to get the other coordinates in $\tilde{m}(\tilde{f}_{(x,s,\sigma)})$ as well and using the formula (8.2) we get the commutative diagram (8.3) above of G-equivariant maps. It follows that $\bar{F}$ is the pull-back of the bundle $\gamma : EG \to BG$, hence $\bar{m}$ is the classifying map for $\bar{F} \to \||\bar{P}_{.}S_{.}|\|$.

In particular for a simplicial complex K we get the following (cf. [19]).

COROLLARY 8.4 (Phillips-Stone).
(1) *A set of compatible transition functions* $\{v_\sigma\}$ *for* K *a simplicial complex there is a natural prismatic map*

$$P_{.}\operatorname{St}(K^s)_{.} \to P_{.}BG$$

(2) *The induced map on geometric realization gives a classifying map for the bundle* F *pulled back to* $|\operatorname{St}(K)| \subseteq |K| \times |K|$.

PROOF. In the second part of Theorem 5.1, we have showed that $\bar{p} : \bar{P}_{.}K^s \to P_{.}\operatorname{St}(K^s)_{.}$ is an isomorphism. On the other hand in the previous proposition, we have defined the classifying map m. This is also valid when $S = K^s$. So the p.t.f. v_σ will determine a natural map

$$m : P_{.}\operatorname{St}(K^s)_{.} \to P_{.}BG$$

Furthermore $\pi_1 : P_{.}\operatorname{St}(K^s)_{.} \to K$ is a homotopy equivalence.

REMARK. The point of the corollary is that there is a connection in the universal bundle in the simplicial sense (see Dupont-Ljungmann [7]) which thus pulls back to a connection in the bundle over the star complex. We shall return to this elsewhere.

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