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THE UNICITY OF BEST APPROXIMATION IN A SPACE OF COMPACT OPERATORS

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Abstract

Chebyshev subspaces of $\mathcal{K}(c_0, c_0)$ are studied. A k-dimensional non-interpolating Chebyshev subspace is constructed. The unicity of best approximation in non-Chebyshev subspaces is considered.

1. Introduction

Let K be the field of real or complex numbers and let $(X, \|\cdot\|)$ be a normed space over K. Let $\operatorname{ext} S_{X^\star}$ denote the set of all extreme points of $S_{X^\star}$, where $S_{X^\star}$ is the unit sphere in $X^\star$.

For every $x \in X$ we put

$$E(x) = \{f \in \operatorname{ext} S_{X^\star} : f(x) = \|x\|\}. \tag{1}$$

By the Hahn-Banach and the Krein-Milman Theorems, $E(x) \neq \emptyset$.

Let for $Y \subset X$,

$$P_Y(x) = \{y \in Y : \|x - y\| = \operatorname{dist}(x, Y)\}.$$

A linear subspace $Y \subset X$ is called *a Chebyshev subspace* if for every $x \in X$ the set $P_Y(x)$ contains one and only one element.

THEOREM 1 (see [3]). *Assume X is a normed space, $Y \subset X$ is a linear subspace, and let $x \in X \setminus Y$. Then $y_0 \in P_Y(x)$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $\Re f(y) \leq 0$.*

DEFINITION (see, e.g., [8]). An element $y_0 \in Y$ is called a strongly unique best approximation for $x \in X$ if there exists $r > 0$ such that for every $y \in Y$,

$$\|x - y\| \geq \|x - y_0\| + r\|y - y_0\|.$$

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The biggest constant r satisfying the above inequality is called *a strong unicity constant*. There exist two main applications of a strong unicity constant:

- the error estimate of the Remez algorithm (see e.g. [13]),
- the Lipschitz continuity of the best approximation mapping at x_0 (assuming that there exists a strongly unique best approximation to x_0) (see e.g. [5], [9], [11]).

THEOREM 2 (see [17]). *Let $x \in X \setminus Y$ and let Y be a linear subspace of X. Then $y_0 \in Y$ is a strongly unique best approximation for x with a constant $r > 0$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $\Re f(y) \leq -r\|y\|$.*

Recall that a k-dimensional subspace $\mathscr{V}$ of a normed space X is called *an interpolating subspace* if for any linearly independent $f_1, f_2, \ldots, f_k \in \operatorname{ext} S_{X^\star}$ and for every $v \in \mathscr{V}$ the following holds:

$$\text{if} \quad f_i(v) = 0, \quad i = 1, 2, \ldots, k \quad \text{then} \quad v = 0.$$

Every interpolating subspace is a finite dimensional Chebyshev subspace. If $\mathscr{V} \subset X$ is an interpolating subspace then every $x \in X$ has a strongly unique best approximation in $\mathscr{V}$ (see [2]).

In this paper we consider $X = \mathscr{K}(c_0, c_0)$ (the space of all compact operators from c_0 to c_0 equipped with the operator norm). Here c_0 denotes the space of all real sequences convergent to zero. For any $x = (x_k) \in c_0$ we put

$$\|x\|_\infty = \sup_k |x_k|.$$

In [8, Theorem 3.1] it has been proved that if $\mathscr{V} \subset \mathscr{K}(c_0, c_0)$ is a finite-dimensional Chebyshev subspace then every $A \in \mathscr{K}(c_0, c_0)$ has a strongly unique best approximation in $\mathscr{V}$. However, in [8] no example of a non-interpolating Chebyshev subspace has been proposed. If it were true that any finite-dimensional Chebyshev subspace of $\mathscr{K}(c_0, c_0)$ is an interpolating subspace we would have obtained the proof of Theorem 3.1, [8] immediately (see [2] for more details).

The aim of this paper is to show that for every $k < \infty$ there exists a k-dimensional non-interpolating Chebyshev subspace of $\mathscr{K}(c_0, c_0)$. This result is quite different from the result obtained in [7]. In the space $\mathscr{L}(l_1^n, c_0)$ any finite-dimensional Chebyshev subspace is an interpolating subspace.

Additionally, we discuss the strong unicity of best approximation in some (not necessarily Chebyshev) subspaces of $\mathscr{K}(c_0, c_0)$.

2. k-dimensional Chebyshev subspaces of $\mathscr{K}(c_0, c_0)$

Let $A \in \mathscr{K}(c_0, c_0)$ be represented by a matrix $[a_{ij}]_{i,j \in \mathsf{N}}$. Note that

$$(a_{ij})_{i=1}^{\infty} \in c_0 \qquad \text{for every} \quad j \in \mathsf{N}.$$

Since each row of a matrix $[a_{ij}]_{i,j \in \mathsf{N}}$ corresponds to a linear functional on c_0,

$$(a_{ij})_{j=1}^{\infty} \in l^1 \qquad \text{for every} \quad i \in \mathsf{N}.$$

Moreover, by the Schur Theorem (see [6])

$$\lim_{i \to \infty} \left(\sum_{j=1}^{\infty} |a_{ij}| \right) = 0.$$

Recall (see [4]) that ext $S_{\mathscr{K}^*(c_0,c_0)}$ consists of functionals of the form $e_i \otimes x$, where $x \in \operatorname{ext} S_{l^\infty}$ and

$$(e_i \otimes x)(A) = \sum_{j=1}^{\infty} x_j a_{ij}. \tag{2}$$

It is easy to see that

$$\|A\| = \sup_{i \geq 1} \sum_{j=1}^{\infty} |a_{ij}|.$$

REMARK 1. Let X be a Banach space and let $\mathscr{V}$ be a finite-dimensional subspace with $V_1, V_2, \ldots, V_k$ as a basis. Then $\mathscr{V}$ is an interpolating subspace if and only if for any linearly independent $f_1, f_2, \ldots, f_k \in \operatorname{ext} S_{X^\star}$ the determinant of $[f_i(V_j)]_{i,j=1,2,\ldots,k}$ is not equal to zero.

PROOF. We apply the definition of a k-dimensional interpolating subspace and the theory of linear equations. This completes the proof.

In the sequel, we denote by $\operatorname{lin}\{V_1, V_2, \ldots, V_k\}$ the k-dimensional subspace of $\mathscr{K}(c_0, c_0)$ with $V_1, V_2, \ldots, V_k$ as a basis.

EXAMPLE 1. Let $V = [v_{ij}]_{i,j \in \mathsf{N}}$, where $v_{i1} = \frac{1}{2^i}$, $v_{ij} = 0, i, j \in \mathsf{N}, j \geq 2$. It is obvious that $\mathscr{V} = \operatorname{lin}\{V\}$ is a one-dimensional interpolating subspace of $\mathscr{K}(c_0, c_0)$.

THEOREM 3. *Let* $\mathscr{V} = \operatorname{lin}\{V_1, V_2, \ldots, V_n\}$. *Let* $V_m = [(v_m)_{ij}]_{i,j\in\mathsf{N}}$, $m = 1, 2, \ldots, n$. *If* $\mathscr{V}$ *is a Chebyshev subspace then*

$$\left|\begin{array}{ccccc} f_1(V_1) & . & . & . & f_1(V_n) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ f_n(V_1) & . & . & . & f_n(V_n) \end{array}\right| \neq 0 \tag{3}$$

for any $f_1, \ldots, f_n \in \operatorname{ext} S_{\mathscr{K}^*(c_0,c_0)}$ *such that* $f_m = e_{i_m} \otimes x^{i_m}$, $m = 1, 2, \ldots, n$, *where* $i_m \neq i_k$ *for* $m \neq k$.

PROOF. Assume (3) does not hold. Therefore there exist $f_1, \ldots, f_n \in \operatorname{ext} S_{\mathscr{K}^*(c_0,c_0)}$, $f_m = e_{i_m} \otimes x^{i_m}$, $m = 1, 2, \ldots, n$, where $i_m \neq i_k$ for $m \neq k$ such that $\det D = 0$, where

$$D = \begin{bmatrix} f_1(V_1) & . & . & . & f_1(V_n) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ f_n(V_1) & . & . & . & f_n(V_n) \end{bmatrix}.$$

Since $\det D = \det D^T$, there exists $y = (y_1, y_2, \ldots, y_n) \neq 0$ such that $D^T y = 0$. Consequently,

$$\sum_{m=1}^{n} y_m f_m\big|_{\mathscr{V}} = 0. \tag{4}$$

Since $y \neq 0$, replacing f_m by $-f_m$ if necessary, we may assume $y_m \geq 0$ for $m = 1, 2, \ldots, n$ and

$$\sum_{m=1}^{n} y_m = 1.$$

Set $\mathscr{C} = \{l \in \{1, 2, \ldots, n\} : y_l > 0\}$.

Fix $(d_j)_{j\in\mathsf{N}}$ with the following properties:

$$d_j > 0, \qquad j \in \mathsf{N} \qquad \text{and} \qquad \sum_{j=1}^{\infty} d_j = 1.$$

Define $A = [a_{i_p j}]_{i_p, j\in\mathsf{N}} \in \mathscr{K}(c_0, c_0)$ by

$$a_{i_p j} = 0 \qquad \text{for} \quad p \notin \mathscr{C},\ j \in \mathsf{N},$$
$$a_{i_p j} = d_j \cdot \operatorname{sgn} x^{i_p}(j) \qquad \text{for} \quad p \in \mathscr{C},\ j \in \mathsf{N}.$$

Note that $\|A\| = 1$ and

$$E(A) = \{f_p : p \in \mathscr{C}\}.$$

By (4) and Theorem 1, $0 \in \mathscr{P}_{\mathscr{V}}(A)$.

Since $\det D = 0$, there exists $x = (x_1, x_2, \dots, x_n) \neq 0$ such that $Dx = 0$. Put

$$V = \sum_{m=1}^{n} x_m V_m.$$

Note that $V \neq 0$ and $f_m(V) = 0$, $m = 1, 2, \dots, n$. By Theorem 2, 0 is not a strongly unique best approximation for A in $\mathscr{V}$. By [8, Theorem 3.1], $\mathscr{V}$ is not a Chebyshev subspace and the proof is complete.

THEOREM 4. *Let $\mathscr{V} = \operatorname{lin}\{V\}$, $V \in \mathscr{K}(c_0, c_0)$, $V \neq 0$. Then $\mathscr{V}$ is a Chebyshev subspace if and only if $\mathscr{V}$ is an interpolating subspace.*

PROOF. The classical work here is [12]. In l^1, the one-dimensional subspace $\operatorname{lin}\{v\}$ is Chebyshev iff for every $x \in \operatorname{ext} S_{l^\infty}$ the following holds

$$\sum_{j=1}^{\infty} x(j)v(j) \neq 0.$$

Note that for any $x \in c_0$ we obtain $V(x) = [f_1(x), f_2(x), \dots]$, where the functionals f_i correspond to elements of l^1.

It is obvious that if for any j, $\operatorname{lin}\{f_j\}$ is not a Chebyshev subspace of l^1, then $\operatorname{lin}\{V\}$ is not a Chebyshev subspace of $\mathscr{K}(c_0, c_0)$. This proves the theorem.

Note that by a result of Malbrock (see [10], Theorem 3.3) each one-dimensional subspace $\mathscr{V} = \operatorname{lin}\{V\} \subset \mathscr{L}(c_0, c_0)$ is a Chebyshev subspace iff there exists $\delta > 0$ such that

$$\left| \sum_{j=1}^{\infty} x(j)v_{ij} \right| \geq \delta,$$

where $|x(j)| = 1$, $j \in \mathsf{N}$.

COROLLARY. *Let $\mathscr{V} \subset \mathscr{K}(c_0, c_0)$ be a one-dimensional Chebyshev subspace. Every operator $A \in \mathscr{K}(c_0, c_0)$ has a strongly unique best approximation in $\mathscr{V}$.*

PROOF. Obvious. For more details we refer the reader to [2].

It is clear that (3) is satisfied for any n-dimensional interpolating subspace. However, (3) is not sufficient for an n-dimensional ($n \geq 2$) subspace to be Chebyshev.

EXAMPLE 2. Let $\mathscr{V} = \text{lin}\{V_1, V_2\}$, where

$$V_1 = \begin{bmatrix} 1 & 0 & . & . & . \\ \frac{1}{2} & 0 & . & . & . \\ \frac{1}{4} & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 1 & 0 & . & . & . \\ \frac{1}{3} & 0 & . & . & . \\ \frac{1}{9} & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}.$$

Note that $\mathscr{V}$ satisfies (3). We claim that $\mathscr{V}$ is a non-Chebyshev subspace.

Indeed, define $A = [a_{ij}]_{i,j\in\mathsf{N}}$ by

$$a_{12} = 100, \quad a_{ij} = 0 \qquad \text{for each} \quad (i, j) \neq (1, 2), \quad i, j \in \mathsf{N}.$$

It follows that

$$A - (\alpha_1 V_1 + \alpha_2 V_2) = \begin{bmatrix} -\alpha_1 - \alpha_2 & 100 & 0 & . & . \\ -\frac{1}{2}\alpha_1 - \frac{1}{3}\alpha_2 & 0 & . & . & . \\ -\frac{1}{4}\alpha_1 - \frac{1}{9}\alpha_2 & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}, \qquad \alpha_1, \alpha_2 \in \mathsf{R}.$$

Hence

$$\|A\| = \|A - (600V_1 - 600V_2)\| = 100 = \inf_{\alpha_1,\alpha_2\in\mathsf{R}} \|A - (\alpha_1 V_1 + \alpha_2 V_2)\|.$$

THEOREM 5. *Let $V_1, V_2, \ldots, V_n$ be given by*

$$V_j = \begin{bmatrix} 0 & 0 & . & . & v_{1j} & 0 & . & . \\ 0 & 0 & . & . & v_{2j} & 0 & . & . \\ 0 & 0 & . & . & v_{3j} & 0 & . & . \\ . & . & . & . & . & & & \\ . & . & . & . & . & & & \\ . & . & . & . & . & & & \\ . & . & . & . & . & & & \end{bmatrix},$$

where $v_{ij} \neq 0$ for each $i \in \mathsf{N}$, $j \in \{1, 2, \ldots, n\}$ and

$$\lim_{i\to\infty} v_{ij} = 0 \qquad \textit{for each} \quad j \in \{1, 2, \ldots, n\}.$$

The following statements are equivalent:

(i) *For every choice of distinct $j_1, \ldots, j_k$ from $\{1, 2, \ldots, n\}$, $\mathscr{V}(j_1, \ldots, j_k) := \text{lin}\{V_{j_1}, \ldots, V_{j_k}\}$ is a Chebyshev subspace of $\mathscr{K}(c_0, c_0)$*

(ii)
$$\forall\, 1 \le k \le n, \quad \forall\, 1 \le j_1 < j_2 < \cdots < j_k \le n,$$
$$\forall\, 1 \le i_1 < i_2 < \cdots < i_k,$$
$$\forall\, x_{ml} \in \mathsf{R} : |x_{ml}| = 1, \quad m, l = 1, 2, \ldots, k$$
$$\det[x_{ml} v_{i_m j_l}]_{m=1,2,\ldots,k,\ l=1,2,\ldots,k} \neq 0.$$

Proof. First, we assume that (ii) holds.

If $k = 1$ then $\mathscr{V}(j_1)$ is an interpolating subspace for every $j_1 \in \{1, 2, \ldots, n\}$. Let $1 < k < n$ and assume that for any $j_1, \ldots, j_k \in \{1, 2, \ldots, n\}$, $j_p \neq j_q$, $p \neq q$, $\mathscr{V}_k := \mathscr{V}(j_1, \ldots, j_k)$ is a Chebyshev subspace.

Suppose that there exist $1 \le j_1 < j_2 < \cdots < j_k < j_{k+1} \le n$ such that
$$\mathscr{V}_{k+1} := \mathscr{V}(j_1, \ldots, j_k, j_{k+1})$$
is a non-Chebyshev subspace. Without loss of generality we can assume that for any $k+1 \in \{1, 2, \ldots, n\}$, $j_m = m$, $m = 1, 2, \ldots, k+1$. This means precisely that $V_{j_m} = [(V_{j_m})_{ij}]_{i,j \in \mathsf{N}}$, where
$$(V_{j_m})_{ij} = \begin{cases} v_{ij_m}, & j = m \\ 0, & j \neq m \end{cases}.$$
for $i \in \mathsf{N}$, $m \in \{1, 2, \ldots, k, k+1\}$.

Since $\mathscr{V}_{k+1}$ is a non-Chebyshev subspace, there exists $A = [a_{ij}]_{i,j \in \mathsf{N}} \in \mathscr{K}(c_0, c_0)$ such that $\sharp \mathscr{P}_{\mathscr{V}_{k+1}}(A) > 1$. We can assume that $0, W \in \mathscr{P}_{\mathscr{V}_{k+1}}(A)$, where $W \neq 0$. Let $\mathscr{U} = \{i : \|e_i \circ A\| = \|A\|\}$. Since $A \in \mathscr{K}(c_0, c_0)$, $\sharp \mathscr{U} < \infty$. For every $i \in \mathscr{U}$ we put
$$E_i = \{x \in \operatorname{ext} S_{l^\infty} : (e_i \otimes x)(A) = \|A\|\}.$$
Since $0, W \in \mathscr{P}_{\mathscr{V}_{k+1}}(A)$, we conclude that for all $i \in \mathscr{U}$ and $x \in E_i$
$$(e_i \otimes x)(W) \ge 0. \tag{5}$$
Let
$$\mathscr{U}_1 = \{i \in \mathscr{U} : \exists\, x \in E_i : (e_i \otimes x)(W) = 0\}.$$
Since $0 \in \mathscr{P}_{\mathscr{V}_{k+1}}(A)$, $\mathscr{U}_1 \neq \emptyset$.

We will prove that for any $i \in \mathscr{U}_1$ and $x, y \in E_i$ such that
$$\begin{aligned} &(e_i \otimes x)(W) = (e_i \otimes y)(W) = 0, \\ &\qquad x(l) = y(l), \qquad l = 1, 2, \ldots, k+1. \end{aligned} \tag{6}$$
On the contrary, suppose that (6) does not hold. Let $x, y \in E_i$ be such that
$$(e_i \otimes x)(W) = 0, \qquad (e_i \otimes y)(W) = 0,$$

and

$$x(l) \neq y(l) \qquad \text{for some} \quad l \in \{1, 2, \ldots, k+1\}.$$

Without loss of generality we can assume

$$x(j) = y(j) \qquad \text{for} \quad j = 1, 2, \ldots, p, \quad p < k+1$$

and

$$x(j) = -y(j) \qquad \text{for} \quad j = p+1, p+2, \ldots, k+1.$$

Hence

$$\sum_{j=1}^{p} x(j)w_{ij} = 0, \qquad \sum_{j=p+1}^{k+1} x(j)w_{ij} = 0. \tag{7}$$

As

$$x(j) = -y(j) \qquad \text{for} \quad j = p+1, p+2, \ldots, k+1$$

we obtain

$$a_{ij} = 0 \qquad \text{for} \quad j = p+1, p+2, \ldots, k+1.$$

By (5),

$$\sum_{j=p+1}^{k} x(j)w_{ij} - x(k+1)w_{i,k+1} \geq 0$$
$$\sum_{j=p+1}^{k} -x(j)w_{ij} + x(k+1)w_{i,k+1} \geq 0.$$

Therefore

$$\sum_{j=p+1}^{k} x(j)w_{ij} = x(k+1)w_{i,k+1}.$$

By (7), $x(k+1)w_{i,k+1} = 0$. Consequently, $w_{i,k+1} = 0$. Hence $W \in \mathcal{V}_k$. Since $0 \in \mathcal{V}_k$ and $\mathcal{V}_k$ is a Chebyshev subspace, (6) is proved.

We will show that there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$,

$$E(A-\alpha W) = \{e_i \otimes x \,:\, i \in \mathcal{U}_1,\ (e_i \otimes x)(W) = 0,\ (e_i \otimes x)(A) = \|A\|\}. \tag{8}$$

We first prove that

$$\begin{aligned} \sup\{f(A) : f = e_i \otimes x, i \in \mathcal{U} : f(W) < 0\} \\ \leq \|A\| - 2\min\{|a_{ij}| : i \in \mathcal{U}, j \in \{1, 2, \ldots, n\}, a_{ij} \neq 0\}, \end{aligned} \tag{9}$$

where $A = [a_{ij}]_{i,j\in\mathrm{N}}$.

Let $i \in \mathcal{U}$, $f = e_i \otimes x$, $f(W) < 0$. Hence there exists $j_0 \in \{1, 2, \dots, n\}$ satisfying

$$x(j_0) \neq \operatorname{sgn}(a_{ij_0}) \qquad \text{for} \quad a_{ij_0} \neq 0.$$

Now, we will show

$$\begin{aligned} f(A) = \sum_{j=1}^{\infty} x(j)a_{ij} &\leq \|A\| - 2|a_{ij_0}| \\ &\leq \|A\| - 2\min\{|a_{ij}| : i \in \mathcal{U},\, j = 1, 2, \dots, n,\, |a_{ij}| \neq 0\}, \end{aligned}$$

and (9) is proved.

We conclude from (9) that there exist $\alpha_0 > 0$, $b > 0$ such that for every $\alpha \in (0, \alpha_0]$,

$$f(A - \alpha W) < b < \|A\|,$$

where $f \in \operatorname{ext} S_{\mathcal{K}^*(c_0,c_0)}$, $f(W) < 0$.

Assume α_0 is so small that

$$\sup_{i\in\mathrm{N}\setminus\mathcal{U}} \|e_i \circ (A - \alpha_0 W)\| < \|A\|.$$

Consequently, if $f \in E(A - \alpha_0 W)$ then $f = e_i \otimes x$, where $i \in \mathcal{U}_1$ and $f(W) = 0$. Since

$$\|A - \alpha_0 W\| = \|A\| = \operatorname{dist}(A, \mathcal{V}_{k+1}),$$

(8) is proved.

Since $\alpha_0 W \in \mathcal{P}_{\mathcal{V}_{k+1}}(A)$, we conclude (see [16]) that

$$\exists\, 1 \leq q \leq k+2, \quad \exists\, \lambda_1, \dots, \lambda_q > 0, \quad \sum_{m=1}^{q} \lambda_m = 1$$

such that

$$\sum_{m=1}^{q} \lambda_m (e_{i_m} \otimes x^{i_m})\big|_{\mathcal{V}_{k+1}} = 0, \tag{10}$$

where $(e_{i_m} \otimes x^{i_m})(A - \alpha_0 W) = \|A - \alpha_0 W\|$. Let q be the smallest number having property (10). By (6), $i_j \neq i_l$ for $j \neq l$, $j, l \in \{1, 2, \dots, q\}$. If $q = k+2$ then (see [18]) $\alpha_0 W$ is the strongly unique best approximation for A in $\mathcal{V}_{k+1}$, a contradiction. Suppose that $1 \leq q \leq k+1$. This contradicts (ii).

Let us assume that $\mathscr{V}_k$ is a Chebyshev subspace of $\mathscr{K}(c_0, c_0)$ for every $1 \le k \le n$. Suppose that (ii) is false. Consequently, there exist

$$1 \le k \le n, \quad 1 \le j_1 < j_2 < \cdots < j_k \le n,$$
$$1 \le i_1 < i_2 < \cdots < i_k,$$
$$x_{ml} \in \mathsf{R} : |x_{ml}| = 1, \quad m, l = 1, 2, \ldots, k$$

satisfying

$$\det[x_{ml} v_{i_m j_l}]_{m=1,2,\ldots,k,\ l=1,2,\ldots,k} = 0.$$

It follows that there exist

$$\lambda_1, \ldots, \lambda_k \in \mathsf{R}, \qquad \sum_{m=1}^{k} |\lambda_m| > 0$$

such that

$$\sum_{m=1}^{k} \lambda_m (e_{i_m} \otimes x^{i_m})\big|_{\mathscr{V}_k} = 0, \tag{11}$$

where $x^{i_m} = (x^{i_m}(1), x^{i_m}(2), \ldots)$, $x^{i_m}(l) = x_{ml}$.

Without loss of generality we can assume

$$\lambda_m > 0, \quad m = 1, 2, \ldots, k, \quad \sum_{m=1}^{k} \lambda_m = 1.$$

We define an operator $B = [b_{ij}]_{i,j \in \mathsf{N}}$ by

$$b_{ij} = \frac{\operatorname{sgn} x^i(j)}{2^j}, \qquad i \in \{i_1, i_2, \ldots, i_k\},$$
$$b_{ij} = 0, \qquad i \notin \{i_1, i_2, \ldots, i_k\}, \quad j \in \mathsf{N}.$$

Hence $(e_{i_m} \otimes x^{i_m})(B) = \|B\|$, $m = 1, 2, \ldots, k$. By (11), $0 \in \mathscr{P}_{\mathscr{V}_k}(B)$ and

$$\dim \operatorname{span}\{e_{i_m} \otimes x^{i_m} \mid \mathscr{V}_k\} < k,$$

where $\dim \mathscr{V}_k = k$. Therefore there exists $V \in \mathscr{V}_k \setminus \{0\}$ such that

$$(e_{i_m} \otimes x^{i_m})(V) = 0, \qquad m = 1, 2, \ldots, k.$$

Note that (see the proof of the formula (9))

$$\sup\{f(B) : f = e_{i_m} \otimes x,\ m = 1, 2, \ldots, k,\ f(V) < 0\}$$
$$< \|B\| - \min\{|\, b_{ij}\,| : i = i_1, i_2, \ldots, i_k,\ j = 1, 2, \ldots, n\}.$$

Hence there exist $\alpha_0 > 0$, $b > 0$ such that

$$f(B - \alpha_0 V) \leq b < \|B\|, \qquad f \in \operatorname{ext} S_{\mathscr{K}^*(c_0,c_0)}, \qquad f(V) \leq 0.$$

Consequently, $\|B - \alpha_0 V\| = \|B\|$, a contradiction. The proof is complete.

EXAMPLE 3. We will construct an n-dimensional Chebyshev subspace $\mathscr{V} \subset \mathscr{K}(c_0, c_0)$. Let $0 < t_1 < t_2 < \cdots < t_{n-1}$ be such that

$$\lim_{i\to\infty} \frac{1}{2^i} t_m^i = 0, \qquad m = 1, 2, \ldots, n-1.$$

Define $V_m = [(v_m)_{ij}]_{i,j\in\mathsf{N}}$ by

$$(v_m)_{im} = \frac{1}{2^i} t_m^i, \qquad (v_m)_{ij} = 0, \;\; i \in \mathsf{N}, \;\; j \neq m.$$

Hence $V_m \in \mathscr{K}(c_0, c_0)$ for every $m = 1, 2, \ldots, n-1$.

Let $\mathscr{V}_{n-1} := \operatorname{lin}\{V_1, V_2, \ldots, V_{n-1}\}$ satisfy the formula (ii) for every $1 \leq k \leq n-1$.

We will construct an operator $V_n \in \mathscr{K}(c_0, c_0)$ such that $\mathscr{V}_n := \operatorname{lin}\{V_1, V_2, \ldots, V_{n-1}, V_n\}$ satisfies the formula (ii) for every $1 \leq k \leq n$. Our goal is to find $x \in \mathsf{R}$ such that

$$\lim_{i\to\infty} \frac{1}{2^i} x^i = 0 \tag{12}$$

and

$$W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots m_{k-1}) \tag{13}$$
$$:= \begin{vmatrix} y_1^1 \frac{1}{2^{i_1}} t_{m_1}^{i_1} & \cdot\;\cdot\;\cdot & y_1^{k-1} \frac{1}{2^{i_1}} t_{m_{k-1}}^{i_1} & y_1^k \frac{1}{2^{i_1}} x^{i_1} \\ \cdot & \cdot\;\cdot\;\cdot & & \\ \cdot & \cdot\;\cdot\;\cdot & & \\ \cdot & \cdot\;\cdot\;\cdot & & \\ y_k^1 \frac{1}{2^{i_k}} t_{m_1}^{i_k} & \cdot\;\cdot\;\cdot & y_k^{k-1} \frac{1}{2^{i_k}} t_{m_{k-1}}^{i_k} & y_k^k \frac{1}{2^{i_k}} x^{i_k} \end{vmatrix} \neq 0,$$

where $k \in \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_k \in \mathsf{N}$, $y^1, \ldots, y^k \in \{-1, 1\}^k$, $m_1, m_2, \ldots, m_{k-1} \in \{1, 2, \ldots, n-1\}$. Since $W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots, m_{k-1})$ is not totally equal to zero, we conclude that the set of roots of $W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots m_{k-1})$ is finite for arbitrary but fixed $y^1, \ldots, y^k$, $i_1, \ldots, i_k, m_1, \ldots m_{k-1}$. Therefore for all $y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots m_{k-1}$ as above, the set of roots of $W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots m_{k-1})$ is countable. Since R is not countable we see that there exists $x \in \mathsf{R}$ satisfying (12) and (13).

REMARK 2. An n-dimensional Chebyshev subspace proposed in Example 3 is a non-interpolating subspace of $\mathcal{K}(c_0, c_0)$.

PROOF. Let us assume that $\mathcal{V}_n = \text{lin}\{V_1, V_2, \dots, V_n\}$ is an n-dimensional Chebyshev subspace, where V_m, $m = 1, 2, \dots, n$ are defined in Example 3.

Put $V = \frac{1}{t_1}V_1 - \frac{1}{t_2}V_2$. Note that $V \neq 0$ and $v_{ij} = 0$, $j \geq 3$, $i \in \mathsf{N}$, where $V = [v_{ij}]_{i,j \in \mathsf{N}}$. It is obvious that there exist $x^1, x^2, \dots, x^n \in \text{ext}\, S_{l^\infty}$ such that $x^m(1) = x^m(2) = 1$, $m = 1, 2, \dots, n$ and $f_m := e_1 \otimes x^m$, $m = 1, 2, \dots, n$ are linearly independent. Note that

$$f_m(V) = 0, \qquad m = 1, 2, \dots, n.$$

This completes the proof.

LEMMA. *Let X be a normed space and let $\mathcal{V}$ be a finite-dimensional subspace of X. Let $T \in X$. If $0 \in \mathcal{P}_{\mathcal{V}}(T)$ and 0 is not a strongly unique best approximation for T in $\mathcal{V}$ then*

$$\exists\, V \in \mathcal{V}, \quad V \neq 0 \quad : \quad \forall\, f \in E(T) \quad f(V) \geq 0.$$

PROOF. Let us assume that

$$\forall\, V \in \mathcal{V}, \quad V \neq 0, \quad \exists\, f \in E(T) \quad : \quad f(V) < 0.$$

Set for any $V \in \mathcal{V}$, $\|V\| = 1$,

$$-r_V = \inf\{f(V) : f \in E(T)\},$$
$$-r = \sup\{-r_V : V \in \mathcal{V},\ \|V\| = 1\}.$$

We show that $r > 0$. If not, there exists $(V_n) \subset S_{\mathcal{V}}$ such that $-r_{V_n} \geq -\frac{1}{n}$. Since $\mathcal{V}$ is a finite-dimensional subspace, we may assume that $V_n \to V \in S_{\mathcal{V}}$. Take $f \in E(T)$, $f(V) < 0$. Hence for $n \geq n_0$ there exists $d > 0$ such that

$$-\frac{1}{n} \leq -r_{V_n} \leq f(V_n) < f(V) + d < 0,$$

a contradiction. Therefore

$$\forall\, V \in \mathcal{V}, \quad V \neq 0, \quad \exists\, f \in E(T) \quad : \quad f\left(\frac{V}{\|V\|}\right) < -r.$$

By the above,

$$\forall\, V \in \mathcal{V}, \quad V \neq 0, \quad \exists\, f \in E(T) \quad : \quad f(V) \leq -r\|V\|.$$

Hence 0 is a strongly unique best approximation for T, a contradiction. This proves the lemma.

THEOREM 6. *Let $\mathscr{V} \subset \mathscr{K}(c_0, c_0)$ be an n-dimensional subspace such that*

$$\forall\, V \in \mathscr{V}, \quad \forall\, i \in \mathsf{N} \quad \sharp\{j \in \mathsf{N} : v_{ij} \neq 0\} < \infty,$$

where $V = [v_{ij}]_{i,j\in\mathsf{N}}$ and let $T \in \mathscr{K}(c_0, c_0)$. Then T has a unique best approximation in $\mathscr{V}$ if and only if T has a strongly unique best approximation in $\mathscr{V}$.

PROOF. Let us assume that 0 is the unique best approximation for T in $\mathscr{V}$. Suppose that 0 is not a strongly unique best approximation. Hence (see Lemma)

$$\exists\, V \in \mathscr{V}, \quad V \neq 0 \quad : \quad \forall\, f \in E(T) \quad f(V) \geq 0,$$

where $f = e_i \otimes x^i$ for some $x^i \in \operatorname{ext} S_{l^\infty}$.

Put

$$\mathscr{N} = \{i \in \mathsf{N} : \exists\, x^i \in \operatorname{ext} S_{l^\infty} : e_i \otimes x^i \in E(T)\}.$$

Since T is compact, we conclude that $\sharp\mathscr{N} < \infty$.

For every $i \in \mathscr{N}$ we set

$$E_i = \{x^i \in \operatorname{ext} S_{l^\infty} : (e_i \otimes x^i)(T) = \|T\|\}.$$

Let $i \in \mathsf{N} \setminus \mathscr{N}$. Hence there exists $b > 0$ such that

$$(e_i \otimes x)(T) < b < \|T\|, \qquad x \in \operatorname{ext} S_{l^\infty}.$$

Consequently, there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$,

$$\mid (e_i \otimes x)(T - \alpha V) \mid < b.$$

Therefore

$$\sup_{i\in\mathsf{N}\setminus\mathscr{N}} \mid (e_i \otimes x)(T - \alpha V) \mid \leq b < \|T\|.$$

Let $i \in \mathscr{N}$ and let $x^i \notin E_i$. From this we conclude that there exists $j_0 \in \mathsf{N}$ such that

$$\operatorname{sgn} x^i(j_0) \neq \operatorname{sgn}(t_{ij_0}), \qquad t_{ij_0} \neq 0,$$

where $T = [t_{ij}]_{i,j\in\mathsf{N}}$.

Set $J = \{j \in \mathsf{N} : v_{ij} \neq 0\}$. If $\operatorname{sgn} x^i(j) = \operatorname{sgn}(t_{ij})$ for any $j \in J$, then there exists $y^i \in E_i$ such that

$$(e_i \otimes y^i)(T) = \|T\|, \qquad (e_i \otimes y^i)(V) = (e_i \otimes x^i)(V).$$

By the above,

$$(e_i \otimes x^i)(T - \alpha V) \leq \|T\| - (e_i \otimes y^i)(\alpha V) \leq \|T\|.$$

Let $\operatorname{sgn} x^i(j_0) \neq \operatorname{sgn}(t_{ij_0})$ for some $j_0 \in J$, where $t_{ij_0} \neq 0$. Since J is finite, there exists $\alpha_0 > 0$ such that

$$\|\alpha_0 V\| < \min\{|t_{ij}| : j \in J, t_{ij} \neq 0\}.$$

Let $\alpha \in (0, \alpha_0]$. Hence

$$\begin{aligned}(e_i \otimes x^i)(T - \alpha V) &= \sum_{j \in J} x^i(j)(t_{ij} - \alpha v_{ij}) + \sum_{j \notin J} x^i(j)(t_{ij} - \alpha v_{ij}) \\ &\leq \sum_{j \in J} |t_{ij}| - 2|t_{ij_0}| + \sum_{j \notin J} |t_{ij}| + \alpha\|V\| \\ &= \|T\| + \alpha\|V\| - 2|t_{ij_0}| < \|T\|.\end{aligned}$$

Finally,

$$\|T - \alpha V\| = f(T - \alpha V),$$

where $f = e_i \otimes x^i$, $i \in \mathcal{N}$, $x^i \in E_i$. Hence

$$\|T - \alpha V\| = f(T - \alpha V) \leq \|T\|.$$

The proof is complete.

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