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CHARACTERIZATIONS OF RIESZ SETS

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Abstract

Let G be a compact abelian group, $M(G)$ its measure algebra and $L^1(G)$ its group algebra. For a subset E of the dual group $\widehat{G}$, let $M_E(G) = \{\mu \in M(G) : \widehat{\mu} = 0 \text{ on } \widehat{G}\setminus E\}$ and $L^1_E(G) = \{a \in L^1(G) : \widehat{a} = 0 \text{ on } \widehat{G}\setminus E\}$. The set E is said to be a Riesz set if $M_E(G) = L^1_E(G)$. In this paper we present several characterizations of the Riesz sets in terms of Arens multiplication and in terms of the properties of the Gelfand transform $\Gamma : L^1_E(G) \to c_0(E)$.

Introduction

Let G be a compact infinite abelian group equipped with its normalized Haar measure, $L^1(G)$ its group algebra and $M(G)$ its measure algebra. The multiplication is of course the convolution in both algebras. As usual, by $C(G)$ we denote the space of the continuous functions on G and by $L^\infty(G)$ the dual space of the group algebra $L^1(G)$. All these spaces are considered as Banach spaces over the field of the complex numbers. Let E be a subset of the dual group $\widehat{G}$ of G, and let $\Re$ be one of these four Banach spaces. By $\Re_E$ we denote the following subspace

$$\Re_E = \{a \in \Re : \widehat{a}(\gamma) = 0 \text{ for } \gamma \in \widehat{G}\setminus E\}$$

of $\Re$. Here, as usual, the cap denotes the Fourier (or Fourier-Stieltjes) transform. In the case where $\Re = L^1(G)$, the space $\Re_E$ is a closed ideal of $L^1(G)$; and in the case where $\Re = M(G)$, the space $\Re_E$ is a weak-star (i.e., $\sigma(M(G), C(G))$) closed ideal of $M(G)$. We recall that the set E is said to be a Riesz set if

$$M_E(G) = L^1_E(G) \qquad \text{(via the Radon-Nikodym Theorem).}$$

Let now $\mathsf{T} = \{z \in \mathsf{C} : |z| = 1\}$ be the unit circle group. Then $\widehat{\mathsf{T}} = \mathsf{Z}$ and, for $n \in \mathsf{Z}$, $a \in L^1(\mathsf{T})$ and $\mu \in M(\mathsf{T})$,

$$\widehat{a}(n) = \frac{1}{2\pi}\int_0^{2\pi} a(t)e^{-inx}\,dx \qquad \text{and} \qquad \widehat{\mu}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\,d\mu(x),$$

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respectively. A classical result due to Riesz brothers ([12], p. 47) says that any measure $\mu \in M(\mathsf{T})$ whose Fourier-Stieltjes transform vanishes on the negative integers is absolutely continuous with respect to the Lebesgue measure on the unit circle. In other words, the equality $M_{\mathsf{N}}(\mathsf{T}) = L^1_{\mathsf{N}}(\mathsf{T})$ holds. It is this classical result that is behind the notion of Riesz sets. As far as we know, the term of "Riesz set" has been introduced and studied for its own sake in the seminal paper by Y. Meyer [17]. Later on the Riesz sets have been studied by several mathematicians. In [14] and [15] Lust-Piquard has characterized the Riesz sets in terms of the Radon-Nikodym property, proving that the set E is a Riesz set iff the space $L^1_E(G)$ has the RNP. In the papers [8] and [9] G. Godefroy has studied the Riesz sets in connection with Havin-Moorey theorem. The paper [9] contains a wealth of examples, counterexamples and methods of constructions of Riesz sets and the so-called nicely placed sets. For a more recent work on this subject we refer the reader to the paper [13] of Lefèvre and Rodriguez-Piazza. The reader can find further works on the Riesz sets in the references of the papers [9] and [13]. Finally we mention that, as proved by V. Tardivel in [19], the set Σ of the Riesz subsets of $\widehat{G}$, considered as a subset of the product space $2^{\widehat{G}}$, is a non-Borelian coanalytic set so that the collection Σ is extremely rich and no concrete characterization of the Riesz subsets of $\widehat{G}$ is possible.

We also recall that, endowed with one of the two Arens multiplications, the second dual of any Banach algebra can be made into a Banach algebra. In general the second dual algebra of a commutative Banach algebra A need not be commutative. If it is commutative then it is said to be Arens regular. In the next section we shall give more information on this notion. For the moment let us recall that the algebra $L^1(G)$ is Arens regular iff the group G is finite. In this paper we present several characterizations of the Riesz sets in terms of Arens multiplication and also in terms of the properties of the Gelfand transform $\Gamma : L^1_E(G) \to c_0(E)$.

To explain the main results of the paper, equip the second dual $L^1(G)^{**}$ of the algebra $L^1(G)$ with the first Arens multiplication (the one which is weak-star continuous on the left-hand variable). Let e be an arbitrary but fixed right identity in $L^1(G)^{**}$. This is just a weak-star cluster point in $L^1(G)^{**}$ of some bounded approximate identity $(e_i)_{i\in I}$ of $L^1(G)$ (see for instance [4], p. 310, Proposition 2.9.16). Thus, for all $m \in L^1(G)^{**}$, $me = m$ but in general $em \neq m$. Let E be an arbitrary nonempty subset of the dual group $\widehat{G}$ and $R_E = L^1_E(G)^{**} \cap C(G)^{\perp}$. Here $C(G)^{\perp}$ is the annihilator of the subspace $C(G)$ of $L^{\infty}(G)$ in $L^1(G)^{**}$. The main results of the paper can be summarized as follows. The undefined terms used here will be defined in the next section.

a) The set E is a Riesz set iff $eL^1_E(G)^{**} = L^1_E(G)$.

b) The set E is a Riesz set iff $L^1_E(G)^{**} = L^1_E(G) \oplus R_E$.

c) The set E is a Riesz set iff, for each $m \in L^1_E(G)^{**}$, the operator $\tau_m : L^\infty(G) \to L^\infty(G)$, defined by $\tau_m(\varphi) = m.\varphi$, is weakly completely continuous. Here $m.\varphi$ is the functional on $L^1(G)$ defined by $\langle m.\varphi, a\rangle = \langle \varphi, am\rangle$.

d) The set E is a Riesz set iff the Gelfand transform $\Gamma : L^1_E(G) \to c_0(E)$ is a weakly compact semi-embedding.

e) The set E is a Riesz set iff, for each $\mu \in M_E(G)$, the set $L^1_E(G)_1 * \mu$ is weak-star closed in the space $M(G)$. Here $L^1_E(G)_1$ is the closed unit ball of the space $L^1_E(G)$.

The paper also contains several corollaries of these results. The proofs are mostly functional analytic and use only the well-known Banach space and Banach algebra properties of the involved spaces or algebras.

1. Notation and Preliminary Results

Our notation and terminology are standard and some of it has already been introduced in the preceding section. In addition to these, we need a couple of facts related to the *Arens multiplications*.

Let A be a commutative Banach algebra. We always regard A as canonically embedded into its second dual A^{**}. By $\langle a, \varphi\rangle$ we denote the natural duality between A and A^*, and by A_1 the closed unit ball of A. On A^{**} there are two Banach algebra multiplications extending that of A, under each of which A^{**} is a Banach algebra. These multiplications are known as the first and second Arens multiplications of A^{**}. The first of these is constructed in three steps as follows. For a, b in A, φ in A^* and n, m in A^{**}, the elements $a.\varphi$, $n.\varphi$ of A^*, and mn of A^{**} are defined, respectively, as follows.

$$\langle a.\varphi, b\rangle = \langle \varphi, ab\rangle, \quad \langle n.\varphi, a\rangle = \langle n, a.\varphi\rangle \quad \text{and} \quad \langle mn, \varphi\rangle = \langle m, n.\varphi\rangle.$$

For $m \in A^{**}$ fixed, the multiplication operator $n \mapsto nm$ is weak-star to weak-star continuous on A^{**} but, for n fixed, the multiplication operator $m \mapsto nm$ is in general not weak-star to weak-star continuous on A^{**}. Although under this multiplication A^{**} is in general not commutative, the algebra A being commutative, for each $m \in A^{**}$ and $a \in A$, we have $am = ma$. The reader can find ample information on the Arens multiplications in the books [4] and [18].

Let now G be a compact abelian group with dual group $\widehat{G}$. The second dual $L^1(G)^{**}$ of the group algebra $L^1(G)$ will be equipped with the first Arens multiplication as defined above. Although the multiplication on the algebra $L^1(G)$

is the convolution, to be consistent with the Arens multiplication on $L^1(G)^{**}$, we shall use the Arens multiplication notation. This is very convenient when we consider the product am of an element m of $L^1(G)^{**}$ with an element a of $L^1(G)$. Thus, for $\varphi \in L^\infty(G)$ and $a \in L^1(G)$, the functional $a.\varphi$ is defined by $\langle a.\varphi, b\rangle = \langle \varphi, a*b\rangle$ so that $a.\varphi = \varphi * a^\vee$, where $a^\vee(t) = a(t^{-1})$. In particular the function $a.\varphi$ is continuous on G.

Let now $(e_i)_{i\in I}$ be a bounded approximate identity (=BAI) in the algebra $L^1(G)$. Then any weak-star cluster point e of the net $(e_i)_{i\in I}$ in $L^1(G)^{**}$ is a *right identity* in $L^1(G)^{**}$ (i.e., $me = m$, for all $m \in L^1(G)^{**}$) ([4], Proposition 2.9.16). In particular, for each $a \in L^1(G)$, $ea = ae = a$. Such a right identity e will play an important role in the proofs of the results presented in this paper.

If $a \in L^1(G)$ and the support of the function $\widehat{a}$ is compact (so finite), the multiplication operator $L_a : L^1(G) \to L^1(G)$, defined by $L_a(b) = a * b$, has a finite dimensional range. So it is a compact operator. Since the elements $a \in L^1(G)$ whose Fourier transforms have compact supports form a norm dense set in $L^1(G)$ (since the algebra $L^1(G)$ is Tauberian), a simple density argument shows that, for each $a \in L^1(G)$, the operator L_a is compact. Since for each $a \in L^1(G)$, the multiplication operator L_a is compact and since for $m \in L^1(G)^{**}$, $L_a^{**}(m) = am = ma$, the algebra $L^1(G)$ is a two-sided closed ideal in the algebra $L^1(G)^{**}$. Thus, for any $m \in L^1(G)^{**}$ and $a \in L^1(G)$, the product am is in $L^1(G)$. We have sketched the proof of this well-known fact because we shall use it constantly throughout the paper.

We shall also need the following two notions.

Weakly Completely Continuous (=wcc) *Linear Operators*. We recall that a bounded linear operator $T : X \to Y$ from a Banach space X into another one Y is said to be wcc if it sends the weakly Cauchy sequences in X to weakly convergent ones in Y. Every weakly compact operator is of course wcc.

Semi-embeddings. A bounded linear operator $T : X \to Y$ is said to be a semi-embedding if it is one-to-one and the image $T(X_1)$ of the closed unit ball X_1 of X under T is closed in Y. The reader can find ample information on this notion in the paper [2] of Bourgain and Rosenthal and the paper [16] of Lotz, Peck and Porta.

Throughout the paper G will be an arbitrary infinite compact abelian group, e a fixed right identity of $L^1(G)^{**}$ and E a nonempty subset of the dual group $\widehat{G}$.

2. Riesz Sets and Arens Multiplication

In this section our aim is to present some characterizations of the Riesz sets in terms of the Arens multiplication of $L^1(G)^{**}$ and related notions. We start with a couple of preliminary results.

The right identity e being an idempotent, the mapping $P_e : L^1(G)^{**} \to L^1(G)^{**}$ defined by $P_e(m) = em$ is a bounded projection. This projection induces the direct sum

$$L^1(G)^{**} = P_e(L^1(G)^{**}) \oplus (I - P_e)(L^1(G)^{**}).$$

This direct sum will be denoted simply as

$$L^1(G)^{**} = eL^1(G)^{**} \oplus (1-e)L^1(G)^{**}.$$

The annihilator $C(G)^{\perp}$ of the subspace $C(G)$ of $L^{\infty}(G)$ in $L^1(G)^{**}$ identifies naturally with the subspace $(1-e)L^1(G)^{**}$ of $L^1(G)^{**}$. Thus, in the preceding direct sum, the second component does not depend on the right unit e that we have chosen. The first factor $eL^1(G)^{**}$, as a Banach algebra, is isomorphic (isometrically isomorphic if $\|e\| = 1$) to the measure algebra $M(G)$ of G. A specific isomorphism from the algebra $M(G)$ into the algebra $L^1(G)^{**}$ that implements this isomorphism is the mapping

$$j : M(G) \to L^1(G)^{**}, \qquad \text{defined by} \quad j(\mu) = L_{\mu}^{**}(e).$$

Here $L_{\mu} : L^1(G) \to L^1(G)$ is the multiplier defined by $L_{\mu}(a) = a * \mu$, and L_{μ}^{**} is its second adjoint. This j is an isomorphism since $C(G) = \{a * f : a \in L^1(G) \text{ and } f \in L^{\infty}(G)\}$ and $\langle L_{\mu}^{**}(e), a^{\vee} * f\rangle = \langle a * \mu, f\rangle$. Thanks to this isomorphism, we can go from $M(G)$ to $L^1(G)^{**}$ and return back. This permits us to transform the problems about the algebra $M(G)$ into problems about the algebra $L^1(G)^{**}$, which is in some cases easier to deal with.

Since the algebra $L^1(G)$ is an ideal in $L^1(G)^{**}$, for each $m \in L^1(G)^{**}$, the operator $L_m : L^1(G) \to L^1(G)$ defined by $L_m(a) = am$ is also a multiplier. Actually, since $j : M(G) \to eL^1(G)^{**}$ is an onto isomorphism, given any $m \in L^1(G)^{**}$, there is unique measure $\mu \in M(G)$ such that $em = L_{\mu}^{**}(e)$. Hence, for $a \in L^1(G)$,

$$am = a(em) = aL_{\mu}^{**}(e) = L_{\mu}(a)$$

so that $L_m(a) = L_{\mu}(a)$. Moreover, for $n \in L^1(G)$, one has $L_m^{**}(n) = nm$ since the first Arens multiplication is weak-star continuous on the left hand variable. We shall use these facts in the proofs of the following results.

LEMMA 2.1. *Let $m \in L^1(G)^{**}$ be a given element, and let $L_m : L^1(G) \to L^1(G)$ be the multiplier defined by $L_m(a) = am$. Then the operator L_m is weakly compact iff the product em is in the algebra $L^1(G)$.*

PROOF. Suppose first that the operator L_m is weakly compact. Let $(e_i)_{i\in I}$ be a BAI in the algebra $L^1(G)$ that converges to e in the weak-star topology of $L^1(G)^{**}$. Then, in the weak-star topology of $L^1(G)^{**}$,

$$L_m(e_i) = e_i m \to em = L_m^{**}(e),$$

by the left-hand continuity of the Arens multiplication in the weak-star topology of $L^1(G)^{**}$. Since L_m is weakly compact, the product $em = L_m^{**}(e)$ is in $L^1(G)$.

Conversely, suppose that the product em is in $L^1(G)$. Then, since $L^1(G)$ is an ideal in its second dual, for all $n \in L^1(G)^{**}$, the product $nm = n(em)$ is in $L^1(G)$. This means that L_m^{**} maps the space $L^1(G)^{**}$ into $L^1(G)$ so that L_m is weakly compact.

Identifying the elements of the space $L^1(G)$ with the measures which are absolutely continuous with respect to the Haar measure of G, we can (and always do) consider $L^1(G)$ as a closed ideal of the measure algebra $M(G)$. The next lemma is almost the same as the preceding lemma; we have included it since we shall use it in this form.

LEMMA 2.2. *Let $\mu \in M(G)$ be a given measure. Then the multiplier operator $L_\mu : L^1(G) \to L^1(G)$ defined by $L_\mu(a) = a * \mu$ is weakly compact iff μ is in $L^1(G)$.*

PROOF. If μ is in $L^1(G)$ then the operator L_μ is weakly compact since $L^1(G)$ is an ideal in its second dual. Conversely, suppose that L_μ is weakly compact. Then the element $a = L_\mu^{**}(e)$ is in the algebra $L^1(G)$. Now let $(e_i)_{i\in I}$ be a BAI in $L^1(G)$ that converges to e in the weak-star topology of $L^1(G)^{**}$. Then, for all $i \in I$,

$$e_i * a = L_\mu(e_i) = e_i * \mu.$$

Passing to the limit in the space $(M(G), w^*)$, we get that $\mu = a$, so that μ is in $L^1(G)$.

The next theorem is the first main result of this section.

THEOREM 2.3. *The set E is a Riesz set iff $eL_E^1(G)^{**} = L_E^1(G)$.*

PROOF. Suppose first that E is a Riesz set so that $L_E^1(G) = M_E(G)$. Let $m \in L_E^1(G)^{**}$ be a given element. Our aim is to prove that the multiplier $L_m : L^1(G) \to L^1(G)$, defined by $L_m(a) = am$, is weakly compact. As seen above, there is a unique measure $\mu \in M(G)$ such that, for all $a \in L^1(G)$, we have $am = a * \mu$. That is, $L_m = L_\mu$. So, since $m \in L^1_E(G)^{**}$, for each $\gamma \in \widehat{G}\backslash E$, $\langle m, \gamma \rangle = 0$. Hence, the equality $am = a * \mu$, which valid for all $a \in L^1(G)$, implies that μ is in $M_E(G)$. As $M_E(G) = L^1_E(G)$, we have that $\mu \in L^1_E(G)$. Hence the multiplier L_μ, so the multiplier L_m, is weakly compact (actually compact). Hence, by Lemma 2.1, the product em is in the space $L^1(G)$, therefore em is in the ideal $L^1_E(G)$. Hence the inclusion $eL^1_E(G)^{**} \subseteq L^1_E(G)$ holds. As $eL^1_E(G) = L^1_E(G)$ and $L^1_E(G) \subseteq L^1_E(G)^{**}$, the equality $eL^1_E(G)^{**} = L^1_E(G)$ holds.

Conversely, suppose that the equality $eL^1_E(G)^{**} = L^1_E(G)$ holds. Let us see that each measure $\mu \in M_E(G)$ is in $L^1_E(G)$. To this end, let $\mu \in M_E(G)$ be a given measure and $L_\mu : L^1(G) \to L^1(G)$ be the multiplier defined by $L_\mu(a) = a * \mu$. Then the element $m = L^{**}_\mu(e)$ is in $L^1_E(G)^{**}$ so that the product $em = eL^{**}_\mu(e) = L^{**}_\mu(e)$ is in the ideal $L^1_E(G)$. Hence, for all $n \in L^1(G)^{**}$, the product $nm = n(em)$ is in $L^1(G)$. Since

$$nm = n(em) = nL^{**}_\mu(e) = L^{**}_\mu(ne) = L^{**}_\mu(n),$$

we see that L^{**}_μ maps $L^1(G)^{**}$ into $L^1(G)$ so that the multiplier L_μ is weakly compact. Hence $\mu \in L^1(G)$. Since $\mu \in M_E(G)$, we conclude that $\mu \in L^1_E(G)$. Hence E is a Riesz set.

Let $R_E = L^1_E(G)^{**} \cap C(G)^\perp$. The space $R_E = L^1_E(G)^{**} \cap C(G)^\perp$ is a weak-star closed subspace of $L^1(G)^{**}$. With this notation, we also have the following characterization of the Riesz sets, which is the second main result of this section.

THEOREM 2.4. *The set E of $\widehat{G}$ is a Riesz set iff the direct sum $L^1_E(G)^{**} = L^1_E(G) \oplus R_E$ holds.*

PROOF. First observe that an element m of $L^1(G)^{**}$ is in $C(G)^\perp$ iff $em = 0$. This follows from the fact that $C(G) = \{a.f : a \in L^1(G) \text{ and } f \in L^\infty(G)\}$ and

$$\langle em, a.f \rangle = \langle am, f \rangle = \langle m, a.f \rangle.$$

Hence the intersection of the spaces $L^1_E(G)$ and R_E is always trivial. Now suppose that E is a Riesz set. Let $P : L^1_E(G)^{**} \to L^1_E(G)^{**}$ be the linear operator defined by $P(m) = em$. This linear operator is bounded, idempotent and, by the preceding theorem, $P(L^1_E(G)^{**}) = L^1_E(G)$. Moreover P is the identity on $L^1_E(G)$. Hence P is a bounded projection from $L^1_E(G)^{**}$ onto $L^1_E(G)$. The kernel of P is the set $\{m \in L^1_E(G)^{**} : em = 0\}$, which is just the space R_E. Hence the direct sum $L^1_E(G)^{**} = L^1_E(G) \oplus R_E$ holds.

Conversely, if this direct sum holds then the equality $eL_E^1(G)^{**} = L_E^1(G)$ holds obviously (since $er = 0$ for $r \in C(G)^{\perp}$) so that, by the preceding theorem, E is a Riesz set.

Suppose now that the set E is a Riesz set. Then the direct sum $L_E^1(G)^{**} = L_E^1(G) \oplus R_E$ holds so that each element p of $L_E^1(G)^{**}$ is of the form $p = a + r$, where $a \in L_E^1(G)$ and $r \in R_E$. So, if $p = a + r$ and $q = b + s$ are any two elements of $L_E^1(G)^{**}$, the product of them in $L^1(G)^{**}$ is given by:

$$pq = (a + r)(b + s) = a * b + as + rb + rs.$$

Since the terms r and s are in the space $C(G)^{\perp}$, we have $as = rb = rs = 0$. This is due to the fact that, for any $\varphi \in L^{\infty}(G)$ and $r \in C(G)^{\perp}$, $r.\varphi = 0$ since, for each $a \in L^1(G)$, $\langle a, r.\varphi\rangle = \langle r, a.\varphi\rangle = 0$. Thus $pq = a * b$. As $L^1(G)$ is commutative, this calculation shows that $pq = qp = a * b$ so that the algebra $L_E^1(G)^{**}$ is commutative. This proves the next result.

Corollary 2.5. *For any Riesz set E, the Banach algebra $A = L_E^1(G)$ is Arens regular.*

Whether the converse of this corollary is true or not seems to be unknown. This is an important open problem in this area.

As another corollary of Theorem 2.4 we present the following result. At this point we recall that, for each Riesz set E, the question whether the quotient space $L^1(G)/L_E^1(G)$ is weakly sequentially complete or not seems to be open (see [9], p. 325). A weakly sequentially complete space cannot contain an isomorphic copy of c_0. Since, as of the today, whether the space $L^1(G)/L_E^1(G)$ is weakly sequentially complete or not is not known, the next result is not trivial.

Corollary 2.6. *If E is a Riesz set then the quotient space $L^1(G)/L_E^1(G)$ does not contain an isomorphic copy of c_0.*

Proof. Suppose that the set E is a Riesz set. Let $q : L^1(G) \to L^1(G)/L_E^1(G)$ be the natural surjection. Then the kernel of q is $L_E^1(G)$ and this space, by Theorem 2.4 above, is complemented in its second dual. Hence, by a result of Kalton and Pelczynski ([11], Proposition 2.2), the quotient space $L^1(G)/L_E^1(G)$ does not contain an isomorphic copy of c_0.

As explained in the preceding section, for $m \in L^1(G)^{**}$ and $\varphi \in L^{\infty}(G)$, the functional $m.\varphi : L^1(G) \to \mathsf{C}$ is defined by $\langle m.\varphi, a\rangle = \langle\varphi, ma\rangle$, so that $\|m.\varphi\|_{\infty} \le \|m\|.\|\varphi\|_{\infty}$. It follows that the functional $m.\varphi$ is in $L^{\infty}(G)$ and the linear operator $\tau_m : L^{\infty}(G) \to L^{\infty}(G)$, defined by $\tau_m(\varphi) = m.\varphi$, is a bounded linear operator. Actually $\tau_m = L_m^*$. The third main result of the paper is the following theorem.

THEOREM 2.7. *The set E is a Riesz set iff, for each m in the space $L^1_E(G)^{**}$, the linear operator $\tau_m : L^\infty(G) \to L^\infty(G)$ is weakly completely continuous.*

PROOF. Suppose first that the set E is a Riesz set. By Theorem 2.3, for each m in $L^1_E(G)^{**}$, the product em is in $L^1_E(G)$. Hence, by Lemma 2.1, the multiplier $L_m : L^1(G) \to L^1(G)$, defined by $L_m(a) = am$, is weakly compact. So $\tau_m = L^*_m$ is weakly compact. Hence it is wcc.

Conversely, let $m \in L^1_E(G)^{**}$ and τ_m be wcc. Let $(\varphi_n)_{n\in\mathsf{N}}$ be a weakly Cauchy sequence in $L^\infty(G)$. Let $\varphi = \sigma(L^\infty(G), L^1(G)) - \lim_{n\to\infty} \varphi_n$ be its weak-star limit. By hypothesis, the sequence $(m.\varphi_n)_{n\in\mathsf{N}}$ converges weakly to some element ψ of $L^\infty(G)$. Since $L^1(G)$ is an ideal in its second dual, for $a \in L^1(G)$, am is in $L^1(G)$ and

$$\langle a, m.\varphi_n\rangle = \langle am, \varphi_n\rangle \to \langle am, \varphi\rangle = \langle a, m.\varphi\rangle$$

so that $\psi = m.\varphi$. Thus $m.\varphi_n \to m.\varphi$ weakly. In particular,

$$\langle em, \varphi_n\rangle = \langle e, m.\varphi_n\rangle \to \langle e, m.\varphi\rangle = \langle em, \varphi\rangle.$$

Since the space $L^\infty(G)$ is a von Neumann algebra, by a result of Godefroy ([10], pp. 155–161, Theorems V.I, V.3 and Example 4 on p. 161), the preceding convergence implies that the product em is in $L^1(G)$. As m is in $L^1_E(G)^{**}$, the product em is in the ideal $L^1_E(G)$ so that, by Theorem 2.3, the set E is a Riesz set.

The next corollary is now obvious. In this corollary the operator τ_m is the same as in the preceding theorem. At this point we recall that, for each $a \in L^1(G)$, the operator τ_a is compact, as seen in Section 1.

COROLLARY 2.8. *For a nonempty subset E of $\widehat{G}$, the following assertions are equivalent.*

a) *E is a Riesz set.*
b) *For each $m \in L^1_E(G)^{**}$, the operator $\tau_m : L^\infty(G) \to L^\infty(G)$ is compact.*
c) *For each $m \in L^1_E(G)^{**}$, the operator $\tau_m : L^\infty(G) \to L^\infty(G)$ is weakly compact.*
d) *For each $m \in L^1_E(G)^{**}$, the operator $\tau_m : L^\infty(G) \to L^\infty(G)$ is wcc.*

In [14] Theorem 3(b) Lust-Piquard proves that if the quotient space $L^1(G)/L^1_{\widehat{G}\setminus E}(G)$ does not contain an isomorphic copy of ℓ^1 then the set E is a Riesz set. The next corollary greatly improves this result of Lust-Piquard.

COROLLARY 2.9. *If the quotient space $L^1(G)/L^1_{\widehat{G}\setminus E}(G)$ does not contain a complemented copy of ℓ^1 then the set E is a Riesz set.*

PROOF. We first recall that the space $L^1(G)/L^1_{\widehat{G}\setminus E}(G)$ does not contain a complemented copy of ℓ^1 iff its dual $L^\infty_E(G)$ does not contain an isomorphic copy of ℓ^∞ ([5], p. 48, Theorem). Now suppose that the space $L^\infty_E(G)$ does not contain an isomorphic copy of ℓ^∞. Then observe that, for each $m \in L^1_E(G)^{**}$ and $\varphi \in L^\infty(G)$, the functional $m.\varphi$ is in the space $L^\infty_E(G)$. Indeed, for $a \in L^1_E(G)$, $a.\varphi = \varphi * a^\vee$ is in the space $L^\infty_E(G)$, which is weak-star closed in $L^\infty(G)$. On the other hand, if a bounded net $(a_i)_{i\in I}$ converges to m in the weak-star topology of $L^1(G)^{**}$, $a_i.\varphi \to m.\varphi$ in the weak-star topology of $L^\infty(G)$. This proves that for each $m \in L^1_E(G)^{**}$ and $\varphi \in L^\infty(G)$, the functional $m.\varphi$ is in the subspace $L^\infty_E(G)$. Thus, for each $m \in L^1_E(G)^{**}$, the linear operator τ_m maps the space $L^\infty(G)$ into the space $L^\infty_E(G)$. Since the space $L^\infty(G)$ is a von Neumann algebra and the space $L^\infty_E(G)$ does not contain an isomorphic copy of ℓ^∞, every bounded linear operator from $L^\infty(G)$ into $L^\infty_E(G)$ is weakly compact. (See [1], Proposition 2.10). Hence, by the preceding corollary, E is a Riesz set.

3. Riesz Sets and Semi-embeddings

The Gelfand spectrum of the algebra $L^1_E(G)$ is the set E so that the Gelfand transform Γ maps the algebra $L^1_E(G)$ into the space $c_0(E)$, the Banach space of the functions $f : E \to \mathsf{C}$ that vanish at infinity. (This means: given any $\varepsilon > 0$ there is a finite subset F of E such that $|f| < \varepsilon$ on $E\setminus F$). The norm on the space $c_0(E)$ is of course the supremum norm. In this section we are interested with questions such as:

- When is Γ weakly compact?
- When is Γ semi-embedding?
- When, for each $\mu \in M_E(G)$, is the set $\mu * L^1_E(G)_1$ closed in the weak-star topology of the space $M(G)$?

It has turned out that these questions are very closely connected with Riesz sets.

As proved by N. Wiener and A. Wintner [20], on the unit circle group $\mathsf{T} = \{z \in \mathsf{C} : |z| = 1\}$, there exists a measure μ which is singular with respect to the Lebesgue measure but the product measure $\mu * \mu$ is absolutely continuous with respect to the same measure. The Fourier-Stieltjes transform $\widehat{\mu}$ of such a measure μ is in the space $c_0(\widehat{\mathsf{T}})$. More generally, if E is a small-2 subset of $\widehat{G}$ (i.e., for each $\mu \in M_E(G)$, $\mu^2 = \mu * \mu$ is in $L^1_E(G)$) then, for each $\mu \in M_E(G)$, the function $\widehat{\mu}$ is in the space $c_0(E)$. Whether every small-2 set is a Riesz set or not is a long standing open problem that goes back to Glicksberg's paper [7]. At any case, the following definition will facilitate the discourse:

We shall say that "the set E is a c_0-set" if for each $\mu \in M_E(G)$, the Fourier-Stieltjes transform $\widehat{\mu}$ of μ is in the space $c_0(E)$.

Of course, every Riesz set, more generally, every small-2 set, is a c_0-set. The next result characterizes the c_0-sets.

LEMMA 3.1. *The Gelfand transform $\Gamma : L^1_E(G) \to c_0(E)$ is weakly compact iff the set E is a c_0-set.*

PROOF. Suppose first that Γ is weakly compact. Let $\mu \in M_E(G)$ be a given measure. Consider the element $m = L^{**}_\mu(e)$ of $L^1_E(G)^{**}$. Since Γ is weakly compact, the function $\widehat{m} = \Gamma^{**}(m)$ is in the space $c_0(E)$. Since $m = L^{**}_\mu(e)$, $\Gamma^{**}(m) = \widehat{\mu}$ on E. From this we see that $\widehat{\mu}$ is in $c_0(E)$. Hence E is a c_0-set.

Conversely, suppose that E is a c_0-set. Let $m \in L^1_E(G)^{**}$ be a given element. Then, the mapping $j : M(G) \to eL^1(G)^{**}$, $j(\mu) = L^{**}_\mu(e)$, being an onto isomorphism, there exists a measure $\mu \in M(G)$ such that, $L_m = L_\mu$. So, for all $a \in L^1(G)$, $am = a * \mu$. This measure μ is in $M_E(G)$ since m is in $L^1_E(G)^{**}$. From the equality $am = a * \mu$, which is valid for all $a \in L^1(G)$, we conclude that $\Gamma^{**}(m) = \widehat{\mu}$ on E. This proves that Γ^{**} maps the second dual of $L^1_E(G)$ into $c_0(E)$ so that Γ is weakly compact.

The first main result of this section is the following theorem.

THEOREM 3.2. *The set E is a Riesz set iff the Gelfand transform $\Gamma : L^1_E(G) \to c_0(E)$ is a weakly compact semi-embedding.*

PROOF. We first recall that, the algebra $L^1(G)$ being semisimple, the algebra $L^1_E(G)$ is semisimple so that the Gelfand transform Γ is one-to-one. Suppose now that the set E is a Riesz set. Then, since E is a c_0-set, Γ is weakly compact. To prove that the set $\Gamma(L^1_E(G)_1)$ is closed in $c_0(E)$, let $\widehat{a_n} = \Gamma(a_n)$ be a sequence in the set $\Gamma(L^1_E(G)_1)$ that converges to some element f of the space $c_0(E)$ in the norm of this space. Let m be any weak-star cluster point of the sequence $(a_n)_{n\in N}$ in $L^1_E(G)^{**}$. That is, $m = \sigma(L^1(G)^{**}, L^\infty(G)) - \lim_i a_{n_i}$ for some subnet $(a_{n_i})_{i\in I}$ of the sequence $(a_n)_{n\in \mathsf{N}}$. Then, since $m \in L^1_E(G)^{**}$, by Theorem 2.3, the product em is in the set $L^1_E(G)_1$ (we can always assume that $\|e\| = 1$ so that $\|em\| \leq \|e\|.\|m\| \leq 1$). On the other hand, since $\widehat{G} \subseteq L^\infty(G)$, for each $\gamma \in \widehat{G}$,

$$f(\gamma) = \lim_i \widehat{a_{n_i}}(\gamma) = \lim_i \langle a_{n_i}, \gamma\rangle = \langle m, \gamma\rangle = \langle em, \gamma\rangle = \widehat{em}(\gamma)$$

so that $\widehat{em} = f$ on $\widehat{G}$. Since the element $a = em$ is in $L^1_E(G)_1$, this proves that the set $\Gamma(L^1_E(G)_1)$ is closed in $c_0(E)$. Hence Γ is a weakly compact semi-embedding.

Conversely, suppose that Γ is a weakly compact semi-embedding. Then, as Γ is a semi-embedding, the set $\Gamma(L^1_E(G)_1)$ is norm closed, so, being convex, weakly closed. As Γ is also weakly compact, the set $\Gamma(L^1_E(G)_1)$ is weakly compact. As $L^1_E(G)_1$ is weak-star dense in the set $L^1_E(G)^{**}_1$ and the set $\Gamma(L^1_E(G)_1)$ is weakly compact, the sets $\Gamma^{**}(L^1_E(G)^{**}_1)$ and $\Gamma(L^1_E(G)_1)$ are equal. Now let m be an element of $L^1_E(G)^{**}_1$. Then $\Gamma^{**}(m)$ is in the set $\Gamma(L^1_E(G)_1)$. So, for some a in $L^1_E(G)_1$, we have $\Gamma^{**}(m) = \Gamma(a)$. Now let $(e_i)_{i\in I}$ be a BAI in $L^1(G)$ that converges to e in the weak-star topology of $L^1(G)^{**}$. The equality $\Gamma^{**}(m) = \Gamma(a)$ implies that

$$\widehat{e_i}.\Gamma^{**}(m) = \widehat{e_i}.\Gamma(a)$$

As $\widehat{e_i} = \Gamma(e_i)$ and Γ is an injective homomorphism, we have $e_i m = e_i * a$, for all $i \in I$. Passing to the limit in the space $(L^1(G)^{**}, w^*)$, we get that $em = a$. It follows that $eL^1_E(G)^{**} \subseteq L^1_E(G)$. As the reverse inclusion is always true, by Theorem 2.3, E is a Riesz set.

As an illustration of this result, let $G = \mathsf{T}$ be the unit circle group and B be the closed unit ball of the Hardy space $H^1 = L^1_{\mathsf{N}}(\mathsf{T})$. The spaces B and $\Gamma(B)$ are Polish spaces, i.e., they are separable and complete metric spaces. The preceding theorem combined with Proposition 1.9 of [2] shows that the inverse of the restriction of the Gelfand transform Γ to B, that is, $\Gamma^{-1} : \Gamma(B) \to B$, is a Baire-1 function. Hence the set D of the points $\widehat{a}$ in $\Gamma(B)$ at which Γ^{-1} is continuous is a dense G_δ-subset of $\Gamma(B)$. So, if $\widehat{a}$ is one of the elements of this set then, for any sequence $(a_n)_{n\in\mathsf{N}}$ in B, the convergence $\widehat{a_n} \to \widehat{a}$ in $c_0(\mathsf{N})$ implies that $||a_n - a||_1 \to 0$. Thus, on the set D, the function Γ^{-1} acts as a homeomorphism. This result is probably new.

The next characterization of the Riesz sets relies on a characterization due to Lust-Piquard, which states that the set E is a Riesz set iff the space $L^1_E(G)$ has the RNP [14].

THEOREM 3.3. *The set E is a Riesz set iff, for each $\mu \in M_E(G)$, the set $\mu * L^1_E(G)_1$ is weak-star closed in the space $M(G)$.*

PROOF. We first recall that, in the algebra $M(G)$ the multiplication is separately weak-star continuous. That is, for $\mu \in M(G)$ fixed, the multiplication operator $\lambda \mapsto \lambda * \mu$ is continuous from $(M(G), w^*)$ into itself. This is due to the fact that, for each $f \in C(G)$, $\mu * f$ is also in $C(G)$. So, for any μ in $M_E(G)$, the set $\mu * M_E(G)_1$ is weak-star compact, so weak-star closed in $M(G)$. This proves the direct implication.

To prove the converse implication, suppose that for each $\mu \in M_E(G)$, the set $\mu * L^1_E(G)_1$ is weak-star closed in the space $M(G)$. Our aim is to prove that the space $L^1_E(G)$ has the RNP. Since a Banach space has the RNP iff each of its separable subspace has the RNP, we can assume that E is countable and the space $L^1_E(G)$ is separable. Now let us choose a measure $\mu \in M_E(G)$ such that $\widehat{\mu}$ does not vanish on the set E. This can be done since E is countable. Then the multiplier $L_\mu : L^1_E(G) \to L^1_E(G)$, defined by $L_\mu(a) = a * \mu$, is one-to-one. By hypothesis, the set $\mu * L^1_E(G)_1$ is weak-star closed in the space $M(G)$, so in $M_E(G)$. Hence L_μ is a semi-embedding. As the weak-star compact norm separable subsets of any dual space have the RNP ([3], p. 71, Proposition 4.1.1), the set $\mu * L^1_E(G)_1$ has the RNP. Since the space $L^1_E(G)$ is separable and L_μ is a semi-embedding, by Theorem 1.1′ in the paper [2] of Bourgain and Rosenthal, the set $L^1_E(G)_1$ has the RNP. So the space $L^1_E(G)$ has the RNP. Hence, by the above mentioned result of Lust-Piquard, the set E is a Riesz set.

The next corollary is now obvious.

COROLLARY 3.4. *For a subset E of $\widehat{G}$, the following assertions are equivalent.*

a) *E is a Riesz set.*
b) *For each $\mu \in M_E(G)$, the set $\mu * L^1_E(G)_1$ is norm compact.*
c) *For each $\mu \in M_E(G)$, the set $\mu * L^1_E(G)_1$ is weakly compact.*
d) *For each $\mu \in M_E(G)$, the set $\mu * L^1_E(G)_1$ is weak-star compact as a subset of $M(G)$.*

Now let E be a countable small-2 set. Theorem 3.3 shows that the set E is a Riesz set iff, for some measure μ in the space $M_E(G)$ with $\widehat{\mu}(\gamma) \neq 0$ for each $\gamma \in E$, the set $\mu * L^1_E(G)_1$ is weak-star closed in the space $M(G)$. We do not know if such a measure exists. Since the set E is countable, finding a measure $\mu \in M(G)$ such that $\widehat{\mu} \neq 0$ is not a problem; the problem is to find such a μ in $M_E(G)$ for which the set $\mu * L^1_E(G)_1$ is weak-star closed in the space $M(G)$.

REMARK 3.5. The noncommutative versions of the above results (except maybe Theorem 3.3 and its corollary) are also valid. Indeed, let G be a nonabelian discrete amenable group and $A(G)$ be its Fourier algebra as defined by Eymard in [6]. The algebra $A(G)$ has a BAI and is an ideal in its second dual. The multiplier algebra of $A(G)$ is the Fourier-Stieltjes algebra $B(G)$, and $B(G)$ is the dual of the group C^*-algebra $C^*(G)$. The Gelfand spectrum of $A(G)$ is G and each element x of G acts on $A(G)$ as a Dirac measure on $A(G)$. For a subset E of G, let $A_E(G) = \{a \in A(G) : a = 0 \text{ on } G\backslash E\}$ and $B_E(G) = \{u \in B(G) : u = 0 \text{ on } G\backslash E\}$. Say that E is a Riesz set if the equality $B_E(G) = A_E(G)$ holds. Then, all the proofs given above being essentially functional analytic, the exact analogues of the results proved above (except maybe Theorem 3.3 and its corollary) are also valid for the algebra $A(G)$.

QUESTIONS. We finish the paper with some open questions.

1. If the ideal $L^1_E(G)$ is Arens regular, can we say that E is a Riesz set? If the answer to this question is positive this will characterize Arens regular ideals of $L^1(G)$ in term of Riesz sets, and vice versa.
2. For a small-2 set E, is the algebra $L^1_E(G)$ Arens regular? If the answer to this question is negative, this will solve the small-2 set problem in the negative.
3. If E is a c_0-set, is then the ideal $L^1_E(G)$ Arens regular?
4. If E and F are two disjoint Riesz sets such that the set $E \cup F$ is a c_0-set then, is then $E \cup F$ a Riesz set?
5. Is the converse of Corollary 2.6 true?

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