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EXTENSIONS OF THE CLASSICAL CESÀRO OPERATOR ON HARDY SPACES

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Abstract

For each $1 \leq p < \infty$, the classical Cesàro operator $\mathscr{C}$ from the Hardy space H^p to itself has the property that there exist analytic functions $f \notin H^p$ with $\mathscr{C}(f) \in H^p$. This article deals with the identification and properties of the (Banach) space $[\mathscr{C}, H^p]$ consisting of *all* analytic functions that $\mathscr{C}$ maps into H^p. It is shown that $[\mathscr{C}, H^p]$ contains classical Banach spaces of analytic functions X, genuinely bigger that H^p, such that $\mathscr{C}$ has a continuous H^p-valued extension to X. An important feature is that $[\mathscr{C}, H^p]$ is the *largest* amongst all such spaces X.

1. Introduction

The classical Cesàro operator, given by

$$\mathscr{C}(f)(z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \tag{1}$$

with $f(z) = \sum_0^\infty a_k z^k$ an analytic function on the open unit disc D, is bounded on the Hardy space H^p, for every $0 < p < \infty$. For $1 < p < \infty$, this follows from a result of Hardy concerning trigonometric series together with M. Riesz's theorem. The boundedness on H^1 was proved by Siskakis, who also gave an alternative proof for $1 < p < \infty$, [7], [8].

Observe that $\mathscr{C}$ is injective, but not surjective on H^p (as 0 belongs to the spectrum of $\mathscr{C}$, [7]), that is, $\mathscr{C}$ is not an isomorphism on H^p. However, $\mathscr{C}$ is an isomorphism on the Fréchet space $H(\mathsf{D})$ of all analytic functions on D. So, there exist analytic functions $f \notin H^p$ such that $\mathscr{C}(f) \in H^p$. Accordingly, the domain of $\mathscr{C}\colon H^p \to H^p$ is, in a certain natural sense, lacking in size. This raises the question of whether there exist *Banach* spaces of analytic functions, always meant over D (i.e., vector subspaces of $H(\mathsf{D})$ which are complete for some norm), larger than H^p and which $\mathscr{C}$ maps continuously into H^p? If so, does there exist a "largest" such space and what properties would it have?

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The first observation is that such a space cannot be H^q, for any $q \in [1, p)$. Indeed, Aleman and Cima have considered operators T_g determined by an analytic symbol g via $T_g f(z) := \int_0^z f(\xi)g'(\xi)\,d\xi$. They have shown, for $1 < q < p < \infty$, that T_g maps H^q into H^p if and only if g is in the Lipschitz class Λ_α with $\alpha := (1/q) - (1/p)$, [1, Theorem 1(iii)]. Since this is not the case for $g(z) = -\log(1-z)$, which corresponds to the Cesàro operator, it follows that $\mathscr{C}$ (taking values in H^p) cannot be extended to any larger H^q space as its domain.

In Section 2 we characterize precisely when $\mathscr{C}$ is bounded from a weighted Hardy space $H^p(w)$ into H^p, which allows us to exhibit a class of weights w, with certain growth conditions, for which $H^p \subsetneqq H^p(\omega)$ and such that $\mathscr{C}(H^p(w)) \subseteq H^p$. Such weights ω_1 and ω_2 exist for which $H^p(\omega_1)$ and $H^p(\omega_2)$ are not comparable. In Section 3 we show that there actually does *exist* a largest Banach space of analytic functions (denoted by $[\mathscr{C}, H^p]$) to which $\mathscr{C}$ has a continuous extension and maps into H^p. In particular, for the above mentioned weights ω we have that $H^p(\omega) \subseteq [\mathscr{C}, H^p]$. This containment is actually proper (as is $H^p \subseteq [\mathscr{C}, H^p]$). It is precisely this feature, i.e., that $[\mathscr{C}, H^p]$ contains classical Banach spaces of analytic functions which are genuinely larger than H^p, which makes the space $[\mathscr{C}, H^p]$ interesting. Just as interesting is the *optimality* of the space $[\mathscr{C}, H^p]$ relative to $\mathscr{C}$, in the sense that it is also the largest Banach space of analytic functions f on D for which the formula (1) produces an element of H^p and such that the extended Cesàro operator $\mathscr{C}: [\mathscr{C}, H^p] \to H^p$ is still continuous. Section 3 is also devoted to exposing certain Banach space properties of $[\mathscr{C}, H^p]$, to studying various properties of individual functions from $[\mathscr{C}, H^p]$, which can behave quite differently to those from H^p, and to identifying the space of all multipliers for $[\mathscr{C}, H^p]$.

2. Extensions of the Cesàro operator

A weight is any function ω on the unit circle T such that $\omega > 0$ a.e. and with $\log \omega$ integrable. Let ψ be an outer function corresponding to ω, that is, ψ is analytic on D and $|\psi| = \omega$ a.e. on T, [5, §2.4]. The weighted Hardy space $H^p(\omega)$ associated to ω is then the Banach space $\psi^{-1/p} \cdot H^p = \{f \in H(\mathsf{D}) : \psi^{1/p} f \in H^p\}$ with norm $\|f\|_{p,\omega} := \|\psi^{1/p} f\|_p$; see, for example, [6].

We state for further reference the following facts.

Proposition 2.1. *Let $1 \le p < \infty$ and ω be a weight with ψ an outer function corresponding to ω.*

(i) *Given $\varphi \in H(\mathsf{D})$ the multiplication operator $M_\varphi(f) := \varphi \cdot f$ is well defined (and hence, continuous) from H^p to H^p if and only if $\varphi \in H^\infty$.*

(ii) *$H^p \subseteq H^p(\omega)$ if and only if ψ is bounded.*

(iii) *$H^p(\omega) \subseteq H^p$ if and only if ψ^{-1} is bounded.*

PROOF. (i) If $\varphi f \in H^p$ for all $f \in H^p$, then a closed graph argument shows that M_φ is continuous. We may assume that $\|M_\varphi\| = 1$. Since $\varphi = M_\varphi(1) \in H^p$, it follows by iteration that $\{\varphi^n\}$ is contained in the closed unit ball of H^p. Accordingly, $\frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{i\theta})|^{np}\,d\theta \le 1$ for all $n \in \mathsf{N}$. By splitting these integrals over the sets $|\varphi|^{-1}([0,1])$ and $|\varphi|^{-1}((1,\infty))$ it follows that $|\varphi| \le 1$ a.e.

(ii) $H^p \subseteq H^p(\omega)$ is equivalent to $f \mapsto \psi^{1/p} f$ being bounded on H^p which, by (i), is equivalent to $\psi^{1/p}$, and hence ψ, being bounded.

(iii) $f \in H^p(\omega)$ precisely when $f = \psi^{-1/p} g$ for some $g \in H^p$. Hence, $H^p(\omega) \subseteq H^p$ is equivalent to $g \mapsto \psi^{-1/p} g$ being bounded on H^p which, by (i), is equivalent to $\psi^{-1/p}$, hence also to ψ^{-1}, being bounded.

The following result characterizes those weights ω with the property that $\mathscr{C}$ maps $H^p(\omega)$ continuously into H^p.

THEOREM 2.2. *Let $1 \le p < \infty$ and ω be a weight with ψ an outer function corresponding to w. The following conditions are equivalent.*

(i) $\mathscr{C}\colon H^p(\omega) \to H^p$ *continuously.*

(ii) *The operator which sends $g \in H(\mathsf{D})$ to the function*

$$z \longmapsto \int_0^z g(\xi)\frac{\psi^{-1/p}(\xi)}{1-\xi}\,d\xi, \qquad z \in \mathsf{D},$$

maps H^p into itself continuously.

(iii) *The function*

$$\rho_\psi\colon z \longmapsto \int_0^z \frac{\psi^{-1/p}(\xi)}{1-\xi}\,d\xi, \qquad z \in \mathsf{D}, \tag{2}$$

belongs to the space BMOA.

PROOF. Let $f \in H^p(\omega)$. Then $f = \psi^{-1/p}\cdot g$, for some unique $g \in H^p$. We require the following well known integral expression for $\mathscr{C}$, namely, for each $h \in H(\mathsf{D})$,

$$\mathscr{C}(h)(z) = \frac{1}{z}\int_0^z \frac{h(\xi)}{1-\xi}\,d\xi, \qquad z \in \mathsf{D}, \tag{3}$$

which yields

$$\mathscr{C}(f)(z) = \frac{1}{z}\int_0^z g(\xi)\frac{\psi^{-1/p}(\xi)}{1-\xi}\,d\xi.$$

Following [2], for analytic functions ρ and h on D, we consider

$$T_\rho(h)(z) := \frac{1}{z}\int_0^z h(\xi)\rho'(\xi)\,d\xi, \qquad z \in \mathsf{D}.$$

Then $\mathscr{C}(f) = T_{\rho_\psi}(g)$ with ρ_ψ given by (2). Consequently, $\mathscr{C}$ maps $H^p(\omega)$ continuously into H^p if and only if T_{ρ_ψ} maps H^p into itself continuously. Furthermore, [1, Theorem 1(ii)] asserts that this last condition is equivalent to $\rho_\psi \in \mathrm{BMOA}$.

Theorem 2.2 makes no assertion concerning any (possible) relationship between $H^p(\omega)$ and H^p. However, combining it with Proposition 2.1 we can deduce, for appropriate ω, that the Cesàro operator has a *genuine* extension to a larger space $H^p(\omega)$ with values still in H^p.

COROLLARY 2.3. *Let $1 \le p < \infty$ and ω be a weight with ψ an outer function corresponding to w. Suppose that*

(i) *ψ is bounded and ψ^{-1} is unbounded, and*

(ii) *the function $z \mapsto \int_0^z \frac{\psi^{-1/p}(\xi)}{1-\xi}\, d\xi$ belongs to* BMOA.

Then $\mathscr{C} \colon H^p(\omega) \to H^p$ continuously and $H^p \subsetneqq H^p(\omega)$.

The following result specifies growth conditions on a weight ω which are sufficient to ensure that $\mathscr{C}$ maps $H^p(\omega)$ continuously into H^p and $H^p \subsetneqq H^p(\omega)$.

COROLLARY 2.4. *Let $1 \le p < \infty$ and ω be a weight with ψ an outer function corresponding to w. Suppose that*

(i) *ψ is bounded and ψ^{-1} is unbounded, and*

(ii) *there exist distinct points $a_1, \dots, a_m \in \mathsf{T} \setminus \{1\}$ such that*

$$\psi^{-1}(z) = O\left(\frac{1}{\prod_{k=1}^m |z - a_k|^p}\right), \qquad |z| \to 1^-.$$

Then $\mathscr{C} \colon H^p(\omega) \to H^p$ continuously and $H^p \subsetneqq H^p(\omega)$.

PROOF. We could apply condition (iii) of Theorem 2.2. However, we prefer a direct argument which highlights the operator-theoretic approach.

Let $f \in H^p(\omega)$. Then $f = \psi^{-1/p} \cdot g$, for some $g \in H^p$. Using (3) and setting $h(z) := \psi^{-1/p}(z) \prod_{k=1}^m (z - a_k)$ yields

$$\mathscr{C}(f)(z) = \frac{1}{z} \int_0^z \frac{\psi^{-1/p}(\xi)\, g(\xi)}{1-\xi}\, d\xi = \frac{1}{z} \int_0^z \frac{h(\xi)\, g(\xi)}{(1-\xi) \prod_{k=1}^m (\xi - a_k)}\, d\xi.$$

So, for suitable constants $A_0, A_1, \dots, A_m \in \mathsf{C}$ (with $a_0 := 1$) we have

$$\begin{aligned}\mathscr{C}(f)(z) &= \sum_{k=0}^m \frac{A_k}{z} \int_0^z \frac{h(\xi)\, g(\xi)}{a_k - \xi}\, d\xi = \sum_{k=0}^m A_k \frac{1}{z} \int_0^{z/a_k} \frac{h(a_k\eta)\, g(a_k\eta)}{1-\eta}\, d\eta \\ &= \sum_{k=0}^m \frac{A_k}{a_k} \mathscr{C}\big(h(a_k \cdot)\, g(a_k \cdot)\big)(z/a_k).\end{aligned}$$

The function h is, by condition (ii), bounded. Since $z \mapsto z/a_k$ and $z \mapsto a_k z$ are automorphisms of D, each function $g_k(z) := h(a_k z)\, g(a_k z)$ is in H^p and so, $\mathscr{C}(g_k)(\cdot/a_k) \in H^p$, for $0 \le k \le m$. Consequently, $\mathscr{C}(f) \in H^p$. Hence, $\mathscr{C}(H^p(\omega)) \subseteq H^p$. This, Lemma 2.5 below, and the fact that point evaluations are continuous linear functionals on both H^p and $H^p(\omega)$, imply that $\mathscr{C}\colon H^p(\omega) \to H^p$ continuously.

Condition (i) and Proposition 2.1 imply that $H^p \subsetneq H^p(\omega)$.

If X and Y are Banach spaces of analytic functions, then the vector space containment $X \subseteq Y$ is equivalent to continuity of the inclusion $X \hookrightarrow Y$, provided that point evaluations are continuous on both X and Y. Moreover, since point evaluations are continuous on $H(\mathsf{D})$, we always have a continuous inclusion $X \hookrightarrow H(\mathsf{D})$. A similar result holds for the Cesàro operator.

LEMMA 2.5. *Let X, Y be Banach spaces of analytic functions such that point evaluations are continuous on both X, Y. Then $\mathscr{C}(X) \subseteq Y$ if and only if $\mathscr{C}$ maps X into Y continuously.*

PROOF. We can apply the Closed Graph Theorem. Let $f_n \to 0$ in X and $\mathscr{C}(f_n) \to g$ in Y. By the discussion prior to the lemma, $f_n \to 0$ in $H(\mathsf{D})$. Fix $z \in \mathsf{D} \setminus \{0\}$. Since $f_n(\xi)/(1-\xi)$ converges to zero uniformly on the segment $[0, z]$, it follows that $\mathscr{C}(f_n)(z) \to 0$. But, $\mathscr{C}(f_n)(z) \to g(z)$. Consequently, $g = 0$.

3. Further extensions of the Cesàro operator

Can the Cesàro operator be extended beyond the already larger spaces $H^p(\omega)$, while still remaining H^p-valued? Yes, and genuinely. Let us see how to proceed.

As already noted, $\mathscr{C}$ is a topological isomorphism from $H(\mathsf{D})$ onto itself. For $1 \le p < \infty$, define the linear space

$$[\mathscr{C}, H^p] := \{ f \in H(\mathsf{D}) : \mathscr{C}(f) \in H^p \}, \tag{4}$$

which is then complete with respect to the norm

$$\|f\|_{[\mathscr{C}, H^p]} := \|\mathscr{C}(f)\|_{H^p}. \tag{5}$$

Moreover, we have that

$$H^p \subseteq [\mathscr{C}, H^p] \subseteq H(\mathsf{D}). \tag{6}$$

The first containment follows from $\mathscr{C}(H^p) \subseteq H^p$. Moreover, both inclusions are continuous. This follows from Lemma 2.5 and the fact that point evaluations are continuous on $[\mathscr{C}, H^p]$. To see this, fix $z_0 \in \mathsf{D}$ and let $f \in [\mathscr{C}, H^p]$. Taking into account the identity

$$g(z) = (1-z)\big(z\mathscr{C}(g)(z)\big)', \qquad g \in H(\mathsf{D}), \tag{7}$$

which follows from (3), we have, for $|z_0| < r < 1$, that

$$\begin{aligned} |f(z_0)| &= |1-z_0|\cdot\Big|\big(z\mathscr{C}(f)(z)\big)'(z_0)\Big| \\ &= |1-z_0|\cdot\left|\frac{1}{2\pi i}\int_{|\xi|=r}\frac{\xi\mathscr{C}(f)(\xi)}{(\xi-z_0)^2}\,d\xi\right| \\ &\le \frac{r^2|1-z_0|}{2\pi(r-|z_0|)^2}\int_0^{2\pi}|\mathscr{C}(f)(re^{i\theta})|\,d\theta. \end{aligned}$$

Consequently,

$$|f(z_0)| \le \frac{|1-z_0|}{2\pi(1-|z_0|)^2}\,\|\mathscr{C}(f)\|_{H^p} = \frac{|1-z_0|}{2\pi(1-|z_0|)^2}\,\|f\|_{[\mathscr{C},H^p]}. \tag{8}$$

Note that the first containment in (6), namely $H^p \subseteq [\mathscr{C}, H^p]$, is strict. Indeed, $f(z) := 1/(1+z) \notin H^1$ but, $\mathscr{C}(f)(z) = (1/2z)\log\big((1+z)/(1-z)\big)$ belongs to every H^p, $1 \le p < \infty$. Accordingly, $f \in [\mathscr{C}, H^p]$, for all $1 \le p < \infty$. Clearly, the second containment in (6) is also strict.

REMARK 3.1. Fix $1 \le p < \infty$. Let X be a Banach space of analytic functions. If $\mathscr{C}\colon X \to H^p$ is continuous, then $\mathscr{C}(X) \subseteq H^p$ and so $X \subseteq [\mathscr{C}, H^p]$. On the other hand, suppose that $X \subseteq [\mathscr{C}, H^p]$. Then $\mathscr{C}(X) \subseteq H^p$ and hence, if point evaluations are continuous on X, it follows from Lemma 2.5 that $\mathscr{C}\colon X \to H^p$ continuously. This means that $\|\mathscr{C}(f)\|_{H^p} \le M\|f\|_X$, $f \in X$, for some constant $M > 0$, that is, $\|f\|_{[\mathscr{C},H^p]} \le M\|f\|_X$, $f \in X$. Thus, the natural inclusion $X \subseteq [\mathscr{C}, H^p]$ is necessarily continuous. Consequently, since $\mathscr{C}\colon [\mathscr{C}, H^p] \to H^p$ is clearly continuous, the space $[\mathscr{C}, H^p]$ can be considered as the "optimal domain" for the operator $\mathscr{C}$, with $\mathscr{C}$ still taking its values in H^p. That is, $[\mathscr{C}, H^p]$ is the *largest* of all Banach spaces of analytic functions X such that $\mathscr{C}$ maps X continuously into H^p. Equivalently, $[\mathscr{C}, H^p]$ can be interpreted as the largest Banach space of analytic functions to which the Cesàro operator $\mathscr{C}\colon H^p \to H^p$ can be extended, still with all its values in H^p.

In view of the previous comments, Corollaries 2.3 and 2.4 imply that $H^p(\omega) \subseteq [\mathscr{C}, H^p]$ continuously, for all weights ω satisfying the conditions of these results.

To better understand the nature of individual functions from $[\mathscr{C}, H^p]$ we begin with the following description.

PROPOSITION 3.2. *For each* $1 \leq p < \infty$ *we have, as vector spaces, that*

$$[\mathscr{C}, H^p] = \{f \in H(\mathsf{D}) : f(z) = (1-z)g'(z) \text{ for some } g \in H^p\}. \tag{9}$$

PROOF. If $f \in [\mathscr{C}, H^p]$, then $\mathscr{C}(f)$ and hence, also $h(z) := z\mathscr{C}(f)(z)$, belongs to H^p. According to (7) we have $f(z) = (1-z)h'(z)$ and so f belongs to the right-hand-side of (9).

Conversely, suppose that $f \in H(\mathsf{D})$ has the form $f(z) = (1-z)g'(z)$ for some $g \in H^p$. Then

$$\mathscr{C}(f)(z) = \frac{1}{z}\int_0^z \frac{(1-\xi)g'(\xi)}{1-\xi}\,d\xi = \frac{g(z)-g(0)}{z}, \qquad z \neq 0,$$

with $\mathscr{C}(f)(0) = g'(0)$. Choose $0 < \varepsilon < 1$ such that $\left|\frac{g(z)-g(0)}{z} - g'(0)\right| < \varepsilon$ for $0 < |z| < \varepsilon$, in which case $\left|\frac{g(z)-g(0)}{z}\right| < \varepsilon + |g'(0)|$. Moreover, for $\varepsilon \leq |z| < 1$ we have $\left|\frac{g(z)-g(0)}{z}\right| < \frac{|g(z)|+|g(0)|}{\varepsilon}$. Hence, $|\mathscr{C}(f)(z)| \leq \alpha|g(z)| + \beta$ for $z \in \mathsf{D}$ and constants $\alpha, \beta > 0$. Since $g \in H^p$, this implies that $\mathscr{C}(f) \in H^p$, that is, $f \in [\mathscr{C}, H^p]$.

Let us deduce some consequences of the previous result. We begin with an alternative function theoretic description of $[\mathscr{C}, H^p]$. By applying to the function $z \mapsto z\mathscr{C}(f)(z) = \int_0^z \frac{f(\xi)}{1-\xi}\,d\xi$ (which has value 0 at $z = 0$ and whose derivative equals $\frac{f(z)}{1-z}$) the criterion for membership of H^p based on the Littlewood-Paley g-function, [9, Ch. XIV, Theorems (3.5) and (3.19)], we obtain from Proposition 3.2 the following fact.

COROLLARY 3.3. *Let* $1 < p < \infty$. *Then* $f \in [\mathscr{C}, H^p]$ *if and only if*

$$\int_0^{2\pi}\left(\int_0^1 \frac{|f(re^{i\theta})|^2}{|1-re^{i\theta}|^2}(1-r)\,dr\right)^{p/2} d\theta < \infty.$$

Note that, for $p = 1$, the above condition is only necessary.

Recall that every element of H^p, for $1 \leq p < \infty$, has boundary values a.e. on T.

COROLLARY 3.4. *Let* $1 \leq p < \infty$. *Then there exists a function in* $[\mathscr{C}, H^p]$ *which fails to have a.e. boundary values. In particular,* $H^p \subsetneqq [\mathscr{C}, H^p]$.

PROOF. According to [5, p.92] there exists $g \in H^\infty \subseteq H^p$ such that g' fails to have a.e. boundary values. Then $f(z) := (1-z)g'(z)$ belongs to $[\mathscr{C}, H^p]$ (c.f. Proposition 3.2) and f fails to have a.e. boundary values.

REMARK 3.5. (i) The proof of Corollary 3.4 shows there actually exists a function in $\bigcap_{1\leq p<\infty}[\mathscr{C}, H^p]$ which fails to have a.e. boundary values.

(ii) Let ω be a weight as in Section 2 with ψ a corresponding outer function. It is clear from $H^p(\omega) = \{\psi^{-1/p} f : f \in H^p\}$ that every function in $H^p(\omega)$ has a.e. boundary values. So, whenever $H^p(\omega) \subseteq [\mathscr{C}, H^p]$, Corollary 3.4 implies that the inclusion is proper.

Let $\mathrm{Aut}(\mathsf{D})$ denote the group of all automorphisms on D, in which case each space H^p, for $1 \le p < \infty$, is invariant under composition with $\mathrm{Aut}(\mathsf{D})$. That is, $\{f \circ \rho : f \in H^p\} \subseteq H^p$ for all $\rho \in \mathrm{Aut}(\mathsf{D})$.

PROPOSITION 3.6. *There exists* $\rho \in \mathrm{Aut}(\mathsf{D})$ *and* $f \in \bigcap_{1\le p<\infty}[\mathscr{C}, H^p]$ *such that* $f \circ \rho \notin [\mathscr{C}, H^1]$. *In particular,* $f \circ \rho \notin [\mathscr{C}, H^p]$ *for* $1 \le p < \infty$.

PROOF. The function $f(z) := 1/(1+z)$ satisfies $\mathscr{C}(f)(z) = (2z)^{-1}\log\big(\frac{1+z}{1-z}\big)$ and so $f \in [\mathscr{C}, H^p]$ for all $1 \le p < \infty$, that is, $f \in \bigcap_{1\le p<\infty}[\mathscr{C}, H^p]$. Of course, $f \notin H^1$. Let $\rho(z) := -z$, for $z \in \mathsf{D}$. Then $(f \circ \rho)(z) = 1/(1-z)$ and $\mathscr{C}(f \circ \rho)(z) = 1/(z-1) \notin H^1$, that is $f \circ \rho \notin [\mathscr{C}, H^1]$. According to Proposition 3.8(v) below, also $f \circ \rho \notin [\mathscr{C}, H^p]$ for all $1 \le p < \infty$.

Corollary 3.4 and Proposition 3.6 show that certain "nice" properties of functions from H^p fail to be inherited by functions in the larger space $[\mathscr{C}, H^p]$. This is not always the case. Proposition 2.1(i) asserts, for $\varphi \in H(\mathsf{D})$, that the operator M_φ of multiplication by φ is defined and continuous from H^p into itself precisely when $\varphi \in H^\infty$. The same conclusion holds for the spaces $[\mathscr{C}, H^p]$ in place of H^p.

THEOREM 3.7. *Let* $1 \le p < \infty$. *Given* $\varphi \in H(\mathsf{D})$ *the multiplication operator* $M_\varphi(f) := \varphi\cdot f$ *is well defined* (*and hence, continuous*) *from* $[\mathscr{C}, H^p]$ *to* $[\mathscr{C}, H^p]$ *if and only if* $\varphi \in H^\infty$.

PROOF. Suppose first that $\varphi \in H^\infty$. Fix $f \in [\mathscr{C}, H^p]$. By Proposition 3.2 there exists $g \in H^p$ such that $f(z) = (1-z)g'(z)$. Observe that $G \in H(\mathsf{D})$ defined by

$$G(z) := \varphi(z)g(z) - \int_0^z \varphi'(\xi)g(\xi)\,d\xi, \qquad z \in \mathsf{D},$$

satisfies

$$\varphi(z)f(z) = (1-z)\varphi(z)g'(z) = (1-z)G'(z), \qquad z \in \mathsf{D},$$

and so, again by Proposition 3.2, we see that $\varphi f \in [\mathscr{C}, H^p]$ provided that $G \in H^p$. Since $\varphi g \in H^p$, to verify that $G \in H^p$ it suffices to verify that $T_\varphi(g)\colon z \mapsto \int_0^z \varphi'(\xi)g(\xi)\,d\xi \in H^p$. But, $\varphi \in H^\infty \subseteq \mathrm{BMOA}$ and so indeed $T_\varphi(g) \in H^p$ for every $g \in H^p$, [2, Theorem 1], [1, Theorem 1(ii)]. Accordingly, φ has the property that $M_\varphi(f) := \varphi \cdot f$ belongs to $[\mathscr{C}, H^p]$ whenever $f \in [\mathscr{C}, H^p]$. Using continuity of the point evaluations on $[\mathscr{C}, H^p]$, a closed graph argument shows that $M_\varphi\colon [\mathscr{C}, H^p] \to [\mathscr{C}, H^p]$ is actually continuous.

Of course, for $p \neq 1$, the above proof can be replaced by a direct appeal to Corollary 3.3.

Conversely, let $\varphi \in H(\mathsf{D})$ be such that $M_\varphi\colon [\mathscr{C}, H^p] \to [\mathscr{C}, H^p]$ is well defined (and hence, continuous). We may assume that the operator norm of M_φ satisfies $\|M_\varphi\| = 1$. Note, for every $n \geq 1$, that the operator $M_{\varphi^n} = (M_\varphi)^n$ maps $[\mathscr{C}, H^p]$ into $[\mathscr{C}, H^p]$ and moreover, that $\|M_{\varphi^n}\| \leq 1$. Accordingly, $\varphi^n \in [\mathscr{C}, H^p]$ and hence, also $(1-z)\varphi^n(z) \in [\mathscr{C}, H^p]$ (because $(1-z) \in H^\infty$). Then

$$\begin{aligned}\left\| z \mapsto \int_0^z \varphi^n \right\|_{H^p} &= \|z\mathscr{C}((1-z)\varphi^n(z))\|_{H^p} = \|\mathscr{C}((1-z)\varphi^n(z))\|_{H^p} \\ &= \|(1-z)\varphi^n(z)\|_{[\mathscr{C},H^p]} = \|M_{\varphi^n}(1-z)\|_{[\mathscr{C},H^p]} \\ &\leq \|M_{\varphi^n}\|\cdot\|1-z\|_{[\mathscr{C},H^p]} \leq 1.\end{aligned}$$

From [9, Ch. XIV, Theorem (3.5)] it follows, for some constant $A_p > 0$ and all $n \geq 1$, that

(10)
$$\int_0^{2\pi} \left(\int_0^1 |\varphi^n(re^{i\theta})|^2 (1-r)\, dr \right)^{p/2} d\theta \leq A_p \left\| z \mapsto \int_0^z \varphi^n \right\|_{H^p} \leq A_p.$$

Suppose there exists $z \in \mathsf{D}$ such that $|\varphi(z)| > 1$. Then there exists $0 \leq r_0 < r_1 < 1$ and $0 \leq \theta_0 < \theta_1 < 2\pi$ such that $|\varphi(re^{i\theta})| \geq a$ for some $a > 1$ and all $r_0 \leq r \leq r_1$ and $\theta_0 \leq \theta \leq \theta_1$. From (10) we conclude, for all $n \geq 1$, that

$$\begin{aligned} A_p &\geq \int_0^{2\pi} \left(\int_0^1 |\varphi(re^{i\theta})|^{2n} (1-r)\, dr \right)^{p/2} d\theta \\ &\geq \int_{\theta_0}^{\theta_1} \left(\int_{r_0}^{r_1} |\varphi(re^{i\theta})|^{2n} (1-r)\, dr \right)^{p/2} d\theta \\ &\geq (\theta_1 - \theta_0) \left((r_1 - r_0) a^{2n} (1-r_1) \right)^{p/2}. \end{aligned}$$

Since $a > 1$, this is impossible. Hence, $|\varphi(z)| \leq 1$ for all $z \in \mathsf{D}$ and so $\varphi \in H^\infty$.

Despite the general lack of regularity concerning individual functions from $[\mathscr{C}, H^p]$, the *spaces* $[\mathscr{C}, H^p]$ exhibit rather good structural properties. Indeed, various Banach space properties of $[\mathscr{C}, H^p]$ follow directly from the fact that $\mathscr{C}$ maps $[\mathscr{C}, H^p]$ linearly and isometrically onto H^p, for $1 \leq p < \infty$; see (5). Some immediate consequences are as follows.

PROPOSITION 3.8. *Let* $1 \le p < \infty$ *and* $[\mathscr{C}, H^p]$ *be the optimal domain for the Cesàro operator* $\mathscr{C}$ *on* H^p.

(i) $[\mathscr{C}, H^p]$ *is separable.*

(ii) $[\mathscr{C}, H^p]$ *is uniformly convex* (*in particular, reflexive*) *for* $p \neq 1$.

(iii) *For* $p = 2$, $[\mathscr{C}, H^2]$ *is a Hilbert space. In particular,*

$$f(z) = \sum_0^\infty a_n z^n \in [\mathscr{C}, H^2] \iff \left(\frac{1}{n+1}\sum_0^n a_k\right) \in \ell^2.$$

(iv) *Polynomials are dense in* $[\mathscr{C}, H^p]$.

(v) $[\mathscr{C}, H^{p_2}] \subsetneq [\mathscr{C}, H^{p_1}]$ *whenever* $1 \le p_1 < p_2 < \infty$.

PROOF. The isometry between $[\mathscr{C}, H^p]$ and H^p immediately yields (i), (ii) and (v). For (iii), observe that $f \in [\mathscr{C}, H^2]$ if and only if $\mathscr{C}(f) \in H^2$ if and only if $\left(\frac{1}{n+1}\sum_0^n a_k\right) \in \ell^2$; see (1). Finally, for (iv), let $f \in [\mathscr{C}, H^p]$ and $\varepsilon > 0$. Since $\mathscr{C}(f) \in H^p$, choose N and $(b_k)_0^N \subseteq \mathbb{C}$ so that $\left\|\mathscr{C}(f) - \sum_0^N b_k z^k\right\|_{H^p} < \varepsilon$. Taking into account that $\mathscr{C}(z^k - z^{k+1}) = z^k/(k+1)$ for $k \ge 0$, we can write

$$\begin{aligned}\left\|\mathscr{C}(f) - \sum_0^N b_k z^k\right\|_{H^p} &= \left\|\mathscr{C}(f) - \sum_0^N b_k (k+1)\mathscr{C}(z^k - z^{k+1})\right\|_{H^p} \\ &= \left\|\mathscr{C}\left(f - \sum_0^N b_k (k+1)(z^k - z^{k+1})\right)\right\|_{H^p} \\ &= \left\|f - \sum_0^N b_k (k+1)(z^k - z^{k+1})\right\|_{[\mathscr{C}, H^p]}.\end{aligned}$$

REMARK 3.9. Concerning $p = \infty$, the definition given in (4) still makes sense and generates the space $[\mathscr{C}, H^\infty]$ for which (5) is again a complete norm. Since $[\mathscr{C}, H^\infty] \subsetneq [\mathscr{C}, H^p]$ continuously, for all $1 \le p < \infty$ (see (5)), it follows from (8) that point evaluations are continuous on $[\mathscr{C}, H^\infty]$ and

$$|f(z_0)| \le \frac{|1 - z_0|}{2\pi(1 - |z_0|)^2} \, \|f\|_{[\mathscr{C}, H^\infty]}.$$

However, since $\mathscr{C}$ is not continuous on H^∞, we do *not* have the inclusion $H^\infty \subseteq [\mathscr{C}, H^\infty]$ corresponding to (6) for $p = \infty$.

Optimal domains exhibit good behaviour with respect to interpolation via the Petree K-method; see [3, Ch.5.§1].

PROPOSITION 3.10. *Let* $1 < p < \infty$ *and* $[\mathscr{C}, H^p]$ *be the optimal domain for the Cesàro operator* $\mathscr{C}$ *on* H^p. *Then*

$$\big([\mathscr{C}, H^1], [\mathscr{C}, H^\infty]\big)_{1-\frac{1}{p},p} = [\mathscr{C}, H^p].$$

PROOF. Note that $[\mathscr{C}, H^\infty] \subseteq [\mathscr{C}, H^1]$ since $H^\infty \subseteq H^1$. Fix $f \in [\mathscr{C}, H^1]$. Let $f = g_1 + g_2$ with $g_1 \in [\mathscr{C}, H^1]$ and $g_2 \in [\mathscr{C}, H^\infty]$. This is equivalent to $\mathscr{C}(f) = \mathscr{C}(g_1) + \mathscr{C}(g_2)$ with $\mathscr{C}(g_1) \in H^1$ and $\mathscr{C}(g_2) \in H^\infty$ which, in turn, is equivalent to $\mathscr{C}(f) = h_1 + h_2$ with $h_1 \in H^1$ and $h_2 \in H^\infty$ (since $\mathscr{C}$ is an isomorphism between $[\mathscr{C}, H^p]$ and H^p, $1 \le p \le \infty$). The isometry between $[\mathscr{C}, H^p]$ and H^p then gives

$$K(f, t; [\mathscr{C}, H^1], [\mathscr{C}, H^\infty]) = K(\mathscr{C}(f), t; H^1, H^\infty), \qquad t > 0.$$

REMARK 3.11. Following the procedure given in Remark 3.9 for defining the space $[\mathscr{C}, H^\infty]$, we can also consider the Banach space $[\mathscr{C}, \mathrm{BMOA}]$ consisting of those functions $h \in H(\mathrm{D})$ such that $\mathscr{C}(h) \in \mathrm{BMOA}$. We have $H^\infty \subseteq [\mathscr{C}, \mathrm{BMOA}]$ and $\mathrm{BMOA} \subsetneq [\mathscr{C}, \mathrm{BMOA}]$; see [4, Section 3].

The space $[\mathscr{C}, \mathrm{BMOA}]$ arises naturally as the space $M(H^p, [\mathscr{C}, H^p])$ of functions generating continuous multiplication operators from H^p into $[\mathscr{C}, H^p]$. Indeed, $\varphi \in M(H^p, [\mathscr{C}, H^p])$ means precisely that $\varphi f \in [\mathscr{C}, H^p]$ for every $f \in H^p$, that is, $\mathscr{C}(\varphi f) \in H^p$ for every $f \in H^p$. Hence, we have the bounded operator (mapping into H^p) given by

$$f \mapsto \mathscr{C}(\varphi f)\colon z \mapsto \frac{1}{z}\int_0^z f(\xi)\frac{\varphi(\xi)}{1-\xi}\,d\xi, \qquad f \in H^p.$$

It follows from [2, Theorem 1] that the function $z \mapsto \int_0^z \frac{\varphi(\xi)}{1-\xi}\,d\xi$ belongs to BMOA. Consequently, $\mathscr{C}(\varphi) \in \mathrm{BMOA}$, showing that

$$M(H^p, [\mathscr{C}, H^p]) = [\mathscr{C}, \mathrm{BMOA}].$$

The optimal domain space $[\mathscr{C}, H^p]$ of the Cesàro operator (for $1 \le p < \infty$) has been identified not just as a linear space properly containing H^p, but also as a Banach space of analytic functions in its own right possessing various properties. Moreover, for certain weights ω, the weighted Hardy space $H^p(\omega)$ is properly and continuously included in $[\mathscr{C}, H^p]$. It would be interesting to find further examples of classical Banach spaces of analytic functions X such that $H^p \subsetneq X \subsetneq [\mathscr{C}, H^p]$ continuously.

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