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HYPERSURFACES OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract

In this paper we study the invariant and noninvariant hypersurfaces of (1, 1, 1) almost contact manifolds, Lorentzian almost paracontact manifolds and Lorentzian para-Sasakian manifolds, respectively. We show that a noninvariant hypersurface of an (1, 1, 1) almost contact manifold admits an almost product structure. We investigate hypersurfaces of affinely cosymplectic and normal (1, 1, 1) almost contact manifolds. It is proved that a noninvariant hypersurface of a Lorentzian almost paracontact manifold is an almost product metric manifold. Some necessary and sufficient conditions have been given for a noninvariant hypersurface of a Lorentzian para-Sasakian manifold to be locally product manifold. We establish a Lorentzian para-Sasakian structure for an invariant hypersurface of a Lorentzian para-Sasakian manifold. Finally we give some examples for invariant and noninvariant hypersurfaces of a Lorentzian para-Sasakian manifold.

1. Introduction

Hypersurfaces of an almost contact manifold have been studied by D. E. Blair [2], S. S. Eum [5], S. I. Goldberg and K. Yano [7], G. D. Ludden [8] and others. In 1970, S. I. Goldberg and K. Yano [7] defined noninvariant hypersurfaces of almost contact manifolds. A hypersurface such that the transform of a tangent vector of the hypersurface by the tensor φ defining the almost contact structure is never tangent to the hypersurface is called a noninvariant hypersurface of the almost contact manifold [7]. The authors [7] showed that a noninvariant hypersurface of an almost contact manifold admits an almost complex structure and a distinguished 1-form induced by the contact form of the manifold. They also investigated noninvariant hypersurfaces of an almost contact metric manifold.

In 1976, I. Sato [13] studied a structure similar to the almost contact structure, namely almost paracontact structure. In [1], T. Adati studied hypersurfaces of an almost paracontact manifold. A. Bucki [3] considered hypersurfaces of an almost r-paracontact Riemannian manifold. Some properties of invariant hypersurfaces of an almost r-paracontact Riemannian manifold were investigated in [4] by A. Bucki and A. Miernowski. Moreover, in [10], I. Mihai and K. Matsumoto studied submanifolds of an almost r-paracontact Riemannian

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manifold of P-Sasakian type. In [6] the authors studied invariant and noninvariant hypersurfaces of almost r-paracontact manifolds. R. Singh [14] defined (e_1, e_2, r) almost contact structure as a generalization of many known structures, which are obtained by taking particular values of (e_1, e_2) and r (see also [15]). The study of Lorentzian almost paracontact manifolds was initiated by K. Matsumoto in 1989 [9]. Also he introduced the notion of Lorentzian para-Sasakian (for short, LP-Sasakian) manifold. I. Mihai and R. Rosca [11] defined the same notion independently and thereafter many authors ([18], [12], [16], [17]) studied Lorentzian para-Sasakian manifolds and their submanifolds.

In the present paper, we study the invariant and noninvariant hypersurfaces of $(1, 1, 1)$ almost contact manifolds, Lorentzian almost paracontact manifolds and Lorentzian para-Sasakian manifolds, respectively. We investigate the invariant hypersurfaces with two different conditions: when the characteristic vector field ξ is everywhere tangent to the hypersurfaces and when the characteristic vector field ξ is not tangent the hypersurfaces. Section 2 is devoted to preliminaries. In Section 3 we show that a noninvariant hypersurface of an $(1, 1, 1)$ almost contact manifold with the characteristic vector field ξ nowhere tangent to the hypersurface admits an almost product structure. In Section 4 we study hypersurfaces of affinely cosymplectic and normal $(1, 1, 1)$ almost contact manifolds. In Section 5 it is proved that a noninvariant hypersurface of a Lorentzian almost paracontact manifold is an almost product metric manifold. We also find a necessary and sufficient condition for a noninvariant hypersurface of a Lorentzian para-Sasakian manifold to be locally product manifold. Moreover, in this section we establish a Lorentzian para-Sasakian structure for an invariant hypersurface of a Lorentzian para-Sasakian manifold with the characteristic vector field ξ tangent to the hypersurface. In the last section we give some examples for invariant and noninvariant hypersurfaces of an $(1, 1, 1)$ almost contact manifold, a Lorentzian almost paracontact manifold and a Lorentzian para-Sasakian manifold.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional differentiable manifold. If there exist a tensor field φ of type $(1, 1)$, r-linearly independent vector fields ξ_α and r 1-forms η^α on $\overline{M}$ such that [15]

$$\varphi(\xi_\alpha) = 0, \tag{2.1}$$

$$\varphi^2 = e_1 I + e_2 \eta^\alpha \otimes \xi_\alpha, \tag{2.2}$$

where e_1, e_2 take values ± 1 independently, I denotes the identity map of $\Gamma(T\overline{M})$ and $\otimes$ is the tensor product, then the structure $(\varphi, \xi_\alpha, \eta^\alpha)$ is said to be an almost (e_1, e_2)-r-contact structure or in short (e_1, e_2, r) AC-structure and the manifold $\overline{M}$ with the (e_1, e_2, r) AC-structure is called an (e_1, e_2, r) AC-manifold.

Let $\overline{M}$ be an (e_1, e_2, r) AC-manifold. Then the following relations hold on $\overline{M}$ [14]:

$$\eta^\alpha \circ \varphi = 0, \tag{2.3}$$

$$\eta^\alpha(\xi_\beta) = -e_1 e_2 \delta^\alpha_\beta, \tag{2.4}$$

$$\operatorname{rank}(\varphi) = n - r. \tag{2.5}$$

Now, consider that $\overline{M}$ is an $(1, 1, 1)$ AC-manifold. Then $\overline{M}$ admits a Lorentzian metric $\overline{g}$, such that

$$\eta(\overline{X}) = \overline{g}(\overline{X}, \xi), \tag{2.6}$$

$$\overline{g}(\varphi\overline{X}, \varphi\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) + \eta(\overline{X})\eta(\overline{Y}), \tag{2.7}$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$. In this case $\overline{M}$ is said to admit a Lorentzian almost paracontact structure $(\varphi, \xi, \eta, \overline{g})$. Then we get

$$\Phi(\overline{X}, \overline{Y}) \equiv \overline{g}(\overline{X}, \varphi\overline{Y}) \equiv \overline{g}(\varphi\overline{X}, \overline{Y}) \equiv \Phi(\overline{Y}, \overline{X}), \tag{2.8}$$

$$(\overline{\nabla}_{\overline{X}}\Phi)(\overline{Y}, \overline{Z}) = \overline{g}(\overline{Y}, (\overline{\nabla}_{\overline{X}}\varphi)Z) = (\overline{\nabla}_{\overline{X}}\Phi)(\overline{Z}, \overline{Y}), \tag{2.9}$$

where $\overline{\nabla}$ is the Levi-Civita connection with respect to $\overline{g}$. It is clear that the Lorentzian metric $\overline{g}$ makes ξ a timelike unit vector field, i.e., $\overline{g}(\xi, \xi) = -1$. The manifold $\overline{M}$ equipped with a Lorentzian almost paracontact structure $(\phi, \xi, \eta, \overline{g})$ is called a Lorentzian almost paracontact manifold (for short, LAP-manifold) [9], [19].

A Lorentzian almost paracontact manifold $\overline{M}$ endowed with the structure $(\varphi, \xi, \eta, \overline{g})$ is called a Lorentzian paracontact manifold (for short, LP-manifold) [9] if

$$\Phi(\overline{X}, \overline{Y}) = \tfrac{1}{2}\big((\overline{\nabla}_{\overline{X}}\eta)\overline{Y} + (\overline{\nabla}_{\overline{Y}}\eta)\overline{X}\big). \tag{2.10}$$

A Lorentzian almost paracontact manifold $\overline{M}$ endowed with the structure $(\varphi, \xi, \eta, \overline{g})$ is called a Lorentzian para Sasakian manifold (for short, LP-Sasakian) [9] if

$$(\overline{\nabla}_{\overline{X}}\varphi)\overline{Y} = \eta(\overline{Y})\overline{X} + \overline{g}(\overline{X}, \overline{Y})\xi + 2\eta(\overline{X})\eta(\overline{Y})\xi. \tag{2.11}$$

We note that in a LP-Sasakian manifold the 1-form η is closed.

Let $\overline{M} \times R$ be a product manifold, where $\overline{M}$ is an $(1, 1, 1)$ AC-manifold. The tensor field J' of type $(1, 1)$ on $\overline{M} \times R$ defined by

$$J'\left(\overline{X}, f\frac{d}{dt}\right) = \left(\varphi\overline{X} - f\xi, \eta(\overline{X})\frac{d}{dt}\right), \tag{2.12}$$

where f is a C^∞ real-valued function and $\overline{X} \in \Gamma(T\overline{M})$, satisfies $J'^2 = I$ and thus provides an almost product structure on $\overline{M} \times R$. If the induced almost product structure on $\overline{M} \times R$ is integrable then the (1, 1, 1) AC-structure on $\overline{M}$ is said to be normal [15]. Since the vanishing of the Nijenhuis tensor $[J', J']$ is a necessary and sufficient condition for integrability, the condition of the normality in terms of the Nijenhuis tensor $[\varphi, \varphi]$ of φ is (see [15])

$$[\varphi, \varphi] + d\eta \otimes \xi = 0, \tag{2.13}$$

where

$$[\varphi, \varphi](\overline{X}, \overline{Y}) = [\varphi\overline{X}, \varphi\overline{Y}] - \varphi[\varphi\overline{X}, \overline{Y}] - \varphi[\overline{X}, \varphi\overline{Y}] + \varphi^2[\overline{X}, \overline{Y}], \tag{2.14}$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$.

3. Noninvariant Hypersurfaces of (1, 1, 1) AC-Manifolds

Let $\overline{M}$ be an (1, 1, 1) AC-manifold. Consider an $(n-1)$-dimensional manifold M imbedded in $\overline{M}$ with the immersion $i : M \to \overline{M}$ and assume that for each $p \in M$ the vector $\xi_{i(p)}$ is not tangent to the hypersurface. Then we have

$$\varphi i_* X = i_* JX + \alpha(X)\xi, \tag{3.1}$$

where J is a tensor field of type (1, 1), α is a 1-form on M and i_* is the differential of the immersion i of M into $\overline{M}$. If $\alpha \neq 0$, then the submanifold $i(M)$ is called a noninvariant hypersurface of $\overline{M}$. On the other hand, if the 1-form α vanishes, then $i(M)$ is called an invariant hypersurface of $\overline{M}$ (see [7]). A hypersurface may, of course, be neither invariant nor noninvariant. Throughout this section, unless specified otherwise $i(M)$ will be a noninvariant hypersurface of the (1, 1, 1) AC-manifold $\overline{M}$.

THEOREM 3.1. *If M is a noninvariant hypersurface of an* (1, 1, 1) *AC-manifold $\overline{M}$ with ξ nowhere tangent to M, then M admits an almost product structure.*

PROOF. By applying φ to (3.1) and using (2.1)–(2.4), we have

$$i_* X + \eta(i_* X)\xi = i_*(J^2 X) + \alpha(JX)\xi. \tag{3.2}$$

Then from (3.1), we get

$$J^2 X = X$$

and

$$\alpha(JX) = \eta(i_* X) = i^*(\eta X), \tag{3.3}$$

where $X \in \Gamma(TM)$ and i^* is the dual map of i_*. So J acts as an almost product structure on M. This completes the proof.

If we define a 1-form $C\alpha$ on M by $C\alpha(X) = \alpha(JX)$ then from (3.3) we can write

$$C\alpha = i^*\eta.$$

Thus, the hypersurface M admits a 1-form α whose vanishing means that the tangent hyperplane of the hypersurface is invariant under φ.

Now, let $\overline{\nabla}$ be a symmetric affine connection on $\overline{M}$ and define an affine connection ∇ on M with respect to the affine normal ξ by

$$\overline{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + h(X, Y)\xi, \tag{3.4}$$

where h is a symmetric tensor field of type $(0, 2)$ on M which is called the second fundamental form of M with respect to ξ.

Suppose that the $(1, 1, 1)$ AC-structure is normal. Then, the torsion field S of type $(1, 2)$ on M which is defined by

$$S(\overline{X}, \overline{Y}) = [\varphi\overline{X}, \varphi\overline{Y}] - \varphi[\varphi\overline{X}, \overline{Y}] - \varphi[\overline{X}, \varphi\overline{Y}] + \varphi^2[\overline{X}, \overline{Y}] + d\eta(\overline{X}, \overline{Y})\xi, \tag{3.5}$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$, vanishes. By taking $\overline{Y} = \xi$ in (3.5), we get

$$L_\xi\varphi = 0 \qquad \text{and} \qquad L_\xi\eta = 0,$$

where L_ξ is the Lie derivative operator with respect to ξ. From (3.5) the tensor field S is also expressed by

$$\begin{aligned} S(\overline{X}, \overline{Y}) = {} & \overline{\nabla}_{\varphi\overline{X}}(\varphi\overline{Y}) - \overline{\nabla}_{\varphi\overline{Y}}(\varphi\overline{X}) - \varphi(\overline{\nabla}_{\varphi\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}(\varphi\overline{X})) \\ & - \varphi(\overline{\nabla}_{\overline{X}}(\varphi\overline{Y}) - \overline{\nabla}_{\varphi\overline{Y}}\overline{X}) + \varphi^2(\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}) \\ & + (\overline{\nabla}_{\overline{X}}\eta(\overline{Y}) - \overline{\nabla}_{\overline{Y}}\eta(\overline{X}) - \eta([\overline{X}, \overline{Y}]))\xi, \end{aligned} \tag{3.6}$$

or

$$\begin{aligned} S(\overline{X}, \overline{Y}) = {} & (\overline{\nabla}_{\varphi\overline{X}}\varphi)\overline{Y} - (\overline{\nabla}_{\varphi\overline{Y}}\varphi)\overline{X} + \varphi(\overline{\nabla}_{\overline{Y}}\varphi)\overline{X} \\ & - \varphi(\overline{\nabla}_{\overline{X}}\varphi)\overline{Y} + [(\overline{\nabla}_{\overline{X}}\eta)\overline{Y} - (\overline{\nabla}_{\overline{Y}}\eta)\overline{X}]\xi. \end{aligned} \tag{3.7}$$

By using (3.1) and (3.4), we obtain

$$\begin{aligned} S(i_*X, i_*Y) = {} & i_*[J, J](X, Y) + L_\xi\varphi\{\alpha(X)i_*Y - \alpha(Y)i_*X\} \\ & + \{d\alpha(JX, Y) + d\alpha(X, JY) - 2i^*\eta([X, Y])\}\xi. \end{aligned} \tag{3.8}$$

Therefore, we have the following:

THEOREM 3.2. *A noninvariant hypersurface of a normal* $(1, 1, 1)$ *AC-manifold* $\overline{M}$ *is a locally product manifold which has a* 1*-form* $\alpha = C^{-1}i^*\eta$ *such that its differential satisfies*

$$d\alpha(JX, Y) + d\alpha(X, JY) = 2C\alpha([X, Y]). \tag{3.9}$$

COROLLARY 3.3. *An invariant hypersurface of an* $(1, 1, 1)$ *AC-manifold is an almost product manifold. If the* $(1, 1, 1)$ *AC-manifold is normal, then the almost product structure is integrable.*

THEOREM 3.4. *Let* ξ *be an infinitesimal automorphism of the* $(1, 1, 1)$ *AC-manifold* $\overline{M}$*. If, for every noninvariant hypersurface, the induced almost product structure* J *is integrable and the differential of the induced* 1*-form* $\alpha = C^{-1}i^*\eta$ *satisfies* (3.9) *then* $\overline{M}$ *is normal.*

4. Hypersurfaces of affinely cosymplectic and normal (1, 1, 1) AC-manifolds

Let $\overline{M}$ be an $(1, 1, 1)$ AC-manifold with a symmetric affine connection $\overline{\nabla}$ and ∇ denotes the induced connection on the noninvariant hypersurface M. If we write

$$(\nabla_X i_*)Y = \overline{\nabla}_{i_*X} i_* Y - i_*(\nabla_X Y), \tag{4.1}$$

then the Gauss and Weingarten equations are

$$(\nabla_X i_*)Y = h(X, Y)\xi, \qquad h(X, Y) = h(Y, X) \tag{4.2}$$

and

$$\overline{\nabla}_{i_*X}\xi = -i_* AX + w(X)\xi, \tag{4.3}$$

where h and A are the second fundamental tensors of type $(0, 2)$ and $(1, 1)$, respectively of M with respect to ξ, and w is a 1-form on M defining the connection on the affine normal bundle.

By using (3.1), (4.2) and (4.3) we get

$$\begin{aligned}(\overline{\nabla}_{i_*X}\varphi)i_*Y &= \overline{\nabla}_{i_*X}\varphi i_*Y - \varphi(\overline{\nabla}_{i_*X} i_*Y)\\ &= [h(X, JY) + (\nabla_X\alpha)(Y) + w(X)\alpha(Y)]\xi\\ &\quad + i_*[(\nabla_X J)Y - \alpha(Y)AX].\end{aligned} \tag{4.4}$$

Then we will investigate the following two cases:

Case I: Let $\overline{M}$ be an affinely cosymplectic (1, 1, 1) AC-manifold, that is, $\overline{M}$ be an (1, 1, 1) AC-manifold with a symmetric affine connection $\overline{\nabla}$ such that

$$\overline{\nabla}\varphi = 0, \qquad \overline{\nabla}\eta = 0. \tag{4.5}$$

From (3.7) we can easily see that an affinely cosymplectic (1, 1, 1) AC-manifold is normal. Also by using (2.1) and (2.2), we can show that (4.5) implies that

$$\overline{\nabla}\xi = 0.$$

Therefore, by (4.3), we have

$$AX = 0 \qquad \text{and} \qquad w(X) = 0.$$

Moreover, since $\overline{\nabla}\varphi = 0$ then from (4.4) we have

$$\nabla J = 0 \qquad \text{and} \qquad (\nabla_X \alpha)(Y) = -h(X, JY).$$

Case II: Let $\overline{M}$ be a normal (1, 1, 1) AC-manifold such that $\varphi = \overline{\nabla}\xi$. Then by using (3.1) and (4.3), we have

$$i_* JX + \alpha(X)\xi = -i_* AX + w(X)\xi,$$

that is, $J = -A$ and $\alpha = w$.

If $AX = 0$, for all $X \in \Gamma(TM)$, then from (4.3) it is obvious that $\overline{\nabla}_{i_* X}\xi$ and ξ are proportional. So affine normals are parallel along the hypersurface. In this case, the hypersurface M is said to be totally flat.

PROPOSITION 4.1. *Let M be a noninvariant hypersurface of an affinely cosymplectic* (1, 1, 1) *AC-manifold. Then M is totally flat and*

$$\nabla J = 0, \qquad (\nabla_X \alpha)(Y) = -h(X, JY), \qquad w = 0.$$

COROLLARY 4.2. *Let M be an invariant hypersurface of an affinely cosymplectic* (1, 1, 1) *AC-manifold. Then*

$$\nabla J = 0, \qquad h = 0, \qquad w = 0.$$

PROPOSITION 4.3. *Let M be a noninvariant hypersurface of a normal* (1,1,1) *AC-manifold such that $\varphi = \overline{\nabla}\xi$. Then*

$$J = -A \qquad \textit{and} \qquad \alpha = w.$$

5. Hypersurfaces of Lorentzian almost paracontact manifolds

An (1, 1, 1) AC-manifold $\overline{M}$ admitting a Lorentzian metric $\overline{g}$ such that

$$\overline{g}(\overline{X}, \xi) = \eta(\overline{X}), \tag{5.1}$$

$$\overline{g}(\overline{X}, \varphi\overline{Y}) \equiv \overline{g}(\varphi\overline{X}, \overline{Y}), \tag{5.2}$$

where $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$, is called Lorentzian almost paracontact manifold and denoted by $(\overline{M}, \varphi, \eta, \overline{g})$.

PROPOSITION 5.1. *Let (M, J, α, g) be a noninvariant hypersurface of $(\overline{M}, \varphi, \eta, \overline{g})$ where g is the induced metric on M, that is, $i^*\overline{g} = g$. Then the hypersurface (M, J, α, g) admits an almost product metric*

$$G = g + \alpha \otimes \alpha. \tag{5.3}$$

PROOF. From (5.2), we can write

$$\overline{g}(\varphi i_* X, i_* Y) = \overline{g}(\varphi i_* X, i_* Y). \tag{5.4}$$

By using (2.1) in (5.4), we obtain

$$\overline{g}(i_* JX, i_* Y) + \alpha(X)\eta(i_* Y) = \overline{g}(i_* X, i_* JY) + \alpha(Y)\eta(i_* X). \tag{5.5}$$

The induced metric g on (M, J, α) can be defined by

$$g(X, Y) = \overline{g}(i_* X, i_* Y).$$

So if we use (3.3) and (5.4) in (5.5), then we have

$$g(JX, Y) + \alpha(X)C\alpha(Y) = g(X, JY) + \alpha(Y)C\alpha(X),$$

that is,

$$(g + \alpha \otimes \alpha)(JX, Y) = (g + \alpha \otimes \alpha)(X, JY).$$

If we denote $g + \alpha \otimes \alpha$ by G, the proof is completed.

COROLLARY 5.2. *A noninvariant hypersurface of a Lorentzian almost paracontact manifold is an almost product metric manifold.*

Now, let define 2-forms

$$\Phi(\overline{X}, \overline{Y}) = \overline{g}(\varphi\overline{X}, \overline{Y}), \qquad \overline{X}, \overline{Y} \in \Gamma(T\overline{M})$$

and

$$\Omega(X, Y) = G(JX, Y), \qquad X, Y \in \Gamma(TM).$$

Φ and Ω are called the fundamental forms of the Lorentzian almost paracontact manifold $(\overline{M}, \varphi, \eta, \overline{g})$ and the submanifold (M, J, G) of $\overline{M}$, respectively. Then we have the following:

LEMMA 5.3. *Let Φ and Ω be the fundamental forms of $(\overline{M}, \varphi, \eta, \overline{g})$ and (M, J, α, G), respectively. Then*

$$i^*\Phi = \Omega - C\alpha \wedge \alpha. \tag{5.6}$$

PROOF. For $X, Y \in \Gamma(TM)$, by using definitions of the fundamental forms, (3.1) and (5.3), we get

$$\Phi(i_*X, i_*Y) = \Omega(X, Y) - (C\alpha \wedge \alpha)(X, Y).$$

Hence, we obtain

$$i^*\Phi(X, Y) = (\Omega - C\alpha \wedge \alpha)(X, Y).$$

THEOREM 5.4. *Let (M, J, α, G) be a noninvariant hypersurface of the Lorentzian para-Sasakian manifold $(\overline{M}, \varphi, \eta, \overline{g})$. Then*

(a) $J = -A$,
(b) $\alpha = w$.

PROOF. Since $(\overline{M}, \varphi, \eta, \overline{g})$ is a Lorentzian para-Sasakian manifold, we have

$$\overline{\nabla}_{i_*X}\xi = \varphi i_* X.$$

By using (4.3) and (3.1), we get

$$-i_*AX + w(X)\xi = i_*JX + \alpha(X)\xi,$$

which completes the proof.

THEOREM 5.5. *If M is a noninvariant hypersurface of a Lorentzian para-Sasakian manifold $(\overline{M}, \varphi, \eta, \overline{g})$, then*

(a) $(\nabla_X J)(Y) = \alpha(Y)JX - C\alpha(Y)X$,
(b) $\overline{g}(i_*X, i_*Y) + 2C\alpha(X)C\alpha(Y) = h(X, JY) + (\nabla_X\alpha)(Y) + \alpha(X)\alpha(Y)$.

PROOF. By using (3.1) and (4.1) we obtain

$$\begin{aligned}(\overline{\nabla}_{i_*X}\varphi)(i_*Y) = {} & [i_*(\nabla_X J)(Y) + \alpha(Y)i_*JX] \\ & + [h(X, JY) + (\nabla_X\alpha)(Y) + \alpha(X)\alpha(Y)]\xi.\end{aligned} \tag{5.7}$$

On the other hand, since $(\overline{M}, \varphi, \eta, \overline{g})$ is a Lorentzian para-Sasakian manifold, from (2.11) we also have

$$(\overline{\nabla}_{i_*X}\varphi)(i_*Y) = \eta(i_*Y)i_*X + \overline{g}(i_*X, i_*Y)\xi + 2\eta(i_*X)\eta(i_*Y)\xi. \tag{5.8}$$

By considering $C\alpha = i^*\eta$ and equating the components of (5.7) and (5.8), we get (a) and (b) in the assertion theorem. This completes the proof.

As an immediate consequence we have the following:

COROLLARY 5.6. *Let M be a noninvariant hypersurface of the Lorentzian para-Sasakian manifold $(\overline{M}, \varphi, \eta, \overline{g})$ with the induced almost product structure J. Then M is a locally product manifold if and only if*

$$\alpha(Y)JX = \alpha(JY)X. \tag{5.9}$$

Now, let $\overline{M}$ be a (1, 1, 1) AC-manifold and M be an invariant hypersurface of $\overline{M}$. Assume that for each $p \in M$ the vector $\xi_{i(p)}$ belongs to the tangent hyperplane of the hypersurface. For an invariant hypersurface of an (1, 1, 1) AC-manifold we can write

$$\varphi i_*X = i_*\psi X, \tag{5.10}$$

where ψ is a tensor of type $(1, 1)$ on the hypersurface M and $X \in \Gamma(TM)$. Applying φ to the both sides of the equation (5.10), we get

$$i_*\psi^2 X = \varphi^2 i_*X = i_*X + \eta(i_*X)\xi. \tag{5.11}$$

If we denote

$$i_*\xi^* = \xi \tag{5.12}$$

and

$$\eta^*(X) = \eta(i_*X), \tag{5.13}$$

then we have

$$\psi^2 X = X + \eta^*(X)\xi^*. \tag{5.14}$$

Furthermore,

$$\eta^*(\psi X) = \eta(i_*\psi X) = \eta(\varphi i_*X) = 0, \tag{5.15}$$

$$\eta^*(\xi^*) = \eta(i_*\xi^*) = \eta(\xi) = -1 \tag{5.16}$$

and

$$i_*\psi\xi^* = \varphi i_*\xi^* = \varphi\xi = 0,$$

that is,

$$\psi\xi^* = 0. \tag{5.17}$$

Thus we have

THEOREM 5.7. *Let M be an invariant hypersurface of an $(1, 1, 1)$ AC-manifold $(\overline{M}, \varphi, \eta, \xi)$ and $\xi \in \Gamma(TM)$. Then M is an $(1, 1, 1)$ AC-manifold with the structure (ψ, ξ^*, η^*) where $i_*\xi^* = \xi$ and $\eta^*(X) = \eta(i_*X)$, for all $X \in \Gamma(TM)$.*

THEOREM 5.8. *Let M be an invariant hypersurface of an $(1, 1, 1)$ AC-manifold $(\overline{M}, \varphi, \eta, \xi)$ with $\xi \in \Gamma(TM)$. If $\overline{M}$ is normal, then M is also normal.*

PROOF. By using (3.5), we can write

$$\begin{aligned} S(i_*X, i_*Y) &= [\varphi, \varphi](i_*X, i_*Y) + d\eta(i_*X, i_*Y)\xi \\ &= [\varphi i_*X, \varphi i_*Y] - \varphi[\varphi i_*X, i_*Y] - \varphi[i_*X, \varphi i_*Y] \\ &\quad + \varphi^2[i_*X, i_*Y] + d\eta(i_*X, i_*Y)\xi. \end{aligned} \tag{5.18}$$

for all $X, Y \in \Gamma(TM)$. If we use (5.10), (5.12) and (5.13) in (5.18), we get

$$\begin{aligned} S(i_*X, i_*Y) &= i_*\psi^2[X, Y] + [i_*\psi X, i_*\psi Y] - i_*\psi[X, \psi Y] - i_*\psi[\psi X, Y] \\ &\quad + \{(i_*X)(\eta^*(Y)) - (i_*Y)(\eta^*(X)) - \eta^*([X, Y])\}i_*\xi^* \\ &= i_*\{[\psi, \psi](X, Y) + d\eta^*(X, Y)\xi^*\}. \end{aligned}$$

Hence, we have the assertion of the theorem.

THEOREM 5.9. *Let M be an invariant hypersurface of a Lorentzian almost paracontact manifold $(\overline{M}, \varphi, \eta, \overline{g})$ where $\xi \in \Gamma(TM)$. Then M is also a Lorentzian almost paracontact manifold.*

PROOF. From Theorem 5.7 it follows that an invariant hypersurface M of $\overline{M}$ is an $(1, 1, 1)$ AC-manifold with the structure (ψ, ξ^*, η^*). Let g^* be the induced metric on M. Then we have

$$g^*(\psi X, \psi Y) = \overline{g}(i_*\psi X, i_*\psi Y) = \overline{g}(\varphi i_*X, \varphi i_*Y). \tag{5.19}$$

Since $\overline{M}$ is a Lorentzian almost paracontact manifold, then by using (5.13) in (5.19) we get

$$g^*(\psi X, \psi Y) = g^*(X, Y) + \eta^*(X)\eta^*(Y). \tag{5.20}$$

Moreover,

$$g^*(X,\xi^*) = \overline{g}(i_*X, i_*\xi^*) = \eta(i_*X) = \eta^*(X), \tag{5.21}$$

which completes the proof.

THEOREM 5.10. *Let $(\overline{M},\varphi,\eta,\overline{g})$ be a Lorentzian para Sasakian manifold. Then an invariant hypersurface with $\xi \in \Gamma(TM)$ of $\overline{M}$ is also a Lorentzian para Sasakian manifold.*

PROOF. Let $\overline{M}$ be a Lorentzian para-Sasakian manifold. Then we have

$$\overline{\nabla}_{i_*X}\xi = \varphi i_*X,$$

where $\overline{\nabla}$ is a Levi-Civita connection with respect to $\overline{g}$. From (5.10) and (5.12), we can write

$$\overline{\nabla}_{i_*X} i_*\xi^* = i_*\psi X.$$

By using (3.4) in the last equation, we obtain

$$i_*\nabla_X\xi^* + h(X,\xi^*)N = i_*\psi X,$$

where ∇ is the induced connection on M and N is normal to M. If we consider normal and tangent components of above equation we get

$$\begin{aligned} \nabla_X\xi^* &= \psi X, \\ h(X,\xi^*) &= 0. \end{aligned}$$

Since $\overline{M}$ be a Lorentzian para Sasakian manifold from (2.11), we have

$$(\overline{\nabla}_{i_*X}\varphi)i_*Y = \eta(i_*Y)i_*X + \overline{g}(i_*X, i_*Y)\xi + 2\eta(i_*X)\eta(i_*Y)\xi, \tag{5.22}$$

for all $X, Y \in \Gamma(TM)$. By using (5.10), (5.12) and (5.13) in (5.22), we obtain

$$(\overline{\nabla}_{i_*X}\varphi)i_*Y = i_*\{\eta^*(Y)X + \overline{g}(X,Y)\xi^* + 2\eta^*(X)\eta^*(Y)\xi^*\}. \tag{5.23}$$

On the other hand, from (3.4) and (5.10), one can get

$$\begin{aligned} (\overline{\nabla}_{i_*X}\varphi)i_*Y &= \overline{\nabla}_{i_*X}\varphi i_*Y - \varphi(\overline{\nabla}_{i_*X} i_*Y) \\ &= \overline{\nabla}_{i_*X} i_*\psi Y - \varphi(i_*\nabla_X Y + h(X,Y)N) \\ &= i_*(\nabla_X\psi Y - \psi(\nabla_X Y)) + h(X,\psi Y)N - h(X,Y)\varphi N, \end{aligned} \tag{5.24}$$

where ∇ is the induced connection on M and N is normal to M. By equating right hand sides of equations (5.23) and (5.24), we have

$$(\nabla_X\psi)Y = \eta^*(Y)X + \overline{g}(X,Y)\xi^* + 2\eta^*(X)\eta^*(Y)\xi^*.$$

This completes the proof.

6. Examples

EXAMPLE 6.1. Let $\overline{M}$, be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). Defining

$$\eta = ds - dx - dz, \qquad \xi = -\frac{\partial}{\partial s},$$
$$\varphi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - \frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}$$
$$\varphi\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z} - \frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \qquad \varphi\left(\frac{\partial}{\partial s}\right) = 0,$$

the set (φ, ξ, η) becomes a $(1, 1, 1)$ AC-structure in $\overline{M}$.

Let M_1 be a hypersurface of $\overline{M}$ which is given by $s = x$ with the immersion $i : M_1 \to \overline{M}$. Then

$$\{u_1 = (1, 0, 0, 0, 1), u_2 = (0, 1, 0, 0, 0),$$
$$u_3 = (0, 0, 1, 0, 0), u_4 = (0, 0, 0, 1, 0)\}$$

is a local basis for the tangent hyperplane of M_1 and $N_1 = (1, 0, 0, 0, -1)$ is the normal vector field of the hypersurface. It is obvious that the characteristic vector field $\xi_{i(p)}$, $p \in M_1$, is not tangent to hypersurface of M_1. A tangent vector field of the hypersurface can be written by $X \equiv i_*X = f_1u_1 + f_2u_2 + f_3u_3 + f_4u_4$ for some smooth functions f_i, $1 \leq i \leq 4$, on M. Then we have

$$\varphi i_*X = -f_1u_1 - f_2u_2 - f_3u_3 - f_4u_4 + f_3\xi,$$

which shows that M_1 is a noninvariant hypersurface of $\overline{M}$.

Now let us consider the hypersurface M_2 of the $(1, 1, 1)$ AC-manifold $\overline{M}$ defining by $x = y$ and let $i : M_2 \to \overline{M}$ be the immersion of M_2 into M. In this case the set

$$\{v_1 = (1, 1, 0, 0, 0), v_2 = (0, 0, 1, 0, 0),$$
$$v_3 = (0, 0, 0, 1, 0), v_4 = (0, 0, 0, 0, 1)\}$$

is a local basis for the tangent hyperplane and $N_2 = (1, -1, 0, 0, 0)$ is the normal vector field of M_2. The characteristic vector field is tangent to the the hypersurface. For any tangent vector field $X \equiv i_*X = h_1v_1 + h_2v_2 + h_3v_3 + h_4v_4$ of the hypersurface we have

$$\varphi i_*X = -h_1v_1 - h_2v_2 - f_3v_3 + (h_1 + h_2)\xi, \tag{6.1}$$

where h_i, $1 \leq i \leq 4$, are some smooth functions on M_2. From (6.1) we see that M_2 is an invariant hypersurface of $\overline{M}$.

EXAMPLE 6.2. Let $\overline{M}$ be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). In $\overline{M}$ we define

$$\eta = ds - dx, \qquad \xi = -\frac{\partial}{\partial s},$$

$$\varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} + \frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y},$$

$$\varphi\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}, \qquad \varphi\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}, \qquad \varphi\left(\frac{\partial}{\partial s}\right) = 0,$$

$$g = (dx)^2 + (dy)^2 + (dz)^2 + (dt)^2 - \eta \otimes \eta.$$

Then (φ, ξ, η, g) is a Lorentzian almost paracontact structure in $\overline{M}$.

Let M be a hypersurface of $\overline{M}$ which is defined by $s = x$ with the immersion $i : M \to \overline{M}$. Then the set

$$\{u_1 = (1, 0, 0, 0, 1), u_2 = (0, 1, 0, 0, 0), \\ u_3 = (0, 0, 1, 0, 0), u_4 = (0, 0, 0, 1, 0)\}$$

is a local basis for the tangent hyperplane of M and $N = (1, 0, 0, 0, -1)$ is the normal vector field of the hypersurface. Since $\xi_{i(p)} = \frac{1}{2}(u_1 - N)_{i(p)}$, it can be easily seen that the characteristic vector field $\xi_{i(p)}$, $p \in M$, is not tangent to M. Moreover, since $\varphi u_1 = u_1$, $\varphi u_2 = u_2$, $\varphi u_3 = u_3$, $\varphi u_4 = u_4$, then M is an invariant hypersurface of $\overline{M}$ with the characteristic vector field $\xi_{i(p)}$, $p \in M$, which is not tangent to the hypersurface.

EXAMPLE 6.3. Let $\overline{M}$ be the 3-dimensional real number space with a coordinate system (x, y, z). If we define

$$\eta = dz, \qquad \xi = -\frac{\partial}{\partial z},$$

$$\varphi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \qquad \varphi\left(\frac{\partial}{\partial s}\right) = 0,$$

$$g = (dx)^2 + (dy)^2 - \eta \otimes \eta.$$

on $\overline{M}$, then (φ, ξ, η, g) is a Lorentzian almost paracontact structure in $\overline{M}$.

Assume that M be a surface of $\overline{M}$ given by $x = \arcsin y$ with the immersion $i : M \to \overline{M}$. Then

$$\left\{u_1 = \left(1, \sqrt{1 - y^2}, 0\right), u_2 = (0, 0, 1)\right\}$$

forms a local basis for the tangent plane of M and $N = \left(\sqrt{1-y^2}, -1, 0\right)$ is the normal vector field of the surface. For any tangent vector field X of the surface we have

$$\varphi i_* X = -f_1 u_1, \tag{6.2}$$

where $X \equiv i_* X = f_1 u_1 + f_2 u_2$ for some smooth functions f_1, f_2 on M. From (6.2) we obtain that M is an invariant surface of $\overline{M}$ with the characteristic vector field $\xi_{i(p)}$, $p \in M$, belonging to the tangent plane of the surface.

EXAMPLE 6.4. Let $\overline{M} = R^3$ be the 3-dimensional real number space with a coordinate system (x, y, z). We define

$$\begin{gathered} \eta = dz, \qquad \xi = -\frac{\partial}{\partial z}, \\ \varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \qquad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \\ g = e^{-2z}(dx)^2 + e^{2z}(dy)^2 - (dz)^2. \end{gathered} \tag{6.3}$$

Then (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on $\overline{M}$.

Let M_1 be a surface of $\overline{M}$ with the immersion $i : M_1 \to \overline{M}$ which is given by

$$z = x + y.$$

Then $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 1)$ is a local basis for the tangent plane of the surface. The vector field

$$N = (e^{2(x+y)}, e^{2(x+y)}, 1)$$

is a normal vector field of M_1. Since

$$\xi = -\frac{1}{e^{2(x+y)} + e^{-2(x+y)} - 1}\left((e^{2(x+y)})u_1 + (e^{-2(x+y)})u_2 - N\right)$$

then for each $p \in M_1$ the characteristic vector field $\xi_{i(p)}$ is not tangent to the surface. A tangent vector field of the surface can be written by $X \equiv i_* X = f_1 u_1 + f_2 u_2$ for some smooth functions f_1, f_2 on M. By using (6.3) we have

$$\varphi i_* X = f_1 u_1 - f_2 u_2 + (f_1 - f_2)\xi. \tag{6.4}$$

From (3.1) and (6.4) we get

$$i_* J X = f_1 u_1 - f_2 u_2$$

and

$$\alpha(X) = f_1 - f_2,$$

where J acts an almost product structure on M_1. Consequently, M_1 is a noninvariant surface of the Lorentzian para-Sasakian manifold $\overline{M}$ with ξ nowhere tangent to M_1.

Let M_2 be another surface of $\overline{M}$ which is given by

$$x = \arctan y.$$

Then $v_1 = \left(\frac{1}{1+y^2}, 1, 0\right)$, $v_2 = (0, 0, 1)$ forms a local orthogonal basis for the tangent plane of the surface and

$$N = \left(e^{2z}, -\frac{1}{1+y^2}e^{-2z}, 0\right)$$

is a normal vector field of M_2. It is obvious that the characteristic vector field of the manifold is tangent to the surface M_2. For any tangent vector field $i_*Y \equiv Y$ of the surface where $i : M_2 \to \overline{M}$ is an immersion into the Lorentzian para-Sasakian manifold $\overline{M}$ we can write $i_*Y = \gamma_1 v_1 + \gamma_2 v_2$ for some smooth functions $\gamma_1,\ \gamma_2$ on M_2. By using (6.3) we have

$$\varphi i_* Y = -\gamma_1\left(v_1 - \frac{2(1+y^2)}{(1+y^2)^2 e^{2z} - e^{-2z}} N\right),$$

which shows that M_2 is a noninvariant surface of the Lorentzian para-Sasakian manifold $\overline{M}$ with ξ tangent to the surface.

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