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C^*-ALGEBRAS ARISING FROM DYCK SYSTEMS OF TOPOLOGICAL MARKOV CHAINS

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Abstract

Let A be an $N \times N$ irreducible matrix with entries in $\{0, 1\}$. We define the topological Markov Dyck shift D_A to be a nonsofic subshift consisting of bi-infinite sequences of the $2N$ brackets $(_1, \dots, (_N,)_1, \dots,)_N$ with both standard bracket rule and Markov chain rule coming from A. It is regarded as a subshift defined by the canonical generators $S_1^*, \dots, S_N^*, S_1, \dots, S_N$ of the Cuntz-Krieger algebra $\mathcal{O}_A$. We construct an irreducible λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ that presents the subshift D_A so that we have an associated simple purely infinite C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$. We prove that $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ is a universal unique C^*-algebra subject to some operator relations among $2N$ generating partial isometries.

1. Introduction

Let Σ be a finite set with its discrete topology, that is called an alphabet. Each element of Σ is called a symbol. Let Σ^{Z} be the infinite product space $\prod_{i=-\infty}^{\infty} \Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on Σ^{Z} given by $\sigma((x_i)_{i\in\mathsf{Z}}) = (x_{i+1})_{i\in\mathsf{Z}}$ is called the full shift over Σ. Let Λ be a closed subset of Σ^{Z} such that $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift or a symbolic dynamical system. It is written as Λ for brevity. There is a class of subshifts called sofic shifts, that contains the class of topological Markov shifts. Sofic shifts are presented by finite labeled graphs, called λ-graphs. In [19], the author has introduced a notion of λ-graph system as a generalization of λ-graph. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \dots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \dots$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota_{l,l+1} : V_{l+1} \to V_l$ for each $l \in \mathsf{Z}_+$, where Z_+ denotes the set of all nonnegative integers. An edge $e \in E_{l,l+1}$ has its source vertex $s(e)$ in V_l, its terminal vertex $t(e)$ in V_{l+1} and its label $\lambda(e)$ in Σ.

The theory of symbolic dynamical system has a close relationship with formal language theory. In the theory of formal language, there is a class of universal languages due to W. Dyck. The symbolic dynamics generated by the

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languages are called the Dyck shifts D_N (cf. [3], [10], [11], [12]). They are non-sofic subshifts. Its alphabet consists of the $2N$ brackets: $(_1, \dots, (_N,)_1, \dots,)_N$. The forbidden words consist of words that do not obey the standard bracket rules. In [14], a λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ that presents the subshift D_N has been introduced. The λ-graph system is called the Cantor horizon λ-graph system for the Dyck shift D_N. The K-groups for $\mathfrak{L}^{\mathrm{Ch}(D_N)}$, that are invariant under topological conjugacy of the subshift D_N, have been computed ([14]).

In [20], a nuclear C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a λ-graph system $\mathfrak{L}$ has been introduced. The class of the C^*-algebras contain the class of the Cuntz-Krieger algebras. They are universal unique concrete C^*-algebras generated by finite families of partial isometries and sequences of projections subject to certain operator relations encoded by structure of the λ-graph systems. Its K-groups $K_i(\mathcal{O}_{\mathfrak{L}}), i = 0, 1$ are realized as the K-groups of the λ-graph system $\mathfrak{L}$. The results of [14] imply that the C^*-algebras $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}$ for $N = 2, 3, \dots$ are unital, simple and purely infinite whose K-groups are

$$(1.1) \qquad K_0(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{K}, \mathbb{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}) \cong 0$$

where $C(\mathfrak{K}, \mathbb{Z})$ denotes the abelian group of all integer valued continuous functions on a Cantor discontinuum $\mathfrak{K}$. Let $u_1, \dots, u_N$ be the canonical generating isometries of the Cuntz algebra $\mathcal{O}_N$ that satisfy the relations: $\sum_{j=1}^N u_j u_j^* = 1$, $u_i^* u_i = 1$ for $i = 1, \dots, N$. Then the bracket rule of the symbols $(_1, \dots, (_N,)_1, \dots,)_N$ of the Dyck shift D_N may be interpreted as the relations

$$(1.2) \qquad u_i^* u_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, \dots, N$ in $\mathcal{O}_N$ through the correspondence $(_i \longrightarrow u_i^*,)_i \longrightarrow u_i$ (cf. (2.1)).

In the present paper, we consider a generalization of the Dyck shifts D_N by using the canonical generating partial isometries of the Cuntz-Krieger algebras $\mathcal{O}_A$ for $N \times N$ matrices A with entries in $\{0, 1\}$. The generalized Dyck shift is denoted by D_A and called the topological Markov Dyck shift for A (cf. [7], [11], [15], [16]). Let $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N$ be the alphabet of D_A, corresponding to the brackets $(_1, \dots, (_N,)_1, \dots,)_N$ respectively. Let $t_1, \dots, t_N$ be the canonical generating partial isometries of $\mathcal{O}_A$ satisfying the relations

$$\sum_{j=1}^N t_j t_j^* = 1, \quad t_i^* t_i = \sum_{j=1}^N A(i, j) t_j t_j^* \qquad \text{for} \quad i = 1, \dots, N.$$

Consider the correspondence $\varphi(\alpha_i) = t_i^*, \varphi(\beta_i) = t_i, i = 1, \dots, N$. Then a word w of $\{\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N\}$ is defined to be admissible for the subshift

D_A precisely if the corresponding element to w through φ in $\mathcal{O}_A$ is not zero. If A satisfies condition (I) in the sense of [5], the subshifts D_A are not sofic (Proposition 2.1). If all entries of A are 1's, D_A is reduced to D_N. We consider the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ for the topological Markov Dyck shift D_A. The λ-graph system will be proved to be λ-irreducible with λ-condition (I) in the sense of [23] if the matrix A is irreducible with condition (I) (Proposition 2.5). Hence the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ is simple and purely infinite. We will show:

THEOREM 1.1. *Let A be an $N \times N$ matrix with entries in* $\{0, 1\}$. *Suppose that A is irreducible with condition* (I). *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ associated with the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ is unital, separable, nuclear, simple and purely infinite. It is the unique C^*-algebra generated by $2N$ partial isometries $S_i, T_i, i = 1, \dots, N$ subject to the following relations:*

$$\sum_{j=1}^{N}(S_j S_j^* + T_j T_j^*) = 1, \tag{1.3}$$

$$\sum_{j=1}^{N} S_j^* S_j = 1, \tag{1.4}$$

$$T_i^* T_i = \sum_{j=1}^{N} A(i, j) S_j^* S_j, \qquad i = 1, 2, \dots, N, \tag{1.5}$$

$$E_{\mu_1\dots\mu_k} = \sum_{j=1}^{N} A(j, \mu_1) S_j S_j^* E_{\mu_1\dots\mu_k} S_j S_j^* + T_{\mu_1} E_{\mu_2\dots\mu_k} T_{\mu_1}^*, \qquad k > 1 \tag{1.6}$$

where $E_{\mu_1\dots\mu_k} = S_{\mu_1}^* \dots S_{\mu_k}^* S_{\mu_k} \dots S_{\mu_1}$ for $\mu_1, \dots, \mu_k \in \{1, \dots, N\}$.

Let X_A be the right one-sided topological Markov shift

$$X_A = \{(x_i)_{i\in\mathsf{N}} \in \{1, \dots, N\}^{\mathsf{N}} \mid A(x_i, x_{i+1}) = 1, i \in \mathsf{N}\}$$

for the matrix A and σ_A the shift on X_A defined by $\sigma_A((x_i)_{i\in\mathsf{N}}) = (x_{i+1})_{i\in\mathsf{N}}$ for $(x_i)_{i\in\mathsf{N}} \in X_A$. Let σ_{Λ_A} and λ_{Λ_A} be endomorphisms of the abelian group $C(X_A, \mathsf{Z})$ of all Z-valued continuous functions on X_A defined by

$$\sigma_{\Lambda_A}(f)(x) = f(\sigma_A(x)), \qquad \lambda_{\Lambda_A}(f)(x) = \sum_{j=1}^{N} A(j, x_1) f(jx)$$

for $f \in C(X_A, \mathsf{Z})$ and $x = (x_i)_{i\in\mathsf{N}} \in X_A$, where $jx = (j, x_1, x_2, \dots) \in X_A$ for $A(j, x_1) = 1$. Then we will show:

THEOREM 1.2.

(i) $K_0(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) = C(X_A, \mathbb{Z})/(\mathrm{id} - (\sigma_{\Lambda_A} + \lambda_{\Lambda_A}))C(X_A, \mathbb{Z})$.

(ii) $K_1(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) = \mathrm{Ker}(\mathrm{id} - (\sigma_{\Lambda_A} + \lambda_{\Lambda_A}))$ *in* $C(X_A, \mathbb{Z})$.

If all entries of A are 1's, the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ becomes $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ so that the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ goes to $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}$. If A is the Fibonacci matrix $F = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_F)}}$ is simple and purely infinite. Its K-groups are $K_0(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_F)}}) \cong \mathbb{Z} \oplus C(\mathfrak{K}, \mathbb{Z})^\infty$, $K_1(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_F)}}) \cong 0$ where $C(\mathfrak{K}, \mathbb{Z})^\infty$ denotes the countable infinite direct sum of the group $C(\mathfrak{K}, \mathbb{Z})$ (cf. [25]). In general, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a λ-graph system $\mathfrak{L}$ has an infinite family of generators. Both of the C^*-algebras $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}$, $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_F)}}$ are finitely generated, and their K_0-groups however are not finitely generated. Therefore they are not semiprojective whereas Cuntz algebras and Cuntz-Krieger algebras are semiprojective (cf. [1], [2], [21], [26]).

2. The topological Markov Dyck shifts

Throughout the paper N is a fixed positive integer larger than 1.

We consider the Dyck shift D_N with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \dots, \alpha_N\}$, $\Sigma^+ = \{\beta_1, \dots, \beta_N\}$. The symbols α_i, β_i correspond to the brackets $(_i,)_i$ respectively, and have the relations

$$\alpha_i \beta_j = \begin{cases} \mathbf{1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for $i, j = 1, \dots, N$ (cf. (1.2), [11],[12]). A word $\gamma_1 \dots \gamma_n$ of Σ is defined to be admissible for D_N precisely if $\prod_{m=1}^n \gamma_m \neq 0$, where $\prod_{m=1}^n \gamma_m$ means the product $\gamma_1 \dots \gamma_n$ obtained by applying (2.1).

Let $A = [A(i, j)]_{i,j=1,\dots,N}$ be an $N \times N$ matrix with entries in $\{0, 1\}$ having no zero rows or columns. Consider the Cuntz-Krieger algebra $\mathcal{O}_A$ for the matrix A that is the universal C^*-algebra generated by N partial isometries $t_1, \dots, t_N$ subject to the following relations ([5]):

$$\sum_{j=1}^N t_j t_j^* = 1, \quad t_i^* t_i = \sum_{j=1}^N A(i, j) t_j t_j^* \qquad \text{for} \quad i = 1, \dots, N. \tag{2.2}$$

Define a correspondence $\varphi_A : \Sigma \longrightarrow \{t_1^*, \dots, t_N^*, t_1, \dots, t_N\}$ by setting

$$\varphi_A(\alpha_i) = t_i^*, \quad \varphi_A(\beta_i) = t_i \qquad \text{for} \quad i = 1, \dots, N.$$

We denote by Σ^* the set of all words $\gamma_1 \dots \gamma_n$ of elements of Σ. Define the set

$$\mathfrak{F}_A = \{\gamma_1 \dots \gamma_n \in \Sigma^* \mid \varphi_A(\gamma_1) \dots \varphi_A(\gamma_n) = 0\}.$$

Let D_A be the subshift over Σ whose forbidden words are $\mathfrak{F}_A$. The subshift is called the topological Markov Dyck shift defined by A. These kinds of subshifts have first appeared in [7] in semigroup setting and in [15] in more general setting without using C^*-algebras. If all entries of A are 1's, the partial isometries $\varphi_A(\alpha_1), \dots, \varphi_A(\alpha_N), \varphi_A(\beta_1), \dots, \varphi_A(\beta_N)$ satisfy the same relations as (2.1) so that the subshift D_A becomes the Dyck shift D_N. We note the fact that $\alpha_i\beta_j \in \mathfrak{F}_A$ if $i \neq j$, and $\alpha_{i_n} \dots \alpha_{i_1} \in \mathfrak{F}_A$ if and only if $\beta_{i_1} \dots \beta_{i_n} \in \mathfrak{F}_A$. Consider the following two subsystems of D_A

$$D_A^+ = \{(\gamma_i)_{i\in\mathsf{Z}} \in D_A \mid \gamma_i \in \Sigma^+ \text{ for all } i \in \mathsf{Z}\},$$
$$D_A^- = \{(\gamma_i)_{i\in\mathsf{Z}} \in D_A \mid \gamma_i \in \Sigma^- \text{ for all } i \in \mathsf{Z}\}.$$

The subshift D_A^+ is identified with the topological Markov shift

$$\Lambda_A = \{(x_i)_{i\in\mathsf{Z}} \in \{1, \dots, N\}^{\mathsf{Z}} \mid A(x_i, x_{i+1}) = 1, i \in \mathsf{Z}\}$$

defined by A through the one block code $\beta_i \longrightarrow i$. Similarly D_A^- is identified with the topological Markov shift Λ_{A^t} defined by the transposed matrix A^t of A. Hence the subshift D_A contains copies of both of the topological Markov shifts Λ_A and Λ_{A^t}. The following proposition implies that most irreducible matrices A yield non Markov subshifts D_A.

PROPOSITION 2.1. *If A satisfies condition* (I) *in the sense of [5], the subshift D_A is not sofic.*

PROOF. Recall that X_A is the right one-sided topological Markov shift $\{(x_i)_{i\in\mathsf{N}} \mid (x_i)_{i\in\mathsf{Z}} \in \Lambda_A\}$ for A. Put $X_{D_A^+} = \{(\gamma_i)_{i\in\mathsf{N}} \mid (\gamma_i)_{i\in\mathsf{Z}} \in D_A^+\}$. Since A satisfies condition (I), we can find elements $(n(i))_{i\in\mathsf{N}} \in X_A$ such that $n(i)_{i\in\mathsf{N}} \neq k(i)_{i\in\mathsf{N}}$ for $n \neq k$. Put $x(n) = (\beta_{n(i)})_{i\in\mathsf{N}} \in X_{D_A^+}$ for $n \in \mathsf{N}$. Let $\Gamma^-(x(n))$ be the predecessor set of $x(n)$ in D_A, that is,

$$\Gamma^-(x(n)) = \{(\dots, y_{-2}, y_{-1}, y_0) \mid (\dots, y_{-2}, y_{-1}, y_0, \beta_{n(1)}, \beta_{n(2)}, \dots) \in D_A\}.$$

The left one-sided sequence $(\dots, \alpha_{k(2)}, \alpha_{k(1)})$ belongs to $\Gamma^-(x(n))$ if and only if $n = k$. Thus the predecessor sets $\Gamma^-(x(n))$, $n = 1, 2, \dots$ are mutually distinct, so that D_A is not sofic (cf. [18, Theorem 3.2.10]).

A λ-graph system $\mathfrak{L}$ is said to present a subshift Λ if the set of all admissible words of Λ coincides with the set of all finite labeled sequences appearing in concatenating edges of $\mathfrak{L}$. There are many λ-graph systems that present a given subshift. Among them the canonical λ-graph system is a generalization of the left-Krieger cover graph for a sofic shift([19]). The canonical λ-graph system $\mathfrak{L}^{C(D_N)}$ for the Dyck shift D_N together with its K-groups has been studied in [22]. One however sees that the λ-graph system $\mathfrak{L}^{C(D_N)}$ is not irreducible, so that the resulting C^*-algebra $\mathcal{O}_{\mathfrak{L}^{C(D_N)}}$ is not simple. The Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ for D_N is an irreducible component of $\mathfrak{L}^{C(\Lambda)}$ so that the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}$ is simple and purely infinite whose K-groups have been computed as (1.1) [24].

In the paper we will study the Cantor horizon λ-graph systems $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ for the topological Markov Dyck shifts D_A and its associated C^*-algebras $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$. In what follows we fix an $N \times N$ matrix A with entries in $\{0, 1\}$ having no zero rows or columns. We denote by $B_l(D_A)$ and $B_l(\Lambda_A)$ the set of admissible words of length l of D_A and that of Λ_A respectively. Let $m(l)$ be the cardinal number of $B_l(\Lambda_A)$. We use lexicographic order from the left on the words of $B_l(\Lambda_A)$, so that we assign a word $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$ the number $N(\mu_1 \dots \mu_l)$ from 1 to $m(l)$. For example, if $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, then

$$\begin{aligned} &B_1(\Lambda_A) = \{1, 2\}, && N(1) = 1,\ N(2) = 2, \\ &B_2(\Lambda_A) = \{11, 12, 21\}, && N(11) = 1,\ N(12) = 2,\ N(21) = 3, \end{aligned}$$

and so on. Hence the set $B_l(\Lambda_A)$ bijectively corresponds to the set of natural numbers less than or equal to $m(l)$. Let us now describe the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ of D_A. The vertices V_l at level l for $l \in \mathsf{Z}_+$ are given by the admissible words of length l consisting of the symbols of Σ^+. We regard V_0 as a one point set of the empty word $\{\emptyset\}$. Since V_l is identified with $B_l(\Lambda_A)$, we may write V_l as

$$V_l = \{v^l_{N(\mu_1 \dots \mu_l)} \mid \mu_1 \dots \mu_l \in B_l(\Lambda_A)\}.$$

The mapping $\iota\,(= \iota_{l,l+1}) : V_{l+1} \to V_l$ is defined by deleting the rightmost symbol of a corresponding word such as

$$\iota(v^{l+1}_{N(\mu_1 \dots \mu_{l+1})}) = v^l_{N(\mu_1 \dots \mu_l)} \qquad \text{for} \quad v^{l+1}_{N(\mu_1 \dots \mu_{l+1})} \in V_{l+1}.$$

We define an edge labeled α_j from $v^l_{N(\mu_1 \dots \mu_l)} \in V_l$ to $v^{l+1}_{N(\mu_0 \mu_1 \dots \mu_l)} \in V_{l+1}$ precisely if $\mu_0 = j$, and an edge labeled β_j from $v^l_{N(j \mu_1 \dots \mu_{l-1})} \in V_l$ to $v^{l+1}_{N(\mu_1 \dots \mu_{l+1})} \in V_{l+1}$. For $l = 0$, we define an edge labeled α_j from v^0_1 to $v^1_{N(j)}$, and an edge labeled β_j from v^0_1 to $v^1_{N(i)}$ if $A(j, i) = 1$. We denote by $E_{l,l+1}$ the set of edges from V_l to V_{l+1}. Set $E = \bigcup_{l=0}^{\infty} E_{l,l+1}$. It is easy to see that the resulting labeled Bratteli diagram with ι-map becomes a λ-graph system over Σ, that is denoted by $\mathfrak{L}^{\mathrm{Ch}(D_A)}$.

In the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$, we consider two λ-graph subsystems $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ and $\mathfrak{L}^{\mathrm{Ch}(D_A^-)}$. Both of the λ-graph subsystems have the same vertex sets as $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ together with the same ι-maps as $\mathfrak{L}^{\mathrm{Ch}(D_A)}$. The edge set of $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ consists of edges labeled Σ^+ in the edges of $\mathfrak{L}^{\mathrm{Ch}(D_A)}$, whereas that of $\mathfrak{L}^{\mathrm{Ch}(D_A^-)}$

consists of edges labeled Σ^-. Hence $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ and $\mathfrak{L}^{\mathrm{Ch}(D_A^-)}$ are λ-graph systems over Σ^+ and over Σ^- respectively. The latter λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A^-)}$ is called the word λ-graph system in [15]. Since the union of the edge sets of $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ and $\mathfrak{L}^{\mathrm{Ch}(D_A^-)}$ coincides with the edge set of $\mathfrak{L}^{\mathrm{Ch}(D_A)}$, we may write $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ as

$$\mathfrak{L}^{\mathrm{Ch}(D_A)} = \mathfrak{L}^{\mathrm{Ch}(D_A^+)} \sqcup \mathfrak{L}^{\mathrm{Ch}(D_A^-)}.$$

We will prove that $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ presents the subshift D_A.

LEMMA 2.2. *For $\gamma_1 \dots \gamma_k \in B_k(D_A)$ and $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$, if the word $\gamma_1 \dots \gamma_k \beta_{\mu_2} \dots \beta_{\mu_l}$ is admissible in D_A, so is the word $\gamma_1 \dots \gamma_k \alpha_{\mu_1} \beta_{\mu_1} \beta_{\mu_2} \dots \beta_{\mu_l}$.*

PROOF. As the word $\gamma_1 \dots \gamma_k \beta_{\mu_2} \dots \beta_{\mu_l}$ is admissible in D_A, one has

$$\varphi_A(\gamma_1) \dots \varphi_A(\gamma_k) t_{\mu_2} \dots t_{\mu_l} t_{\mu_l}^* \dots t_{\mu_2}^* \neq 0.$$

By the condition $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$ with the relations (2.2), one sees

$$t_{\mu_1}^* t_{\mu_1} t_{\mu_2} \dots t_{\mu_l} t_{\mu_l}^* \dots t_{\mu_2}^* = t_{\mu_2} \dots t_{\mu_l} t_{\mu_l}^* \dots t_{\mu_2}^*$$

so that

$$\varphi_A(\gamma_1) \dots \varphi_A(\gamma_k) t_{\mu_1}^* t_{\mu_1} t_{\mu_2} \dots t_{\mu_l} t_{\mu_l}^* \dots t_{\mu_2}^* \neq 0$$

and hence the word $\gamma_1 \dots \gamma_k \alpha_{\mu_1} \beta_{\mu_1} \beta_{\mu_2} \dots \beta_{\mu_l}$ is admissible in D_A.

For $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$ and $k \leq l$ we set

$$\Gamma_{D_A}^k(\beta_{\mu_1} \dots \beta_{\mu_l}) = \{\gamma_1 \dots \gamma_k \in B_k(D_A) \mid \gamma_1 \dots \gamma_k \beta_{\mu_1} \dots \beta_{\mu_l} \in B_{k+l}(D_A)\}$$

the k-predecessor set of the word $\beta_{\mu_1} \dots \beta_{\mu_l}$ in D_A and

$$\Gamma_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}^k(v_{N(\mu_1 \dots \mu_l)}^l)$$
$$= \big\{\gamma_1 \dots \gamma_k \in B_k(D_A) \mid \text{there exist } e_i \in E, i = 1, \dots, k \text{ such that } \gamma_i = \lambda(e_i) \text{ for } i = 1, \dots, k,\ t(e_i) = s(e_{i+1}) \text{ for } i = 1, \dots, k-1 \text{ and } t(e_k) = v_{N(\mu_1 \dots \mu_l)}^l\big\}$$

the k-predecessor set of the vertex $v_{N(\mu_1 \dots \mu_l)}^l$ in $\mathfrak{L}^{\mathrm{Ch}(D_A)}$.

LEMMA 2.3. $\Gamma_{D_A}^k(\beta_{\mu_1} \dots \beta_{\mu_l}) = \Gamma_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}^k(v_{N(\mu_1 \dots \mu_l)}^l)$.

PROOF. We will prove the desired equality by induction on the length k.
(1) Assume that k is 1.

For $\mu_0 \in \{1, \dots, N\}$, one sees $\alpha_{\mu_0} \in \Gamma_{D_A}^1(\beta_{\mu_1} \dots \beta_{\mu_l})$ if and only if $\mu_0 = \mu_1$, which is equivalent to $\alpha_{\mu_0} \in \Gamma_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}^1(v_{N(\mu_1 \dots \mu_l)}^l)$. Similarly $\beta_{\mu_0} \in$

$\Gamma^1_{D_A}(\beta_{\mu_1}\dots\beta_{\mu_l})$ if and only if $A(\mu_0,\mu_1)=1$, which is equivalent to $\beta_{\mu_0}\in\Gamma^1_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^l_{N(\mu_1\dots\mu_l)})$.

(2) Assume next that the desired equality holds for a fixed k with $k+1\le l$. For a word $\gamma_1\dots\gamma_{k+1}\in B_{k+1}(D_A)$, we have two cases.

Case 1: $\gamma_{k+1}=\alpha_{\mu_0}$ for some $\mu_0\in\{1,\dots,N\}$.

Assume $\gamma_1\dots\gamma_{k+1}\in\Gamma^{k+1}_{D_A}(\beta_{\mu_1}\dots\beta_{\mu_l})$ and hence $\gamma_1\dots\gamma_k\alpha_{\mu_0}\beta_{\mu_1}\dots\beta_{\mu_l}$ is admissible in D_A so that $\mu_0=\mu_1$. Since $t^*_{\mu_0}t_{\mu_0}$ is a projection in the algebra $\mathcal{O}_A$, the word $\gamma_1\dots\gamma_k\beta_{\mu_2}\dots\beta_{\mu_l}$ is admissible in D_A. Hence

$$\gamma_1\dots\gamma_k\in\Gamma^k_{D_A}(\beta_{\mu_2}\dots\beta_{\mu_l}).$$

By the hypothesis of induction, one has

$$\gamma_1\dots\gamma_k\in\Gamma^k_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^{l-1}_{N(\mu_2\dots\mu_l)}).$$

Since $\mu_1\mu_2\dots\mu_l$ is admissible in Λ_A, there exists an edge $e\in E_{l-1,l}$ in $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ such that $\lambda(e)=\alpha_{\mu_1}$ and $s(e)=v^{l-1}_{N(\mu_2\dots\mu_l)}$, $t(e)=v^l_{N(\mu_1\dots\mu_l)}$. Hence we know that

$$\gamma_1\dots\gamma_{k+1}\in\Gamma^{k+1}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^l_{N(\mu_1\dots\mu_l)}).$$

Conversely assume $\gamma_1\dots\gamma_{k+1}\in\Gamma^{k+1}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^l_{N(\mu_1\dots\mu_l)})$ so that $\mu_0=\mu_1$. Hence

$$\gamma_1\dots\gamma_k\in\Gamma^k_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^{l-1}_{N(\mu_2\dots\mu_l)}).$$

By the hypothesis of induction, the word $\gamma_1\dots\gamma_k\beta_{\mu_2}\dots\beta_{\mu_l}$ is admissible in D_A. By the preceding lemma, $\gamma_1\dots\gamma_k\alpha_{\mu_1}\beta_{\mu_1}\beta_{\mu_2}\dots\beta_{\mu_l}$ is admissible in D_A so that

$$\gamma_1\dots\gamma_{k+1}\in\Gamma^{k+1}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(\beta_{\mu_1}\dots\beta_{\mu_l}).$$

Case 2: $\gamma_{k+1}=\beta_{\mu_0}$ for some $\mu_0\in\{1,\dots,N\}$.

Assume $\gamma_1\dots\gamma_{k+1}\in\Gamma^{k+1}_{D_A}(\beta_{\mu_1}\dots\beta_{\mu_l})$. Then

$$\gamma_1\dots\gamma_k\in\Gamma^k_{D_A}(\beta_{\mu_0}\beta_{\mu_1}\dots\beta_{\mu_{l-2}}).$$

By the hypothesis of induction, we have

$$\gamma_1\dots\gamma_k\in\Gamma^k_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^{l-1}_{N(\mu_0\dots\mu_{l-2})}).$$

Since $\mu_0\mu_1\dots\mu_l$ is admissible in Λ_A, there exists an edge $e\in E_{l-1,l}$ in $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ such that $\lambda(e)=\beta_{\mu_0}$ and $s(e)=v^{l-1}_{N(\mu_0\dots\mu_{l-2})}$, $t(e)=v^l_{N(\mu_1\dots\mu_l)}$. Hence we know

$$\gamma_1\dots\gamma_{k+1}\in\Gamma^{k+1}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^l_{N(\mu_1\dots\mu_l)}).$$

Conversely assume $\gamma_1 \dots \gamma_{k+1} \in \Gamma^{k+1}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^l_{N(\mu_1\dots\mu_l)})$. Hence

$$\gamma_1 \dots \gamma_k \in \Gamma^k_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^{l-1}_{N(\mu_0\dots\mu_{l-2})}).$$

As $\Gamma^k_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^{l-1}_{N(\mu_0\dots\mu_{l-2})}) = \Gamma^k_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(v^{l+1}_{N(\mu_0\dots\mu_l)})$, by the hypothesis of induction

$$\gamma_1 \dots \gamma_k \in \Gamma^k_{D_A}(\beta_{\mu_0} \dots \beta_{\mu_l}).$$

Hence we have

$$\gamma_1 \dots \gamma_{k+1} \in \Gamma^{k+1}_{D_A}(\beta_{\mu_1} \dots \beta_{\mu_l}).$$

Therefore the desired equality holds for all k with $k \le l$.

PROPOSITION 2.4. *The λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ presents the subshift D_A.*

PROOF. Put $X_A = \{(\mu_i)_{i\in\mathbb{N}} \mid (\mu_i)_{i\in\mathbb{Z}} \in \Lambda_A\}$ and $X_{D_A} = \{(\gamma_i)_{i\in\mathbb{N}} \mid (\gamma_i)_{i\in\mathbb{Z}} \in D_A\}$. Let $\mathfrak{H}$ be the Hilbert space $\mathfrak{H}$ whose complete orthonormal basis are given by the vectors

$$e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots \qquad \text{for} \quad (\mu_1, \mu_2, \dots) \in X_A.$$

We faithfully represent $\mathcal{O}_A$ on $\mathfrak{H}$ by using the creation operators t_i, $i = 1, \dots, N$ defined by

$$t_i(e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots) = \begin{cases} e_i \otimes e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots & \text{if } A(i, \mu_1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We may identify $\varphi_A(\alpha_i)$ and $\varphi_A(\beta_i)$ with the operators t_i^* and t_i on $\mathfrak{H}$ respectively. For a word $\gamma_1 \dots \gamma_k \in \Sigma^*$, it follows that $\gamma_1 \dots \gamma_k$ is admissible in D_A if and only if there exists a sequence $(\mu_1, \mu_2, \dots) \in X_A$ such that $\varphi_A(\gamma_1) \dots \varphi_A(\gamma_k) e_{\mu_1} \otimes e_{\mu_2} \otimes \dots$ is a nonzero vector. The latter condition is equivalent to the condition $(\gamma_1, \dots, \gamma_k, \mu_1, \mu_2, \dots) \in X_{D_A}$. This is equivalent to the condition $\gamma_1 \dots \gamma_k \in \Gamma^k_{D_A}(\beta_{\mu_1}\beta_{\mu_2} \dots \beta_{\mu_l})$ for all $l \ge k$. Therefore by the preceding lemma, the subshift $\Lambda_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ presented by the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ is D_A.

We automatically know that the λ-graph systems $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ and $\mathfrak{L}^{\mathrm{Ch}(D_A^-)}$ present the subshifts D_A^+ and D_A^- respectively. A λ-graph system $\mathfrak{L}$ satisfies λ-condition (I) if for every vertex $v \in V_l$ of $\mathfrak{L}$ there exist at least two paths with distinct label sequences starting with the vertex v and terminating with a same vertex. $\mathfrak{L}$ is said to be λ-irreducible if for an ordered pair of vertices $u, v \in V_l$, there exists a number $L_l(u, v) \in \mathbb{N}$ such that for a vertex $w \in V_{l+L_l(u,v)}$ with $\iota^{L_l(u,v)}(w) = u$, there exists a path ξ in $\mathfrak{L}$ such that $s(\xi) = v$, $t(\xi) = w$, where $\iota^{L_l(u,v)}$ means the $L_l(u, v)$-times compositions of ι, and $s(\xi)$, $t(\xi)$ denote the source vertex, the terminal vertex of ξ respectively ([23]).

PROPOSITION 2.5. *Let A be an $N \times N$ matrix with entries in* $\{0, 1\}$.

(i) *If A satisfies condition* (I) *in the sense of [5], the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ satisfies λ-condition* (I).

(ii) *If A is irreducible, the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ is λ-irreducible.*

Hence if A is an irreducible matrix with condition (I)*, then both the λ-graph systems $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ and $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ are λ-irreducible with λ-condition* (I).

PROOF. (i) Suppose that A satisfies condition (I). In the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$, let v_i^l be a vertex in V_l. We write $i = N(i_1 \dots i_l)$ for $i_1 \dots i_l \in B_l(\Lambda_A)$. By condition (I) for A, there exist $\mu = \mu_1 \dots \mu_r,\ \nu = \nu_1 \dots \nu_r \in B_r(\Lambda_A)$ such that $\mu \neq \nu$, $\mu_1 = \nu_1 = i_l$ and $\mu_r = \nu_r$. Take $\eta_{r+1} \dots \eta_{2l+2r-1} \in B_{2l+r-1}(\Lambda_A)$ such that $\mu_r \eta_{r+1} \dots \eta_{2l+2r-1} \in B_{2l+r}(\Lambda_A)$. We put $\mu_n = \nu_n = \eta_n$ for $n = r+1, \dots, 2l+2r-1$ and $L' = 2l+2r-1$. Let $v_j^{L'} \in V_{L'}$ be the vertex in $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ such that $j = N(\mu_r \mu_{r+1} \dots \mu_{2l+2r-2}) (= N(\nu_r \nu_{r+1} \dots \nu_{2l+2r-2}))$. Then there exist two paths labeled $\beta_{i_1} \dots \beta_{i_l} \beta_{\mu_1} \dots \beta_{\mu_{r-1}}$ and $\beta_{i_1} \dots \beta_{i_l} \beta_{\nu_1} \dots \beta_{\nu_{r-1}}$ whose sources are both v_i^l and terminals are both $v_j^{L'}$. Hence $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ satisfies λ-condition (I).

(ii) In the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$, let v_i^l, v_j^l be vertices in V_l. We write $i = N(i_1 \dots i_l), j = N(j_1 \dots j_l)$ for $i_1 \dots i_l, j_1 \dots j_l \in B_l(\Lambda_A)$ respectively. As Λ_A is irreducible, there exists a word $\eta_1 \dots \eta_L \in B_L(\Lambda_A)$ such that $j_1 \dots j_l \eta_1 \dots \eta_L i_1 \dots i_l \in B_{2l+L}(\Lambda_A)$. We may assume $L \geq l$. For $v_h^{2l+L} \in V_{2l+L}$ with $\iota^{l+L}(v_h^{2l+L}) = v_i^l, h = 1, \dots, m(2l+L)$ we have $h = N(i_1 \dots i_l \mu_{l+1} \dots \mu_{2l+L})$ for some $\mu_{l+1} \dots \mu_{2l+L} \in B_{l+L}(\Lambda_A)$. Then there exists a path labeled $\beta_{j_1} \dots \beta_{j_l} \beta_{\eta_1} \dots \beta_{\eta_L}$ whose source is v_j^l and whose terminal is v_h^{2l+L}. This means that $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$ is λ-irreducible.

Therefore we have by [23]

THEOREM 2.6. *Let A be an $N \times N$ matrix with entries in* $\{0, 1\}$. *If A is an irreducible matrix with condition* (I)*, then the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ is simple and purely infinite.*

We note that the λ-graph systems $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ are examples of λ-synchronizing λ-graph systems for D_A introduced in [17].

3. The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$

This section is devoted to studying operator relations among generators of the algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ to prove Theorem 1.1. A general structure for the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a λ-graph system $\mathfrak{L}$ has been studied in [20]. For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, Let $\{v_1^l, \dots, v_{m(l)}^l\}$ be the vertex set V_l. We set for $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+1)$, $\gamma \in \Sigma$,

$$A_{l,l+1}(i,\gamma,j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \gamma, t(e) = v_j^{l+1} \\ & \text{for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.1 ([20, Theorem A and B], cf. [23]). *Suppose that a λ-graph system $\mathfrak{L}$ satisfies λ-condition* (I). *Then the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is the unique C^*-algebra generated by nonzero partial isometries s_γ, $\gamma \in \Sigma$ and nonzero projections e_i^l, $i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$ satisfying the following operator relations:*

$$\sum_{\gamma \in \Sigma} s_\gamma s_\gamma^* = 1, \tag{3.1}$$

$$\sum_{j=1}^{m(l)} e_j^l = 1, \qquad e_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) e_j^{l+1}, \tag{3.2}$$

$$s_\gamma s_\gamma^* e_i^l = e_i^l s_\gamma s_\gamma^*, \tag{3.3}$$

$$s_\gamma^* e_i^l s_\gamma = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\gamma,j) e_j^{l+1}, \tag{3.4}$$

for $i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$, $\gamma \in \Sigma$. If in particular $\mathfrak{L}$ is λ-irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite.

We first consider the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A^+)}}$ for the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A^+)}$.

PROPOSITION 3.2. *Suppose that A satisfies condition* (I). *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A^+)}}$ is canonically isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$.*

PROOF. Both the algebras $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A^+)}}$ and $\mathcal{O}_A$ are uniquely determined by certain operator relations of their canonical generators. We write the canonical generating partial isometries and the projections in $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A^+)}}$ as s_{β_i}, $i = 1, \dots, N$ and $e^l_{N(i_1\dots i_l)}$, $i_1 \dots i_l \in B_l(\Lambda_A)$, $l \in \mathbb{Z}_+$ respectively. By (3.1), (3.3) and (3.4), one has

$$e^l_{N(i_1\dots i_l)} = \sum_{i_{l+1}, i_{l+2}=1}^{N} s_{\beta_{i_1}} e^{l+1}_{N(i_2\dots i_{l+1} i_{l+2})} s^*_{\beta_{i_1}}.$$

For $l = 1$, one sees that by (3.2)

$$e^1_{N(i_1)} = \sum_{i_2,i_3=1}^{N} s_{\beta_{i_1}} e^2_{N(i_2 i_3)} s^*_{\beta_{i_1}} = s_{\beta_{i_1}} s^*_{\beta_{i_1}}.$$

As $\iota^2(v^{l+1}_{N(i_2\ldots i_{l+1}i_{l+2})}) = v^{l-1}_{N(i_2\ldots i_l)}$, (3.2) implies the equality

$$\sum_{i_{l+1},i_{l+2}=1}^{N} e^{l+1}_{N(i_2\ldots i_{l+1}i_{l+2})} = e^{l-1}_{N(i_2\ldots i_l)}$$

so that by induction one obtains

$$e^l_{N(i_1\ldots i_l)} = s_{\beta_{i_1}} \ldots s_{\beta_{i_l}} s^*_{\beta_{i_l}} \ldots s^*_{\beta_{i_1}}.$$

One also sees that (3.4) implies the equality

$$s^*_{\beta_i} s_{\beta_i} = \sum_{j=1}^{N} A(i, j) s_{\beta_j} s^*_{\beta_j}.$$

As the equality $\sum_{i=1}^{N} s_{\beta_i} s^*_{\beta_i} = 1$ holds, the C^*-algebra generated by partial isometries s_{β_i}, $i = 1, \ldots, N$ is canonically isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$.

In what follows, an $N \times N$ matrix A is assumed to be irreducible with entries in $\{0, 1\}$, and satisfy condition (I). We will describe concrete operator relations among the canonical generators of the algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$. Let $A_{l,l+1}$, $I_{l,l+1}$ be the matrices as in Lemma 3.1 for the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$. We denote by $m(l)$ the number of the vertex set $V_l = \{v^l_1, \ldots, v^l_{m(l)}\}$ of $\mathfrak{L}^{\mathrm{Ch}(D_A)}$. Let s_γ, $\gamma \in \Sigma$ and e^l_i, $i = 1, \ldots, m(l)$, $l \in \mathsf{Z}_+$ be the canonical generating partial isometries and projections of $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$. They satisfy the relations (3.1), (3.2), (3.3) and (3.4) for $\mathfrak{L}^{\mathrm{Ch}(D_A)}$. Define the operators S_i, , T_i, $i = 1, \ldots, N$ by setting

$$S_i := s_{\alpha_i}, \quad T_i := s_{\beta_i} \qquad \text{for} \quad i = 1, \ldots, N.$$

PROPOSITION 3.3. *The operators S_i, T_i, $i = 1, \ldots, N$ satisfy the relations* (1.3)*,* (1.4)*,* (1.5) *and* (1.6)*, and generate the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$.*

PROOF. The equality (1.3) is nothing but (3.1). To prove (1.4), by the equality (3.4) and the first equality of (3.2), one has for a fixed $l \in \mathsf{Z}_+$,

$$\sum_{j=1}^{N} S^*_j S_j = \sum_{j=1}^{N} \sum_{i=1}^{m(l)} \sum_{k=1}^{m(l+1)} A_{l,l+1}(i, \alpha_j, k) e^{l+1}_k.$$

For $k = 1, \dots, m(l+1)$, there exists a unique edge in $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ labeled a symbol in Σ^- whose terminal is v_k^{l+1}. Hence we have $\sum_{j=1}^N \sum_{i=1}^{m(l)} A_{l,l+1}(i, \alpha_j, k) = 1$ so that

$$\sum_{j=1}^{N} S_j^* S_j = \sum_{k=1}^{m(l+1)} e_k^{l+1} = 1.$$

For (1.5), one similarly has

$$T_i^* T_i = \sum_{k=1}^{m(l)} s_{\beta_i}^* e_k^l s_{\beta_i} = \sum_{h=1}^{m(l+1)} \sum_{k=1}^{m(l)} A_{l,l+1}(k, \beta_i, h) e_h^{l+1}.$$

On the other hand,

$$\sum_{j=1}^{N} A(i, j) S_j^* S_j = \sum_{h=1}^{m(l+1)} \left(\sum_{k=1}^{m(l)} \sum_{j=1}^{N} A(i, j) A_{l,l+1}(k, \alpha_j, h) \right) e_h^{l+1}.$$

Let h be written as $N(h_1 \dots h_{l+1})$. Then the condition $A_{l,l+1}(k, \beta_i, h) = 1$ is equivalent to the condition that $ih_1 \in B_2(\Lambda_A)$ and $k = N(ih_1 \dots h_{l-1})$. On the other hand, the condition $\sum_{j=1}^N A(i, j) A_{l,l+1}(k, \alpha_j, h) = 1$ is equivalent to the condition that $j = h_1$, $A(i, j) = 1$ for some j and $k = N(h_2 \dots h_{l+1})$. Hence one has

$$\sum_{k=1}^{m(l)} A_{l,l+1}(k, \beta_i, h) = \sum_{k=1}^{m(l)} \sum_{j=1}^{N} A(i, j) A_{l,l+1}(k, \alpha_j, h).$$

This implies the equality (1.5). For (1.6), we put

$$E_{\mu_1 \dots \mu_k} = S_{\mu_1}^* \dots S_{\mu_k}^* S_{\mu_k} \dots S_{\mu_1}.$$

By using the first equality of (3.2), (3.3) and (3.4) recursively, $E_{\mu_1 \dots \mu_k}$ commutes with $S_j S_j^*$ and $T_j T_j^*$ for $j = 1, \dots, N$, so that by (1.3)

$$E_{\mu_1 \dots \mu_k} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1 \dots \mu_k} S_j S_j^* + \sum_{j=1}^{N} T_j T_j^* E_{\mu_1 \dots \mu_k} T_j T_j^*.$$

As $S_j^* E_{\mu_1 \dots \mu_k} S_j = A(j, \mu_1) S_j^* E_{\mu_1 \dots \mu_k} S_j$ and $S_{\mu_1} T_j = s_{\alpha_{\mu_1}} s_{\beta_j} = 0$ if $\mu_1 \neq j$, one has

$$E_{\mu_1 \dots \mu_k} = \sum_{j=1}^{N} A(j, \mu_1) S_j S_j^* E_{\mu_1 \dots \mu_k} S_j S_j^* + T_{\mu_1} T_{\mu_1}^* E_{\mu_1 \dots \mu_k} T_{\mu_1} T_{\mu_1}^*.$$

Since $A_{0,1}(1, \alpha_{\mu_1}, j) = 1$ if and only if $j = \mu_1$, it follows that by (3.4),

$$E_{\mu_1} = s^*_{\alpha_{\mu_1}} s_{\alpha_{\mu_1}} = \sum_{j=1}^{m(1)} A_{0,1}(1, \alpha_{\mu_1}, j) e^1_j = e^1_{\mu_1}.$$

By (3.2) and (3.4), we similarly have

$$\begin{aligned} E_{\mu_1 \dots \mu_k} &= s^*_{\alpha_{\mu_1}} \dots s^*_{\alpha_{\mu_k}} s_{\alpha_{\mu_k}} \dots s_{\alpha_{\mu_1}} \\ &= \sum_{i_1=1}^{m(1)} \dots \sum_{i_k=1}^{m(k)} A_{0,1}(1, \alpha_{\mu_k}, i_1) \dots A_{k-1,k}(i_{k-1}, \alpha_{\mu_1}, i_k) e^k_{i_k}. \end{aligned}$$

As $\sum_{i_1=1}^{m(1)} \dots \sum_{i_{k-1}=1}^{m(k-1)} A_{0,1}(1, \alpha_{\mu_k}, i_1) \dots A_{k-1,k}(i_{k-1}, \alpha_{\mu_1}, i_k) = 1$ if and only if $i_k = N(\mu_1 \dots \mu_k)$, one knows $E_{\mu_1 \dots \mu_k} = e^k_{N(\mu_1 \dots \mu_k)}$. Hence we have

$$T^*_{\mu_1} E_{\mu_1 \dots \mu_k} T_{\mu_1} = \sum_{j=1}^{m(k+1)} A_{k,k+1}(N(\mu_1 \dots \mu_k), \beta_{\mu_1}, j) e^{k+1}_j.$$

Since $A_{k,k+1}(N(\mu_1 \dots \mu_k), \beta_{\mu_1}, j) = 1$ if and only if $j = N(\mu_2 \dots \mu_k \mu_{k+1} \mu_{k+2})$ for some $\mu_{k+1}, \mu_{k+2} = 1, \dots, N$, and the equality

$$\sum_{\mu_{k+1}, \mu_{k+2} = 1, \dots, N} E_{\mu_2 \dots \mu_k \mu_{k+1} \mu_{k+2}} = E_{\mu_2 \dots \mu_k}$$

holds, we have $T^*_{\mu_1} E_{\mu_1 \dots \mu_k} T_{\mu_1} = E_{\mu_2 \dots \mu_k}$. Thus we conclude that (1.6) holds. Consequently the operators $S_i, T_i, i = 1, \dots, N$ satisfy the relations (1.3), (1.4), (1.5) and (1.6).

In the above discussions, we have proved the equality

$$e^k_{N(\mu_1 \dots \mu_k)} = E_{\mu_1 \dots \mu_k} (= S^*_{\mu_1} \dots S^*_{\mu_k} S_{\mu_k} \dots S_{\mu_1})$$

for $\mu_1 \dots \mu_k \in B_k(\Lambda_A)$. Hence $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ is generated by $S_1, \dots, S_N, T_1, \dots, T_N$.

We next show that the relations (1.3), (1.4), (1.5) and (1.6) imply the relations (3.1), (3.2), (3.3) and (3.4). Let $S_i, T_i, i = 1, \dots, N$ be partial isometries satisfying the relations (1.3), (1.4), (1.5) and (1.6). In the relation (1.6) for $k = 2$, by summing up μ_2 over $\{1, \dots, N\}$ and using (1.4), we have

$$S^*_i S_i = \sum_{j=1}^{N} A(j, i) S_j S^*_j S^*_i S_i S_j S^*_j + T_i T^*_i, \qquad i = 1, \dots, N. \tag{3.5}$$

LEMMA 3.4.
(i) $T_i^* S_j^* S_j T_i = \begin{cases} T_i^* T_i & \textit{if } i = j, \\ 0 & \textit{otherwise.} \end{cases}$

(ii) $T_i^* E_{\mu_1 \dots \mu_l} T_i = \begin{cases} A(i, \mu_2) E_{\mu_2 \dots \mu_l} & \textit{if } i = \mu_1, \\ 0 & \textit{otherwise} \end{cases}$

for $l > 1$, *where* $E_{\mu_1 \dots \mu_l} = S_{\mu_1}^* \dots S_{\mu_l}^* S_{\mu_l} \dots S_{\mu_1}$ *for* $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$.

PROOF. (i) By (3.5), we have

$$T_i^* S_i^* S_i T_i = \sum_{j=1}^{N} A(j, i) T_i^* S_j S_j^* S_i^* S_i S_j S_j^* T_i + T_i^* T_i T_i^* T_i.$$

The equality (1.3) implies $T_i^* S_j = 0$ for $i, j = 1, \dots, N$ and hence we have $T_i^* S_i^* S_i T_i = T_i^* T_i$. By (1.4), one has

$$\sum_{j=1}^{N} T_i^* S_j^* S_j T_i = T_i^* T_i$$

so that $T_i^* S_j^* S_j T_i = 0$ for $i \neq j$.

(ii) By (1.6), we have

$$T_i^* E_{\mu_1 \dots \mu_l} T_i = \sum_{j=1}^{N} A(j, \mu_1) T_i^* S_j S_j^* E_{\mu_1 \dots \mu_l} S_j S_j^* T_i + T_i^* T_{\mu_1} E_{\mu_2 \dots \mu_l} T_{\mu_1}^* T_i$$

for $l > 1$. Since $T_i^* S_j = 0$ for $i, j = 1, \dots, N$ and $T_i^* T_{\mu_1} = 0$ for $i \neq \mu_1$, we have

$$T_i^* E_{\mu_1 \dots \mu_l} T_i = T_i^* T_{\mu_1} E_{\mu_2 \dots \mu_l} T_{\mu_1}^* T_i = \begin{cases} T_i^* T_i E_{\mu_2 \dots \mu_l} T_i^* T_i & \text{if } i = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

By (1.5) one has

$$T_i^* T_i E_{\mu_2 \dots \mu_l} T_i^* T_i = \sum_{j=1}^{N} \sum_{k=1}^{N} A(i, j) A(i, k) S_j^* S_j S_{\mu_2}^* \dots S_{\mu_l}^* S_{\mu_l} \dots S_{\mu_2} S_k^* S_k.$$

By (1.4), one sees that $S_j^* S_j S_{\mu_2}^* = 0$ for $j \neq \mu_2$, and $S_{\mu_2}^* S_{\mu_2} S_k^* S_k = 0$ for $k \neq \mu_2$. It then follows that

$$T_i^* E_{\mu_1 \dots \mu_l} T_i = A(i, \mu_2) E_{\mu_2 \dots \mu_l}.$$

LEMMA 3.5. *Keep the above notations. The projection* $E_{\mu_1 \dots \mu_l}$ *commutes with both* $S_j S_j^*$ *and* $T_j T_j^*$.

PROOF. By (1.6), we have for $l > 1$

$$S_i S_i^* E_{\mu_1 \dots \mu_l} = \sum_{j=1}^{N} A(j, \mu_1) S_i S_i^* S_j S_j^* E_{\mu_1 \dots \mu_l} S_j S_j^* + S_i S_i^* T_{\mu_1} E_{\mu_2 \dots \mu_l} T_{\mu_1}^*.$$

By (1.3), one has $S_i^* T_{\mu_1} = 0$ for all i, μ_1, and $S_i^* S_j = 0$ for $i \neq j$. Hence $S_i S_i^* E_{\mu_1 \dots \mu_l} = A(i, \mu_1) S_i S_i^* E_{\mu_1 \dots \mu_l} S_i S_i^*$ and similarly $E_{\mu_1 \dots \mu_l} S_i S_i^* = A(i, \mu_1) S_i S_i^* E_{\mu_1 \dots \mu_l} S_i S_i^*$ so that $S_i S_i^*$ commutes with $E_{\mu_1 \dots \mu_l}$. By (1.6) and (1.3), we have

$$\begin{aligned} T_i T_i^* E_{\mu_1 \dots \mu_l} &= \sum_{j=1}^{N} A(j, \mu_1) T_i T_i^* S_j S_j^* E_{\mu_1 \dots \mu_l} S_j S_j^* + T_i T_i^* T_{\mu_1} E_{\mu_2 \dots \mu_l} T_{\mu_1}^* \\ &= \begin{cases} T_{\mu_1} E_{\mu_2 \dots \mu_l} T_{\mu_1}^* & \text{if } i = \mu_1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We similarly have the same equality for $E_{\mu_1 \dots \mu_l} T_i T_i^*$ as above so that $T_i T_i^*$ commutes with $E_{\mu_1 \dots \mu_l}$. For $l = 1$, the equality $E_{\mu_1} = \sum_{\mu_2=1}^{N} E_{\mu_1 \mu_2}$ from (1.4) implies that E_{μ_1} commutes with both $S_j S_j^*$ and $T_j T_j^*$ by the above discussions.

LEMMA 3.6. *Keep the above notations. For* $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$ *we have* $E_{\mu_1 \dots \mu_l} = 0$ *if* $\mu_1 \dots \mu_l \notin B_l(\Lambda_A)$.

PROOF. As we are assuming that the matrix A has no zero rows or columns, one sees $B_1(\Lambda_A) = \{1, \dots, N\}$. By (3.5) one has for $\mu_1 = 1, \dots, N$

$$S_{\mu_1}^* S_{\mu_1} = \sum_{j=1}^{N} A(j, \mu_1) S_j S_j^* S_{\mu_1}^* S_{\mu_1} S_j S_j^* + T_{\mu_1} T_{\mu_1}^*$$

so that for $\mu_0 = 1, \dots, N$

$$S_{\mu_0}^* S_{\mu_1}^* S_{\mu_1} S_{\mu_0} = A(\mu_0, \mu_1) S_{\mu_0}^* S_{\mu_1}^* S_{\mu_1} S_{\mu_0}$$

because $S_{\mu_0}^* S_j = 0$ if $\mu_0 \neq j$, and $S_{\mu_0}^* T_{\mu_1} = 0$ by (1.3). This means that $E_{\mu_0 \mu_1} = 0$ if $\mu_0 \mu_1 \notin B_2(\Lambda_A)$.

Suppose next that the assertion holds for $l = k > 1$. By (1.6) one has for $\mu_1 \dots \mu_k \in B_k(\Lambda_A)$ and $\mu_0 = 1, \dots, N$

$$S_{\mu_0}^* E_{\mu_1 \dots \mu_k} S_{\mu_0} = \sum_{j=1}^{N} A(j, \mu_1) S_{\mu_0}^* S_j S_j^* E_{\mu_1 \dots \mu_k} S_j S_j^* S_{\mu_0} + S_{\mu_0}^* T_{\mu_1} E_{\mu_2 \dots \mu_k} T_{\mu_1}^* S_{\mu_0}$$

so that we have

$$E_{\mu_0 \mu_1 \dots \mu_k} = A(\mu_0, \mu_1) E_{\mu_0 \mu_1 \dots \mu_k}.$$

For $\mu_1 \dots \mu_k \in B_k(\Lambda_A)$, we have $\mu_0\mu_1 \dots \mu_k \notin B_{k+1}(\Lambda_A)$ if and only if $A(\mu_0, \mu_1) = 0$. Hence the assertion holds for $l = k + 1$, so that it holds for all $l \in \mathsf{Z}_+$.

PROPOSITION 3.7. *Keep the above notations. Put*

$$\begin{aligned} s_{\alpha_i} &:= S_i, \quad s_{\beta_i} := T_i \qquad \textit{for} \quad i = 1, \dots, N, \quad \textit{and} \\ e^0_1 &:= 1, \\ e^l_{N(\mu_1\dots\mu_l)} &:= E_{\mu_1\dots\mu_l} (= S^*_{\mu_1} \dots S^*_{\mu_l} S_{\mu_l} \dots S_{\mu_1}) \qquad \textit{for} \quad \mu_1 \dots \mu_l \in B_l(\Lambda_A). \end{aligned}$$

Then the family of operators s_γ, $\gamma \in \Sigma$, $e^l_{N(\mu_1\dots\mu_l)}$, $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$ *satisfies the relations* (3.1), (3.2), (3.3) *and* (3.4) *for the λ-graph system* $\mathfrak{L}^{\mathrm{Ch}(D_A)}$.

PROOF. The relation (3.1) is nothing but the equality (1.3). The equality (1.4) implies $\sum_{\mu_1 \in B_1(\Lambda_A)} e^1_{N(\mu_1)} = 1$. Suppose that $\sum_{\mu_1\dots\mu_l \in B_l(\Lambda_A)} e^l_{N(\mu_1\dots\mu_l)} = 1$ holds for $l = k$. As

$$S^*_{\mu_1} \dots S^*_{\mu_k} S_{\mu_k} \dots S_{\mu_1} = \sum_{h=1}^{N} S^*_{\mu_1} \dots S^*_{\mu_k} S^*_h S_h S_{\mu_k} \dots S_{\mu_1},$$

the equality $\sum_{\mu_1\dots\mu_l \in B_l(\Lambda_A)} e^l_{N(\mu_1\dots\mu_l)} = 1$ holds for $l = k + 1$ by Lemma 3.6 and hence for all l. The above equality with the equality

$$I_{l,l+1}(N(\mu_1 \dots \mu_l), N(\nu_1 \dots \nu_{l+1})) = \begin{cases} 1 & \text{if } \nu_1 \dots \nu_l = \mu_1 \dots \mu_l, \\ 0 & \text{otherwise} \end{cases}$$

for $\nu_1 \dots \nu_{l+1} \in B_{l+1}(\Lambda_A)$ implies the second relation of (3.2) by using Lemma 3.6. The equality (3.3) comes from Lemma 3.5.

We will finally show the equality (3.4). For $l = 0$, one has $e^0_1 = 1$ by definition. If $\gamma = \alpha_k$ for some $k = 1, \dots, N$, one has $A_{0,1}(1, \alpha_k, j) = 1$ if and only if $j = k$. Hence

$$s^*_{\alpha_k} e^0_1 s_{\alpha_k} = s^*_{\alpha_k} s_{\alpha_k} = e^1_k = \sum_{j=1}^{m(1)} A_{0,1}(1, \alpha_k, j) e^1_j.$$

If $\gamma = \beta_k$ for some $k = 1, \dots, N$, one has $A_{0,1}(1, \beta_k, j) = A(k, j)$. Hence by the relation (1.5) one has

$$s^*_{\beta_k} e^0_1 s_{\beta_k} = T^*_k T_k = e^1_k = \sum_{j=1}^{N} A(k, j) S^*_j S_j = \sum_{j=1}^{N} A_{0,1}(1, \beta_k, j) e^1_j.$$

For $l = 1$, one sees that $e_i^1 = e_{N(i)}^1$. If $\gamma = \alpha_k$ for some $k = 1, \dots, N$, as $A_{1,2}(i, \alpha_k, j) = 1$ if and only if $j = N(ki)$, one has

$$s_{\alpha_k}^* e_i^1 s_{\alpha_k} = S_k^* S_i^* S_i S_k = e_{N(ki)}^2 = \sum_{j=1}^{m(2)} A_{1,2}(i, \alpha_k, j) e_j^2.$$

If $\gamma = \beta_k$ for some $k = 1, \dots, N$, one has by Lemma 3.4(i) and (1.5)

$$s_{\beta_k}^* e_i^1 s_{\beta_k} = T_k^* S_i^* S_i T_k = \begin{cases} \sum_{j=1}^{N} A(i, j) S_j^* S_j & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

By Lemma 3.6 and (1.4), one has

$$\sum_{j=1}^{N} A(i, j) S_j^* S_j = \sum_{\mu_1\mu_2 \in B_2(\Lambda_A)} A(i, \mu_1) A(\mu_1, \mu_2) S_{\mu_1}^* S_{\mu_2}^* S_{\mu_2} S_{\mu_1}.$$

Since $A_{1,2}(i, \beta_k, N(\mu_1\mu_2)) = 1$ if and only if $k = i$, $A(i, \mu_1) = A(\mu_1, \mu_2) = 1$, it follows that by $S_{\mu_1}^* S_{\mu_2}^* S_{\mu_2} S_{\mu_1} = e_{N(\mu_1\mu_2)}^2$,

$$s_{\beta_k}^* e_i^1 s_{\beta_k} = \sum_{j=1}^{m(2)} A_{1,2}(i, \beta_k, j) e_j^2.$$

For $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$ with $l > 1$ and $\alpha_k \in \Sigma^-$, the relation (1.6) implies

$$\begin{aligned} s_{\alpha_k}^* e_{N(\mu_1\dots\mu_l)}^l s_{\alpha_k} &= A(k, \mu_1) S_k^* S_{\mu_1}^* \dots S_{\mu_l}^* S_{\mu_l} \dots S_{\mu_1} S_k \\ &= A_{l,l+1}(N(\mu_1 \dots \mu_l), \alpha_k, N(k\mu_1 \dots \mu_l)) e_{N(k\mu_1\dots\mu_l)}^{l+1}. \end{aligned}$$

Since $A_{l,l+1}(N(\mu_1 \dots \mu_l), \alpha_k, i) = 0$ if $i \neq N(k\mu_1 \dots \mu_l)$, one has

$$\begin{aligned} & s_{\alpha_k}^* e_{N(\mu_1\dots\mu_l)}^l s_{\alpha_k} \\ & \quad = \sum_{\nu_1\dots\nu_{l+1} \in B_{l+1}(\Lambda_A)} A_{l,l+1}(N(\mu_1 \dots \mu_l), \alpha_k, N(\nu_1 \dots \nu_{l+1})) e_{N(\nu_1\dots\nu_{l+1})}^{l+1}. \end{aligned}$$

We also have by Lemma 3.4 for $j = \mu_1$

$$\begin{aligned} & s_{\beta_j}^* e_{N(\mu_1\dots\mu_l)}^l s_{\beta_j} \\ & \quad = T_j^* E_{\mu_1\dots\mu_l} T_j \\ & \quad = A(j, \mu_2) E_{\mu_2\dots\mu_l} \\ & \quad = \sum_{\mu_{l+1}\mu_{l+2} \in B_2(\Lambda_A)} A(j, \mu_2) S_{\mu_2}^* \dots S_{\mu_l}^* S_{\mu_{l+1}}^* S_{\mu_{l+2}}^* S_{\mu_{l+2}} S_{\mu_{l+1}} S_{\mu_l} \dots S_{\mu_2} \end{aligned}$$

and for $j \neq \mu_1$
$$s_{\beta_j}^* e_{N(\mu_1\dots\mu_l)}^l s_{\beta_j} = T_j^* E_{\mu_1\dots\mu_l} T_j = 0.$$

Since one has

$$\begin{aligned} &A_{l,l+1}(N(\mu_1 \dots \mu_l), \beta_j, N(\nu_1 \dots \nu_{l+1})) \\ &\qquad = \begin{cases} 1 & \text{if } j = \mu_1,\ A(j, \mu_2) = 1 \text{ and } \nu_i = \mu_{i+1} \text{ for } i = 1, \dots, l-1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

we have

$$\begin{aligned} &s_{\beta_j}^* e_{N(\mu_1\dots\mu_l)}^l s_{\beta_j} \\ &\qquad = \sum_{\nu_1\dots\nu_{l+1} \in B_{l+1}(\Lambda_A)} A_{l,l+1}(N(\mu_1 \dots \mu_l), \beta_j, N(\nu_1 \dots \nu_{l+1})) e_{N(\nu_1\dots\nu_{l+1})}^{l+1}. \end{aligned}$$

Therefore (3.4) holds.

PROOF OF THEOREM 1.1. By a general theory of the C^*-algebras associated with λ-graph systems [20], the algebras $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ are nuclear. By Proposition 3.3 and Proposition 3.7, the family of the operator relations (1.3), (1.4), (1.5) and (1.6) is equivalent to the family of the operator relations (3.1), (3.2), (3.3) and (3.4). Thus by Lemma 3.1 and Theorem 2.6, we conclude Theorem 1.1.

4. K-Theory

In this section, we will present K-theory formulae of the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ in terms of the topological Markov shift defined by the matrix A. We will prove Theorem 1.2. Recall that the right one-sided topological Markov shift X_A for the matrix A is naturally identified with $X_{D_A^+}$ as in the proof of Proposition 2.1. Let $S_i, T_i, i = 1, \dots, N$ be the generating partial isometries of the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ as in Theorem 1.1. Let $\mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ be the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ generated by the projections $E_{\mu_1\dots\mu_l} = S_{\mu_1}^* \dots S_{\mu_l}^* S_{\mu_l} \dots S_{\mu_1}$, $\mu_1 \dots \mu_l \in B_l(\Lambda_A)$, $l \in \mathsf{Z}_+$. Define two endomorphisms λ_{Σ^-} and λ_{Σ^+} on it by

$$\lambda_{\Sigma^-}(a) = \sum_{j=1}^N S_j^* a S_j, \quad \lambda_{\Sigma^+}(a) = \sum_{j=1}^N T_j^* a T_j \qquad \text{for} \quad a \in \mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$$

Let $C(X_A, \mathsf{C})$ be the abelian C^*-algebra of all C-valued continuous functions on X_A. We note that its K_0-group $K_0(C(X_A, \mathsf{C}))$ is naturally identified with $C(X_A, \mathsf{Z})$.

LEMMA 4.1. *Let* $\Phi : \mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}} \longrightarrow C(X_A, \mathsf{C})$ *be a map defined by*

$$\Phi(E_{\mu_1\dots\mu_l}) = \chi_{\mu_1\dots\mu_l} \qquad \textit{for} \quad \mu_1 \dots \mu_l \in B_l(\Lambda_A), l \in \mathsf{Z}_+$$

where $\chi_{\mu_1\dots\mu_l}$ *is the characteristic function for the word* $\mu_1 \dots \mu_l$ *on* X_A *defined by*

$$\chi_{\mu_1\dots\mu_l}((x_i)_{i\in\mathsf{N}}) = \begin{cases} 1 & \text{if } (x_1,\dots,x_l) = (\mu_1,\dots,\mu_l), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

(i) Φ *gives rise to an isomorphism from* $\mathscr{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ *onto* $C(X_A, \mathsf{C})$.

(ii) *Both of the diagrams*

$$\begin{array}{ccc} K_0(\mathscr{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) & \xrightarrow{\Phi_*} & C(X_A,\mathsf{Z}) \\ {\scriptstyle \lambda_{\Sigma_*^-}}\downarrow & & \downarrow{\scriptstyle \sigma_{\Lambda_A}} \\ K_0(\mathscr{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) & \xrightarrow{\Phi_*} & C(X_A,\mathsf{Z}) \end{array} \quad , \quad \begin{array}{ccc} K_0(\mathscr{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) & \xrightarrow{\Phi_*} & C(X_A,\mathsf{Z}) \\ {\scriptstyle \lambda_{\Sigma_*^+}}\downarrow & & \downarrow{\scriptstyle \lambda_{\Lambda_A}} \\ K_0(\mathscr{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) & \xrightarrow{\Phi_*} & C(X_A,\mathsf{Z}) \end{array}$$

are commutative, where Φ_* *is the induced isomorphism from* $K_0(\mathscr{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}})$ *to* $K_0(C(X_A,\mathsf{C}))(= C(X_A,\mathsf{Z}))$, *and* $\lambda_{\Sigma_*^-}$, $\lambda_{\Sigma_*^+}$ *are induced endomorphisms on* $K_0(\mathscr{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}})$ *by* λ_{Σ^-}, λ_{Σ^+} *respectively.*

PROOF. (i) The assertion is straightforward.
(ii)The equality

$$\Phi(\lambda_{\Sigma^-}(E_{\mu_1\dots\mu_l})) = \sum_{j=1}^{N} \chi_{j\mu_1\dots\mu_l}$$

is immediate. As

$$\sigma_{\Lambda_A}(\chi_{\mu_1\dots\mu_l})(x) = \begin{cases} 1 & \text{if } (x_2,\dots,x_{l+1}) = (\mu_1,\dots,\mu_l), \\ 0 & \text{otherwise,} \end{cases}$$

for $x = (x_i)_{i\in\mathsf{N}} \in X_A$, the equality

$$\sigma_{\Lambda_A}(\chi_{\mu_1\dots\mu_l}) = \Phi(\lambda_{\Sigma^-}(E_{\mu_1\dots\mu_l}))$$

is clear. Hence the first diagram is commutative.

For the second diagram, as

$$T_j^* E_{\mu_1\dots\mu_l} T_j = \begin{cases} A(j,\mu_2)E_{\mu_2\dots\mu_l} & \text{if } j = \mu_1, \\ 0 & \text{if } j \neq \mu_1 \end{cases}$$

by Lemma 3.4, it follows that

$$\begin{aligned} \Phi(T_j^* E_{\mu_1\dots\mu_l} T_j)(x) &= \begin{cases} A(j,\mu_2)\chi_{\mu_2\dots\mu_l}(x) & \text{if } j = \mu_1, \\ 0 & \text{if } j \neq \mu_1 \end{cases} \\ &= \begin{cases} 1 & \text{if } (\mu_1,\dots,\mu_l) = (j, x_1, x_2, \dots, x_{l-1}), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, one sees for $x = (x_i)_{i\in\mathsf{N}} \in X_A$

$$\lambda_{\Lambda_A}(\chi_{\mu_1\dots\mu_l})(x) = \sum_{j=1}^{N} \chi_{\mu_1\dots\mu_l}(jx)$$
$$= \begin{cases} 1 & \text{if } (\mu_1,\dots,\mu_l) = (j, x_1, x_2, \dots, x_{l-1}) \\ & \text{for some } j = 1,\dots,N \\ 0 & \text{otherwise} \end{cases}$$

so that one obtains

$$\lambda_{\Lambda_A}(\chi_{\mu_1\dots\mu_l}) = \sum_{j=1}^{N} \Phi(T_j^* E_{\mu_1\dots\mu_l} T_j) = \Phi(\lambda_{\Sigma^+}(E_{\mu_1\dots\mu_l})).$$

Hence the second diagram is commutative.

Therefore we have

THEOREM 4.2.
(i) $K_0(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) = C(X_A, \mathsf{Z})/(\mathrm{id} - (\sigma_{\Lambda_A} + \lambda_{\Lambda_A}))C(X_A, \mathsf{Z})$.
(ii) $K_1(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) = \mathrm{Ker}(\mathrm{id} - (\sigma_{\Lambda_A} + \lambda_{\Lambda_A}))$ *in* $C(X_A, \mathsf{Z})$.

PROOF. By discussions in [20, Theorem 5.5], one knows

$$K_0(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) = K_0(\mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}})/(\mathrm{id} - \lambda_{\mathfrak{L}^{\mathrm{Ch}(D_A)}*})K_0(\mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}),$$
$$K_1(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}) = \mathrm{Ker}(\mathrm{id} - \lambda_{\mathfrak{L}^{\mathrm{Ch}(D_A)}*}) \text{ in } K_0(\mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}})$$

where $\lambda_{\mathfrak{L}^{\mathrm{Ch}(D_A)}*}$ is an endomorphism on $K_0(\mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}})$ induced by the map $\lambda_{\mathfrak{L}^{\mathrm{Ch}(D_A)}} : \mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}} \to \mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$ defined by

$$\lambda_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(a) = \sum_{\gamma\in\Sigma^-\cup\Sigma^+} S_\gamma^* a S_\gamma \qquad \text{for} \quad a \in \mathcal{A}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}.$$

As $\lambda_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}(a) = \lambda_{\Sigma^-}(a) + \lambda_{\Sigma^+}(a)$, one sees the desired formulae by the previous lemma.

5. Examples

EXAMPLE 1 (Dyck shifts). For the matrix A all of whose entries are 1's, Theorem 1.1 goes to:

PROPOSITION 5.1 ([24]). *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}$ associated with the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ for the Dyck shift D_N is unital, separable, nuclear, simple and purely infinite. It is the unique C^*-algebra generated by N*

partial isometries S_i, $i = 1, \dots, N$ and N isometries T_i, $i = 1, \dots, N$ subject to the following operator relations:

$$\sum_{j=1}^{N} S_j^* S_j = 1, \qquad E_{\mu_1\dots\mu_k} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1\dots\mu_k} S_j S_j^* + T_{\mu_1} E_{\mu_2\dots\mu_k} T_{\mu_1}^*$$

where $E_{\mu_1\dots\mu_k} = S_{\mu_1}^* \dots S_{\mu_k}^* S_{\mu_k} \dots S_{\mu_1}$, $\mu_1, \dots, \mu_k \in \{1, \dots, N\}$. The K-groups are

$$K_0(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}) \cong \mathsf{Z}/N\mathsf{Z} \oplus C(\mathfrak{K}, \mathsf{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}) \cong 0.$$

Proof. The relation (1.5) implies that T_i, $i = 1, \dots, N$ are isometries. By summing up μ_2 over $\{1, \dots, N\}$ in the second relation above for $k = 2$, one has the equalities

$$S_i^* S_i = \sum_{j=1}^{N} S_j S_j^* S_i^* S_i S_j S_j^* + T_i T_i^*, \qquad i = 1, \dots, N$$

by using the first relation above. By summing up $i = 1, 2, \dots, N$ in the above equalities, one sees the relation (1.3).

Example 2 (Fibonacci Dyck shift). Let F be the 2×2 matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. It is the smallest matrix in the irreducible square matices with condition (I) such that the associated topological Markov shift Λ_F is not conjugate to any full shift. The topological entropy of Λ_F is $\log \frac{1+\sqrt{5}}{2}$ the logarithm of the Perron eigenvalue of F. We call the subshift D_F the Fibonacci Dyck shift. As the matrix is irreducible with condition (I), the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_F)}}$ is simple and purely infinite.

Proposition 5.2. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_F)}}$ associated with the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_F)}$ is unital, separable, nuclear, simple and purely infinite. It is the unique C^*-algebra generated by one isometry T_1 and three partial isometries S_1, S_2, T_2 subject to the following operator relations:*

$$\sum_{j=1}^{2}(S_j S_j^* + T_j T_j^*) = \sum_{j=1}^{2} S_j^* S_j = 1, \qquad T_2^* T_2 = S_1^* S_1,$$

$$E_{\mu_1\dots\mu_k} = \sum_{j=1}^{2} F(j, \mu_1) S_j S_j^* E_{\mu_1\dots\mu_k} S_j S_j^* + T_{\mu_1} E_{\mu_2\dots\mu_k} T_{\mu_1}^*, \qquad k > 1$$

where $E_{\mu_1\dots\mu_k} = S_{\mu_1}^* \dots S_{\mu_k}^* S_{\mu_k} \dots S_{\mu_1}$, $(\mu_1, \dots, \mu_k) \in B_k(\Lambda_F)$ and $B_k(\Lambda_F)$ is the set of admissible words of the topological Markov shift Λ_F defined by *the matrix F. The K-groups are*

$$K_0(\mathcal{O}_{\mathfrak{Q}^{\mathrm{Ch}(D_F)}}) \cong \mathsf{Z} \oplus C(\mathfrak{K}, \mathsf{Z})^{\infty}, \qquad K_1(\mathcal{O}_{\mathfrak{Q}^{\mathrm{Ch}(D_F)}}) \cong 0.$$

PROOF. The operator relations above directly come from Theorem 1.1. The K-group formulae above are not direct. Its computations need some tecnichal steps as in [25]. The full proof of the above K-group formulae are written in [25].

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REFERENCES

1. Blackadar, B., *Shape theory for C^*-algebras*, Math. Scand. 56 (1978), 249–275.
2. Blackadar, B., *Semiprojectivity in simple C^*-algebras*, pp. 1–17 in: Operator Algebras and Applications, Adv. Stud. Pure Math. 38, Math. Soc. Japan, Tokyo 2004.
3. Chomsky, N., and Schützenberger, M. P., *The algebraic theory of context-free languages*, pp. 118–161 in: Computer Programing and Formal Systems, North-Holland, Amsterdam 1963.
4. Cuntz, J., *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. 57 (1977), 173–185.
5. Cuntz, J., and Krieger, W., *A class of C^*-algebras and topological Markov chains*, Invent. Math. 56 (1980), 251–268.
6. Effros, E. G., and Kaminker, J., *Homotopy continuity and shape theory for C^*-algebras*, pp. 152–180 in: Geometric Methods in Operator Algebras, Proc. Kyoto 1983, Pitman Res. Notes Math. 23, Longman, Harlow 1986.
7. Hamachi, T., Inoue, K., and Krieger, W., *Subsystems of finite type and semigroup invariants of subshifts*, J. Reine Angew. Math. 632 (2009), 37–61.
8. Hopcroft, J. E., and Ullman, J. D., *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, Reading, MA 1979.
9. Kitchens, B. P., *Symbolic Dynamics*, Springer, Berlin 1998.
10. Krieger, W., *On the uniqueness of the equilibrium state*, Math. Systems Theory 8 (1974), 97–104.
11. Krieger, W., *On subshifts and topological Markov chains*, pp. 453–472 in: Numbers, Information and Complexity, Proc. Bielefeld 1998, Kluwer, Boston 2000.
12. Krieger, W., *On subshift presentations*, preprint.
13. Krieger, W., *On a class of algebraic subshifts*, preprint.
14. Krieger, W., and Matsumoto, K., *A lambda-graph system for the Dyck shift and its K-groups*, Doc. Math. 8 (2003), 79–96.
15. Krieger, W., and Matsumoto, K., *A class of invariants of the topological conjugacy of subshifts*, Ergodic Theory Dynam. Systems 24 (2004), 1155–1172.
16. Krieger, W., and Matsumoto, K., *Zeta functions and topological entropy of the Markov Dyck shifts*, preprint, arXiv:0706.3262.
17. Krieger, W., and Matsumoto, K., *A notion of synchronization for subshifts and a class of C^*-algebras*, preprint 2009.

18. Lind, D., and Marcus, B., *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge (1995).
19. Matsumoto, K., *Presentations of subshifts and their topological conjugacy invariants*, Doc. Math. 4 (1999), 285–340.
20. Matsumoto, K., *C^*-algebras associated with presentations of subshifts*, Doc. Math. 7 (2002), 1–30.
21. Matsumoto, K., *A simple purely infinite C^*-algebra associated with a lambda-graph system of the Motzkin shift*, Math. Z. 248 (2004), 369–394.
22. Matsumoto, K., *K-theoretic invariants and conformal measures of the Dyck shifts*, Internat. J. Math. 16 (2005), 213–248.
23. Matsumoto, K., *Construction and pure infiniteness of C^*-algebras associated with lambda-graph systems*, Math. Scand. 97 (2005), 73–88.
24. Matsumoto, K., *On the simple C^*-algebras arising from Dyck systems*, J. Operator Theory 58 (2007), 205–226.
25. Matsumoto, K., *K-theory for the simple C^*-algebra of the Fibonacci Dyck system*, preprint, arXiv:math.OA/0607519.
26. Szymański, W., *On semiprojectivity of C^*-algebras of directed graphs*, Proc. Amer. Math. Soc. 130 (2001), 1391–1399.

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