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CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH FRACTIONAL q-CALCULUS OPERATORS

SUNIL DUTT PUROHIT and RAVINDER KRISHNA RAINA

Abstract

We first define the q-analogue operators of fractional calculus which are then used in defining certain classes of functions analytic in the open disk. The results investigated for these classes of functions include the coefficient inequalities and some distortion theorems. The results provide extensions of various known results in the q-theory of analytic functions. Special cases of our results are pointed out briefly.

1. Introduction, preliminaries and definitions

The q-shifted factorial is defined for $\alpha, q \in \mathsf{C}$ as a product of n factors by

$$(\alpha; q)_n = \begin{cases} 1; & n = 0 \\ (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}); & n \in \mathsf{N} \end{cases} \tag{1.1}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \qquad (n > 0), \tag{1.2}$$

where the q-gamma function is defined by ([7, p. 16, eq. (1.10.1)])

$$\Gamma_q(x) = \frac{(q; q)_\infty (1-q)^{1-x}}{(q^x; q)_\infty} \qquad (0 < q < 1). \tag{1.3}$$

If $|q| < 1$, the definition (1.1) remains meaningful for $n = \infty$ as a convergent infinite product:

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j).$$

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We recall here the following q-analogue definitions given by Gasper and Rahman [7]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(1+x) = \frac{(1-q^x)\Gamma_q(x)}{1-q} \tag{1.4}$$

and the q-binomial expansion is given by

$$\begin{aligned}(x-y)_\nu = x^\nu(-y/x;q)_\nu &= x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right] \\ &= x^\nu\, {}_1\Phi_0[q^{-\nu};-;q,yq^\nu/x].\end{aligned} \tag{1.5}$$

Also, Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [7, pp. 19–22])

$$D_{q,z}f(z) = \frac{f(z)-f(zq)}{z(1-q)} \qquad (z \neq 0, q \neq 0) \tag{1.6}$$

and

$$\int_0^z f(t)\,d(t;q) = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k). \tag{1.7}$$

In view of the relation

$$\operatorname*{Lim}_{q\to 1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.8}$$

we observe that the q-shifted factorial (1.1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$.

2. Fractional q-calculus operators

The fractional calculus operators have been extensively used in describing and solving various problems in applied sciences and also in the Geometric Function Theory of Complex Analysis (see, for example [9] and [14]). The fractional q-calculus is the q-extension of the ordinary fractional calculus. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference and q-integral equations, and in q-transform analysis. One may refer to the book [7], and the recent papers [1], [6], [8] and [11] on the subject.

The Riemann-Liouville fractional q-integral operator $I_q^\alpha f(x)$ of a real-valued function $f(x)$ of arbitrary order α introduced by Al-Salam [4] (see also Agarwal [2]) is defined by

$$\begin{aligned} I_q^\alpha f(x) &= D_{q,x}^{-\alpha} f(x) \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-tq)_{\alpha-1} f(t)\, d(t;q) \qquad (\alpha > 0). \end{aligned} \tag{2.1}$$

For $\alpha = n$ (n any positive integer), the above operator (2.1) was also defined in [5].

The fractional q-differential operator $D_{q,x}^\alpha f(x)$ of a real-valued function $f(x)$ of arbitrary order α [11, p. 319, eq. (19)] (see also Agarwal [2, p. 366, eq. (4)]) is defined by

$$D_{q,x}^\alpha f(x) = \begin{cases} I_q^{-\alpha} f(x), & \alpha < 0, \\ f(x), & \alpha = 0, \\ D_{q,x}^{\lceil \alpha \rceil} I_q^{\lceil \alpha \rceil - \alpha} f(x), & \alpha > 0, \end{cases} \tag{2.2}$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to α.

We now define the fractional q-calculus operators appropriately with a view of applying these operators in the Geometric Function Theory of complex analysis.

Definition 1 (Fractional q-Integral Operator). The fractional q-integral operator $I_{q,z}^\alpha f(z)$ of a function $f(z)$ of order α is defined by

$$\begin{aligned} I_{q,z}^\alpha f(z) &\equiv D_{q,z}^{-\alpha} f(z) \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^z (z-tq)_{\alpha-1} f(t)\, d(t;q) \qquad (\alpha > 0), \end{aligned} \tag{2.3}$$

where $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin. In view of relation (1.5), the q-binomial function $(z-tq)_{\alpha-1}$ can be expressed as

$$(z-tq)_{\alpha-1} = z^{\alpha-1}\, {}_1\Phi_0[q^{-\alpha+1}; -; q, tq^\alpha/z]. \tag{2.4}$$

Following Gasper and Rahman [7], the series ${}_1\Phi_0[\alpha; -; q, z]$ (which is a special case of the series ${}_2\Phi_1[\alpha, \beta; \gamma; q, z]$ for $\gamma = \beta$) is single-valued when $|\arg(z)| < \pi$ and $|z| < 1$ (see for details [7, pp. 104–106]), therefore, the function $(z-tq)_{\alpha-1}$ in (2.3) is single-valued when $|\arg(-tq^\alpha/z)| < \pi$, $|tq^\alpha/z)| < 1$ and $|\arg z| < \pi$.

DEFINITION 2 (Fractional q-Derivative Operator). The fractional q-derivative operator $D^{\alpha}_{q,z} f(z)$ of a function $f(z)$ of order α is defined by

$$\begin{aligned} D^{\alpha}_{q,z} f(z) &= D_{q,z} I^{1-\alpha}_{q,z} f(z) \\ &= \frac{1}{\Gamma_q(1-\alpha)} D_{q,z} \int_0^z (z-tq)_{-\alpha} f(t)\, d(t;q), \end{aligned} \tag{2.5}$$

for $0 \le \alpha < 1$, where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\alpha}$ is removed as in Definition 1 above.

DEFINITION 3 (Extended Fractional q-Derivative Operator). Under the hypotheses of Definition 2, the fractional q-derivative for a function $f(z)$ of order α is defined by

$$D^{\alpha}_{q,z} f(z) = D^{m}_{q,z} I^{m-\alpha}_{q,z} f(z) \qquad (m-1 \le \alpha < m;\, m \in \mathsf{N}_0 = \mathsf{N} \cup \{0\}), \tag{2.6}$$

where N denotes the set of natural numbers.

We give the following image formulas for the function z^{λ} under the fractional q-integral and q-differential operators defined by (2.3) and (2.5).

PROPOSITION 1. *Let $\alpha > 0$ and $\lambda > -1$. Then*

$$I^{\alpha}_{q,z} z^{\lambda} = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\alpha+1)} z^{\alpha+\lambda}. \tag{2.7}$$

PROOF. Using the series representation (1.7) for the operator (2.3), we can write

$$I^{\alpha}_{q,z} f(z) = z^{\alpha}(1-q)^{\alpha} \sum_{k=0}^{\infty} \frac{q^k (q^{\alpha};q)_k}{(q;q)_k} f(zq^k). \tag{2.8}$$

Setting now $f(z) = z^{\lambda}$ in (2.8), the above relation yields

$$\begin{aligned} I^{\alpha}_{q,z} z^{\lambda} &= z^{\alpha}(1-q)^{\alpha} \sum_{k=0}^{\infty} \frac{q^k (q^{\alpha};q)_k}{(q;q)_k} (zq^k)^{\lambda} \\ &= z^{\alpha+\lambda}(1-q)^{\alpha} \sum_{k=0}^{\infty} \frac{q^{(\lambda+1)k} (q^{\alpha};q)_k}{(q;q)_k}. \end{aligned}$$

On summing the series with the help of the formula ([7, p. 7, eq. (1.3.2)])

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_{\infty}}{(z;q)_{\infty}} \qquad (|z| < 1),$$

and performing elementary simplifications, we obtain

$$I^{\alpha}_{q,z} z^{\lambda} = \frac{(1-q)^{\alpha} z^{\alpha+\lambda} (q^{\alpha+\lambda+1}; q)_{\infty}}{(q^{\lambda+1}; q)_{\infty}}$$

which in view of (1.3) yields the desired result (2.7).

PROPOSITION 2. *Let* $\alpha \geq 0$ *and* $\lambda > -1$. *Then*

$$D^{\alpha}_{q,z} z^{\lambda} = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)} z^{\lambda-\alpha}. \tag{2.9}$$

We omit here the proof of Proposition 2 as its details are similar to Proposition 1.

Our aim in this paper is to introduce some new classes of functions defined by using fractional q-calculus operators which are analytic in the open disk. We also derive some results giving various coefficient inequalities and distortion theorems involving the fractional q-calculus operators. Special cases of the results are also pointed out in the concluding section of this paper.

3. New classes of functions

Let $\mathscr{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathsf{N}), \tag{3.1}$$

which are analytic and univalent in the open disc $\mathsf{U} = \{z : z \in \mathsf{C}, |z| < 1\}$. Also, let $\mathscr{A}_n^-$ denote the subclass of $\mathscr{A}_n$ consisting of analytic and univalent functions expressed in the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geq 0, n \in \mathsf{N}). \tag{3.2}$$

For the purpose of this paper, we define (by using the above definitions of the fractional q-calculus operators) a fractional q-differintegral operator $\Omega^{\alpha}_{q,z}$ for a function $f(z)$ of the form (3.1) by

$$\begin{aligned}\Omega^{\alpha}_{q,z} f(z) &= 1 + \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k z^{k-1} \\ &= \frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)} z^{\alpha-1} D^{\alpha}_{q,z} f(z)\end{aligned} \tag{3.3}$$

($-\infty < \alpha < 2$; $n \in \mathsf{N}$; $0 < q < 1$; $z \in \mathsf{U}$) where $D^{\alpha}_{q,z} f(z)$ in (3.3) represents, respectively, a fractional q-integral of $f(z)$ of order α when $-\infty < \alpha < 0$, and a fractional q-derivative of $f(z)$ of order α when $0 \le \alpha < 2$.

We introduce here the following classes of functions involving the operator $\Omega^{\alpha}_{q,z}$ (defined above by (3.3)):

$$\mathscr{J}^{\alpha}_{q,\delta} f(z) = \left\{ f \in \mathscr{A}^{-}_{n}, \left| \frac{\Omega^{\alpha}_{q,z} f(z) - 1}{\Omega^{\alpha}_{q,z} f(z) - 2\delta + 1} \right| < \beta \right\} \tag{3.4}$$

($-\infty < \alpha < 2, 0 \le \delta < 1, 0 \le \beta < 1, 0 < q < 1, z \in \mathsf{U}$) and

$$\begin{aligned} &\mathscr{S}^{\alpha}_{q,\sigma} f(z) \\ &\quad = \left\{ f \in \mathscr{A}^{-}_{n}, \Re\left((1-\sigma)\Omega^{\alpha}_{q,z} f(z) + \sigma\left(\frac{1-q^{1-\alpha}}{1-q}\right)\Omega^{\alpha+1}_{q,z} f(z) \right) > \beta \right\} \end{aligned} \tag{3.5}$$

($-\infty < \alpha < 2, 0 \le \sigma \le 1, 0 \le \beta < 1, 0 < q < 1, z \in \mathsf{U}$).

A function $f(z)$ defined by (3.1) is said to belong to $\mathscr{V}_n(\theta_k)$ if $f(z) \in \mathscr{A}_n$ satisfies the condition that

$$\arg(a_k) = \theta_k \qquad (k = n+1, n+2, \dots; n \in \mathsf{N}).$$

Further, if there exists a real number ρ such that

$$\theta_k + (k-1)\rho \equiv \pi \pmod{2\pi} \qquad (k = n+1, n+2, \dots; n \in \mathsf{N}), \tag{3.6}$$

then $f(z)$ is said to be in the class $\mathscr{V}(\theta_k; \rho)$.

Let $\mathscr{V}_n = \bigcup \mathscr{V}(\theta_k; \rho)$ for all possible sequences $\{\theta_k\}$ and ρ satisfying (3.6), and denote by $\Delta^{\alpha}_{q} f(z)$ the subclass of $\mathscr{V}_n$, i.e.,

$$\begin{aligned} &\Delta^{\alpha}_{q} f(z) \\ &\quad = \left\{ \mathscr{V}_n;\ f \in \mathscr{S}^{\alpha}_{q,0} f(z)\ (-\infty < \alpha < 2, 0 \le \beta < 1, 0 < q < 1, z \in \mathsf{U}) \right\}. \end{aligned} \tag{3.7}$$

We now obtain the following coefficient bounds for functions of the form (3.1) or (3.2) to belong to the classes $\mathscr{J}^{\alpha}_{q,\delta}$, $\mathscr{S}^{\alpha}_{q,\sigma}$ and Δ^{α}_{q} (defined above).

THEOREM 1. *A function f of the form* (3.2) *belongs to the class $\mathscr{J}^{\alpha}_{q,\delta}$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(2)\Gamma_q(k-\alpha+1)}(1+\beta)a_k \le 2\beta(1-\delta). \tag{3.8}$$

The result is sharp.

PROOF. Assume that the inequality (3.8) holds and let $|z| = 1$. Then on using (3.2) and (3.3), we find that

$$\begin{aligned}
&\left|\Omega_{q,z}^{\alpha} f(z) - 1\right| - \beta\left|\Omega_{q,z}^{\alpha} f(z) - 2\delta + 1\right| \\
&= \left|-\sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k z^{k-1}\right| \\
&\quad - \beta\left|2(1-\delta) - \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k z^{k-1}\right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} (1+\beta) a_k - 2\beta(1-\delta) \\
&\leq 0,
\end{aligned}$$

by our hypothesis. This implies that $f(z) \in \mathscr{J}_{q,\delta}^{\alpha}$.

To prove the converse, assume that $f(z)$ is defined by (3.2) and is in the class $\mathscr{J}_{q,\delta}^{\alpha}$. Then it follows that

$$\begin{aligned}
\left|\frac{\Omega_{q,z}^{\alpha} f(z) - 1}{\Omega_{q,z}^{\alpha} f(z) - 2\delta + 1}\right| &= \left|\sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k z^{k-1}\right| \\
&\quad \times \left|2(1-\delta) - \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k z^{k-1}\right|^{-1} < \beta.
\end{aligned} \tag{3.9}$$

Since $|\Re(z)| \leq |z|$ for any z, therefore, choosing values of z on the real axis so that $\Omega_{q,z}^{\alpha} f(z)$ is real, and letting $z \to 1^-$ through real values, we obtain from (3.9) the following inequality:

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k \\
&\qquad \leq 2\beta(1-\delta) - \beta \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k,
\end{aligned} \tag{3.10}$$

which yieds the desired result (3.8).

Finally, we note that the assertion (3.8) of Theorem 1 is sharp and the extremal function is given by

$$f(z) = z - \frac{2\beta(1-\delta)\Gamma_q(2)\Gamma_q(n-\alpha+2)}{(1+\beta)\Gamma_q(n+2)\Gamma_q(2-\alpha)} z^{n+1} \qquad (n \in \mathsf{N}). \tag{3.11}$$

THEOREM 2. *A function f of the form* (3.2) *belongs to the class $\mathcal{S}^{\alpha}_{q,\sigma}$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(2)\Gamma_q(k-\alpha+1)} a_k \left[(1-\sigma)(1-q)+\sigma(1-q^{k-\alpha})\right] \leq (1-\beta-\sigma)(1-q)+\sigma(1-q^{1-\alpha}). \tag{3.12}$$

The result is sharp.

PROOF. To prove Theorem 2, we apply the elementary assertion that

$$\Re\{F(z)\} \geq \beta \iff |1-\beta+F(z)| \geq |1+\beta-F(z)|. \tag{3.13}$$

Making use of (3.2) and (3.3), and setting

$$\begin{aligned} F(z) &= (1-\sigma)\Omega^{\alpha}_{q,z} f(z) + \sigma\left(\frac{1-q^{1-\alpha}}{1-q}\right)\Omega^{\alpha+1}_{q,z} f(z) \\ &= (1-\sigma)+\sigma\left(\frac{1-q^{1-\alpha}}{1-q}\right) \\ &\quad - \frac{1}{(1-q)}\sum_{k=n+1}^{\infty} A_{k,q}(\alpha,\sigma) a_k z^{k-1}, \end{aligned} \tag{3.14}$$

in (3.13), it then suffices to show that

$$\left|2-\beta-\sigma+\sigma\left(\frac{1-q^{1-\alpha}}{1-q}\right)-\frac{1}{(1-q)}\sum_{k=n+1}^{\infty} A_{k,q}(\alpha,\sigma)a_k z^{k-1}\right| - \left|\beta-\sigma-\sigma\left(\frac{1-q^{1-\alpha}}{1-q}\right)+\frac{1}{(1-q)}\sum_{k=n+1}^{\infty} A_{k,q}(\alpha,\sigma)a_k z^{k-1}\right| \geq 0,$$

where $A_{k,q}(\alpha,\sigma)$ is given by

$$A_{k,q}(\alpha,\sigma) = \frac{\left[(1-\sigma)(1-q)+\sigma(1-q^{k-\alpha})\right]\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(k-\alpha+1)\Gamma_q(2)}. \tag{3.15}$$

Now

$$\begin{aligned}
&\left|2-\beta-\sigma+\sigma\left(\frac{1-q^{1-\alpha}}{1-q}\right)-\frac{1}{(1-q)}\sum_{k=n+1}^{\infty}A_{k,q}(\alpha,\sigma)a_k z^{k-1}\right| \\
&\qquad -\left|\beta-\sigma-\sigma\left(\frac{1-q^{1-\alpha}}{1-q}\right)+\frac{1}{(1-q)}\sum_{k=n+1}^{\infty}A_{k,q}(\alpha,\sigma)a_k z^{k-1}\right| \\
&\geq \frac{2}{(1-q)}\left[(1-\beta-\sigma)(1-q)+\sigma(1-q^{1-\alpha})-\sum_{k=n+1}^{\infty}A_{k,q}(\alpha,\sigma)a_k|z|^{k-1}\right] \\
&\geq 0.
\end{aligned}$$

This completes the proof of Theorem 2.

We observe that the assertion (3.12) is sharp. The extremal function is given by

$$f(z)=z-\frac{\left[(1-\beta-\sigma)(1-q)+\sigma(1-q^{1-\alpha})\right]}{A_{n+1,q}(\alpha,\sigma)}z^{n+1} \qquad (n\in\mathbb{N}), \tag{3.16}$$

where $A_{k,q}(\alpha,\sigma)$ is given by (3.15).

THEOREM 3. *Let the function f be defined by* (3.1) *be in the class Δ_q^α. Then*

$$\sum_{k=n+1}^{\infty}\frac{\Gamma_q(k+1)}{\Gamma_q(k-\alpha+1)}|a_k|\leq\frac{(1-\beta)\Gamma_q(2)}{\Gamma_q(2-\alpha)}. \tag{3.17}$$

The result is sharp.

PROOF. Suppose that

$$\Re\left\{\frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)}z^{\alpha-1}D_{q,z}^{\alpha}f(z)\right\}>\beta.$$

Using (2.9) and (3.1), we get

$$\Re\left\{1+\sum_{k=n+1}^{\infty}\frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)}a_k z^{k-1}\right\}>\beta. \tag{3.18}$$

For $f(z)\in\mathscr{V}(\theta_k;\rho)$, we put $z=re^{i\rho}$ in the above inequality (3.18) and let $r\to 1^-$, to get

$$\Re\left\{1+\sum_{k=n+1}^{\infty}\frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)}|a_k|\exp\bigl(i(\theta_k+(k-1)\rho)\bigr)\right\}>\beta.$$

On making use of the conditions (3.6) in the process, we arrive at

$$1-\sum_{k=n+1}^{\infty}\frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)}|a_k|>\beta,$$

which leads to the inequality (3.17).

The equality in (3.17) is attained for the function $f(z)$ given by

(3.19)
$$f(z)=z+\frac{(1-\beta)\Gamma_q(2)\Gamma_q(k-\alpha+1)}{\Gamma_q(k+1)\Gamma_q(2-\alpha)}z^k\exp(i\theta_k)\qquad(k\geq n+1;\,n\in\mathsf{N}).$$

4. Distortion theorems

In this section, we prove three distortion theorems for the function $f(z)$ of the form (3.1) or (3.2) involving the fractional q-calculus operators.

THEOREM 4. *Let the function* $f(z)$ *defined by* (3.2) *be in the class* $\mathscr{J}_{q,\delta}^{\alpha}$ $(-\infty<\alpha<2,\,0<q<1)$. *Then*

(4.1)
$$|z|-2\beta\left(\frac{1-\delta}{1+\beta}\right)B(n,\alpha,q)|z|^{n+1}\leq|f(z)|\leq|z|+2\beta\left(\frac{1-\delta}{1+\beta}\right)B(n,\alpha,q)|z|^{n+1}\qquad(z\in\mathsf{U}),$$

where

(4.2)
$$B(n,\alpha,q)=\frac{\Gamma_q(2)\Gamma_q(n-\alpha+2)}{\Gamma_q(n+2)\Gamma_q(2-\alpha)}.$$

Furthermore

(4.3)
$$|z|-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{n+1}\leq|z\Omega_{q,z}^{\alpha}f(z)|\leq|z|+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{n+1}\qquad(z\in\mathsf{U}).$$

PROOF. Since $f(z)\in\mathscr{J}_{q,\delta}^{\alpha}$, then in view of Theorem 1, we first show that the function

$$\phi(k)=\frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(2)\Gamma_q(k-\alpha+1)}\qquad(k\geq n+1;\,n\in\mathsf{N})$$

is a decreasing function of k for $-\infty<\alpha<2$, $0<q<1$.

It follows that

$$\frac{\phi(k+1)}{\phi(k)} = \frac{\Gamma_q(k+2)\Gamma_q(k+1-\alpha)}{\Gamma_q(k+1)\Gamma_q(k+2-\alpha)} \qquad (k \geq n+1; n \in \mathsf{N}),$$

and it is sufficient to consider here the value $k = n+1$, so that on using (1.4), we get

$$\frac{\phi(n+2)}{\phi(n+1)} = \frac{1-q^{n+2}}{1-q^{n+2-\alpha}} \qquad (0 < q < 1).$$

The function $\phi(k)$ is a decreasing function of k if $\frac{\phi(n+2)}{\phi(n+1)} \leq 1$ $(n \in \mathsf{N})$, and this gives

$$\frac{1-q^{n+2}}{1-q^{n+2-\alpha}} \leq 1 \qquad (0 < q < 1).$$

Multiplying the above inequality both sides by $1-q^{n+2-\alpha}$ (provided that $\alpha < 2$), we are at once lead to the inequality $\alpha \leq 0$. Thus, $\phi(k)$ $(k \geq n+1; n \in \mathsf{N})$ is a decreasing function of k for $-\infty < \alpha < 2$, $0 < q < 1$.

Now (3.8) gives the following inequality:

$$\frac{(1+\beta)}{B(n,\alpha,q)} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} \frac{(1+\beta)}{B(k-1,\alpha,q)} a_k \leq 2\beta(1-\delta),$$

which implies that

$$\sum_{k=n+1}^{\infty} a_k \leq 2\beta\left(\frac{1-\delta}{1+\beta}\right) B(n,\alpha,q),$$

and this last inequality in conjunction with the following inequality (easily obtainable from (3.2)):

$$|z| - |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \leq |f(z)| \leq |z| + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \tag{4.4}$$

yields the assertion (4.1) of Theorem 4.

Next, on using (3.2) and (3.3), we observe that

$$\begin{aligned}
|z\Omega_{q,z}^{\alpha} f(z)| &\geq |z| - \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k |z|^k \\
&\geq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k,
\end{aligned}$$

which on using Theorem 1 gives

$$|z\Omega_{q,z}^{\alpha} f(z)| \geq |z| - 2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{n+1}. \tag{4.5}$$

Similarly, it follows that

$$|z\Omega_{q,z}^{\alpha} f(z)| \leq |z| + 2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{n+1}, \tag{4.6}$$

which establishes the assertion (4.3) of Theorem 4.

In view of (3.3), Theorem 4 gives the following distortion inequality for the function $f(z) \in \mathscr{A}_n^-$ involving fractional q-derivative operator $D_{q,z}^{\alpha}$:

COROLLARY 1. *Let the function* $f(z)$ *defined by* (3.2) *be in the class* $\mathscr{J}_{q,\delta}^{\alpha}$. *Then*

$$\begin{aligned}\frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)}|z|^{1-\alpha}\left\{1-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^n\right\} &\leq |D_{q,z}^{\alpha} f(z)| \\ &\leq \frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)}|z|^{1-\alpha}\left\{1+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^n\right\},\end{aligned} \tag{4.7}$$

where $0 \leq \alpha < 2$, $z \in \mathsf{U}$.

Also, in view of (3.3), Theorem 4 gives the following inequality involving fractional q-integral operator $I_{q,z}^{\alpha}$:

COROLLARY 2. *Let the function* $f(z)$ *be in the class* $\mathscr{J}_{q,\delta}^{\alpha}$. *Then*

$$\begin{aligned}\frac{\Gamma_q(2)}{\Gamma_q(2+\alpha)}|z|^{1+\alpha}\left\{1-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^n\right\} &\leq |I_{q,z}^{\alpha} f(z)| \\ &\leq \frac{\Gamma_q(2)}{\Gamma_q(2+\alpha)}|z|^{1+\alpha}\left\{1+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^n\right\},\end{aligned} \tag{4.8}$$

where $\alpha > 0$, $z \in \mathsf{U}$.

THEOREM 5. *Let the function* $f(z)$ *defined by* (3.2) *be in the class* $\mathscr{S}_{q,\sigma}^{\alpha}$ $(-\infty < \alpha < 2, 0 < q < 1)$. *Then for* $z \in \mathsf{U}$:

$$|z| - B(n,\alpha,q)C|z|^{n+1} \leq |f(z)| \leq |z| + B(n,\alpha,q)C|z|^{n+1}. \tag{4.9}$$

Also

$$|z| - CD|z|^{n+1} \leq |z\Omega_{q,z}^{\lambda} f(z)| \leq |z| + CD|z|^{n+1} \qquad (-\infty < \lambda < 2), \tag{4.10}$$

where C and D are the expressions given, respectively, by

$$C = \frac{(1-\beta-\sigma)(1-q)+\sigma(1-q^{1-\alpha})}{(1-\sigma)(1-q)+\sigma(1-q^{n-\alpha+1})} \tag{4.11}$$

and

$$D = \frac{\Gamma_q(n-\alpha+2)\Gamma_q(2-\lambda)}{\Gamma_q(n-\lambda+2)\Gamma_q(2-\alpha)}, \tag{4.12}$$

and $B(n,\alpha,q)$ is defined by (4.2).

Proof. Since $f(z) \in \mathcal{S}^{\alpha}_{q,\sigma}$, then under the hypotheses of Theorem 2, we have

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\left[(1-\beta-\sigma)(1-q)+\sigma(1-q^{1-\alpha})\right]\Gamma_q(2)\Gamma_q(n-\alpha+2)}{\left[(1-\sigma)(1-q)+\sigma(1-q^{n-\alpha+1})\right]\Gamma_q(n+2)\Gamma_q(2-\alpha)}$$

i.e.,

$$\sum_{k=n+1}^{\infty} a_k \leq B(n,\alpha,q)C, \tag{4.13}$$

where $B(n,\alpha,q)$ and C are given by (4.2) and (4.11), respectively. On using the inequality (4.4), we arrive at the assertion (4.9) of Theorem 5.

From the relation (3.3), we obtain

$$|z\Omega^{\lambda}_{q,z} f(z)| \geq |z| - \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\lambda)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\lambda)} a_k |z|^k, \tag{4.14}$$

and we observe (as in the proof of Theorem 4) that the function $\psi(k)$ defined by

$$\psi(k) = \frac{\Gamma_q(2-\lambda)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k-\lambda+1)} \qquad (-\infty < \lambda < 2, 0 < q < 1) \tag{4.15}$$

is nonincreasing for $k \geq n+1$. Therefore, we assert that

$$0 < \psi(k) < \psi(n+1) = \frac{\Gamma_q(2-\lambda)\Gamma_q(n+2)}{\Gamma_q(2)\Gamma_q(n-\lambda+2)} \tag{4.16}$$

and the desired distortion inequality (4.10) follows now from (4.13)-(4.16).

In view of (2.3), (2.5) and (3.3), the distortion inequality (4.10) of Theorem 5 gives the following corollaries involving the q-fractional calculus (derivative and integral) operators:

COROLLARY 3. *Let* $0 \leq \lambda < 2$, $n \in \mathsf{N}$ *and the function* $f(z)$ *defined by* (3.2) *be in the class* $\mathcal{S}^{\alpha}_{q,\sigma}$. *Then for all* $z \in \mathsf{U}$*:*

$$\begin{aligned}\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\{1-CD|z|^n\} &\leq |D^{\lambda}_{q,z}f(z)| \\ &\leq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\{1+CD|z|^n\}.\end{aligned} \tag{4.17}$$

and

COROLLARY 4. *Let* $\lambda > 0$, $n \in \mathsf{N}$ *and the function* $f(z)$ *be in the class* $\mathcal{S}^{\alpha}_{q,\sigma}$. *Then for all* $z \in \mathsf{U}$*:*

$$\begin{aligned}\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\{1-CD|z|^n\} &\leq |I^{\lambda}_{q,z}f(z)| \\ &\leq \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\{1+CD|z|^n\},\end{aligned} \tag{4.18}$$

where C *is the expression given by* (4.11)*, and* D *is the expression given by* (4.12) (*with* λ *replaced by* $-\lambda$ *therein*)*.*

THEOREM 6. *Let the function* $f(z)$ *defined by* (3.1) *be in the class* Δ^{α}_{q} $(-\infty < \alpha < 2, 0 < q < 1)$. *Then for* $z \in \mathsf{U}$*:*

$$\begin{aligned}|z| - (1-\beta)B(n,\alpha,q)|z|^{n+1} &\leq |f(z)| \\ &\leq |z| + (1-\beta)B(n,\alpha,q)|z|^{n+1},\end{aligned} \tag{4.19}$$

where $B(n,\alpha,q)$ *is given by* (4.2)*. Furthermore*

$$\begin{aligned}|z| - (1-\beta)D|z|^{n+1} &\leq |z\Omega^{\lambda}_{q,z}f(z)| \\ &\leq |z| + (1-\beta)D|z|^{n+1} \qquad (-\infty < \lambda < 2),\end{aligned} \tag{4.20}$$

where D *is given by* (4.12)*.*

The proof of the above distortion theorem is similar to Theorem 5, details are hence omitted.

Using (2.3), (2.5) and (3.3), the distortion inequality (4.20) of Theorem 6 gives the following corollaries involving the q-fractional calculus (derivative and integral) operators:

COROLLARY 5. *If* $0 \leq \lambda < 2$, $n \in \mathsf{N}$ *and the function* $f(z)$ *defined by* (3.1) *is in the class* Δ_q^α, *then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\{1-(1-\beta)D|z|^n\} \leq |D_{q,z}^\lambda f(z)| \leq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\{1+(1-\beta)D|z|^n\}. \tag{4.21}$$

and

COROLLARY 6. *If* $\lambda > 0$, $n \in \mathsf{N}$ *and the function* $f(z)$ *is in the class* Δ_q^α, *then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\{1-(1-\beta)D|z|^n\} \leq |I_{q,z}^\lambda f(z)| \leq \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\{1+(1-\beta)D|z|^n\}, \tag{4.22}$$

for all $z \in \mathsf{U}$, *where* D *is the expression given by* (4.12) (*with* λ *replaced by* $-\lambda$ *therein*).

5. Concluding observations and remarks

We briefly consider now some consequences of the results derived in the preceeding sections. If we let $q \to 1^-$, and make use of the limit formula (1.10), we observe that the function class $\mathscr{J}_{q,\delta}^\alpha$ defined by (3.4), and the inequalities (3.8) of Theorem 1, (4.1) of Theorem 4, (4.7) of Corollary 1 provide, respectively, the q-extensions of the known class and the related inequalities due to Srivastava and Owa [16] (see also Srivastava and Aouf [12], for $p = 1$). Also, the function class $\mathscr{S}_{q,\sigma}^\alpha$ defined by (3.5), Theorem 2 and the distortion inequalities (4.17) of Corollary 3 and (4.18) of Corollary 4 are the q-extensions of the corresponding known function class and related results due to Altintas, Irmak, and Srivastava [3, p. 2, Theorem 1; p. 4 Theorems 3 and 4]. Further, the function class Δ_q^α defined by (3.7), and the inequalities (3.17) of Theorem 3, (4.21) of Corollary 5 and (4.22) of Corollary 6 provide the q-extensions of the known function class and related results due to Raina and Srivastava [10, p. 76, Corollary 1; p. 77, Corollaries 2 and 3].

We conclude with the remark that the fractional q-calculus operators defined in Section 2 can fruitfully be used in investigating several other analytic, multivalent (or meromorphic) function classes and their various geometric properties like, the coefficient estimates, distortion bounds, radii of starlikeness, convexity and close to convexity can be studied for such contemplated classes of functions.

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DEPARTMENT OF BASIC SCIENCES (MATHEMATICS)
COLLEGE OF TECHNOLOGY AND ENGINEERING
M.P. UNIVERSITY OF AGRICULTURE
AND TECHNOLOGY
UDAIPUR-313001, RAJASTHAN
INDIA
E-mail: sunil_a_purohit@yahoo.com

M.P. UNIVERSITY OF AGRICULTURE
AND TECHNOLOGY
UDAIPUR-313001, RAJASTHAN
Current address:
10/11, GANPATHI VIHAR, OPPOSITE SECTOR 5
UDAIPUR-313002, RAJASTHAN
INDIA
E-mail: rkraina_7@hotmail.com