

FOURIER TRANSFORMS OF SPHERICAL DISTRIBUTIONS ON COMPACT SYMMETRIC SPACES

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Abstract

In our previous articles [27] and [28] we studied Fourier series on a symmetric space $M = U/K$ of the compact type. In particular, we proved a Paley-Wiener type theorem for the smooth functions on M , which have sufficiently small support and are K -invariant, respectively K -finite. In this article we extend these results to K -invariant distributions on M . We show that the Fourier transform of a distribution, which is supported in a sufficiently small ball around the base point, extends to a holomorphic function of exponential type. We describe the image of the Fourier transform in the space of holomorphic functions. Finally, we characterize the singular support of a distribution in terms of its Fourier transform, and we use the Paley-Wiener theorem to characterize the distributions of small support, which are in the range of a given invariant differential operator. The extension from symmetric spaces of compact type to all compact symmetric spaces is sketched in an appendix.

Introduction

The Paley-Wiener theorem for \mathbf{R}^n describes (in the version due to L. Schwartz) the image by the Fourier transform of the space of compactly supported smooth functions on \mathbf{R}^n . A similar theorem describes the image of the space of compactly supported distributions. More precisely, let $C_c^\infty(\mathbf{R}^n)$ and $C_c^{-\infty}(\mathbf{R}^n)$ denote the spaces of compactly supported smooth functions and distributions, respectively. Then the Fourier image of $C_c^\infty(\mathbf{R}^n)$ is the space of entire functions F on \mathbf{C}^n of exponential type, that is, for which there exist $r > 0$ and for every $N \in \mathbf{Z}^+ = \{0, 1, 2, \dots\}$ a constant C_N such that

$$|F(\lambda)| \leq C_N (1 + |\lambda|)^{-N} e^{r|\operatorname{Im}(\lambda)|}$$

for all $\lambda \in \mathbf{C}^n$. Furthermore, the Fourier image of $C_c^{-\infty}(\mathbf{R}^n)$ is the space of entire functions F for which there exist $r > 0$ and for some $N \in \mathbf{Z}^+$ a constant C_N such that

$$|F(\lambda)| \leq C_N (1 + |\lambda|)^N e^{r|\operatorname{Im}(\lambda)|}$$

*The research of G. Ólafsson was supported by NSF grants DMS-0402068 and DMS-0801010. Received 27 October 2009.

for all $\lambda \in \mathbb{C}^n$. An important aspect of these theorems is that the smallest exponent r in the estimates matches with the radius of the smallest closed ball $\bar{B}_r = \{x \in \mathbb{R}^n \mid |x| \leq r\}$ containing the support of the function or distribution. Finally, by an analogous result due to Hörmander, the singular support of a compactly supported distribution is contained in \bar{B}_r if and only if its Fourier transform F satisfies the following condition. There exists $N \in \mathbb{Z}^+$ and for every $m \in \mathbb{Z}^+$ a constant C_m such that

$$|F(\lambda)| \leq C_m(1 + |\lambda|)^N e^{r|\operatorname{Im}(\lambda)|}$$

for all $\lambda \in \mathbb{C}^n$ with $|\operatorname{Im}(\lambda)| \leq m \log(1 + |\lambda|)$. See [22], Section 1.7.

There are several generalizations of these theorems to settings where \mathbb{R}^n is replaced by a symmetric space X . The most general results have been obtained for $X = G/K$ a Riemannian symmetric space of the non-compact type, by Gangolli [16] and Helgason [18], [19] for smooth functions, and by Eguchi, Hashizume and Okamoto [15] for distributions. Again the exponent matches with the radius of the support. For functions this is seen in the references just mentioned, and for distributions it is shown by Dadok [12], who gives a proof of the distributional Paley-Wiener theorem different from that of [15]. Combining these results with Hörmander's theorem for \mathbb{R}^n , a characterization of singular supports is easily deduced, see [12] (see also [1]).

In the present paper we investigate the generalization of the theorems for distributions to a Riemannian symmetric space $M = U/K$ of the compact type. In previous papers [27] and [28], we have treated the case of K -invariant, respectively, K -finite smooth functions on M . These papers generalized partial results in [9], [10], [17], [23]. In contrast to the non-compact cases G/K , the results obtained for M are *local* in the sense that they are only valid for functions supported in sufficiently small balls, with an explicit (but not necessarily optimal) upper bound for the radius. In the present paper we use the results of [27] to extend the K -invariant Paley-Wiener theorem to K -invariant distributions on M , including the analogous result for singular support. The more general case of K -finite distributions can be treated similarly, based on [28] (the details are omitted). In an appendix at the end of the paper we briefly discuss the extension to all compact symmetric spaces of the results from [27] as well as those of the present paper.

The Paley-Wiener theorems have also been generalized to non-Riemannian symmetric spaces. General reductive symmetric spaces are treated in [7] and [8]. The case of a reductive Lie group (which can be considered as a symmetric space), was earlier treated in [5], see also [6] and [14]. Hyperbolic spaces were treated in [2]. Some partial results have been obtained for the Fourier-Laplace

transform on causal symmetric spaces, see [3], [4], [25] and the overview in [26].

The article is organized as follows. In Section 1 we introduce the basic notation. In Section 2 we discuss the parametrization of the irreducible unitary K -spherical representations and the related Fourier transform. Let f be a K -spherical smooth function (or distribution) on M , then its Fourier transform $\tilde{f}(\lambda)$ is defined for λ in the semi-lattice $\Lambda^+(M)$ consisting of the highest weights $\lambda \in \alpha_{\mathbb{C}}^*$ of irreducible K -spherical representations (the weights are purely imaginary linear forms on the maximal abelian subspace α). In Section 4 we recall the main result of [27] which, in short, says the following. Assume f is smooth, then the Fourier transform \tilde{f} extends to a holomorphic function on $\alpha_{\mathbb{C}}^*$ of exponential type, and the best exponent of growth is equal to the radius of the smallest closed ball around the origin, which contains the support of f . Here it is required that the support of f is sufficiently small, as explained in the Remark 4.3 in [27].

Section 6 contains the main results of this article. First, we introduce the Paley-Wiener space $\text{PW}_r^*(\alpha)$ of holomorphic functions on $\alpha_{\mathbb{C}}^*$ such that:

- a) There exists $k \in \mathbb{Z}^+$ and $C > 0$ such that for all $\lambda \in \alpha_{\mathbb{C}}^*$,

$$|\Phi(\lambda)| \leq C(1 + |\lambda|)^k e^{r|\text{Re } \lambda|}$$

- b) $\Phi(w(\lambda + \rho) - \rho) = \Phi(\lambda)$ for all w in the Weyl group W .

Let $C_r^{-\infty}(M)^K$ denote the space of K -invariant distributions on M with support in a closed ball of radius r around the origin. Our main results are:

THEOREM 6.4 (Local Paley-Wiener theorem for distributions). *There exists $R > 0$ such that the following holds for each $0 < r < R$.*

- i) *Let $F \in C_r^{-\infty}(M)^K$. Then the Fourier transform $\tilde{F}: \Lambda^+(M) \rightarrow \mathbb{C}$ extends to a function in $\text{PW}_r^*(\alpha)$.*
- ii) *Let $\Phi \in \text{PW}_r^*(\alpha)$. There exists a unique distribution $F \in C_r^{-\infty}(M)^K$ such that $\tilde{F}(\mu) = \Phi(\mu)$ for all $\mu \in \Lambda^+(M)$.*
- iii) *The functions in the Paley-Wiener space $\text{PW}_r^*(\alpha)$ are uniquely determined by their values on $\Lambda^+(M)$.*

Thus, the Fourier transform followed by the extension gives a bijection

$$C_r^{-\infty}(M)^K \simeq \text{PW}_r^*(\alpha).$$

THEOREM 6.6 (Characterization of singular support). *Let R be as above and let $0 < s \leq r < R$. Let $F \in C_r^{-\infty}(M)^K$. Then the singular support of F*

is contained in a closed ball of radius s if and only if there exists $N \in \mathbb{Z}$ such that for each $m \in \mathbb{Z}^+$ the holomorphic extension of \tilde{F} satisfies

$$|\tilde{F}(\lambda)| \leq C_m(1 + |\lambda|)^N e^{s|\operatorname{Re} \lambda|}$$

for all $\lambda \in \alpha_{\mathbb{C}}^*$ with $|\operatorname{Re} \lambda| \leq m \log(1 + |\lambda|)$.

One of the consequences of the Paley-Wiener theorem is a condition for the solvability of invariant differential equations $DT = F$ in the space $C_r^{-\infty}(M)^K$. This is stated in Theorem 6.7.

1. Basic notation

Let M be a connected Riemannian symmetric space of the compact type. Then there exists a compact connected semisimple Lie group U acting on M by isometries and a closed subgroup $K \subset U$ such that $M = U/K$. Furthermore, there exists an involution $\theta : U \rightarrow U$ such that $U_0^\theta \subset K \subset U^\theta$. Here U^θ denotes the subgroup of θ -fixed points, and U_0^θ its identity component. We denote the base point eK in M by o .

Let \mathfrak{u} denote the Lie algebra of U , then θ induces an involution of \mathfrak{u} (denoted by the same symbol). Let $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ be the corresponding decomposition in eigenspaces for θ . Let $\langle \cdot, \cdot \rangle$ be the inner product on \mathfrak{u} defined by $\langle X, Y \rangle = -B(X, Y)$, where B is the Killing form. We assume that the Riemannian metric g of M is normalized such that it agrees with $\langle \cdot, \cdot \rangle$ on the tangent space $\mathfrak{q} \simeq T_oM$. We denote by \exp the exponential map $\mathfrak{u} \rightarrow U$, and by Exp the map $\mathfrak{q} \rightarrow M$ given by $\operatorname{Exp}(X) = \exp(X) \cdot o$. Denote by $B_r(0)$ the open ball in \mathfrak{q} of radius $r > 0$ and centered at 0 and $D_r(o)$ the open metric ball in M of radius $r > 0$ and centered at o . Similarly $\bar{B}_r(0)$ and $\bar{D}_r(o)$ stand for the closed balls. The exponential map Exp is surjective and an analytic diffeomorphism $B_r(0) \rightarrow D_r(o)$ for r sufficiently small.

Let $\alpha \subset \mathfrak{q}$ be a maximal abelian subspace, α^* its dual space, and $\alpha_{\mathbb{C}}^*$ the complexified dual space. Then $\langle \cdot, \cdot \rangle$ defines an inner product on α^* . By sesquilinear extension we obtain inner products on $\alpha_{\mathbb{C}}^*$ and $i\alpha^*$, which we shall denote by the same symbol. The corresponding norm is denoted by $|\cdot|$.

We denote by Σ the set of non-zero (restricted) roots of $\mathfrak{u}_{\mathbb{C}}$ with respect to $\alpha_{\mathbb{C}}$. Then $\Sigma \subset i\alpha^*$. Furthermore, Σ^+ stands for a fixed set of positive roots and $\rho \in i\alpha^*$ denotes half of the sum of the roots in Σ^+ counted with multiplicities. The corresponding Weyl group, generated by the reflections in the roots, is denoted W .

2. Fourier analysis on M

In this section we recall the basic facts on Fourier series on M and the parametrization of K -spherical representations. Let π denote an irreducible unitary representation of U , and V_π the Hilbert space on which π acts. Let

$$V_\pi^K = \{v \in V_\pi \mid (\forall k \in K) \pi(k)v = v\}.$$

If $V_\pi^K \neq \{0\}$ then $\dim V_\pi^K = 1$ and π is said to be K -spherical. If π is K -spherical, then e_π will denote a fixed choice of a unit vector in V_π^K .

Recall the following parametrization of K -spherical irreducible representations of U , due to Helgason (see [20], p. 535). Denote by \tilde{U} the universal covering of U and by κ the canonical projection $\tilde{U} \rightarrow U$. Then θ defines an involution $\tilde{\theta}$ on \tilde{U} , and the group \tilde{K} of $\tilde{\theta}$ -fixed points is connected. If π is a K -spherical representation of U , then $\pi \circ \kappa$ is a \tilde{K} -spherical representation of \tilde{U} .

THEOREM 2.1. *The map $\pi \mapsto \mu$, where $\mu \in i\alpha^*$ is the highest weight of π , induces a bijection between the set of equivalence classes of irreducible \tilde{K} -spherical representations of \tilde{U} and the set*

$$(2.1) \quad \Lambda^+(\tilde{U}/\tilde{K}) = \left\{ \mu \in i\alpha^* \mid (\forall \alpha \in \Sigma^+) \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z}^+ \right\}.$$

For $\mu \in \Lambda^+(\tilde{U}/\tilde{K})$, let (π_μ, V_μ) denote a fixed irreducible unitary representation of \tilde{U} with highest weight μ , and let $e_\mu = e_{\pi_\mu}$. We denote by $\Lambda^+(U/K)$ the set of elements in $\Lambda^+(\tilde{U}/\tilde{K})$, for which the representation π_μ of \tilde{U} descends to a representation of U with a K -fixed vector. Note that if it descends, it will have a K_0 -fixed, but not necessarily a K -fixed vector. This was not made clear in [27], Theorem 3.1, which is only valid as stated under the extra condition that K is connected. As an example take $U = \mathrm{SO}(n)$, $n \geq 3$, and

$$K = \mathrm{O}(n-1) = \left\{ \left(\begin{array}{cc} \det(A) & 0 \\ 0 & A \end{array} \right) \mid A \in \mathrm{O}(n-1) \right\}.$$

Then $K_0 \simeq \mathrm{SO}(n-1)$. The natural representation of $\mathrm{SO}(n)$ acting on \mathbf{C}^n has a K_0 -fixed vector e_1 , but is not K -spherical. However, the restricted validity does not affect the main results of [27], as the exact description of $\Lambda^+(U/K)$ is not used. However, the following property of $\Lambda^+(U/K)$ is used.

LEMMA 2.2. *The subset $\Lambda^+(U/K) \subset i\alpha^*$ is closed under addition, and it has full rank in $i\alpha^*$, that is, there exist $\mu_1, \dots, \mu_n \in i\alpha^*$ linearly independent where $n = \dim \alpha$, such that*

$$(2.2) \quad \mathbf{Z}^+ \mu_1 + \dots + \mathbf{Z}^+ \mu_n \subset \Lambda^+(U/K).$$

PROOF. It follows from Theorem 2.1 that the result holds for $\Lambda^+(\tilde{U}/\tilde{K})$. In fact, in this case equality is attained in (2.2) when μ_1, \dots, μ_n are the fundamental weights (see [34]).

Let $K^* = \kappa^{-1}(K)$ then $\tilde{K} = K_0^*$, and the quotient K^*/\tilde{K} is a finite group, which acts by a homomorphism $\gamma_\mu: K^*/\tilde{K} \rightarrow \mathbb{C}$ on the one-dimensional space $V_{\pi_\mu}^{\tilde{K}}$ for each $\mu \in \Lambda^+(\tilde{U}/\tilde{K})$. In particular, we see that μ belongs to $\Lambda^+(U/K)$ if and only if γ_μ is trivial. We shall see below that

$$(2.3) \quad \gamma_{\mu+\nu} = \gamma_\mu \cdot \gamma_\nu$$

for all $\mu, \nu \in \Lambda^+(\tilde{U}/\tilde{K})$. It follows that $\Lambda^+(U/K)$ is closed under addition, and also that $\gamma_{p\mu} = 1$, where p is the order of K^*/\tilde{K} . Hence $p\Lambda^+(\tilde{U}/\tilde{K}) \subset \Lambda^+(U/K)$, and thus the lemma follows from (2.3).

For each $\mu \in \Lambda^+(\tilde{U}/\tilde{K})$, let $e_\mu = e_{\pi_\mu} \in V_\mu = V_{\pi_\mu}$ denote the chosen \tilde{K} -fixed unit vector, and let $v_\mu \in V_\mu$ be a highest weight vector normalized such that $\langle v_\mu, e_\mu \rangle = 1$. Then $\int_{\tilde{K}} \pi_\mu(k)v_\mu dk = e_\mu$.

Consider the tensor product $V_\mu \otimes V_\nu$. It is well known that the representation $V_{\mu+\nu}$ occurs with multiplicity one in the tensor product, and that $v_\mu \otimes v_\nu$ is a highest weight vector in $V_{\mu+\nu}$. The vector

$$e := \int_{\tilde{K}} \pi_\mu(k)v_\mu \otimes \pi_\nu(k)v_\nu dk \in V_{\mu+\nu}$$

is \tilde{K} -fixed. Using Fubini's theorem and the invariance of Haar measure, we see that

$$(2.4) \quad \int_{\tilde{K}} (\pi_\mu(l) \otimes 1)e dl = e_\mu \otimes e_\nu.$$

In particular, $e \neq 0$ and we can identify e as a multiple of the unit vector $e_{\mu+\nu}$. The desired relation (2.3) follows from (2.4), by using that Haar measure on \tilde{K} is invariant under the adjoint action of K^* .

For $\mu \in \Lambda^+(M) = \Lambda^+(U/K)$ the *spherical function* associated with μ is the matrix coefficient

$$\psi_\mu(x) = (\pi_\mu(x)e_\mu, e_\mu), \quad x \in U.$$

It is left and right K -invariant and can therefore be viewed as a left K -invariant function on M . It is an eigenfunction of $\mathbf{D}(M)$, the algebra of invariant differential operators on M . The *spherical Fourier transform* of a K -invariant L^1 -function f on M is the function $\tilde{f}: \Lambda^+(M) \rightarrow \mathbb{C}$ defined by

$$\tilde{f}(\mu) = \int_M f(x) \overline{\psi_\mu(x)} dx = (f, \psi_\mu),$$

where dx is the normalized invariant measure on M (that is, $\int_M dx = 1$). Notice that if $f \in L^p(M)$ then $|\tilde{f}(\mu)| \leq \|f\|_p$ as $|\psi_\mu(x)| \leq 1$. In particular

$$|\tilde{f}(\mu)| \leq \|f\|_\infty$$

if f is continuous, and hence bounded. It follows from the Schur orthogonality relations that

$$(2.5) \quad \tilde{\psi}_\nu(\mu) = \delta_{\nu,\mu} d(\mu)^{-1}$$

for $\nu, \mu \in \Lambda^+(M)$, where $d(\mu) = \dim(V_\mu)$.

The *spherical Fourier series* for f is the series given by

$$(2.6) \quad \sum_{\mu \in \Lambda^+(M)} d(\mu) \tilde{f}(\mu) \psi_\mu.$$

Denote by Δ_M the negative definite Laplace operator on M . Then

$$(2.7) \quad \Delta_M \psi_\mu = -\langle \mu, \mu + 2\rho \rangle \psi_\mu.$$

Based on (2.7) it can be shown, see Sugiura [32], that f is smooth if and only if the Fourier transform \tilde{f} is *rapidly decreasing*, that is, for each $k \in \mathbb{Z}^+$ there exists a constant C_k such that

$$(2.8) \quad |\tilde{f}(\mu)| \leq C_k (1 + |\mu|)^{-k}$$

for all $\mu \in \Lambda^+(M)$. In this case the Fourier series (2.6) converges pointwise and absolutely to f .

There are different ways to describe the topology on $C^\infty(M)^K$. First, the topology on $C^\infty(U)$ is defined by the seminorms

$$(2.9) \quad \nu_p(f) := \|L_p f\|_\infty$$

where $p \in U(\mathfrak{g})$. Here L denotes the left regular representation and $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} . If \mathcal{C} is a closed subspace of $C^\infty(U)$ then the topology on \mathcal{C} is given by the same family of seminorms. This applies to the space $C^\infty(M)$, viewed as the space of *right* K -invariant smooth functions on U , the space $C^\infty(M)^K$ of *left* K -invariant functions in $C^\infty(M)$, as well as the spaces $C_r^\infty(M)$ and $C_r^\infty(M)^K$, $r > 0$, where the subscript r indicates that the support is contained in $\bar{D}_r(o)$. Note, if r is big enough then $C_r^\infty(M) = C^\infty(M)$.

According to [32] the topology can also be described using Δ_M :

LEMMA 2.3. *The topology of $C^\infty(M)$, $C^\infty(M)^K$, and $C_r^\infty(M)^K$ is given by the seminorms*

$$\tau_m(f) = \|\Delta_M^m f\|_\infty, \quad m \in \mathbb{Z}^+.$$

PROOF. This is the corollary to Theorem 4 in [32].

We shall also need the following fact from [35] Lemma 5.6.7 or [32] Lemma 1.3.

LEMMA 2.4. *There exists $t_0 \in \mathbf{R}$ such that $\sum_{\mu \in \Lambda^+(M)} (1 + |\mu|)^{-t} < \infty$ if $t > t_0$.*

By Weyl's dimension formula, the map $\mu \mapsto d(\mu)$ extends to a polynomial function on $\mathfrak{a}_\mathbb{C}^*$. We derive the following consequence from Lemmas 2.3 and 2.4 together with (2.7) and (2.8).

LEMMA 2.5. *Let $f \in C^\infty(M)$. Then the Fourier series (2.6) converges to f in $C^\infty(M)$.*

It follows from the *KAK*-decomposition of U that the restriction map $f \mapsto f|_{A \cdot o}$ is injective for $f \in C^\infty(M)^K$. We use the topology on $C^\infty(A \cdot o)$ given by the seminorms $\nu_u(f) = \|L_u f\|_\infty$, $u \in U(\alpha)$. Then $C^\infty(A \cdot o)$ is a Fréchet space, and $C^\infty(A \cdot o)^W$ is a closed subspace whose topology is given by the same family of seminorms. The following lemma gives a different way to describe the topology on $C^\infty(M)^K$:

LEMMA 2.6. *The restriction map from $C^\infty(M)^K$ to $C^\infty(A \cdot o)^W$ is a topological isomorphism. It is also a topological isomorphism from $C_r^\infty(M)^K$ onto $C_r^\infty(A \cdot o)^W$, for each $r > 0$.*

PROOF. According to [13], Theorem 1.7, the restriction map is bijective. It is obviously continuous. By the open mapping theorem for Fréchet spaces [33], Theorem 17.1., p. 170, it follows that the restriction map is a topological isomorphism. For the last statement we note first that $C_r^\infty(A \cdot o)^W$ is closed in $C^\infty(A \cdot o)^W$ and similarly for $C_r^\infty(M)^K$ in $C^\infty(M)^K$. Furthermore, the metric ball in $A \cdot o$ of radius r centered at o is $\bar{D}_r(o) \cap A \cdot o$ and $\bar{D}_r(o) = K(\bar{D}_r(o) \cap A \cdot o)$. Hence $f \mapsto f|_{A \cdot o}$ is a bijection $C_r^\infty(M)^K \rightarrow C_r^\infty(A \cdot o)^W$, and it follows from the first statement that it is an isomorphism.

3. The Fourier series of a distribution

The continuous dual of $C^\infty(M)$, denoted by $C^{-\infty}(M)$, is the space of *distributions* on M . Recall that U acts on $C^{-\infty}(M)$ by

$$L_g F(f) = F(L_{g^{-1}} f), \quad g \in U, \quad f \in C^\infty(M), \quad \text{and} \quad F \in C^{-\infty}(M).$$

Denote by $C^{-\infty}(M)^K$ the space of K -invariant distributions on M . Since $C^\infty(M)^K$ is a closed subspace of $C^\infty(M)$, we obtain a map from $C^{-\infty}(M)$ to $[C^\infty(M)^K]^*$, by taking restrictions of linear forms to this subspace. Here

$[C^\infty(M)^K]^*$ denotes the space of continuous linear forms on $C^\infty(M)^K$. We provide $C^{-\infty}(M)$ and $[C^\infty(M)^K]^*$ with the weak $*$ -topology.

LEMMA 3.1. *The restriction defines a linear isomorphism*

$$C^{-\infty}(M)^K \simeq [C^\infty(M)^K]^*.$$

PROOF. The mentioned restriction map is clearly continuous. Let $\text{pr} : C^\infty(M) \rightarrow C^\infty(M)^K$ be the projection $\text{pr}(f)(x) = \int_K f(kx) dk$. It is continuous, hence the transposed pr^t maps $[C^\infty(M)^K]^* \rightarrow C^{-\infty}(M)$ is also continuous. It is easily seen that this provides the inverse to the restriction.

LEMMA 3.2. *Let $F : C^\infty(M)^K \rightarrow \mathbb{C}$ be linear. Then the following statements are equivalent:*

- (1) *F is a K -invariant distribution.*
- (2) *There exist $C > 0$ and $m \in \mathbb{Z}^+$ such that*

$$(3.1) \quad |F(f)| \leq C \max_{j=1, \dots, m} \|\Delta_M^j f\|_\infty \quad (\forall f \in C^\infty(M)^K).$$

- (3) *There exist $C > 0$ and finitely many $u_1, \dots, u_s \in U(\alpha)$ such that*

$$(3.2) \quad |F(f)| \leq C \max_{j=1, \dots, s} \|L_{u_j}(f|_{A \cdot o})\|_\infty \quad (\forall f \in C^\infty(M)^K).$$

PROOF. This follows from Lemmas 2.3 and 2.6.

Let $w^* \in W$ be such that $w^*(\Sigma^+) = -\Sigma^+$. Then $\mu \mapsto \mu^* := -w^*(\mu)$ defines a bijection of $\Lambda^+(M)$, such that π_{μ^*} is the contragradient representation to π_μ . Notice that $\overline{\psi_\mu} = \psi_{\mu^*} = \psi_\mu^\vee$ where $f^\vee(g) = f(g^{-1})$. Furthermore $d(\mu^*) = d(\mu)$. We define the Fourier transform of a spherical distribution $F \in C^{-\infty}(M)^K$ by

$$(3.3) \quad \tilde{F}(\mu) := F(\psi_{\mu^*}) = F(\psi_\mu^\vee).$$

In particular, for smooth K -invariant functions regarded as distributions by means of the pairing with the invariant measure, the two notions of Fourier transform agree.

LEMMA 3.3. *Let $F \in C^{-\infty}(M)^K$. Then $\mu \mapsto \tilde{F}(\mu)$ has at most polynomial growth.*

PROOF. This follows from (3.1) and (2.7).

We can now write down the Fourier series for F .

LEMMA 3.4. *Let $F \in C^{-\infty}(M)^K$ and $f \in C^\infty(M)^K$. Then*

$$(3.4) \quad F(f) = \sum_{\mu \in \Lambda^+(M)} d(\mu) \tilde{f}(\mu^*) \tilde{F}(\mu)$$

with absolute convergence. In particular, the distributional Fourier transform $F \mapsto \tilde{F}$ is injective.

PROOF. It follows from Lemma 2.5 that

$$f = \sum_{\mu \in \Lambda^+(M)} d(\mu^*) \tilde{f}(\mu^*) \psi_{\mu^*}$$

in the topology of $C^\infty(M)^K$. Since F is continuous we can apply it termwise, and since $d(\mu^*) = d(\mu)$ we then obtain (3.4) with convergence in \mathbb{C} . The absolute convergence follows from Lemma 2.4, since $d(\mu)$ and $\tilde{F}(\mu)$ have at most polynomial growth with respect to μ .

4. Local Paley-Wiener Theorem for K -invariant functions on M

We recall the main results from [27].

DEFINITION 4.1 (Paley-Wiener space). For $r > 0$ let $\text{PW}_r(\alpha)$ denote the space of holomorphic functions φ on $\alpha_{\mathbb{C}}^*$ satisfying the following.

a) For each $k \in \mathbb{Z}^+$ there exists a constant $C_k > 0$ such that

$$|\varphi(\lambda)| \leq C_k (1 + |\lambda|)^{-k} e^{r|\text{Re } \lambda|} \quad \text{for all } \lambda \in \alpha_{\mathbb{C}}^*.$$

b) $\varphi(w(\lambda + \rho) - \rho) = \varphi(\lambda)$ for all $w \in W, \lambda \in \alpha_{\mathbb{C}}^*$.

The following is Theorem 4.2 of [27]. As pointed out in [27], Remark 4.3, the known value for the constant R can be different in each part of the theorem.

THEOREM 4.2. *There exists $R > 0$ such that the following holds for each $0 < r < R$.*

- i) *Let $f \in C_r^\infty(M)^K$. Then the Fourier transform $\tilde{f}: \Lambda^+(M) \rightarrow \mathbb{C}$ extends to a function in $\text{PW}_r(\alpha)$.*
- ii) *Let $\varphi \in \text{PW}_r(\alpha)$. There exists a unique function $f \in C_r^\infty(M)^K$ such that $\tilde{f}(\mu) = \varphi(\mu)$ for all $\mu \in \Lambda^+(M)$.*
- iii) *The functions in the Paley-Wiener space $\text{PW}_r(\alpha)$ are uniquely determined by their values on $\Lambda^+(M)$.*

Thus, the Fourier transform followed by the extension gives a bijection

$$C_r^\infty(M)^K \simeq \text{PW}_r(\alpha).$$

REMARK 4.3. The proof of this theorem in [27] is not entirely correct, as an error occurs in the proof of Corollary 10.2. The function $\psi(\lambda) = \varphi(\lambda_1)\varphi_m(\lambda_2)$, constructed in the proof of the corollary is not of exponential type r as stated, but only of type $\sqrt{2}r$. This follows from the estimate $|\lambda_1| + |\lambda_2| \leq \sqrt{2}|\lambda|$, which is sharp. However, one can apply the theorem of [11] to construct an entire function ψ on \mathfrak{h}_C^* , which is of the proper exponential type, and which restricts to φ on α . The rest of the proof is then unchanged.

5. Analytic continuation of spherical functions

We need some details from [27] concerning the analytic continuation of the spherical functions ψ_μ with respect to the parameter μ .

Let $\bar{\Omega}$ be the closure of

$$(5.1) \quad \Omega = \left\{ X \in \alpha \mid (\forall \alpha \in \Sigma) |\alpha(X)| < \frac{\pi}{2} \right\}.$$

As U is compact, it follows that U is contained in a complex Lie group U_C with Lie algebra \mathfrak{u}_C . Denote by $K_{0,C}$ the analytic subgroup of U_C corresponding to \mathfrak{k}_C . Note that we are at this point not assuming that θ extends to an involution on U_C . Let $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{q} \subset \mathfrak{u}_C$ and let G be the corresponding analytic subgroup of U_C , then $K_0 \subset G$ is a maximal compact subgroup. The space $M^d = G/K_0$ is the (noncompact) dual of U/K_0 . Note that the center of G is contained in K_0 so M^d is independent of the choice of the complexification U_C . Let $K_C = K K_{0,C}$. Then $M, M^d \subset M_C := U_C/K_C$. Then K_C is a closed subgroup of U_C . For each $\mu \in \Lambda^+(M)$ the spherical function ψ_μ has an analytic continuation to $M_C = U_C/K_C$, denoted by the same symbol, and $\psi_\mu|_{M^d} = \varphi_{\mu+\rho}$ where φ_λ denotes the spherical function on M^d with spectral parameter λ . According to [24], [29] (see also the proof due to J. Faraut in [9]) the spherical function φ_λ on M^d has a holomorphic extension as a K_C -invariant function on $K_C \exp(2\Omega) \cdot o$ for every $\lambda \in \alpha_C^*$. For each $x \in K_C \exp(2\Omega) \cdot o$ and $\lambda \in \alpha_C^*$ we define

$$(5.2) \quad \psi_\lambda(x) = \varphi_{\lambda+\rho}(x)$$

and thus obtain an extension to α_C^* of the map $\mu \mapsto \psi_\mu(x)$ where $\mu \in \Lambda^+(M)$. The map $(\lambda, x) \mapsto \psi_\lambda(x)$ is holomorphic on the open set $\alpha_C^* \times \exp(\Omega + i\alpha) \cdot o \subset \alpha_C^* \times A_C \cdot o$ and it satisfies the following estimate, cf. [29], Proposition 6.1:

LEMMA 5.1. *There exists a constant C such that*

$$(5.3) \quad |\psi_\lambda(\exp(X + iY) \cdot o)| \leq C e^{\max_{w \in W} \operatorname{Re} w\lambda(X) - \min_{w \in W} \operatorname{Im} w\lambda(Y)}$$

for all $X \in \bar{\Omega}$, $Y \in \alpha$ and $\lambda \in \alpha_C^*$.

COROLLARY 5.2. *Let $X_1, \dots, X_j \in \alpha$ and $X \in \Omega$. There exists a constant C such that*

$$(5.4) \quad |L_{X_1 \dots X_j} \psi_\lambda(\exp(X) \cdot o)| \leq C (1 + |\lambda|)^j e^{|\lambda| \operatorname{Re} \lambda}$$

for all $\lambda \in \alpha_c^*$. The constant C depends locally uniformly on X .

PROOF. Let V be a complex neighborhood of 0 such that $X + V \subset \Omega + i\alpha$. Let Z_1, \dots, Z_ℓ be an orthonormal basis for α . By considering linear combinations and using that α is abelian it is enough to prove the claim for the derivatives

$$\left(\frac{\partial}{\partial x_1} \right)^{m_1} \dots \left(\frac{\partial}{\partial x_\ell} \right)^{m_\ell} \psi_\lambda(\exp(X + x_1 Z_1 + \dots + x_\ell Z_\ell) \cdot o)|_{x_1 = \dots = x_\ell = 0}.$$

To simplify the notation let

$$f_\lambda(x_1, \dots, x_\ell) = \psi_\lambda(\exp(X + x_1 Z_1 + \dots + x_\ell Z_\ell) \cdot o).$$

We will also use the following notation for $m = (m_1, \dots, m_\ell) \in (\mathbb{Z}^+)^{\ell}$ and $\zeta = (\zeta_1, \dots, \zeta_\ell) \in \mathbb{C}^{\ell}$: $m! := m_1! \dots m_\ell!$, $|m| = m_1 + \dots + m_\ell$, $m + 1 = (m_1 + 1, \dots, m_\ell + 1)$, and $\zeta^m = \zeta_1^{m_1} \dots \zeta_\ell^{m_\ell}$. We also set $\partial^m = (\partial/\partial x_1)^{m_1} \dots (\partial/\partial x_\ell)^{m_\ell}$.

Let $\epsilon_0 > 0$ be so small that

$$\{z_1 Z_1 + \dots + z_\ell Z_\ell \mid |z_j| < \epsilon_0 \text{ for } j = 1, \dots, \ell\} \subset V.$$

Then f_λ is holomorphic on $\{z = (z_1, \dots, z_\ell) \mid |z_j| < \epsilon_0 \text{ for } j = 1, \dots, \ell\}$. By Cauchy's integral theorem for the derivatives of f_λ we get for each $\epsilon < \epsilon_0$

$$\partial^m f_\lambda(0) = \frac{m!}{(2\pi i)^\ell} \oint_{|z_1|=\epsilon} \dots \oint_{|z_\ell|=\epsilon} \frac{f_\lambda(\zeta)}{\zeta^{m+1}} d\zeta_1 \dots d\zeta_\ell.$$

Thus (5.3) implies, with the same constant C as in (5.3), that

$$\begin{aligned} |\partial^m f_\lambda(0)| &\leq C m! (2\pi)^{-\ell} \epsilon^{-(|m|+\ell)} e^{|\lambda| \operatorname{Re} \lambda} e^{\epsilon \ell (\operatorname{Re} \lambda + |\operatorname{Im} \lambda|)} (2\pi \epsilon)^\ell \\ &= C m! e^{\epsilon \ell (\operatorname{Re} \lambda + |\operatorname{Im} \lambda|)} \epsilon^{-|m|} e^{|\lambda| \operatorname{Re} \lambda}. \end{aligned}$$

Now, take

$$\epsilon = \frac{\epsilon_0}{\ell(1 + |\lambda|)}$$

then, with a new constant C depending on V , but independent of λ and X , we get

$$|\partial^m f_\lambda(0)| \leq C (1 + |\lambda|)^{|m|} e^{|\lambda| \operatorname{Re} \lambda}$$

as was to be shown.

6. Paley-Wiener Theorem for Distributions

In this section we state and prove the Paley-Wiener theorem for distributions on M .

DEFINITION 6.1 (Paley-Wiener space for distributions). For $r > 0$ let $PW_r^*(\alpha)$ denote the space of holomorphic functions Φ on α_C^* satisfying the following.

- a) There exists a $k \in \mathbb{Z}^+$ and a constant $C_k > 0$ such that

$$|\Phi(\lambda)| \leq C_k(1 + |\lambda|)^k e^{r|\operatorname{Re} \lambda|}$$

for all $\lambda \in \alpha_C^*$.

- b) $\Phi(w(\lambda + \rho) - \rho) = \Phi(\lambda)$ for all $w \in W, \lambda \in \alpha_C^*$.

Let $r > 0$. A distribution F has support in $\bar{D}_r(o)$ if and only if $F(f) = 0$ for all $f \in C^\infty(M)$ with $\operatorname{Supp}(f) \subset M \setminus \bar{D}_r(o)$. Denote by $C_r^{-\infty}(M)$ the space of distributions that are supported on $\bar{D}_r(o)$.

REMARK 6.2. Recall (see (3.2)) that every distribution F on M satisfies an estimate

$$|F(f)| \leq C \sup_{x \in M, j \leq k} |\Delta_M^j f(x)|.$$

If the support of F is contained in some compact subset $S \subset M$, it is tempting to replace the supremum over $x \in M$ by the supremum over $x \in S$, but in general such an estimate is false. The supremum *has* to be taken over an open neighborhood of S (see [31], example p. 95 and discussion p. 98–100). This causes a minor complication in the proof of Theorem 6.4 (this problem appears to be overlooked in [12]).

We need the following elementary result.

LEMMA 6.3. Let $\Omega \subset \mathbb{C}^\ell$ be open and let M be a differentiable manifold. Let $f \in C^\infty(\Omega \times M)$, and assume that f is holomorphic in the first variable. Then $z \mapsto f(z, \cdot)$ is holomorphic as a map $\Omega \rightarrow C^\infty(M)$.

PROOF. We first observe that for $a \in \Omega \subset \mathbb{R}^{2\ell}$ and $f \in C^\infty(\Omega \times M)$, we have

$$\frac{f(a + he_j, \cdot) - f(a, \cdot)}{h} \rightarrow \frac{\partial f}{\partial x_j}(a, \cdot)$$

in $C^\infty(M)$ for $h \rightarrow 0$ and $j = 1, \dots, 2\ell$. Hence, if T is a continuous linear form on $C^\infty(M)$, it follows that $a \mapsto T(f(a, \cdot))$ is differentiable on Ω with

$$\frac{\partial}{\partial x_j} [T(f(a, \cdot))] = T\left(\frac{\partial f}{\partial x_j}(a, \cdot)\right).$$

It follows from this observation that $z \mapsto T(f(z, \cdot))$ is continuously differentiable and satisfies the Cauchy-Riemann equations, for each continuous linear form T on $C^\infty(M)$. Hence $z \mapsto f(z, \cdot)$ is weakly holomorphic into $C^\infty(M)$, and, as this space is Fréchet, also strongly holomorphic.

THEOREM 6.4 (Local Paley-Wiener theorem for distributions). *There exists $R > 0$ such that the following holds for each $0 < r < R$.*

- i) *Let $F \in C_r^{-\infty}(M)^K$. Then the Fourier transform $\tilde{F}: \Lambda^+(M) \rightarrow \mathbb{C}$ extends to a function in $\text{PW}_r^*(\alpha)$.*
- ii) *Let $\Phi \in \text{PW}_r^*(\alpha)$. There exists a unique distribution $F \in C_r^{-\infty}(M)^K$ such that $\tilde{F}(\mu) = \Phi(\mu)$ for all $\mu \in \Lambda^+(M)$.*
- iii) *The functions in the Paley-Wiener space $\text{PW}_r^*(\alpha)$ are uniquely determined by their values on $\Lambda^+(M)$.*

Thus, the Fourier transform followed by the extension gives a bijection

$$C_r^{-\infty}(M)^K \simeq \text{PW}_r^*(\alpha).$$

REMARK 6.5. Note, that as in Theorem 4.2, R can be different in each part of the above theorem.

PROOF. (i) Let $R > 0$ be such that $D_R(o) \subset K \exp \Omega \cdot o$, where Ω is defined in (5.1). Let $r < R$ and let $\epsilon > 0$ be so that $r + \epsilon < R$. Let $\varphi \in C^\infty(M)^K$ be a function which is 1 on a neighborhood of the closed ball $\bar{D}_r(o)$, and supported on $\bar{D}_{r+\epsilon}(o)$. The product $\varphi\psi_\lambda$ is a globally defined smooth function on M , and it belongs to $C^\infty(M)^K$ for all $\lambda \in \alpha_\mathbb{C}^*$. Let $F \in C_r^{-\infty}(M)^K$. We extend the Fourier transform of F to a function on $\alpha_\mathbb{C}^*$ by

$$(6.1) \quad \tilde{F}(\lambda) := F(\varphi\psi_\lambda^\vee).$$

The extension is independent of the choice of φ . Note also that $\tilde{F}(w(\lambda + \rho) - \rho) = \tilde{F}(\lambda)$ and that

$$\tilde{F}(\lambda) = F(\varphi\varphi_{\lambda+\rho}^\vee) = F(\varphi\varphi_{-\lambda-\rho}) = F(\varphi\psi_{-\lambda-2\rho}).$$

Since the map $(\lambda, x) \mapsto \psi_\lambda(x)$ is smooth on the open subset $\alpha_\mathbb{C}^* \times \exp(\Omega) \cdot o$ of $\alpha_\mathbb{C}^* \times A \cdot o$, it follows that $(\lambda, x) \mapsto \varphi(x)\psi_\lambda(x)$ is smooth on $\alpha_\mathbb{C}^* \times A \cdot o$. By Lemma 6.3 it follows that $\lambda \mapsto \varphi\psi_\lambda$ is holomorphic into $C^\infty(A \cdot o)$, and as it is also W -invariant in the A -variable, it follows from Lemma 2.6 that it is holomorphic into $C^\infty(M)^K$. Hence $\lambda \mapsto \tilde{F}(\lambda)$ is holomorphic on $\alpha_\mathbb{C}^*$.

We still need to show that this extension has exponential growth with exponent r . For that we will choose the function φ of (6.1) in such a way that we

can control the right hand side of equation (6.2) below (this is similar to what is done in the Euclidean case, see for example [22]).

As F is a K -invariant distribution, it follows by Lemma 3.2 that there exists finitely many $u_1, \dots, u_s \in U(\alpha)$ and a constant $C > 0$, such that

$$(6.2) \quad |\tilde{F}(\lambda)| \leq C \max_{i=1, \dots, s} \|L_{u_i}(\varphi|_{A \cdot o} \psi_\lambda^\vee|_{A \cdot o})\|_\infty.$$

Let $m = \max_{i=1, \dots, s} \deg u_i$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be such that $0 \leq h \leq 1$, $h|_{(-\infty, 1/3]} = 1$, and $\text{Supp}(h) \subseteq (-\infty, 2/3]$. Let $C > 0$ be such that $\|h^{(j)}\|_\infty \leq C$ for $j = 0, \dots, m$ (where m is as above). Finally, for $\delta > 0$ let $h_\delta(t) = h(t/\delta)$. Then h_δ has the properties that

- (1) $0 \leq h_\delta \leq 1$,
- (2) $h_\delta(t) = 1$ for all $t \leq \delta/3$,
- (3) $h_\delta(t) = 0$ if $2\delta/3 \leq t$,
- (4) $|h_\delta^{(j)}(t)| \leq C\delta^{-j}$ for all $t \in \mathbb{R}$, $j = 0, 1, \dots, m$ and $\delta > 0$.

Recall that $r + \epsilon < R$ and let $\delta \leq \epsilon$ be arbitrary for the moment. Then $r + \delta < R$. Let

$$\varphi(x) = h_\delta(d(x, o) - r)$$

for $x \in M$. Then $\text{Supp}(\varphi) \subset \bar{D}_{r+\epsilon}(o)$ and $\varphi = 1$ on a neighborhood of $\bar{D}_r(o)$. Let $j \leq m$ and let $X_1, \dots, X_j \in \alpha$ with $|X_i| = 1$. By applying the chain and Leibniz rules we obtain

$$(6.3) \quad |X_1 \dots X_j \varphi(x)| \leq C_1 \delta^{-j}$$

for some constant $C_1 > 0$. Note that C_1 is independent of δ . In fact it only depends on the constant C above, and the derivatives of $x \mapsto d(x, o)$ on the compact set $\{x \in M \mid r \leq d(x, o) \leq r + \epsilon\}$. As $d(\cdot, o)$ is smooth away from o it follows that those derivatives are bounded independently of δ .

For the derivatives $X_1 \dots X_j \psi_\lambda^\vee$ we note first that $|\text{Re}(-\lambda - 2\rho)| = |\text{Re} \lambda|$ as $\rho \in i\alpha^*$. By Corollary 5.2 we get for $X \in \alpha$, $|X| < R$:

$$(6.4) \quad |X_1 \dots X_j \psi_\lambda^\vee(\text{Exp } X)| = D(1 + |\lambda|)^j e^{(r+\delta)|\text{Re} \lambda|}$$

for some constant D , independent of λ .

Using (6.1), the estimates (6.2), (6.3), (6.4), and the Leibniz rule, it follows that there exists a constant $C > 0$ such that for all $\lambda \in \alpha_C^*$ and every $\delta \leq \epsilon$ we have

$$|\tilde{F}(\lambda)| \leq C\delta^{-m}(1 + |\lambda|)^m e^{(r+\delta)|\text{Im} \lambda|}.$$

We now specialize to $\delta = (1 + |\lambda|)^{-1}\epsilon$ and conclude that $\tilde{F} \in \text{PW}_r^*(\alpha)$.

(ii) Let $\Phi \in \text{PW}_r^*(\alpha)$. The asserted uniqueness of F follows from Lemma 3.4. Motivated by (3.4) in that lemma we define $F : C^\infty(M)^K \rightarrow \mathbb{C}$ by

$$(6.5) \quad F(f) := \sum_{\mu \in \Lambda^+(M)} d(\mu^*) \tilde{f}(\mu^*) \Phi(\mu).$$

We need to justify the convergence of the sum. Let $\omega(\lambda) = \langle \lambda, \lambda + 2\rho \rangle$. Then $\Delta_M \psi_\mu = -\omega(\mu) \psi_\mu$. Let $\Lambda_1 = \Lambda^+(M) \setminus \{0\}$, and observe that $\omega(\mu) > 0$ for all $\mu \in \Lambda_1$. Let $D_1 > 0$ be such that

$$(\forall \mu \in \Lambda_1) \quad \omega(\mu) \geq D_1(1 + |\mu|).$$

By Weyl's dimension formula, there exists a constant $D_2 > 0$ and $m \in \mathbb{Z}^+$ such that

$$d(\mu^*) \leq D_2(1 + |\mu|)^m.$$

Let $k \in \mathbb{Z}^+$ be such that

$$|\Phi(\lambda)| \leq C(1 + |\lambda|)^k e^{r|\text{Re } \lambda|},$$

and let $s \in \mathbb{Z}^+$ be such that

$$s > m + k + t_0$$

where t_0 is as in Lemma 2.4, ensuring that $\sum_{\mu \in \Lambda_1} (1 + |\mu|)^{m+k-s} < \infty$. Using the fact that $\Delta_M \psi_\mu = -\omega(\mu) \psi_\mu$ we get:

$$(6.6) \quad \sum_{\mu \in \Lambda_1} d(\mu^*) |\tilde{f}(\mu^*)| |\Phi(\mu)| = \sum_{\mu \in \Lambda_1} d(\mu^*) \frac{|(\Delta_M^s \tilde{f})(\mu^*)|}{|\omega(\mu)|^s} |\Phi(\mu)|$$

$$(6.7) \quad \leq C_s \left(\sum_{\mu \in \Lambda_1} (1 + |\mu|)^{m+k-s} \right) \|\Delta_M^s(f)\|_\infty$$

$$(6.8) \quad < \infty.$$

Here $C_s = CD_1D_2$. It follows that

$$|F(f)| \leq |\Phi(0)| \|f\|_\infty + C_s \|\Delta_M^s f\|_\infty < \infty.$$

Thus F is well defined and continuous. It is linear by definition. It follows that $F \in C^{-\infty}(M)^K$. Let $\mu \in \Lambda^+(M)$. By application of (6.5) to $f = \psi_\mu^\vee$ it follows, using (2.5), that $\tilde{F}(\mu) = \Phi(\mu)$.

To finish the proof of (ii), we need to show $\text{Supp}(F) \subseteq \overline{D}_r(o)$. For each $\epsilon > 0$ such that $r + \epsilon < R$ let $f_\epsilon \in C^\infty(M)^K$ be positive with $\text{Supp}(f_\epsilon) \subseteq$

$\bar{D}_\epsilon(o)$ and $\int_M f_\epsilon(x) dx = 1$. Note that $|\tilde{f}_\epsilon(\mu)| \leq 1$ for all $\mu \in \Lambda^+(M)$ and that $\lim_{\epsilon \rightarrow 0} \tilde{f}_\epsilon(\mu) \rightarrow 1$ for each $\mu \in \Lambda^+(M)$.

Denote the holomorphic extension of \tilde{f}_ϵ by the same letter and recall that $\tilde{f}_\epsilon \in \text{PW}_\epsilon(\alpha)$. Let $\phi_\epsilon(\lambda) := \Phi(\lambda)\tilde{f}_\epsilon(\lambda)$. Then $\phi_\epsilon \in \text{PW}_{r+\epsilon}(\alpha)$. By Theorem 4.2, part (ii), there exists $F_\epsilon \in C_{r+\epsilon}^\infty(M)^K$ such that $\tilde{F}_\epsilon = \phi_\epsilon$. In particular

$$\int f(x)F_\epsilon(x) dx = \sum_{\mu \in \Lambda^+(M)} d(\mu^*)\tilde{f}(\mu^*)\tilde{f}_\epsilon(\mu)\Phi(\mu)$$

for all $f \in C^\infty(M)^K$. Hence, using (6.6) to justify the limit,

$$\lim_{\epsilon \rightarrow 0} \int_M f(x)F_\epsilon(x) dx = F(f).$$

As the support of F_ϵ is contained in $\bar{D}_{r+\epsilon}(o)$ it follows that the support of F is contained in

$$\bigcap_{\epsilon > 0} \bar{D}_{r+\epsilon}(o) = \bar{D}_r(o).$$

(iii) follows as in the proof of Theorem 4.2 in [27], given in Section 7 of that paper.

Let $F \in C^{-\infty}(M)^K$. The *singular support* of F , is the complement of the largest open set on which F is given by a smooth function.

THEOREM 6.6 (Characterization of singular support). *Let R be as in Theorem 4.2 and $0 < s \leq r < R$. Let $F \in C_r^{-\infty}(M)^K$. Then the singular support of F is contained in $\bar{D}_s(o)$ if and only if there exists $N \in \mathbb{Z}$ such that for each $m \in \mathbb{Z}^+$ the holomorphic extension of \tilde{F} satisfies*

$$|\tilde{F}(\lambda)| \leq C_m(1 + |\lambda|)^N e^{s|\text{Re } \lambda|}$$

for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ with $|\text{Re } \lambda| \leq m \log(1 + |\lambda|)$.

PROOF. The proof, in which Theorems 4.2 and 6.4 play the crucial roles, is similar to that of Propositions 1.3 and 1.4 in [12], which is a reduction to Hörmander’s theorem for the Euclidean case.

If D is a differential operator on M , then we define the differential operator D^* by

$$\int_M Df(x)g(x) dx = \int_M f(x)D^*g(x) dx$$

for $f, g \in C^\infty(M)$. Recall also the definition of the Harish-Chandra isomorphism $\gamma^* : \mathfrak{D}(M) \rightarrow S(\mathfrak{a}^*)^W$ from Lemma 5.1 in [27].

THEOREM 6.7 (Solvability for distributions). *Let $0 < r < R$, let $F \in C_r^{-\infty}(M)^K$, and let $D \in \mathcal{D}(M)$. Then there exists a $T \in C_r^{-\infty}(M)^K$ such that $DT = F$ if and only if $\lambda \mapsto \tilde{F}(\lambda)/\gamma(D^*, -\mu - \rho)$ is entire. Moreover, in that case T is unique.*

PROOF. The proof is the same as that of Theorem 1.8, p. 419, in [20].

7. Appendix

In this appendix we discuss the extension to compact symmetric spaces of the preceding results. We begin by generalizing the results from [27].

Let U be a connected compact Lie group, not necessarily semisimple. As before, let θ be an involution and let $M = U/K$ where $U_0^\theta \subset K \subset U^\theta$. Let Z denote the center of U . We assume that $Z \cap K = \{e\}$, since otherwise we can replace U by $U/Z \cap K$, noticing that $Z \cap K$ acts trivially on U/K . Let \mathfrak{u} denote the Lie algebra of U , then $\mathfrak{u} = \mathfrak{z} \oplus \mathfrak{u}'$ where \mathfrak{z} is the center of \mathfrak{u} and $\mathfrak{u}' = [\mathfrak{u}, \mathfrak{u}]$ is semisimple. As before we denote by θ also the induced involution of \mathfrak{u} , and by $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ the corresponding Cartan decomposition. Then \mathfrak{z} and \mathfrak{u}' are θ -invariant, and it follows from our assumption that $\mathfrak{z} \subset \mathfrak{q}$ and $\mathfrak{u}' \supset \mathfrak{k}$. Denote by Z_0 and U' the analytic subgroups of U corresponding to \mathfrak{z} and \mathfrak{u}' . Then $U = Z_0 U'$ and $Z_0 \cap U'$ is finite. It follows that $Z_0 \times U'$ is a covering of U by the homomorphism $(z, u) \mapsto zu$. The kernel is $D = \{(z, z^{-1}) \mid z \in Z_0 \cap U'\}$. Thus the covering is

$$Z_0 \times U' \rightarrow U \simeq (Z_0 \times U')/D.$$

The identity component K_0 of K is contained in U' , hence the subgroup $K' = U' \cap K$ is a symmetric subgroup of U . In general K' can be a proper subgroup of K , in spite of the assumption that $Z \cap K = \{e\}$. Let $K^\times \subset Z_0 \times U'$ denote the preimage of K , then it follows that the covering above induces a covering map

$$Z_0 \times U'/K' \rightarrow U/K \simeq (Z_0 \times U')/K^\times.$$

We fix a maximal abelian subspace α of \mathfrak{q} , then $\mathfrak{z} \subset \alpha$. As before, $\Sigma \subset i\alpha^*$ denotes the sets of restricted roots, and Σ^+ denotes a subset of positive roots. We denote by $\Lambda^+(U/K) \subset i\alpha^*$ the set of highest weights of irreducible K -spherical representations. It is now seen that Lemma 2.2 is valid in the generalized situation too: For $Z_0 \times U'/K'$ this is a straightforward extension, otherwise we repeat the proof of the lemma, replacing K^* with K^\times .

We define the spherical Fourier transform of a K -invariant function on U/K as before, and for each $r > 0$ we define the Paley-Wiener space $\text{PW}_r(\alpha)$ exactly as in Definition 4.1. In particular, W is the Weyl group associated with the root system Σ . We can then state:

THEOREM 7.1. *Let U/K be a compact Riemannian symmetric space with assumptions as described above. Then Theorem 4.2 is valid exactly as stated in Section 4.*

PROOF. For $Z_0 \times U'/K'$ this is a straightforward extension of the proof given in [27]. For the general case we apply the covering map above. Here it is used that every smooth function f on U/K supported on a sufficiently small K -invariant neighborhood of eK lifts to a smooth function F on the cover $Z_0 \times U'/K'$, supported in a K' -invariant neighborhood of eK' of the same size. The lifted function F is K' -invariant if and only if f is K -invariant, and the Fourier transform \tilde{F} of the lifted function restricts to the Fourier transform \tilde{f} of the original function on $\Lambda^+(U/K) \subset \Lambda^+(Z_0 \times U'/K')$. Noticing that by definition $\text{PW}_r(\alpha)$ is the same space in the two cases U/K and $Z_0 \times U'/K'$, we thus have a commutative diagram of bijective maps

$$\begin{array}{ccc} C_r^\infty(Z_0 \times U'/K')^{K'} & \longrightarrow & \text{PW}_r(\alpha) \\ \uparrow & & \parallel \\ C_r^\infty(U/K)^K & \longrightarrow & \text{PW}_r(\alpha), \end{array}$$

for r sufficiently small. The horizontal arrows represent Fourier transform followed by holomorphic extension, and it follows from Lemma 2.2 by the argument in [27] Section 7, that functions in $\text{PW}_r(\alpha)$ are uniquely determined by their restriction to $\Lambda^+(U/K)$. The theorem is now easily proved.

The main results of the present paper, Theorems 6.4 and 6.6, can now be generalized to the present setting by a straightforward extension of the previous proof. We omit the details.

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