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APPROXIMATION BY INVERTIBLE ELEMENTS AND THE GENERALIZED E-STABLE RANK FOR $A(\mathbf{D})_{\mathsf{R}}$ AND $C(\mathbf{D})_{\mathrm{sym}}$

RAYMOND MORTINI and RUDOLF RUPP

Abstract

We determine the generalized E-stable ranks for the real algebra, $C(\mathbf{D})_{\mathrm{sym}}$, of all complex valued continuous functions on the closed unit disk, symmetric to the real axis, and its subalgebra $A(\mathbf{D})_{\mathsf{R}}$ of holomorphic functions. A characterization of those invertible functions in $C(E)$ is given that can be uniformly approximated on E by invertibles in $A(\mathbf{D})_{\mathsf{R}}$. Finally, we compute the Bass and topological stable rank of $C(K)_{\mathrm{sym}}$ for real symmetric compact planar sets K.

1. Introduction and notation

Let $A(\mathbf{D})$ denote the disk algebra; that is the algebra of all continuous functions on the closed unit disk $\mathbf{D}$ that are holomorphic in $\{z \in \mathsf{C} : |z| < 1\}$. Its real counter part, $A(\mathbf{D})_{\mathsf{R}}$, is the algebra (over R) of all functions in $A(\mathbf{D})$ that are real on the interval $[-1, 1]$. This algebra plays a prominent role in control theory. We remark that $A(\mathbf{D})$ is the complexification $\overset{c}{A}$ of $A = A(\mathbf{D})_{\mathsf{R}}$. Indeed, by definition $\overset{c}{A} = \{u + iv : u, v \in A\}$. For $f \in A(\mathbf{D})$, let $\check{f}$ be defined as

$$\check{f}(z) = \overline{f(\overline{z})},$$

Choose $u = (f + \check{f})/2$ and $v = (f - \check{f})/(2i)$. Then $u, v \in A(\mathbf{D})_{\mathsf{R}}$ and $f = u + iv$.

In this paper we look upon $A(\mathbf{D})_{\mathsf{R}}$ as a function algebra defined on $\mathbf{D}$ (see [12]). Our goal in section two is to determine the generalized E-stable rank for $A(\mathbf{D})_{\mathsf{R}}$ and to characterize those functions zero free on E that can be uniformly approximated on E by elements invertible in $A(\mathbf{D})_{\mathsf{R}}$. Only recently it has been shown (see [22] and [14]) that the Bass and topological stable rank of $A(\mathbf{D})_{\mathsf{R}}$ coincide:

$$\mathrm{bsr}(A(\mathbf{D})_{\mathsf{R}}) = \mathrm{tsr}(A(\mathbf{D})_{\mathsf{R}}) = 2.$$

The concept of E-stable rank has been introduced in [13]. Let A be a real or complex algebra of functions on $\mathbf{D}$. We say that for a closed set $E \subseteq \mathbf{D}$ the

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generalized E-stable rank of A is one (denoted by $\mathrm{gsr}_E(A) = 1$) if for every pair of elements in A with $|f|+|g| \geq \delta > 0$ on E there exists $h \in A$ such that $f + gh$ has no zeros on E. Similarly, if $\mathrm{gsr}_E(A)$ is not one and if (f_1, f_2, g) is a triple in A with $|f_1|+|f_2|+|g| \geq \delta > 0$ on E, then we say that $\mathrm{gsr}_E(A) = 2$ if there exists (a_1, a_2) and $(h_1, h_2) \in A^2$ such that

$$a_1(f_1 + gh_1) + a_2(f_2 + gh_2) \text{ has no zeros on } E.$$

In [13], it was shown that for any closed subset E of $\mathbf{D}$, one has $\mathrm{gsr}_E(A(\mathbf{D})) = 1$. It turns out that the situation for $A(\mathbf{D})_{\mathsf{R}}$ is quite different (see Theorem 2.4).

In the third section we will be concerned with the real symmetric algebras

$$C(K)_{\mathrm{sym}} = \{f \in C(K),\ f \text{ complex valued and real symmetric}\},$$

where $K \subseteq \mathsf{C}$ is compact and real symmetric. Recall that a closed set $E \subseteq \mathbf{D}$ is *real symmetric* if E coincides with its reflection $E^* = \{z : \overline{z} \in E\}$ with respect to the real axis. The set $E \cup E^*$ is called the symmetrization of E. Real symmetric functions are those that are defined on a real symmetric set E and satisfy $f(z) = \overline{f(\overline{z})}$ for all $z \in E$. It is clear that $A(\mathbf{D})_{\mathsf{R}}$ is the set of real symmetric functions in $A(\mathbf{D})$. In section three we will compute the Bass and topological stable rank of the algebras $C(K)_{\mathrm{sym}}$, thus answering questions in [23]. We also determine the generalized E-stable ranks for $C(\mathbf{D})_{\mathrm{sym}}$.

Next we briefly recall several definitions. Let A be a real or complex Banach function algebra on a compact set X. We assume that A separates the points of X and has as unit the constant function 1. The zero set of $f \in A$ is the set $Z(f) = \{z \in X : f(z) = 0\}$.

An n-tuple $(f_1, \ldots, f_n) \in A^n$ is said to be *invertible* (or unimodular), if there exists $(x_1, \ldots, x_n) \in A^n$ such that $\sum_{j=1}^n x_j f_j = 1$. The set of all invertible n-tuples is denoted by $U_n(A)$. An $(n+1)$-tuple $(f_1, \ldots, f_n, g) \in U_{n+1}(A)$ is called *reducible* if there exists $(a_1, \ldots, a_n) \in A^n$ such that $(f_1 + a_1 g, \ldots, f_n + a_n g) \in U_n(A)$.

Let $E \subseteq X$ be closed. A pair $(f, g) \in A^2$ is said to be *E-reducible* (or *reducible over E*), if there exists $a \in A$ such that $f + ag \neq 0$ on E.

The *Bass stable rank* of A, denoted by $\mathrm{bsr}(A)$, is the smallest integer n such that every element in $U_{n+1}(A)$ is reducible. If no such n exists, then $\mathrm{bsr}(A) = \infty$ (see [2]).

The *topological stable rank*, $\mathrm{tsr}(A)$, of A is the least integer n for which $U_n(A)$ is dense in A^n, or infinite if no such n exists.

The *approximate stable rank*, $\mathrm{appsr}(A)$, of A is the least integer n such that for every A-convex set E in the character space of A the Gelfand transform of any n-tuple $(f_1, \dots, f_n) \in A^n$ satisfying $\sum_{j=1}^n |\hat{f_j}| \geq \delta > 0$ on E can be uniformly approximated on E by the Gelfand transform of n-tuples $(a_1, \dots, a_n)$ that are invertible in A.

In [16] it is mentioned (without proof) that appsr(A) coincides in case of a uniform algebra A over C with the dense stable rank, dsr(A), of A that was introduced by Corach and Suárez [7, p. 542]. Moreover, they noticed that bsr$(A) \leq$ dsr$(A) \leq$ tsr(A). Since we are dealing here with real algebras, it is nice to have a direct proof that bsr$(A) \leq$ appsr$(A) \leq$ tsr(A). This will be given in the appendix.

Since we are using this concept only for the algebras $A = A(\mathbf{D})_{\mathsf{R}}$ and $C(\mathbf{D})_{\text{sym}}$, it suffices to know that the A-convex sets for $C(\mathbf{D})_{\text{sym}}$ are just the compact real symmetric subsets of $\mathbf{D}$ and that the A-convex sets for $A(\mathbf{D})_{\mathsf{R}}$ are those real symmetric compact sets in $\mathbf{D}$ that have no holes. Here, as usual, a hole of a compact set $K \subseteq \mathsf{C}$ is a bounded component of $\mathsf{C} \setminus K$.

We refer the reader to [3], [4], [5], [6], [10], [13], [14], [19], [20], [21], [22], [23], [26] for a glimpse on several aspects and computations of stable ranks.

We conclude this section with the following notation, that will be used throughout the paper. For a subset $E \subseteq X$ and a function defined on X, let $\|f\|_E := \sup\{|f(x)| : x \in E\}$ and $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Finally, let $\mathbf{D}^+ = \{z \in \mathbf{D} : \operatorname{Im} z \geq 0\}$ and $\mathbf{D}^- = \{z \in \mathbf{D} : \operatorname{Im} z \leq 0\}$.

2. The real disk algebra $A(\mathbf{D})_{\mathsf{R}}$

LEMMA 2.1. *Let $f \in A(\mathbf{D})$ and $g \in A(\mathbf{D})_{\mathsf{R}}$. Suppose that $(f, g) \in U_2(A(\mathbf{D}))$. Then there exist $h, u \in A(\mathbf{D})_{\mathsf{R}}$, u invertible in $A(\mathbf{D})_{\mathsf{R}}$, such that $f\check{f} + hg = u$. In particular, if $(f, g) \in U_2(A(\mathbf{D})_{\mathsf{R}})$, then (f^2, g) is reducible in $A(\mathbf{D})_{\mathsf{R}}$.*

PROOF. Since bsr$(A(\mathbf{D})) = 1$, (see [10], [5]), there exists $a \in A(\mathbf{D})$ so that $f + ag =: v$ is invertible in $A(\mathbf{D})$. Note that $g = \check{g}$ since $g \in A(\mathbf{D})_{\mathsf{R}}$. Conjugation then gives $\check{f} + \check{a}g = \check{v}$. Mutiplying both equations yields

$$f\check{f} + [(a\check{f} + \check{a}f) + a\check{a}g]g = v\check{v}.$$

The assertion now follows from the facts that $a\check{f} + \check{a}f$ and $a\check{a} \in A(\mathbf{D})_{\mathsf{R}}$ and that $v\check{v}$ is invertible in $A(\mathbf{D})_{\mathsf{R}}$.

LEMMA 2.2. *Let $(f_0, \dots, f_n)$ be an $(n+1)$-tuple in $A(\mathbf{D})$ satisfying $\sum_{j=0}^n |f_j| \geq \delta > 0$ on some closed set $E \subseteq \mathbf{D}$. Then there exists $(a_1, \dots, a_n) \in A(\mathbf{D})^n$ such that $f_0 + \sum_{j=1}^n a_j f_j \neq 0$ on E.*

PROOF. Since $A(\mathbf{D})$ is a separating algebra, there exists by [24, Lemma 2.5, p. 244] $b_j \in A(\mathbf{D})$ such that $\sum_{j=0}^n b_j f_j \neq 0$ on E. Then $\left(f_0, \sum_{j=1}^n b_j f_j\right)$ is a pair in $A(\mathbf{D})$ satisfying $|f_0|+\left|\sum_{j=1}^n b_j f_j\right| \geq \tilde{\delta} > 0$ on E. Since $\operatorname{gsr}_E(A(\mathbf{D})) = 1$ ([13]), there exists $a \in A(\mathbf{D})$ so that $f_0 + a\sum_{j=1}^n b_j f_j \neq 0$ on E.

Proposition 2.3. $\operatorname{gsr}_E(A(\mathbf{D})_{\mathsf{R}}) \leq 2$ *for any closed set* $E \subseteq \mathbf{D}$.

Proof. Let (f_1, f_2, f_3) be a triple in $A(\mathbf{D})_{\mathsf{R}}$ satisfying $|f_1|+|f_2|+|f_3| \geq \delta > 0$ on E. We may assume that E is real symmetric, otherwise consider the set $E \cup E^*$. By Lemma 2.2, there exists $(a_2, a_3) \in A(\mathbf{D})^2$ so that

$$s := f_1 + a_2 f_2 + a_3 f_3 \neq 0 \qquad \text{on} \quad E.$$

Conjugation gives

$$\check{s} = f_1 + \check{a}_2 f_2 + \check{a}_3 f_3 \neq 0 \qquad \text{on} \quad E.$$

Multiplying s with $\check{s}$ yields

$$s\check{s} = f_1^2 + b_2 f_2 + b_3 f_3 \neq 0 \qquad \text{on} \quad E,$$

where $b_j \in A(\mathbf{D})_{\mathsf{R}}$.

Since $\operatorname{tsr}(A(\mathbf{D})_{\mathsf{R}}) = 2$, we may approximate the pair (f_1, b_2) uniformly on $\mathbf{D}$ by an invertible pair $(u, v) \in U_2(A(\mathbf{D})_{\mathsf{R}})$ so that $uf_1 + vf_2 + b_3 f_3 \neq 0$ on E. Now $b_3 = g_1 u + g_2 v$ for some $g_j \in A(\mathbf{D})_{\mathsf{R}}$. Thus

$$uf_1+vf_2+b_3f_3 = uf_1+vf_2+(g_1u+g_2v)f_3 = u(f_1+g_1f_3)+v(f_2+g_2f_3),$$

a function that does not vanish on E. Hence $\operatorname{gsr}_E(A(\mathbf{D})_{\mathsf{R}}) \leq 2$.

Theorem 2.4. *Let* $E \subseteq \mathbf{D}$ *be closed. Then*

(1) $\operatorname{gsr}_E(A(\mathbf{D})_{\mathsf{R}}) = 1$ *if and only if* $E \cap [-1, 1]$ *is totally disconnected or empty.*

(2) $\operatorname{gsr}_E(A(\mathbf{D})_{\mathsf{R}}) = 2$ *if and only if* $E \cap [-1, 1]$ *contains an interval.*

Proof. In view of Proposition 2.3, assertion (2) is the negation of (1). To show (1), we first note that if $E \cap [-1, 1]$ is not totally disconnected (and not empty), then there exists an interval $[a, b] \subseteq E \cap [-1, 1]$. But then the pair $\left(z - \frac{a+b}{2}, (z-a)(z-b)\right)$ is not reducible over $[a, b]$ nor over E, since any representation $z - \frac{a+b}{2} + h(z-a)(z-b)$ contains a zero within $[a, b]$. Thus $\operatorname{gsr}_E(A(\mathbf{D})_{\mathsf{R}}) \neq 1$ in that case.

Now suppose that $E \cap [-1, 1]$ is totally disconnected or empty. We may assume that E is real symmetric, otherwise consider the set $E \cup E^*$. Let $f, g \in A(\mathbf{D})_{\mathsf{R}}$ satisfy $|f| + |g| \geq \delta > 0$ on E. Since $\operatorname{gsr}_E(A(\mathbf{D})) = 1$ ([13]), there exists $k \in A(\mathbf{D})$ such that $s := f + kg \neq 0$ on E. Note that $f = \check{f}$ as well as $g = \check{g}$. Conjugation then gives $\check{s} = f + \check{k}g \neq 0$ on E. Multiplication yields that

$$s\check{s} = f^2 + [(k + \check{k})f + (k\check{k})g]g \neq 0 \qquad \text{on} \quad E.$$

But $\check{k} + k$ and $k\check{k}$ are in $A(\mathbf{D})_{\mathsf{R}}$. Hence $f^2 + ag \neq 0$ on E for some $a \in A(\mathbf{D})_{\mathsf{R}}$.

Since the real symmetric polynomials are dense in $A(\mathbf{D})_{\mathsf{R}}$ (see for example [12] or [22]), there exists $p \in A(\mathbf{D})_{\mathsf{R}} \cap \mathsf{C}[z]$ such that $pf + ag \neq 0$ on E. By moving a little bit the zeros of p (non real zeros in pairs), we may assume that p and g have no zeros in common in $\mathbf{D}$.

Now we use that $E \cap [-1, 1]$ is totally disconnected. If x_j is a zero of p in $[-1, 1]$, we move it a little bit on the real axis, so that the new polynomial $\tilde{p}$ has no zeros on $E \cap [-1, 1]$, that it has no zeros in common with g in $\mathbf{D}$ and that $\tilde{p}f + ag \neq 0$ on E. If $E \cap [-1, 1] = \emptyset$, then we let $\tilde{p} = p$. Since $\tilde{p} \in A(\mathbf{D})_{\mathsf{R}}$, we know that

$$\tilde{p}(z) = r \prod_{j=1}^{n}(z - t_j) \prod_{j=1}^{m}(z - z_j)(z - \overline{z_j}),$$

where $r \in \mathsf{R}$, $t_j \in \mathsf{R}$, $t_j \notin E$, and $\operatorname{Im} z_j > 0$. If the order of the zero t_j of $\tilde{p}$ is odd, we multiply $\tilde{p}$ as well as a with the factor $z - t_j$. Let $\tilde{a}$ (respectively q), be the product of a (respectively p), with these linear factors. Then $\tilde{a}$ and q belong to $A_{\mathsf{R}}(\mathbf{D})$. Moreover we have that $qf + \tilde{a}g \neq 0$ on E and that $Z(q) \cap Z(g) = \emptyset$.

Note that q can be written as

$$q(z) = r \prod_{j=1}^{n}(z - s_j)^{2m_j} \prod_{j=1}^{m}(z - z_j)(z - \overline{z_j}),$$

where the $s_j \in \mathsf{R}$ are pairwise distinct and do not belong to E.

Next we use the fact that the reducibility of (f_1, g) and (f_2, g) implies that of $(f_1 f_2, g)$. Hence, by Lemma 2.1, we obtain that (q, g) is reducible; that is $u := q + \ell g$ is invertible in $A(\mathbf{D})_{\mathsf{R}}$ for some $\ell \in A(\mathbf{D})_{\mathsf{R}}$. This implies that

$$qf + \tilde{a}g = (u - \ell g)f + \tilde{a}g = uf + (\tilde{a} - \ell f)g.$$

Since $qf + \tilde{a}g \neq 0$ on E, we conclude that $f + Rg$ has no zeros on E for some $R \in A(\mathbf{D})_{\mathsf{R}}$. Thus $\operatorname{gsr}_E(A(\mathbf{D})_{\mathsf{R}}) = 1$.

Next we will solve the following problem: which zero free functions on a closed set $E \subseteq \mathbf{D}$ can be uniformly approximated on E by functions invertible in $A(\mathbf{D})_{\mathsf{R}}$. Let us recall that if $f \in A(\mathbf{D})$ does not vanish on the polynomial convex set E, then $\|f - e^{p_n}\|_E \to 0$ for some sequence of polynomials (see [7] and [13]). Now a first attempt at a proof in $A(\mathbf{D})_{\mathsf{R}}$ could work as follows: let p_n be polynomials with $\|e^{p_n} - f\|_E \to 0$. Then $e^{(p_n+\check{p}_n)/2}$ converges uniformly on E to f. But this does not hold for any f, since for example the identity map z cannot be uniformly approximated on $E = [-1, -1/2] \cup [1/2, 1]$ by invertible functions in $A(\mathbf{D})_{\mathsf{R}}$. The reason for the erroneous approach above is that on E the $*$-transform of the square root is not the square root of the $*$-transform (where the $*$ denotes the operation of taking complex conjugates): $(\sqrt{f})(\sqrt{f})^* \neq (\sqrt{f})(\sqrt{f^*})$. For instance we may take $E = [-1, -1/2]$ and $f(z) = z$. Then on E we have $\sqrt{f}\,(x) = \sqrt{f^*}\,(x) = i\sqrt{|x|}$ and $(\sqrt{f})^*(x) = -i\sqrt{|x|}$; so $(\sqrt{f})(\sqrt{f})^*(x) = |x|$, but $(\sqrt{f})(\sqrt{f^*})(x) = -|x|$.

This led us to the following theorem. Recall that the dense stable rank (or approximate stable rank) of $A(\mathbf{D})$ is one ([7], [13]) and that a compact set $K \subseteq \mathbf{D}$ is polynomial convex (or equivalently $A(\mathbf{D})$-convex) if and only if K has no holes.

THEOREM 2.5. *The following assertions hold:*

(1) $\operatorname{appsr}(A(\mathbf{D})_{\mathsf{R}}) = 2$.

(2) *Suppose that $E \subseteq \mathbf{D}$ is closed. If $E \cup E^*$ is not polynomial convex, then there exists $f \in A(\mathbf{D})_{\mathsf{R}}$, f zero free on E, that cannot be uniformly approximated on E by invertible functions in $A(\mathbf{D})_{\mathsf{R}}$.*

(3) *There exist polynomial convex sets $E \subseteq \mathbf{D}$ for which $E \cup E^*$ is not polynomial convex.*

(4) *There exist closed, non-polynomial convex sets $E \subseteq \mathbf{D}$ for which $E \cup E^*$ is polynomial convex.*

(5) *Let $E \subseteq \mathbf{D}$ be compact and suppose that $E \cup E^*$ is polynomial convex. Then a function $f \in A(\mathbf{D})_{\mathsf{R}}$, zero free on E, can be uniformly approximated on E by invertibles in $A(\mathbf{D})_{\mathsf{R}}$ if and only if f has constant sign on $E \cap [-1, 1]$.*

If $E \cap [-1, 1] = \emptyset$, then this latter condition is redundant.

PROOF. (1) Let $(f, g) \in A(\mathbf{D})_{\mathsf{R}}^2$ and suppose that $|f| + |g| \neq 0$ on the $A(\mathbf{D})_{\mathsf{R}}$-convex set E. Since the topological stable rank of $A(\mathbf{D})_{\mathsf{R}} = 2$ (see [22]), we may approximate (f, g) by an invertible tuple $(u, v) \in U_2(A(\mathbf{D})_{\mathsf{R}})$. Thus $\operatorname{appsr}(A(\mathbf{D})_{\mathsf{R}}) \leq 2$. On the other hand, since the function z cannot be uniformly approximated on the set $E = [-1, -1/2] \cup [1/2, 1]$ by functions invertible in $A(\mathbf{D})_{\mathsf{R}}$, we obtain that $\operatorname{appsr}(A(\mathbf{D})_{\mathsf{R}}) > 1$.

(2) Let C be a hole of $K := E \cup E^*$, that is C is a bounded component of $\mathsf{C} \setminus K$. Then $C \subseteq \mathbf{D}$. If $C \cap [-1, 1] = \emptyset$ we fix a point $a \in C$ and if $C \cap [-1, 1] \neq \emptyset$, we fix a point $r \in C \cap\,]-1, 1[$. The functions f_t given by $f_t(z) = (z - a)(z - \overline{a})$, whenever $t = a$ and $f_t(z) = z - r$ whenever $t = r$,

belong to $A(\mathbf{D})_{\mathsf{R}}$, are zero free on E, but cannot be uniformly approximated on E by functions u_n invertible in $A(\mathbf{D})_{\mathsf{R}}$. In fact, suppose that $u_n \in A(\mathbf{D})_{\mathsf{R}}$ converges to f_t uniformly on E, hence, due to symmetry, on K, too. Then on ∂C, a set that is contained in K, we have

$$|f_t - u_n| \le \max_{\partial C} |f_t - u_n| \le \varepsilon < \min_{\partial C} |f_t| \le |f_t| + |u_n|$$

whenever n is large. So Rouché's theorem implies that u_n has a zero within $C \subseteq \mathbf{D}$.

(3), (4) see the figures.

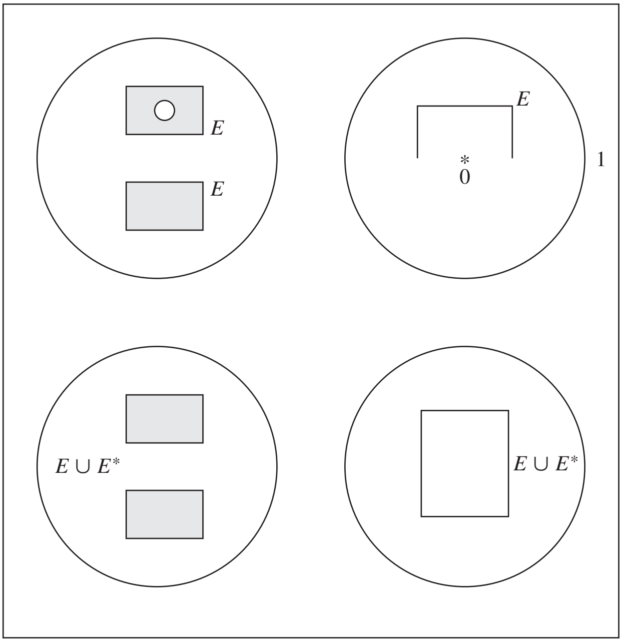

FIGURE 1. Polynomial convexity

(5) First we deal with the case where $E \cap [-1, 1] \neq \emptyset$.

Let $f, u_n \in A(\mathbf{D})_{\mathsf{R}}$ and assume that f has no zeros on E and that u_n is invertible in $A(\mathbf{D})_{\mathsf{R}}$. Suppose that $\|u_n - f\|_E \to 0$. Let $x_0, x_1 \in E \cap [-1, 1]$ satisfy $f(x_0) < 0$ and $f(x_1) > 0$. Then $u_n(x_0) < 0$ and $u_n(x_1) > 0$ for n large enough. Since u_n is real valued on $[-1, 1]$, we see that u_n admits a zero between x_0 and x_1. Hence u_n cannot be invertible in $A(\mathbf{D})_{\mathsf{R}}$. Thus f has constant sign on $E \cap [-1, 1]$.

Conversely, let $K = E \cup E^*$ be polynomial convex. Suppose that $f \in A(\mathbf{D})_{\mathsf{R}}$ has constant sign on $E \cap [-1, 1]$, say $f > 0$ there, and that f does not vanish on E. Due to symmetry, f has no zeros on K, either. Since the polynomials with real coefficients are dense in $A(\mathbf{D})_{\mathsf{R}}$ (see, e.g., [12]), we may assume without loss of generality that f is a polynomial that has no zeros on K and is positive on $K \cap [-1, 1]$. Note that $K \cap [-1, 1] = E \cap [-1, 1]$. Say

$$f(z) = r \prod_{j=1}^{n} (z - r_j)^{n_j} \prod_{j=1}^{m} (z - z_j)(z - \overline{z_j}),$$

where $r_1 < r_2 < \cdots < r_n$, $n_j \in \mathsf{N}$ and $\operatorname{Im} z_j > 0$. Without loss of generality, let $r > 0$. Note that the z_j can appear several times. We claim that in small real symmetric neighborhoods, U, of K the polynomial f can be represented as $f = h\check{h}$ for some holomorphic function h. To this end, let us consider the polynomial

$$p(z) = r \prod_{j=1}^{n} (z - r_j)^{n_j}.$$

Since $\prod_{j=1}^{m} (x - z_j)(x - \overline{z_j}) > 0$ on $[-1, 1]$, $p > 0$ on $K \cap [-1, 1]$.

Let $S = \{x \in \mathsf{R} : p(x) \le 0\}$ and $\Omega = \mathsf{C} \setminus S$. Note that $K \subseteq \Omega$. Let $\Omega^+ = \Omega \cap \{z \in \mathsf{C} : \operatorname{Im} z > 0\}$ and $\Omega^- = \Omega \cap \{z \in \mathsf{C} : \operatorname{Im} z < 0\}$. There is a well defined holomorphic square root $q := \sqrt{p}$ of p on Ω^+ with

$$\lim_{\substack{z \to x \\ z \in \Omega^+}} \operatorname{Im} q(z) = 0$$

whenever $p(x) > 0$.

Now we apply Schwarz's reflection principle (see [18, p. 237]), to deduce that q admits a holomorphic extension to Ω. This function, that we denote by q, too, clearly is symmetric with respect to the real axis; that is $\overline{q(\overline{z})} = q(z)$ for $z \in \Omega$. Hence, using our notation $\check{q}(z) = \overline{q(\overline{z})}$, we see that on small neighborhoods U of K, the polynomial p admits the representation $p = q\check{q}$. Thus, on U,

$$f(z) = \left(q(z) \prod_{j=1}^{m} (z - z_j) \right) \left(\overline{q(\overline{z})} \prod_{j=1}^{m} (z - \overline{z_j}) \right).$$

In other words, on small symmetric neighborhoods, U, of K our polynomial f can be written as $f = h\check{h}$, h holomorphic on U.

Now we continue in the same way as in the paper [13] of the first author. Since K has no holes, we may use Borsuk's theorem (see [1, p. 99]), to conclude that there is a well defined continuous logarithm of h on K. This function, denoted by $\log h$, is holomorphic in the interior of K (see [1, p. 111]). Thus

$\log h \in A(K)$. Hence, by Mergelyan's theorem, $\log h$ can be uniformly approximated by polynomials on K, say $\|p_n - \log h\|_K \to 0$. Thus $u_n := e^{p_n}$ tends uniformly to h on K. Symmetrization shows that $\check{u}_n$ tends uniformly to $\check{h}$ on K. Hence $u_n\check{u}_n$ tends uniformly to f on K. Since $u_n\check{u}_n$ is invertible in $A(\mathbf{D})_{\mathsf{R}}$, we are done.

Now suppose that $E \cap [-1, 1] = \emptyset$ and that $K = E \cup E^*$ is polynomial convex. Then the proof above works, too, by passing if necessary, to $-f$ if $f \le 0$ on $[-1, 1]$.

For a set $E \subseteq \mathsf{C}$, let E° be the set of its interior points. The set of all holomorphic functions on a domain Ω is denoted by $H(\Omega)$.

COROLLARY 2.6. *Let $E \subseteq \mathbf{D}$ be closed and real symmetric. Suppose that E has no holes. Then a function q, invertible in $C(E)$, is the uniform limit (on E) of restrictions to E of invertible functions in $A(\mathbf{D})_{\mathsf{R}}$ if and only if $q \in C(E)_{\text{sym}} \cap H(E^\circ)$, and q has constant sign on $E \cap [-1, 1]$ whenever this set is not empty.*

PROOF. Suppose that $\|f_n - q\|_E \to 0$ for some sequence of invertible functions, (f_n), in $A(\mathbf{D})_{\mathsf{R}}$. Then $q \in C(E)_{\text{sym}} \cap H(E^\circ)$. Now, as in the proof of Theorem 2.5 (5), first paragraph, if $E \cap [-1, 1] \neq \emptyset$, then q has constant sign on $E \cap [-1, 1]$.

Next we prove the converse. Since E has no holes, we may use Mergelyan to approximate q on E by polynomials p_n. Then $(p_n + \check{p}_n)/2$ tends to $(q + \check{q})/2$. But $q \in C(E)_{\text{sym}}$; so we have a sequence, f_n, of polynomials in $A(\mathbf{D})_{\mathsf{R}}$ that converges to q on E. For n large enough, we may assume that f_n does not vanish on E and that f_n has constant sign on $E \cap [-1, 1]$. The assertion now follows by Theorem 2.5 (5).

COROLLARY 2.7. *Let $E \subseteq \mathbf{D}$ be closed. Then* every *function in $A(\mathbf{D})_{\mathsf{R}}$ that does not vanish on E can be uniformly approximated on E by invertible functions in $A(\mathbf{D})_{\mathsf{R}}$* (*we say that E is admissible*) *if and only if $E \cup E^*$ is polynomial convex and exactly one of the following properties holds:*

(1) *$E \cap [-1, 1] = \emptyset$;*

(2) *$E \cap [-1, 1]$ is a singleton;*

(3) *$E \cap [-1, 1]$ is an interval.*

PROOF. We assume that $E \cup E^*$ is polynomial convex. Then each of the properties (1),(2), and (3) imply that any function $f \in A(\mathbf{D})_{\mathsf{R}}$ not vanishing on E has constant sign on $E \cap [-1, 1]$. The assertion that E is admissible, now follows from Theorem 2.5 (5).

To prove the converse, we first note that by Theorem 2.5 (2), $E \cup E^*$ is polynomial convex whenever E is admissible.

Now let us assume that $E \cap [-1, 1]$ is neither empty, nor a singleton. Let $x = \min(E \cap [-1, 1])$ and $y = \max(E \cap [-1, 1])$. Then $x < y$. We claim that $E \cap [-1, 1] = [x, y]$. Indeed, assuming not, there would exist a function $f \in A(\mathbf{D})_{\mathsf{R}}$ of the form $f(z) = z - r$, $r \in [-1, 1] \setminus E$, with $f(x) < 0$, $f(y) > 0$ $f \neq 0$ on E. By Theorem 2.5 (5), that function cannot be approximated on E by invertibles and so E would not be admissible. This contradiction shows that $E \cap [-1, 1]$ is an interval.

We conclude this section with a result connected to the generalized E-stable rank. It extends a special case, where E is the whole algebra spectrum, of Corach and Suárez [5, p. 636].

THEOREM 2.8. *Let $A = (A, \|\cdot\|)$ be a real or complex Banach function algebra on a compact Hausdorff space X. Suppose that $\|\cdot\|$ dominates the sup-norm on X. For $g \in A$ consider the sets*

$$R_E(g) = \{f \in A : (f, g) \text{ is } E\text{-reducible}\}$$

and

$$I_E(g) = \{f \in A : \inf_{x \in E} |f(x)| + |g(x)| > 0\}.$$

Then $R_E(g)$ is open-closed in $I_E(g)$ whenever E is A-convex. In particular, if $\phi : [0, 1] \to I_E(g)$ is a continuous curve and $(\phi(0), g)$ is E-reducible, then $(\phi(1), g)$ is E-reducible.

PROOF. To show that $R_E(g)$ is open inside $I_E(g)$, let $f \in R_E(g)$; that is $f + ag \neq 0$ on E for some $a \in A$. Choose $\tilde{f}$ sufficiently close to f (with respect to the norm on A). Then, on E,

$$|\tilde{f} + ag| \geq |f + ag| - \|\tilde{f} - f\|_\infty \geq |f + ag| - \|\tilde{f} - f\| \geq \delta > 0.$$

Hence $\tilde{f} \in R_E(g)$.

To see that $R_E(g)$ is closed within $I_E(g)$ whenever E is A-convex, we use the facts that the spectrum or character space of the restriction algebra

$$B := (\overline{A_{|E}}, \|\cdot\|_E)$$

equals E (see [8, p. 39] for complex algebras and [11, p. 125] for real algebras). Thus, by [5], [6] or [20][1], the set

$$R(g) = \{b \in B : (b, g) \text{ reducible}\}$$

[1] Note that the proofs given there hold for real function algebras as well.

is open-closed in

$$I(g) = \{r \in B : (r, g) \text{ invertible}\}.$$

Now suppose that $f \in I_E(g)$ and that $f_n \in R_E(g)$ converges to f in the norm of A. The restrictions to E of these functions belong to B. So $f|_E \in I(g)$ and $f_n|_E \in R(g)$. Since $R(g)$ is (uniformly) closed in $I(g)$ and that $\|\cdot\|$ dominates the sup-norm on X, we obtain that $f|_E \in R(g)$. Therefore the element $f + xg$ is invertible in B for some $x \in B$. Now we uniformly approximate $x \in B$ on E by a function $h \in A$; so $f + hg \neq 0$ on E. Hence (f, g) is E-reducible. Thus $R_E(g)$ is closed within $I_E(g)$.

3. The algebra of real symmetric continuous functions on compact planar sets

For a compact real symmetric set $K \subseteq \mathbf{C}$ we recall that $C(K)_{\text{sym}}$ denotes the real algebra of complex valued, continuous functions on K with $f(\overline{z}) = \overline{f(z)}$ for $z \in K$. We may extend these functions in a symmetric way to the whole plane: in fact, let φ be any Tietze extension of f to $\mathbf{C}$ and let

$$F(z) = \frac{\varphi(z) + \overline{\varphi(\overline{z})}}{2}.$$

Then $F(\overline{z}) = \overline{F(z)}$ for all $z \in \mathbf{C}$ and $F = f$ on K.

Let $\mathbf{C}^+ = \{z \in \mathbf{C} : \text{Im } z \geq 0\}$ be the closed upper-half plane and recall that $\mathbf{D}^+ = \mathbf{D} \cap \mathbf{C}^+$. We use the following notation: if (f_1, f_2) is a pair of complex-valued functions, then $\mathbf{f} := (f_1, f_2)$ and $|\mathbf{f}| := \sqrt{|f_1|^2 + |f_2|^2}$. If S^{n-1} is the unit sphere in $\mathbf{R}^n$, then $C(E, S^3)$ is the set of pairs $\mathbf{f} = (f_1, f_2)$ of functions $f_j \in C(E, \mathbf{C})$ such that $|\mathbf{f}| = 1$ on E. Similarly, $C(E, S^1)$ is the set of pairs $\mathbf{f} = (f_1, f_2)$ of functions $f_j \in C(E, \mathbf{R})$ such that $|\mathbf{f}| = 1$ on E.

THEOREM 3.1. $\text{tsr}(C(\mathbf{D})_{\text{sym}}) = 2$.

PROOF. 1. We first note that $\text{tsr}(C(\mathbf{D})_{\text{sym}}) > 1$, since invertible functions in $C(\mathbf{D})_{\text{sym}}$ have constant sign on $[-1, 1]$, hence cannot uniformly approximate the monom z.

2. Next we show that $\text{tsr}(C(\mathbf{D})_{\text{sym}}) \leq 2$. Let $\mathbf{f} = (f_1, f_2) \in (C(\mathbf{D})_{\text{sym}})^2$ and

$$E_n = \{z \in \mathbf{D}^+ : |\mathbf{f}(z)| \geq 1/n\}.$$

Step 1. Suppose that $E_n \cap [-1, 1] \neq \emptyset$. We claim that there is an $\mathbf{R}^2$-valued extension of the tuple $\mathbf{f}/|\mathbf{f}| \in C(E_n \cap [-1, 1], S^1)$ to $\tilde{\mathbf{f}}_{\mathbf{n}} \in C([-1, 1], S^1)$.

In fact, let $g_n \in C([-1, 1], \mathbf{R})$ be a real valued continuous function vanishing exactly on $E_n \cap [-1, 1]$. Then the triple (f_1, f_2, g_n) is invertible in

$C([-1,1],\mathsf{R})$. Since $\operatorname{bsr}(C([-1,1],\mathsf{R}))=2$ (see [26]), there exist $h_{1,n}, h_{2,n} \in C([-1,1],\mathsf{R})$ such that

$$(f_1+h_{1,n}g_n,\ f_2+h_{2,n}g_n)$$

is invertible in $C([-1,1],\mathsf{R})$. Now the pair

$$\tilde{\mathbf{f}}_{\mathbf{n}} := (f_1+h_{1,n}g_n,\ f_2+h_{2,n}g_n)/|(f_1+h_{1,n}g_n,\ f_2+h_{2,n}g_n)|$$

is the desired extension. We point out that $\tilde{\mathbf{f}}_{\mathbf{n}}$ is R^2-valued.

If $E_n \cap [-1,1]=\emptyset$, then we let $\tilde{\mathbf{f}}_{\mathbf{n}}=(1,0)$.

Step 2. Next we claim that there exists an extension of $\mathbf{f}/|\mathbf{f}| \in C(E_n, S^3)$ to $\hat{\mathbf{f}}_{\mathbf{n}} \in C(\mathbf{D}^+, S^3)$.

In fact, define $\mathbf{F}_n=(F_{1,n}, F_{2,n})$ by

$$\mathbf{F}_n(z)=\mathbf{f}(z)/|\mathbf{f}(z)| \text{ whenever } z\in E_n, \tag{3.1}$$

$$\mathbf{F}_n(z)=\tilde{\mathbf{f}}_{\mathbf{n}}(z) \text{ whenever } z\in[-1,1] \tag{3.2}$$

and extend continuously to C by Tietze. Note that $\mathbf{F}_n$ is well defined, due to Step 1. Now let $G_n \in C(\mathbf{D}^+,\mathsf{R})$ be a real valued continuous function vanishing exactly on $E_n \cup [-1,1]$. Then the triple $(F_{1,n}, F_{2,n}, G_n)$ is invertible in $C(\mathbf{D}^+,\mathsf{C})$. Since $\operatorname{bsr}(C(\mathbf{D}^+,\mathsf{C}))=2$ (see [26]), there exist $H_{1,n}, H_{2,n} \in C(\mathbf{D}^+,\mathsf{C})$ such that

$$(F_{1,n}+H_{1,n}G_n,\ F_{2,n}+H_{2,n}G_n)$$

is invertible in $C(\mathbf{D}^+,\mathsf{C})$. Now the pair

$$\hat{\mathbf{f}}_{\mathbf{n}} = (F_{1,n}+H_{1,n}G_n,\ F_{2,n}+H_{2,n}G_n)/|(F_{1,n}+H_{1,n}G_n,\ F_{2,n}+H_{2,n}G_n)|$$

is the desired extension.

Step 3. We claim that $|\mathbf{f}-(|\mathbf{f}|+1/n)\,\hat{\mathbf{f}}_{\mathbf{n}}| \le 3/n$ on $\mathbf{D}^+$.

Indeed, for $z\in E_n$ we have $\mathbf{f}(z)=|\mathbf{f}(z)|\hat{\mathbf{f}}(z)$ for some C^2-valued function $\hat{\mathbf{f}}$. Since on E_n both functions $\hat{\mathbf{f}}$ and $\hat{\mathbf{f}}_{\mathbf{n}}$ coincide, we obtain

$$\left|\mathbf{f}-\left(|\mathbf{f}|+\tfrac{1}{n}\right)\hat{\mathbf{f}}_{\mathbf{n}}\right| = \left||\mathbf{f}|(\hat{\mathbf{f}}-\hat{\mathbf{f}}_{\mathbf{n}})-\tfrac{1}{n}\hat{\mathbf{f}}_{\mathbf{n}}\right| = \tfrac{1}{n}|\hat{\mathbf{f}}_{\mathbf{n}}| = 1/n \qquad \text{on} \quad E_n.$$

For $z\in\mathbf{D}^+\setminus E_n$ and $|\mathbf{f}(z)|\neq 0$, we have $|\mathbf{f}(z)|\le 1/n$; hence for these z

$$\left|\mathbf{f}-\left(|\mathbf{f}|+\tfrac{1}{n}\right)\hat{\mathbf{f}}_{\mathbf{n}}\right| \le |\mathbf{f}|\cdot|\mathbf{f}/|\mathbf{f}|-\hat{\mathbf{f}}_{\mathbf{n}}|+\tfrac{1}{n}|\hat{\mathbf{f}}_{\mathbf{n}}| \tag{3.3}$$

$$\le \tfrac{1}{n}\cdot 2+\tfrac{1}{n}=\tfrac{3}{n}. \tag{3.4}$$

For $z\in\mathbf{D}^+\setminus E_n$ and $|\mathbf{f}(z)|=0$, the assertion is obvious since $|\hat{\mathbf{f}}_{\mathbf{n}}|=1$.

Step 4. In the steps above we have found a C^2-valued function

$$\mathbf{g_n} := (|\mathbf{f}| + 1/n)\hat{\mathbf{f}}_\mathbf{n}$$

with $|\mathbf{f} - \mathbf{g_n}| \leq 3/n$ on $\mathbf{D}^+$. Note that $\mathbf{g_n}$ is R^2-valued on $[-1, 1]$. Thus we can use reflection to define a C^2-valued function Φ_n on $\mathbf{D}$ (whose components are in $C(\mathbf{D})_{\text{sym}}$) so that $|\mathbf{f} - \Phi_n| \leq 3/n$ on $\mathbf{D}$ and such that $|\Phi_n| \geq \frac{1}{n} > 0$ on $\mathbf{D}$. This shows that $\operatorname{tsr}(C(\mathbf{D})_{\text{sym}}) \leq 2$.

For the reader's convenience, we present a second proof, based only on approximation theory, and not on the algebraic theory of the Bass stable rank.

Proof. Let $(f_1, f_2) \in (C(\mathbf{D})_{\text{sym}})^2$. Choose $\varepsilon > 0$. On $[-1, 1]$ we uniformly approximate the real-valued functions f_j by two polynomials p_j with real coefficients so that p_1 and p_2 have no common zeros in C. Then there exists $r > 0$ such that $|p_j - f_j| < \varepsilon$ on $\{z \in \mathbf{D}^+ : 0 \leq \operatorname{Im} z \leq r\}$. Let Φ be a C^∞ function satisfying $\Phi = 0$ on $E := \left\{z \in \mathbf{D}^+ : 0 \leq \operatorname{Im} z \leq \frac{1}{3}r\right\}$, $\Phi = 1$ on $\left\{z \in \mathbf{D}^+ : \operatorname{Im} z \geq \frac{3}{4}r\right\}$ and $0 \leq \Phi \leq 1$. Similarly, let Ψ be such a function with $\Psi = 1$ on $\left\{z \in \mathbf{D}^+ : \operatorname{Im} z \geq \frac{1}{4}r\right\}$, $\Psi = 0$ on $\left\{z \in \mathbf{D}^+ : \operatorname{Im} z \leq \frac{1}{8}r\right\}$ and $0 \leq \Psi \leq 1$.

Define $F_j = \Phi f_j + p_j(1 - \Phi)$, $j = 1, 2$. We note that $F_j = p_j$ on E. Let

$$\delta := \min\{|p_1(z)| + |p_2(z)| : z \in \mathbf{D}\}.$$

Obviously, $\delta > 0$. Note that δ depends on ε. Since $\operatorname{tsr}(C(\mathbf{D}, \mathsf{C})) = 2$ (see [26] and [17]), there exists an invertible pair $(u_1, u_2) \in C(\mathbf{D}, \mathsf{C})$ such that

$$\|F_1 - u_1\|_\infty + \|F_2 - u_2\|_\infty < \min\{\varepsilon, \delta/2\}.$$

Now let $h_j = \Psi u_j + F_j(1 - \Psi)$.

(i) We claim that (h_1, h_2) is an invertible pair in $C(\mathbf{D}^+, \mathsf{C})$. In fact, let $z \in \mathbf{D}^+$, $\operatorname{Im} z \geq r/4$. Then $\Psi(z) = 1$; hence $h_j(z) = u_j(z)$ and so $|h_1(z)| + |h_2(z)| > 0$. If $z \in \mathbf{D}^+$, $\operatorname{Im} z \leq r/3$, then $F_j(z) = p_j(z)$ and so

$$h_j = \Psi u_j + F_j(1 - \Psi) = \Psi(u_j - F_j) + p_j.$$

Hence, for these z,

$$|h_1| + |h_2| \geq |p_1| + |p_2| - (|u_1 - F_1| + |u_2 - F_2|) \geq \delta - \delta/2 = \delta/2 > 0.$$

(ii) Next we show that $\|h_1 - f_1\|_{\mathbf{D}^+} + \|h_2 - f_2\|_{\mathbf{D}^+} \leq 4\varepsilon$.

To see this, we observe that

$$|F_j - f_j| = |\Phi f_j + p_j(1-\Phi) - f_j| = |f_j(\Phi-1) + (1-\Phi)p_j| = (1-\Phi)|f_j - p_j|.$$

For $z \in \mathbf{D}^+$ with $\operatorname{Im} z \le r$, we have that $|f_j - p_j| \le \varepsilon$. For $z \in \mathbf{D}^+$ with $\operatorname{Im} z \ge \frac{3}{4}r$, $\Phi(z) = 1$. Thus $|F_j - f_j| \le \varepsilon$ on $\mathbf{D}^+$.

Next, we have that $|u_j - F_j| < \varepsilon$. Finally,

$$|h_j - F_j| = |\Psi u_j + F_j(1-\Psi) - F_j| = |\Psi||u_j - F_j| \le \varepsilon.$$

To sum up, we get that on $\mathbf{D}^+$

$$|h_j - f_j| \le |h_j - F_j| + |F_j - f_j| \le 2\varepsilon.$$

(iii) Note that $h_j = F_j = p_j$ on $[-1, 1]$; in particular h_j is real there. Reflection and Steps (i) and (ii) above now yield the desired assertion on the topological stable rank of $C(\mathbf{D})_{\mathrm{sym}}$.

We will use Theorem 3.1 to determine the generalized E-stable ranks for $C(\mathbf{D})_{\mathrm{sym}}$.

THEOREM 3.2. $\operatorname{gsr}_E(C(\mathbf{D})_{\mathrm{sym}}) \le 2$ *for any closed set* $E \subseteq \mathbf{D}$.

PROOF. Let $(f_1, f_2, f_3) \in (C(\mathbf{D})_{\mathrm{sym}})^3$. We assume that $|f_1| + |f_2| + |f_3| \neq 0$ on E. Then there exists $(x_1, x_2, x_3) \in C(\mathbf{D})^3_{\mathrm{sym}}$ such that on E one has $1 = \sum_{j=1}^3 x_j f_j$. Indeed, on $K := E \cup E^*$, let

$$X_j := \frac{\overline{f_j}}{\sum_{n=1}^3 |f_n|^2}.$$

Then any symmetric Tietze extension x_j of X_j to $\mathbf{D}$ satisfies on E the Bezout equation above.

By Theorem 3.1, $\operatorname{tsr}(C(\mathbf{D})_{\mathrm{sym}}) = 2$. Hence there exists, for every $\varepsilon > 0$, an invertible pair $(u, v) \in U_2(C(\mathbf{D})_{\mathrm{sym}})$ such that $\|u - x_1\| + \|v - x_2\| < \varepsilon$. When ε is taken sufficiently small, then

$$F := uf_1 + vf_2 + x_3 f_3 \neq 0 \qquad \text{on} \quad E.$$

Now $x_3 = au + bv$ for some $(a, b) \in C(\mathbf{D})^2_{\mathrm{sym}}$. Thus

$$F = uf_1 + vf_2 + (au + bv)f_3 = u(f_1 + af_3) + v(f_2 + bf_3).$$

This shows that $\operatorname{gsr}_E(C(\mathbf{D})_{\mathrm{sym}}) \le 2$.

COROLLARY 3.3. *Let* $K \subseteq \mathbb{C}$ *be compact and real symmetric. Then*

$$1 \le \operatorname{bsr}(C(K)_{\mathrm{sym}}) \le \operatorname{tsr}(C(K)_{\mathrm{sym}}) \le 2.$$

PROOF. The third inequality follows from Theorem 3.1 and the fact every function in $C(K)_{\mathrm{sym}}$, $K \subseteq \mathbf{D}$, admits an extension to $C(\mathbf{D})_{\mathrm{sym}}$. The other inequalities hold for any commutative unital Banach algebra.

We are now able to give a complete determination of the Bass and topological stable ranks for $C(K)_{\mathrm{sym}}$. This answers questions posed in [23].

THEOREM 3.4. *Let $K \subseteq \mathsf{C}$ be compact and real symmetric. Then*

(1) $\operatorname{bsr}(C(K)_{\mathrm{sym}}) = \operatorname{tsr}(C(K)_{\mathrm{sym}}) = 1$ *if and only if $K^\circ = \emptyset$ and $K \cap \mathsf{R}$ is totally disconnected or empty;*

(2) $\operatorname{bsr}(C(K)_{\mathrm{sym}}) = \operatorname{tsr}(C(K)_{\mathrm{sym}}) = 2$ *if and only $K^\circ \neq \emptyset$ or $K \cap \mathsf{R}$ contains an interval.*

PROOF. Note that statement (2) is the negation of (1) in view of Corollary 3.3.

Now statement (1) for the Bass stable rank is Theorem 6.5 in [23]. So it remains to prove the assertion for the topological stable rank. Suppose that $K^\circ = \emptyset$ and that $K \cap \mathsf{R}$ is totally disconnected. Let $f \in C(K)_{\mathrm{sym}}$. Consider the set

$$E_n = \left\{z \in K : |f(z)| \geq \tfrac{1}{n}\right\}.$$

Choose $g_n \in C(K)$ so that $g_n \equiv 0$ on E_n and $g_n \equiv 1$ on $Z(f)$. Then its symmetrization h_n given by $h_n(z) = (g_n(z) + \overline{g_n(\overline{z})})/2$ belongs to $C(K)_{\mathrm{sym}}$. Due to the fact that E_n and $Z(f)$ are real symmetric, $h_n \equiv 0$ on E_n and $h_n \equiv 1$ on $Z(f)$. So the pair (f, h_n) is invertible in $C(K)_{\mathrm{sym}}$. Since $\operatorname{bsr}(C(K)_{\mathrm{sym}}) = 1$, there exists $h \in C(K)_{\mathrm{sym}}$ so that $u_n := f + hh_n$ is invertible. But as in the proof of Theorem 3.1, Step 3,

$$\left| f - \left(|f| + \tfrac{1}{n}\right)\frac{u_n}{|u_n|} \right| \leq \frac{3}{n}.$$

Hence we have approximated f by invertibles. Thus $\operatorname{tsr}(C(K)_{\mathrm{sym}}) = 1$.

On the other hand, if $K^\circ \neq \emptyset$ or if $K \cap \mathsf{R}$ contains an interval I, then

$$\operatorname{tsr}(C(K)_{\mathrm{sym}}) \geq \max\{\operatorname{tsr}(C(I, \mathsf{R})), \operatorname{tsr}(C(S, \mathsf{C}))\},$$

where S is a closed disk contained in $K^\circ \setminus \mathsf{R}$; (this easily follows from the fact that real-valued functions on I and complex-valued functions on S have symmetric extensions to K). But by [26], [17] and [9, p. 44]

$$\operatorname{tsr}(C(I, \mathsf{R})) = 2$$

and

$$\operatorname{tsr}(C(S, \mathsf{C})) = 2.$$

Thus $\operatorname{tsr}(C(K)_{\text{sym}}) \geq 2$. Applying Corollary 3.3, now yields the assertion that $\operatorname{tsr}(C(K)_{\text{sym}}) = 2$ whenever $K \cap \mathsf{R}$ contains an interval or $K^\circ \neq \emptyset$.

The companion to Theorem 2.4 now follows immediately (due the fact that $\operatorname{gsr}_K(C(\mathbf{D})_{\text{sym}}) = \operatorname{bsr}(C(K)_{\text{sym}})$ for real symmetric compact subsets K of $\mathbf{D}$.) Note that the results are quite different when comparing $A(\mathbf{D})_{\mathsf{R}}$ and $C(\mathbf{D})_{\text{sym}}$.

COROLLARY 3.5. *Let $K \subseteq \mathbf{D}$ be compact and real symmetric. Then*

(1) $\operatorname{gsr}_K(C(\mathbf{D})) = 1$ *if and only if $K^\circ = \emptyset$ and $K \cap [-1, 1]$ is totally disconnected or empty;*

(2) $\operatorname{gsr}_K(C(\mathbf{D})) = 2$ *if and only $K^\circ \neq \emptyset$ or $K \cap [-1, 1]$ contains an interval.*

Our final result will be the analogue of Theorem 2.5 for $C(\mathbf{D})_{\text{sym}}$. This will give us a characterization of those zero free functions on $E \subseteq \mathbf{D}$ that cannot only be approximated, but extended by invertible functions in $C(\mathbf{D})_{\text{sym}}$.

We will need the following topological lemma.

LEMMA 3.6. *Let E be a real symmetric, polynomial convex set in $\mathbf{D}$. Then $K := (E \cap \mathbf{D}^+) \cup [-1, 1]$ is polynomial convex.*

Before we begin with the proof, let us note that this assertion is not correct if E is no longer real symmetric. Just take for E the upper half circle.

PROOF. It suffices to show that any given point $\xi \in \mathbf{D} \setminus E$ with $\operatorname{Im} \xi > 0$ can be connected to $w = 2i$ by a continuous arc entirely contained in $\{z \in \mathsf{C} : \operatorname{Im} z > 0\} \setminus E$. This will show that $\mathsf{C} \setminus K$ is connected. Since E has no holes, there exists a piecewise linear arc Γ in $\mathsf{C} \setminus E$ with no pieces parallel to the axis, such that $\Gamma(0) = w$ and $\Gamma(1) = \xi$. If Γ does not meet the real axis, then we are done. If $\Gamma \cap \mathsf{R} \neq \emptyset$, say $\operatorname{Im} \Gamma(t_s) = 0$, then we may assume that $\operatorname{Im} \Gamma(t)$ changes sign at t_s. Now let $0 \leq t_0 < t_1 < t_2 < \cdots < t_{2n+1} = 1$ be chosen so that $\operatorname{Im} \Gamma > 0$ on $X^+ = \bigcup_{j=0}^n]t_{2j}, t_{2j+1}[$ and $\operatorname{Im} \Gamma < 0$ on $X^- = \bigcup_{j=1}^n]t_{2j-1}, t_{2j}[$. Since E is real symmetric, the piecewise reflected arc Γ^* given by $\Gamma^*(t) = \Gamma(t)$ for $t \in X^+$ and $\Gamma^*(t) = \overline{\Gamma(t)}$ for $t \in X^-$ is the desired arc when perturbed a little bit at the points t_j.

THEOREM 3.7. *The following assertions hold:*

(1) $\operatorname{appsr}(C(\mathbf{D})_{\text{sym}}) = 2$.

(2) *Suppose that $E \subseteq \mathbf{D}$ is closed. If $E \cup E^*$ is not polynomial convex, then there exists $f \in C(\mathbf{D})_{\text{sym}}$, f zero free on E, that cannot be uniformly approximated on E by invertible functions in $C(\mathbf{D})_{\text{sym}}$.*

(3) *Let $E \subseteq \mathbf{D}$ be real symmetric and polynomial convex. Then a function $f \in C(E)_{\text{sym}}$, zero free on E, can be extended to an invertible function in $C(\mathbf{D})_{\text{sym}}$ if and only if f has constant sign on $E \cap [-1, 1]$.*

If $E \cap [-1, 1] = \emptyset$, then this latter condition is redundant.

PROOF. (1) This follows from Theorem 3.4 and the fact that $\mathrm{bsr}(A) \leq \mathrm{appsr}(A) \leq \mathrm{tsr}(A)$ (see the appendix).

(2) This works also in the same way as in the case $A(\mathbf{D})_{\mathrm{R}}$; just replace Rouché's theorem by its "continuous" counterpart given in [25].

(3) The sign condition being necessary (proof as in Theorem 2.5 (5)), it suffices to show sufficiency. So assume that $f \in C(E)_{\mathrm{sym}}$ has constant sign on $K := E \cap [-1, 1]$, whenever $K \neq \emptyset$. Say $f > 0$ on K. Then $\log f$ is well defined on K and admits a continuous real-valued extension, L, to $[-1, 1]$. If $K = \emptyset$, we choose $L \equiv 0$ on $[-1, 1]$. Now let

$$\tilde{F} = \begin{cases} e^L & \text{on } [-1, 1] \\ f & \text{on } E \cap \mathbf{D}^+. \end{cases}$$

Then $\tilde{F}$ is well defined and continuous on the polynomial convex set

$$(E \cap \mathbf{D}^+) \cup [-1, 1].$$

Since $\tilde{F}$ has no zeros it admits a zero-free extension $\hat{F}$ to $\mathbf{D}^+$ (Borsuk's theorem [1, p. 99]). Note that $\hat{F}$ is real valued on $[-1, 1]$. The function F given by

$$F(z) = \begin{cases} \hat{F}(z) & \text{if } z \in \mathbf{D}^+ \\ \overline{\hat{F}(\overline{z})} & \text{if } z \in \mathbf{D}^- \end{cases}$$

is now the desired zero free extension of f to $\mathbf{D}$.

Appendix

The goal of this appendix is to show how the dense, respectively approximate stable ranks are related to the Bass and topological stable rank. As noted in Section 1, this is mentioned for complex uniform algebras in [16]. A proof based on the original definitions of the notions of $\mathrm{bsr}(A)$ and $\mathrm{dsr}(A)$ in the category of surjective, respectively dense algebra morphisms, was communicated to me by Daniel Suárez. We present here a different approach that is more adapted to our setting (see also [4], Theorem 1.13).

THEOREM 3.8. *Let A be a (real or complex) commutative, unital Banach algebra and let $M(A)$ denote its character space. Then the following assertions are equivalent:*

(1) $\mathrm{bsr}(A) \leq n$;

(2) *For any $g \in A$, $\varepsilon > 0$, and $\mathbf{f} \in U_n(\overline{\hat{A}|_E})$ with $E = Z(g) := \{x \in M(A) : \hat{g}(x) = 0\}$ there exist $\mathbf{F} \in U_n(A)$ such that $\|\hat{\mathbf{F}} - \mathbf{f}\|_E < \varepsilon$.*

PROOF. "(1) ⇒ (2)" Let $\mathbf{f} \in U_n(\overline{\hat{A}|_E})$. Choose $\mathbf{h} \in A^n$ so that $\|\hat{\mathbf{h}} - \mathbf{f}\|_E < \varepsilon$. Since $M(\overline{\hat{A}|_E}) = E$, the function $|\mathbf{f}|$, and hence $|\hat{\mathbf{h}}|$, do not vanish on E for small $\varepsilon > 0$. Thus $(\mathbf{h}, g) \in U_{n+1}(A)$. Since $\operatorname{bsr}(A) \leq n$, there is $\mathbf{x} \in A^n$ so that $\mathbf{F} := \mathbf{h} + g\mathbf{x} \in U_n(A)$. On E we have that $\hat{\mathbf{F}} = \hat{\mathbf{h}}$ and so $\|\hat{\mathbf{F}} - \mathbf{f}\|_E < \varepsilon$.

"(2) ⇒ (1)" Let $(\mathbf{f}, g) \in U_{n+1}(A)$. Then $\hat{\mathbf{f}} \in U_n(\overline{\hat{A}|_E})$, where $E = Z(g)$. By our hypothesis, there is $\mathbf{F}_j \in U_n(A)$ so that

$$\|\hat{\mathbf{F}}_\mathbf{j} - \hat{\mathbf{f}}\|_E \to 0 \qquad \text{as} \quad j \to \infty. \tag{3.5}$$

Consider the curves $\phi_j(t) = t\mathbf{F}_j + (1 - t)\mathbf{f}$, $0 \leq t \leq 1$. In view of (3.5), we have that for $j \geq j_0$, the $(n + 1)$-tuples $(\phi_j(t), g)$ are invertible and so $\phi_j : [0, 1] \to I_n(g)$, where

$$I_n(g) = \{\mathbf{h} \in A^n : (\mathbf{h}, g) \in U_{n+1}(A)\}.$$

Now fix $j \geq j_0$. For $t = 1$, we get that $(\phi_j(1), g) = (\mathbf{F}_j, g)$ is reducible, since $\mathbf{F}_j$ is invertible. Because the set

$$R_n(g) = \{\mathbf{h} \in A^n : (\mathbf{h}, g) \text{ reducible in } A\}$$

is open-closed in $I_n(g)$ (see [5, p. 636]), $(\phi_j(0), g) = (\mathbf{f}, g)$ is reducible, too.

COROLLARY 3.9. *Let A be a (real or complex) commutative, unital Banach algebra. Then* $\operatorname{bsr}(A) \leq \operatorname{appsr}(A) \leq \operatorname{tsr}(A)$.

PROOF. The second inequality is obvious. The first one follows with the definition of $\operatorname{appsr}(A)$ from Theorem 3.8 and the fact that the zero set $Z(g)$ is always A-convex.

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DÉPARTEMENT DE MATHÉMATIQUES
LMAM, UMR 7122
UNIVERSITÉ PAUL VERLAINE
ILE DU SAULCY
F-57045 METZ
FRANCE
E-mail: mortini@poncelet.univ-metz.fr

FAKULTÄT ALLGEMEINWISSENSCHAFTEN
GEORG-SIMON-OHM-HOCHSCHULE NÜRNBERG
KESSLERPLATZ 12
D-90489 NÜRNBERG
GERMANY
E-mail: Rudolf.Rupp@ohm-hochschule.de