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TOPOLOGICAL EQUIVALENCE OF FINITELY DETERMINED REAL ANALYTIC PLANE-TO-PLANE MAP GERMS

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Abstract

Generic smooth map germs $(\mathsf{R}^2, 0) \to (\mathsf{R}^2, 0)$ are topologically equivalent to cones of mappings $S^1 \to S^1$. We carry out a complete topological classification of smooth stable mappings of the circle and show how this classification leads, via the result mentioned above, to a topological classification of finitely determined real analytic map germs $(\mathsf{R}^2, 0) \to (\mathsf{R}^2, 0)$.

1. Introduction

Let f and g be smooth mappings between smooth manifolds N and P of dimensions n and p, respectively. Let $0 \leq r \leq \infty$. We say that f and g are $\mathscr{A}_r$-equivalent if there is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ N & \xrightarrow{g} & P \end{array}$$

where h and k are C^r diffeomorphisms. Similarly, if f and g are smooth map germs $(N, p) \to (P, q)$, then we say that f and g are $\mathscr{A}_r$-equivalent if there is a commutative diagram

$$\begin{array}{ccc} (N, p) & \xrightarrow{f} & (P, q) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ (N, p) & \xrightarrow{g} & (P, q) \end{array}$$

where h and k are germs of C^r diffeomorphisms. $\mathscr{A}_0$-equivalence is usually referred to as topological equivalence. Let $C^\infty(N, P)$ be the set of proper

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smooth mappings $N \to P$, and let $C^\infty(n, p)$ (resp. $\mathscr{O}(n, p)$) be the set of smooth (resp. real analytic) map germs $(\mathsf{R}^n, 0) \to (\mathsf{R}^p, 0)$.

A subset $\Sigma \subset C^\infty(n, p)$ (resp. $\mathscr{O}(n, p)$) is *proalgebraic* if

$$\Sigma = \bigcap_{r \geq 1} (j^r)^{-1}(\Sigma_r)$$

where each $\Sigma_r \subset J^r(n, p)$ is an algebraic subvariety. A proalgebraic set Σ is of *infinite codimension* if

$$\lim_{r \to \infty} \operatorname{cod} \Sigma_r = \infty.$$

A property of smooth (real analytic resp.) germs is said to hold *in general* if the set of germs not having the property is contained in a proalgebraic set of infinite codimension.

By the cone of a smooth map $f : S^{n-1} \to S^{p-1}$, we mean the map $F : S^{n-1} \times [0, 1)/S^{n-1} \times \{0\} \to S^{p-1} \times [0, 1)/S^{p-1} \times \{0\}$ given by

$$F([(p, t)]) = [(f(p), t)].$$

Consider the space $C^\infty(n, p)$ when $n \leq p, n \neq 4, 5$ and (n, p) is in the 'nice range'. The 'nice range' consists of the pairs of dimensions of N and P such that the set of proper smooth stable mappings $N \to P$ is dense in the set of proper smooth mappings $N \to P$. It is shown in [2] that for germs in $C^\infty(n, p)$, the property of having a realization which is topologically equivalent to the cone of a smooth stable mapping $S^{n-1} \to S^{p-1}$ via homeomorphisms which are diffeomorphisms outside the origin holds in general. We say that map-germs with this property are *generic*. Thus, for n, p in this range, the classification of generic map germs $(\mathsf{R}^n, 0) \to (\mathsf{R}^p, 0)$ with respect to topological equivalence is contained in the classification of the smooth stable mappings $S^{n-1} \to S^{p-1}$ in the sense that the $\mathscr{A}_0$-equivalence class in $C^\infty(n, p)$ of a generic map germ corresponds to an $\mathscr{A}_\infty$-equivalence class in $C^\infty(S^{n-1}, S^{p-1})$.

In this paper we carry out this classification in the real analytic case for $n = p = 2$. In Section 2 we classify the smooth stable mappings $S^1 \to S^1$ and show how to generate complete lists of the $\mathscr{A}_\infty$-equivalence classes of such mappings. In the case of 1-dimensional spheres, the classification is essentially a combinatorical problem. In Section 3 we classify finitely determined real analytic map germs $(\mathsf{R}^2, 0) \to (\mathsf{R}^2, 0)$ using the above strategy. Our method solves the so-called 'recognition problem': Given two finitely determined real analytic map germs $(\mathsf{R}^2, 0) \to (\mathsf{R}^2, 0)$, are they $\mathscr{A}_0$-equivalent?

Some of the results in this article have been obtained independently by Moya-Pérez and Nuño-Ballesteros in [6].

2. Classification of smooth stable mappings $S^1 \to S^1$

In this section we define invariants giving a complete classification of smooth stable mappings $S^1 \to S^1$. Let $f : S^1 \to S^1$ be a smooth stable mapping. Then f has only Morse singularities, $\Sigma(f)$ is finite and f has no singular double points.

2.1. Definition of Ast(f)

Let $P : [0, 2\pi) \to S^1$ be the parametrization given by $P(t) = (\cos t, \sin t)$. If f has no singular points, then we define $\mathrm{Ast}(f) = (p, p, \dots, p)$ where p is repeated $\# f^{-1}(1)$ times. Assume f has singular points $s_i(f), i = 1, \dots, n(f)$ where $k < l \Rightarrow P^{-1}(s_k(f)) < P^{-1}(s_l(f))$. Let $\sigma_i(f) = f(s_i(f))$ and let $f^{-1}(\sigma_i(f)) \setminus \{s_i(f)\} = \{p_{ij}(f)\}_{j=1}^{m_i}$ where $k < l \Rightarrow P^{-1}(p_{ik}(f)) < P^{-1}(p_{il}(f))$. Let

$$A(f) = \{a_1, a_2, \dots, a_N\} = P^{-1}\left(\bigcup_{i=1}^{n}\left(\{s_i(f)\} \cup \{p_{ij}(f)\}_{j=1}^{m_i}\right)\right)$$

where $i < j \Rightarrow a_i < a_j$ and $N = N(f) = n(f) + \sum_{i=1}^{n(f)} m_i$. Let $\Delta(f) = f(\Sigma(f)) = \{\sigma_1(f), \dots, \sigma_n(f)\}$ and define

$$B(f) = \{b_1, b_2, \dots, b_{n(f)}\} = P^{-1}(\Delta(f))$$

where $i < j \Rightarrow b_i < b_j$.

Next, let

$$S = \{s, p\}, \qquad S^* = \bigcup_{i=1}^{\infty} \{s_i, p_i\}$$

and define maps $T : A(f) \to S$ and $T^* : A(f) \to S^*$ given by

$$T(x) = \begin{cases} s, & \text{if } P(x) = s_i(f); \\ p, & \text{if } P(x) = p_{ij}(f) \end{cases}, \qquad T^*(x) = \begin{cases} s_i, & \text{if } P(x) = s_i(f); \\ p_i, & \text{if } P(x) = p_{ij}(f) \end{cases}.$$

Now, define the *associated tuples* of f to be the ordered $N(f)$-tuples

$$\mathrm{Ast}(f) = \left(T(a_1), T(a_2), \dots, T(a_{N(f)})\right)$$

and

$$\mathrm{Ast}^*(f) = \left(T^*(a_1), T^*(a_2), \dots, T^*(a_{N(f)})\right)$$

REMARK 2.1. Given $\mathrm{Ast}^*(f)$ one can obtain $\mathrm{Ast}(f)$ by just forgetting the indices of the s and p in $\mathrm{Ast}^*(f)$. Conversely, given $\mathrm{Ast}(f)$, it is easy to find the right indices for the s in $\mathrm{Ast}^*(f)$ and then we can find the indices of p in

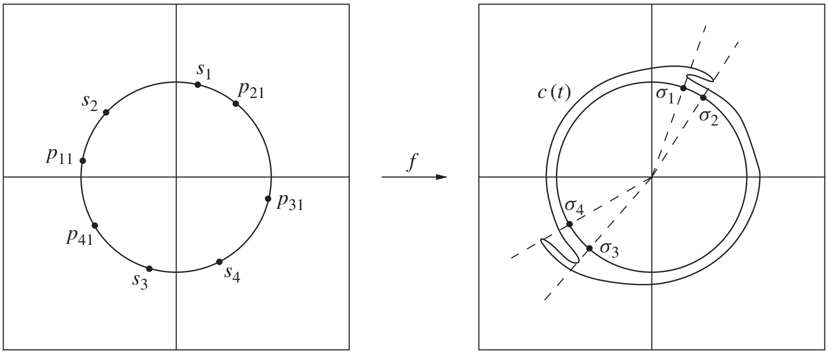

FIGURE 1. Visualization of a map $f : S^1 \to S^1$ with $\mathrm{Ast}(f) = (p, s, s, p, p, s, s, p)$. The curve $c : [0, 2\pi) \to \mathsf{R}^2$ is such that $c(t)/\,\|c(t)\| = f(P(t))$.

$\mathrm{Ast}^*(f)$ as well, using the fact that at a singular point, f changes the behaviour of being orientation preserving or orientation reversing. This enables us to find the correct indices of p.

2.2. *Legal permutations*

Let S_k be the group of permutations of $\mathsf{Z}/k\mathsf{Z}$. Some permutations are of particular interest when trying to classify stable maps under $\mathscr{A}_0$-equivalence. We start with some definitions.

DEFINITION 2.2. An element $\sigma \in S_k$ is a *switch* if there is some $a \in \mathsf{Z}$ such that

$$\sigma([x]) = [x + a].$$

Let Sw_k be the set of switches in S_k.

DEFINITION 2.3. The permutation $r \in S_k$ given by $r([x]) = [-x]$ is called the *reversation*. Let $R_k = \{\mathrm{id}, r\}$.

DEFINITION 2.4. The subgroup $L_k = \{\sigma \circ \tau \mid \sigma \in \mathrm{Sw}_k, \tau \in R_k\}$ of S_k is called the group of *legal permutations*.

Let X be a set. For every k, let $e_k : \{1, 2, \dots, k\} \to \mathsf{Z}/k\mathsf{Z}$ be the bijection $x \mapsto [x]$. We introduce an equivalence relation E_k on X^k by the rule $(t_1, t_2, \dots, t_k) \sim (t'_1, t'_2, \dots, t'_k)$ if there is a permutation $\rho \in L_k$ such that $t_i = t'_{e_k^{-1}(\rho(e_k(i)))}$ for all $i = 1, \dots, k$. Denote the E_k-equivalence class of $t \in X^k$ by $[t]_E$. For $\rho \in S_k$ and $t \in X^k$, let $\rho \cdot t \in X^k$ be defined by $(\rho \cdot t)_i = t_{e_k^{-1}(\rho(e_k(i)))}$ for $i = 1, \dots, k$. For simplicity we write $\rho(i)$ for $e_k^{-1}(\rho(e_k(i)))$.

2.3. *The main theorem of the classification*

The aim of this section is to prove the following theorem.

THEOREM 2.5. *Let $f, g \in C^\infty(S^1, S^1)$ be C^∞-stable. Then*

$$f \sim_{\mathscr{A}_\infty} g \Leftrightarrow N(f) = N(g) \qquad \textit{and} \qquad [\mathrm{Ast}(f)]_E = [\mathrm{Ast}(g)]_E.$$

PROOF. We prove the theorem when $\Sigma(f), \Sigma(g) \neq \emptyset$. The same technique applies when $\Sigma(f) = \Sigma(g) = \emptyset$. The theorem is proved in three steps:

Step 1 is to prove that $\mathrm{Ast}(f) = \mathrm{Ast}(g) \Rightarrow f \sim_{\mathscr{A}_\infty} g$. Suppose $\mathrm{Ast}(f) = \mathrm{Ast}(g)$. After composition with diffeomorphisms in source, we may assume that $A(f) = A(g)$ and that $(1, 0)$ is a regular point of f, and hence also of g. A priori, it may happen that f is orientation preserving on $P([a_1(f), a_2(f)])$ while g is not, but after composition with a diffeomorphism in target, we may assume that f and g are orientation preserving on the same subset of source, and that $\sigma_i(f) = \sigma_i(g)$ for all i as well. Finally, we may assume that $(1, 0) \notin f(\Sigma(f))$.

We are going to define a smooth homotopy f_t of stable mappings of S^1 starting at f and ending at g. The f_t will be smoothly equivalent, and hence, $f \sim_{\mathscr{A}_\infty} g$. The standard technique for producing homotopies between mappings in Euclidean space by taking convex combinations of the mappings is not applicable here, since S^1 is not a vector space. Nevertheless, by choosing appropriate charts, the same strategy may be applied to coordinate neighbourhoods, and our assumptions on f and g ensure that the resulting mapping is in fact a smooth homotopy. The details are as follows.

Let $n = n(f) = n(g)$, and let $N = N(f) = N(g)$. Let $\tau \in S_n$ be such that $b_i = b_i(f) = P^{-1}(\sigma_{\tau(i)}(f))$. Notice that $b_1 > 0$ by the assumption $(1, 0) \notin \Delta(f)$. Let

$$0 < v < \frac{1}{2} \min_i (b_{i+1} - b_i, b_1, 2\pi - b_n)$$

and define

$$\Theta_i : P(b_i - v, b_{i+1} + v) \to (-v, b_{i+1} - b_i + v), \qquad P(x) \mapsto x - b_i$$

for $i = 1, \dots, n - 1$. For $i = n$ we define

$$\Theta_n : S^1 \setminus P([b_1 + v, b_n - v]) \to (-v, b_1 + 2\pi - b_n + v)$$

by

$$P(x) \mapsto \begin{cases} 2\pi - b_n + x, & x \in [0, b_1 + v) \\ x - b_n, & x \in (b_n - v, 2\pi). \end{cases}$$

The mappings Θ_i, $i = 1, \ldots, n$ are well defined by the choice of v. Together with their domains of definition, they cover S^1 with local charts.

Let

$$0 < u < \frac{1}{2} \min_i (a_{i+1} - a_i, a_1, 2\pi - a_N)$$

and let $U_i = P(a_i - u, a_{i+1} + u)$, $i = 1, \ldots, N-1$ and $U_N = S^1 \setminus P([a_1 + u, a_N - u])$. In the same way, let $V_i = P(b_i - v, b_{i+1} + v)$, $i = 1, \ldots, n-1$ and $V_n = S^1 - P([b_1 + v, b_n - v])$. We can now define our homotopy. By continuity of f and g, if u is small enough, then for all i there is a j such that both $f(U_i)$ and $g(U_i)$ are contained in V_j. More precisely; there exists $\rho : \{1, \ldots, N\} \to \{1, \ldots, n\}$ such that for all i, $f(U_i) \cup g(U_i) \subset V_{\rho(i)}$. For even smaller u, we can ensure that $\mathrm{cl}(f(U_i) \cup g(U_i)) \subset V_{\rho(i)}$. Let $F : S^1 \times (-\epsilon, 1 + \epsilon) \to S^1$ be defined by

$$\begin{aligned} F(p, t) &= f_t(p) \\ &= \Theta^{-1}_{\rho(i)}(t\Theta_{\rho(i)}(g(p)) + (1-t)\Theta_{\rho(i)}(f(p))), \qquad p \in U_i. \end{aligned}$$

We need to show that $f_t(p)$ is well defined on S^1 and that $f_t(p)$ is smooth. The continuity of $\Theta_{\rho(i)}$ and the observation that $t\Theta_{\rho(i)}(g(p)) + (1-t)\Theta_{\rho(i)}(f(p))$ lies between $\Theta_{\rho(i)}(g(p))$ and $\Theta_{\rho(i)}(f(p))$ for $0 \le t \le 1$, shows that f_t is well defined on U_i when ϵ is chosen small enough.

Next we show that the definitions of f_t agree on $U_i \cap U_j$. It is enough to check the combinations $(i, j) = (N, 1)$ and $(i, j) = (i, i+1)$ for $i < N$. The other combinations of i and j give $U_i \cap U_j = \emptyset$. We first assume that $1 \le \rho(i) = \rho(j) - 1 < n$. Writing out the definitions,

$$\begin{aligned} &\Theta^{-1}_{\rho(i)}(t\Theta_{\rho(i)}(g(p)) + (1-t)\Theta_{\rho(i)}(f(p))) \\ &\quad = \Theta^{-1}_{\rho(i)}(t[P^{-1}(g(p)) - b_{\rho(i)}] + (1-t)[P^{-1}(f(p)) - b_{\rho(i)}]) \\ &\quad = \Theta^{-1}_{\rho(i)}(t[P^{-1}(g(p))] + (1-t)[P^{-1}(f(p))] - b_{\rho(i)}) \\ &\quad = P(t[P^{-1}(g(p))] + (1-t)[P^{-1}(f(p))]) \\ &\quad = \Theta^{-1}_{\rho(j)}(t\Theta_{\rho(j)}(g(p)) + (1-t)\Theta_{\rho(j)}(f(p))). \end{aligned}$$

For $\rho(i) = n$ and $\rho(j) = 1$, we have

$$\begin{aligned} &\Theta_1^{-1}(t\Theta_1(g(p)) + (1-t)\Theta_1(f(p))) \\ &\qquad = P(t[P^{-1}(g(p))] + (1-t)[P^{-1}(f(p))]) \end{aligned}$$

and

$$
\begin{aligned}
\Theta_n^{-1}&(t\Theta_n(g(p)) + (1-t)\Theta_n(f(p))) \\
&= \Theta_n^{-1}(t[P^{-1}(g(p)) + 2\pi - b_n] + (1-t)[P^{-1}(f(p)) + 2\pi - b_n]) \\
&= \Theta_n^{-1}(t[P^{-1}(g(p))] + (1-t)[P^{-1}(f(p))] + 2\pi - b_n) \\
&= P(t[P^{-1}(g(p))] + (1-t)[P^{-1}(f(p))]).
\end{aligned}
$$

This shows that f_t is well defined on S^1 in this case. The case $1 < \rho(i) = \rho(j) + 1 \leq n$ and the case $\rho(i) = 1$, $\rho(j) = n$ may be checked in a similar way.

It remains to show that f_t has finitely many singularities, all of Morse type, and no singular double points. In fact, f_t has the same singular set and discriminant set as f, g. To actually show this, we need to work with charts in the source too. Let

$$\theta_i : U_i \to (-u, a_{i+1} - a_i + u), \qquad P(x) \mapsto x - a_i$$

for $i = 1, \dots, N-1$. For $i = N$ we define

$$\theta_N : U_N \to (-u, a_1 + 2\pi - a_N + u)$$

by

$$P(x) \mapsto \begin{cases} 2\pi - a_N + x, & x \in [0, a_1 + u) \\ x - a_N, & x \in (a_N - u, 2\pi). \end{cases}$$

The charts (θ_i, U_i) cover S^1. Now, we may compute

$$\Theta_{\rho(i)} \circ f_t \circ \theta_i^{-1}(x) = t\Theta_{\rho(i)}(g(\theta_i^{-1}(x))) + (1-t)\Theta_{\rho(i)}(f(\theta_i^{-1}(x))). \tag{2.1}$$

By our assumptions, f and g are equally oriented at every regular point, and therefore the derivatives with respect to x of two terms on the right side of (2.1) have the same sign, and hence, $\Sigma(f_t) = \Sigma(f) = \Sigma(g)$. Moreover, by definition of Morse singularities, we must have

$$\frac{d^2}{dx^2}\Theta_{\rho(i)}(f(\theta^{-1}(x))) \neq 0 \qquad \text{and} \qquad \frac{d^2}{dx^2}\Theta_{\rho(i)}(g(\theta^{-1}(x))) \neq 0$$

whenever $\theta^{-1}(x) \in \Sigma(f)$ and these second derivatives must have the same sign at singular points. It follows that in these charts, the second derivative of f_t with respect to x is different from 0 at every singular point. Therefore, f_t has only Morse singularities. From the definition of f_t, we see that $f(p) = g(p)$ implies that $f(p) = g(p) = f_t(p)$. It follows that f_t has no singular double points, and hence, f_t is stable.

Step 2. Assume that $\mathrm{Ast}(f) = \rho \cdot \mathrm{Ast}(g)$ for some $\rho = \sigma \cdot \tau \in L_{N(g)}$, where $\sigma \in \mathrm{Sw}_{N(g)}$, $\tau \in R_{N(g)}$. If $\tau = \mathrm{id}$, then there is some $\theta = \theta(\sigma)$ such that if

$$R_\sigma : S^1 \to S^1$$

is given by

$$e^{i\theta} \mapsto e^{i(\theta+\theta(\sigma))},$$

then

$$\mathrm{Ast}(f) = \mathrm{Ast}(g \circ R_\sigma)$$

and by Step 1 there are diffeomorphisms h and k such that the following diagram commutes.

$$\begin{tikzcd}
S^1 \arrow[dr, bend left, "g"] & \\
S^1 \arrow[u, "R_\sigma"] \arrow[r, "g \circ R_\sigma"] & S^1 \\
S^1 \arrow[u, "h"] \arrow[r, "f"] & S^1 \arrow[u, "k"']
\end{tikzcd}$$

Similarly, if $\sigma = \mathrm{id}$ and $\tau = r \in R_{N(g)}$, then, if

$$M : S^1 \to S^1$$

is given by

$$e^{i\theta} \mapsto e^{-i\theta},$$

then

$$\mathrm{Ast}(f) = \mathrm{Ast}(g \circ M)$$

and by Step 1 again, we have a commutative diagram:

$$\begin{tikzcd}
S^1 \arrow[dr, bend left, "g"] & \\
S^1 \arrow[u, "R_M"] \arrow[r, "g \circ M"] & S^1 \\
S^1 \arrow[u, "h"] \arrow[r, "f"] & S^1 \arrow[u, "k"']
\end{tikzcd}$$

If $\rho = \sigma \circ r$ for some $\sigma \in \mathrm{Sw}_{N(g)}$, then

$$\mathrm{Ast}(f) = \sigma \cdot \mathrm{Ast}(g \circ M) = \mathrm{Ast}(g \circ M \circ R_\sigma),$$

which again, by the above arguments, implies that $f \sim_{\mathscr{A}_\infty} g$. Altogether we have shown that $[\mathrm{Ast}(f)]_E = [\mathrm{Ast}(g)]_E \Rightarrow f \sim_{\mathscr{A}_\infty} g$.

Step 3. Suppose that f and g are $\mathscr{A}_\infty$-equivalent. Then there are diffeomorphisms h and k of S^1 such that $k \circ f = g \circ h$. Since a singularity of Morse type is topologically different from a regular germ, it is clear that h maps $\Sigma(f)$ to $\Sigma(g)$, and that k maps $\Delta(f)$ to $\Delta(g)$, and it follows that $f^{-1}(\Delta(f))$ is mapped onto $g^{-1}(\Delta(g))$ by h, and hence, $N(f) = N(g)$. If h is orientation preserving and $h(s_1(f)) = s_i(g)$, then $\mathrm{Ast}(g) = \rho \cdot \mathrm{Ast}(f)$ where $\rho([j]) = [j + i - 1]$. If h is orientation reversing, then $\mathrm{Ast}(g) = \rho' \cdot \mathrm{Ast}(f)$ where $\rho'([j]) = [i - j + 1]$. It follows that $[\mathrm{Ast}(f)]_E = [\mathrm{Ast}(g)]_E$.

2.4. *Feasible tuples*

By Theorem 2.5, the problem of listing all topological equivalence classes of smooth stable maps $S^1 \to S^1$ corresponds to the problem of listing all E_n-equivalence classes of associated tuples to such maps. Every non-singular map $f : S^1 \to S^1$ is clearly equivalent to the map $e^{i\theta} \mapsto e^{in\theta}$ where $n = \# f^{-1}(1)$. To generate such a list for maps with singularities, we will make use of another version of our tuples. If $f : S^1 \to S^1$ is a smooth stable map with $\Sigma(f) \neq \emptyset$, then $[\mathrm{Ast}(f)]_E$ may be represented by a tuple in $\{s, p\}^{N(f)}$ having an s as the last component. This may be done in several different ways. Let A' be such a representation. Let $\rho : \{1, \dots, n(f)\} \to \{1, \dots, N(f)\}$ be such that ρ is increasing and $A'_{\rho(i)} = s$. Set $\rho(0) = 0$ and let $c_i = \rho(i) - \rho(i-1) - 1$ for $i = 1, \dots, n(f)$. Define

$$\mathrm{Ast}^\#(A') = (c_1, c_2, \dots, c_{n(f)}) \in \mathsf{N}_0^{n(f)}.$$

Let

$$\mathrm{Ast}^\#(f) = [\mathrm{Ast}^\#(A')]_E$$

where $A'_{n(f)} = s$. It is not difficult to see that this definition of $\mathrm{Ast}^\#(f)$ is unambiguous.

REMARK 2.6. Clearly, Theorem 2.5 is still valid for maps with singularities if we replace Ast with $\mathrm{Ast}^\#$.

Given an element $(x_1, x_2, \dots, x_n) \in \mathsf{N}_0^n$, we want to determine whether or not there is a smooth stable map $f : S^1 \to S^1$ such that

$$\mathrm{Ast}^\#(f) = [(x_1, \dots, x_n)]_E.$$

Let f be a stable map with $\mathrm{Ast}^{\#}(f) = [(x_1, \dots, x_n)]_E$. We say that f is of type (n, m) if $n(f) = n$ and $N(f) - n(f) = m$. Thus, if f is of type (n, m), then n is an even number and

$$x_1 + x_2 + \cdots + x_n = m.$$

These two properties arise from observing that f has an even number of singular points, and that $N(f) - n(f)$ is the number of regular preimage points of the discriminant set. Another property of f is that f restricted to its singular set is injective, and this fact should be reflected in $[(x_1, \dots, x_n)]_E$. Indeed, the curve $P(x)$, $x \in [0, 2\pi)$, passes x_i points in $f^{-1}(\Delta(f)) \setminus \Sigma(f)$ when x runs through $I = [P^{-1}(s_{i-1}), P^{-1}(s_i))$. Therefore, the curve $f(P(x))$ passes x_i singular values in the same interval of parameters. Thus, if $\sigma_{i-1} = P(b_j)$ and f is orientation preserving on $P(\mathrm{int}\, I)$ and

$$k = (\text{remainder of the division } x_i \text{ by } n) + 1,$$

then $\sigma_i = P(b_{j+k})$.

In general, let $R : \mathsf{Z} \to \{1, 2, \dots, n\}$ be given by $R(x) = (\text{remainder of the division } x \text{ by } n) + 1$. Let $\tau \in S_n$ be as in the proof of Theorem 2.5, i.e. such that $P(b_i) = \sigma_{\tau(i)}$. Assuming that $(x_1, x_2, \dots, x_n) = \mathrm{Ast}^{\#}(\mathrm{Ast}(f))$, $\sigma_1 = P(b_j)$ and that f is orientation reversing on $P(P^{-1}(s_1), P^{-1}(s_2))$, then we see that

$$\sigma_k = \sigma_{\tau(R(j - x_1 - 1 + \sum_{i=1}^{k} (-1)^{i+1}[x_i + 1]))}$$

for $k = 1, \dots, n$. Moreover, since we chose representatives with s in the last component in the definiton of $\mathrm{Ast}^{\#}$, we have

$$\sigma_n = P(b_{R(j - x_1 - 1)}).$$

In order for all these equations to be satisfied, the set

$$R' = \left\{ \sum_{i=1}^{k} (-1)^{i+1}[x_i + 1] \;;\; k = 1, \dots, n \right\}$$

has to be a complete remainder system modulo n, i.e., the canonical map $R' \to \mathsf{Z}/n\mathsf{Z}$ is surjective. Furthermore,

$$\sum_{i=1}^{n} (-1)^{i+1}[x_i + 1] \equiv 0 \mod n.$$

DEFINITION 2.7. An element $A = (x_1, \dots, x_n) \in \mathsf{N}_0^n$ is feasible of type (n, m) if n is an even number and the following condtions are satisfied:

(1) $\sum_{i=1}^{n} x_i = m$.
(2) $\sum_{i=1}^{n}(-1)^{i+1}x_i \equiv 0 \mod n$.
(3) $\left\{\sum_{i=1}^{k}(-1)^{i+1}[x_i+1]\ ;\ k=1,\ldots,n\right\}$ is a complete remainder system modulo n.

REMARK 2.8. There are no feasible tuples of type (n, m) if m is odd, because the numbers $m = \sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n}(-1)^{i+1}x_i$ have the same parity, and by 2 in the definition, the latter number is even, since n is even.

PROPOSITION 2.9. *There are no feasible tuples of type* (n, m) *if* $n \equiv 0 \mod 4$ *and* $m \equiv 2 \mod 4$.

PROOF. Assume that $(x_1, \ldots, x_n)$ is feasible of type (n, m). Let $L_k = \sum_{i=1}^{k}(-1)^{i+1}(x_i+1)$. Notice that

$$2\left(\sum_{i=1}^{n-1}(-1)^{i+1}L_i\right) - L_n = x_1 + \cdots + x_n + n = m + n.$$

Since $L_n \equiv 0 \mod n$,

$$2\sum_{i=1}^{n-1}(-1)^{i+1}L_i \equiv m \mod n. \tag{2.2}$$

Since $\{L_k; k = 1, \ldots, n\}$ is a complete remainder system modulo n, we have

$$2\sum_{i=1}^{n} L_i \equiv 2\sum_{i=0}^{n-1} i \equiv n(n-1) \equiv 0 \mod n. \tag{2.3}$$

Addition of (2.2) and (2.3) yields

$$4(L_1 + L_3 + L_5 + \cdots + L_{n-1}) \equiv m \mod n. \tag{2.4}$$

Hence, there is an integer K such that

$$4(L_1 + L_3 + L_5 + \cdots + L_{n-1}) - m = Kn.$$

It follows that $4|n \Rightarrow 4|m$.

The next theorem justifies the term 'feasible tuple'.

THEOREM 2.10. *Let* $A \in \mathsf{N}_0^n$. *There exists a smooth stable map* $f : S^1 \to S^1$ *with* $\mathrm{Ast}^{\#}(f) = A$ *if and only if* A *is feasible of type* (n, m) *for some number* m.

PROOF. The forward implication follows from the above discussion. For the other implication, let $A = (x_1, \dots, x_n) \in \mathsf{N}_0^n$ be a feasible tuple of type (n, m). We need to construct a smooth stable map $f : S^1 \to S^1$ with $\mathrm{Ast}^\#(f) = A$. We construct a smooth map $f_A : [0, 2\pi) \to \mathsf{R}$ such that $f = P \circ f_A \circ P^{-1}$ is smooth and stable and satisfies $\mathrm{Ast}^\#(f) = A$. It is natural to define f_A to consist of line segments outside some small open intervals about the singular points and consist of a modified parabola around the singular points. This strategy calls for some kind of gluing process, but we can not use a standard partition of unity, because we must have full control over the singularities of $\tilde{f}$, and a partition of unity might introduce unwanted singularites. Instead, we will construct f_A explicitly, using smooth "bump functions" to glue the different parts of the function together.

Let
$$j(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}}, & x \in (-1, 1) \\ 0, & \text{otherwise} \end{cases}$$
and let
$$k(x) = \frac{\int_{-1}^{x} j(t)dt}{\int_{-1}^{1} j(t)dt}.$$
Define
$$l(x) = \begin{cases} x, & x \leq -1 \\ x - 2xk(x), & x \in (-1, 1) \\ -x, & x \geq 1. \end{cases}$$
Then
$$l'(x) = \begin{cases} 1, & x \leq -1 \\ 1 - 2k(x) - 2xk'(x), & x \in (-1, 1) \\ -1, & x \geq 1, \end{cases}$$
and
$$l''(x) = \begin{cases} 0, & x \leq -1 \\ -4k'(x) - 2xk''(x), & x \in (-1, 1) \\ 0, & x \geq 1, \end{cases}$$
Since k is flat at -1 and 1, l is a C^∞ function on R. Also, l is increasing for $x \leq 0$ and decreasing for $x \geq 0$. Since $l'(0) = 0$ and $l''(0) = -4k'(0) < 0$, this means that l has its only extreme point at $x = 0$ and this is a global maximum and a Morse singularity. The definition of f_A is the following. For $k = 1, \dots, n$, let
$$X_k = \sum_{i=1}^{k}(x_i + 1), \quad Y_k = \sum_{i=1}^{k}(-1)^{i+1}(x_i + 1), \quad J_k = \left[X_k - \tfrac{1}{2}, X_k + \tfrac{1}{2}\right).$$

Let
$$I_0 = \left[\tfrac{1}{2}, X_1 - \tfrac{1}{2}\right)$$
and for $k = 1, \dots, n-1$ let
$$I_k = \left[X_k + \tfrac{1}{2}, X_{k+1} - \tfrac{1}{2}\right).$$
For a set $B \in \mathsf{R}$, let χ_B be the corresponding characteristic function which is 1 on B and 0 elsewhere. Put $X_0 = Y_0 = 0$. For $k = 1, \dots, n$, let
$$F_k(x) = \left[Y_{k-1} + (-1)^{k-1}(x - X_{k-1})\right]\chi_{I_{k-1}}$$
$$G_k(x) = \left[Y_k + \frac{(-1)^{k+1}}{2} l(2(x - X_k))\right]\chi_{J_k}.$$
Let
$$H(x) = \sum_{i=1}^{n}(F_k(x) + G_k(x)).$$
Finally, let
$$f_A(x) = \frac{2\pi}{n} H\left(\frac{X_n}{2\pi}x + \frac{1}{2}\right).$$
With this definition of f_A, let $f = P \circ f_A \circ P^{-1}$. It is messy, but straightforward to see that f is smooth and that $\mathrm{Ast}^{\#}(f) = A$.

Let f be a smooth stable map of the circle. All the topological properties of f are coded in $\mathrm{Ast}^{\#}$. We show how $|\deg f|$ can be retrieved from $\mathrm{Ast}^{\#}(f)$.

Proposition 2.11. *Let* $\mathrm{Ast}^{\#}(f) = (x_1, x_2, \dots, x_n)$. *Then*
$$|\deg f| = \left|\frac{1}{n}\sum_{i=1}^{n}(-1)^{i+1}x_i\right|.$$

Proof. Let $A = \mathrm{Ast}^{\#}(f)$, and let f_A be as in the proof of Theorem 2.10. Then $\deg(f) = \deg(P \circ f_A \circ P^{-1})$. Certainly, f_A is homotopic to $\tilde{f}_A$ given by
$$\tilde{f}_A(x) = \frac{1}{n}\left(\sum_{i=1}^{n}(-1)^{i+1}x_i\right)x.$$
by the homotopy $F(x,t) = tf_A(x) + (1-t)\tilde{f}_A(x)$. Clearly, $p \circ \tilde{f}_A \circ P^{-1}$ has degree $\frac{1}{n}\sum_{i=1}^{n}(-1)^{i+1}x_i$, and this finishes the proof.

2.5. Tables of feasible tuples

A complete classification of smooth stable maps $S^1 \to S^1$ can be given by listing all the feasible tuples up to legal permutations. This task is well suited for recursive computer programming. Table 1 and Table 2 give MATLAB generated lists of feasible tuples and numbers of topological types for different (n, m).

(n, m)	Number of topological types	Feasible tuples
(4, 4)	2	(1, 2, 1, 0), (2, 0, 2, 0)
(4, 8)	5	(5, 2, 1, 0), (1, 6, 1, 0), (2, 4, 2, 0), (6, 0, 2, 0), (4, 1, 2, 1)
(4, 12)	12	(9, 2, 1, 0), (5, 6, 1, 0), (1, 10, 1, 0), (6, 4, 2, 0), (2, 8, 2, 0), (5, 2, 5, 0), (8, 1, 2, 1), (4, 5, 2, 1), (6, 1, 4, 1), (10, 0, 2, 0), (6, 0, 6, 0), (4, 2, 4, 2)
(6, 6)	1	(2, 0, 2, 0, 2, 0)
(6, 8)	2	(3, 1, 0, 3, 1, 0), (2, 0, 1, 4, 1, 0)
(6, 10)	3	(3, 0, 4, 2, 1, 0), (1, 4, 0, 4, 1, 0), (3, 1, 2, 1, 3, 0)

TABLE 1. Table of topological types

(n, m)	Number of topological types	(n, m)	Number of topological types
(4, 16)	21	(8, 8)	1
(4, 20)	36	(8, 12)	12
(4, 24)	54	(8, 16)	34
(4, 28)	80	(10, 10)	1
(6, 12)	9	(10, 12)	0
(6, 14)	10	(10, 14)	3
(6, 16)	16	(10, 16)	6

TABLE 2. Number of topological types

Our tables lack the number of feasible tuples of type $(2, m)$ because of the next proposition.

PROPOSITION 2.12. *The number of E_2-equivalence classes of feasible tuples of type $(2, m)$ is $\left\lfloor \frac{m}{4} \right\rfloor + 1$.*

PROOF. Assume (x_1, x_2) is feasible of type $(2, m)$. Then

$$\begin{aligned} x_1 + 1 &\equiv 1 \quad \text{mod } 2 \\ x_1 - x_2 &\equiv 0 \quad \text{mod } 2. \end{aligned}$$

These equations are satisfied if and only if x_1 is even and x_1 and x_2 have the same parity. The feasible tuples of type $(2, m)$ are therefore $\{(2i, m-2i); i = 0, 1, \ldots, \frac{m}{2}\}$. There are $\frac{m}{2} + 1$ elements in this set, and $(2i, m-2i) \sim_{E_2} (m-2i, 2i)$ for all i. If $m = 4k$ for some $k \in \mathsf{N}$, then $\frac{m}{2} + 1 = 2k + 1$ is odd, and the number of E_2-equivalence classes is $k + 1 = \lfloor \frac{m}{4} \rfloor + 1$. If $m = 4k + 2$, then $\frac{m}{2} + 1 = 2k + 2$ is even, and the number of equivalence classes is still $k + 1 = \lfloor \frac{m}{4} \rfloor + 1$.

3. Classification of finitely determined real analytic map germs $(\mathsf{R}^2, 0) \to (\mathsf{R}^2, 0)$

Let $\mathcal{O} = \mathcal{O}(2, 2)$ be the set of real analytic map germs $(\mathsf{R}^2, 0) \to (\mathsf{R}^2, 0)$. Let $\mathcal{O}_g = \mathcal{O}_g(2, 2) \subset \mathcal{O}(2, 2)$ be the set of finitely determined map germs. By Theorem 0.5 of [7], finite determinacy holds in general in $\mathcal{O}(2, 2)$.

3.1. Geometric properties

Finitely determined real analytic plane-to-plane germs have the following well known geometric properties.

PROPOSITION 3.1. *For every $f \in \mathcal{O}_g$ there is an open neighbourhood U of 0 in R^2 and a real analytic representative of f, $\hat{f} : U \to \mathsf{R}^2$ such that*

(1) $\hat{f}^{-1}(0) = \{0\}$,

(2) $\hat{f}|(\Sigma(\hat{f}) \setminus \{0\})$ *is injective,*

(3) *every $p \in \Sigma(\hat{f}) \setminus \{0\}$ is a fold point.*

PROOF. The proof of (2) and (3) goes as the proof of Lemma 6.2 in [1] with semialgebraic replaced by semianalytic. To prove (1), note that $\hat{f}^{-1}(0) \setminus \{0\}$ is a semianalytic set. If 0 is in its closure, then by the Curve Selection Lemma, there is a real analytic curve $\gamma : [0, \epsilon) \to \mathsf{R}^2$ with $\gamma(0) = 0$, $\gamma(0, \epsilon) \in \hat{f}^{-1}(0) \setminus \{0\}$. Hence, $\hat{f}$ is identically 0 along γ, but this contradicts both (2) and (3).

For the rest of this section, let $f \in \mathcal{O}_g$, let U be a small ball around 0 and let $\hat{f} : U \to \mathsf{R}^2$ be a real analytic representative of f such that (1)–(3) of Proposition 3.1 hold.

LEMMA 3.2. *If U is small enough, then $\Sigma(\hat{f}) \setminus \{0\}$ is empty or a 1-dimensional manifold which has only finitely many topological components.*

PROOF. By (3), if $p \in \Sigma(\hat{f}) \setminus \{0\}$, then p is a fold point, and the singular set is diffeomorphic to the real line in a neighbourhood of a fold point. Also, $\Sigma(\hat{f}) \setminus \{0\}$ is a semianalytic set, and hence, its intersection with a small neighbourhood of 0 has only finitely many topological components.

Let $D_\epsilon = \{p \in \mathsf{R}^2 \mid \|p\| \leq \epsilon\}$ and let $S_\epsilon = \{p \in \mathsf{R}^2 \mid \|p\| = \epsilon\} = \partial D_\epsilon$. Define $\tilde{S}_\epsilon(\hat{f}) = \hat{f}^{-1}(S_\epsilon)$ and $\tilde{D}_\epsilon(\hat{f}) = \hat{f}^{-1}(D_\epsilon)$.

LEMMA 3.3. *If U is small enough, then $\hat{f} \pitchfork S_\delta$ for small enough $\delta > 0$.*

PROOF. By Lemma 3.2 there are only finitely many branches of $\Sigma(\hat{f}) \setminus \{0\}$. By the Curve Selection Lemma, for each component B_i of $\Sigma(\hat{f}) \setminus \{0\}$ we may choose an analytic curve $\gamma_i : [0, \epsilon) \to \mathsf{R}^2$ such that $\gamma(0) = 0$ and $\gamma_i(0, \epsilon) \subset B_i$. The curves $\hat{f} \circ \gamma_i$ are analytic and by (1) of Proposition 3.1, $(\hat{f} \circ \gamma_i)(t) \neq 0$ when $t > 0$ and therefore $(\hat{f} \circ \gamma_i) \pitchfork S_{\delta_i}$ for small $\delta_i > 0$. If $\delta < \min_i \delta_i$, then $\hat{f}|\Sigma(\hat{f}) \pitchfork S_\delta$. This proves the lemma, since $\hat{f} \pitchfork S_\delta$ at any regular point of $\hat{f}$ because the dimensions of source and target are equal.

The proof of Lemma 3.3 actually gives us more information. Let $\Delta(\hat{f}) = \hat{f}(\Sigma(\hat{f}))$.

COROLLARY 3.4. *If U is small enough, then $\Delta(\hat{f}) \setminus \{0\}$ is empty or a 1-dimensional smooth manifold such that $\Delta(\hat{f}) \pitchfork S_\delta$ for small δ.*

Let $\theta : \mathsf{R}^2 \to \mathsf{R}$ be given by $\theta(p) = \|p\|^2$.

LEMMA 3.5. *If δ is small enough, then $\nabla(\theta \circ \hat{f})(p) \neq 0$ for all $p \in D_\delta \setminus \{0\}$.*

PROOF. If $\hat{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, then $\theta \circ \hat{f} = f_1^2 + f_2^2$. We compute

$$\begin{aligned}\nabla(\theta \circ \hat{f})(p) &= 2\left(f_1 \frac{\partial f_1}{\partial x} + f_2 \frac{\partial f_2}{\partial x}, f_1 \frac{\partial f_1}{\partial y} + f_2 \frac{\partial f_2}{\partial y} \right)(p) \\ &= 2\begin{pmatrix} f_1(p) & f_2(p) \end{pmatrix} \cdot D\hat{f}(p)\end{aligned}$$

If $p \notin \Sigma(\hat{f})$, then $\hat{f}(p) \neq 0$ and $D\hat{f}(p)$ is invertible, and hence, $\nabla(\theta \circ \hat{f})(p) \neq 0$. Assume that $p \in \Sigma(\hat{f})$ and $\|p\| \neq 0$. By (1), $\hat{f}(p) \neq 0$ and by the above,

$$\begin{aligned}\nabla(\theta \circ \hat{f})(p) = 0 &\Leftrightarrow \hat{f}^T(p) D\hat{f}(p) = 0 \\ &\Leftrightarrow \hat{f}(p) \perp \operatorname{Im} D\hat{f}(p) \\ &\Leftrightarrow D\hat{f}(p)(\mathsf{R}^2) + \mathsf{R}\left\{\begin{pmatrix} -f_2(p) \\ f_1(p) \end{pmatrix}\right\} \neq \mathsf{R}^2.\end{aligned}$$

Note that $\begin{pmatrix} -f_2(p) \\ f_1(p) \end{pmatrix}$ is a tangent vector at $\hat{f}(p)$ to the circle $S_{\|\hat{f}(p)\|}$. It therefore follows from Lemma 3.3 that $D\hat{f}(p)(\mathsf{R}^2) + \mathsf{R}\left\{\begin{pmatrix} -f_2(p) \\ f_1(p) \end{pmatrix}\right\} = \mathsf{R}^2$. This proves the lemma.

LEMMA 3.6 (Lojasiewicz). *There is a* $\rho > 0$ *and constants* $C, r > 0$ *such that for* $p \in D_\rho$, $\|\hat{f}(p)\| \geq C\,\|p\|^r$.

PROOF. Remember that 0 is an isolated zero of $\hat{f}$ and apply IV 4.1 of [5].

LEMMA 3.7. *For small* $\epsilon > 0$, $\tilde{S}_\epsilon(\hat{f})$ *is a compact 1-manifold diffeomorphic to* S^1 *and* 0 *is in the bounded component of* $\mathsf{R}^2 \setminus \tilde{S}_\epsilon(\hat{f})$.

PROOF. Let $\rho > 0$ be such that $\|\hat{f}(p)\| \geq C\,\|p\|^r$ for all $p \in D_\rho$. Such a ρ exists by Lemma 3.6. If $\epsilon \leq C\rho^r$, then $\tilde{S}_\epsilon \subset D_\rho$ is closed and bounded, i.e. compact. By Lemma 3.3, if ρ is small enough, then $f \pitchfork S_{C\rho^r}$ in which case $\tilde{S}_\epsilon$ is a 1-dimensional smooth manifold.

Every component of $\tilde{S}_\epsilon$ is diffeomorphic to S^1 by the classification of smooth compact 1-manifolds. Let C be one such component. Then C is an equipotensial curve of $\theta \circ \hat{f}$. If 0 is not in the bounded component of $\mathsf{R}^2 \setminus C$, then $\theta \circ \hat{f}$ has an extremal point p in the bounded component of $\mathsf{R}^2 \setminus C$, and hence, $\nabla(\theta \circ \hat{f})(p) = 0$. According to Lemma 3.5, this is not possible for small ρ. It follows that 0 is in the bounded component of $\mathsf{R}^2 \setminus C$.

Assume that C and D are different components of $\tilde{S}_\epsilon(\hat{f})$. Then there are two bounded components of $\mathsf{R}^2 \setminus (C \cup D)$, one of them containing 0. The other component must contain an extremal point of $\theta \circ \hat{f}$ which is impossible for small ρ.

Figure 2 below illustrates some of the properties we have proven so far.

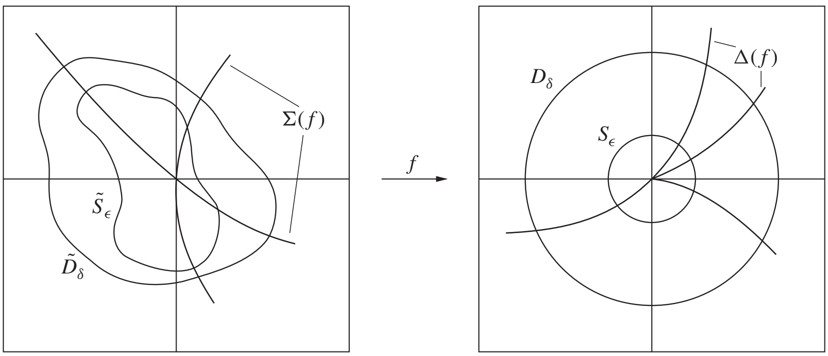

FIGURE 2. Illustration of Lemma 3.2, Lemma 3.3, Corollary 3.4 and Lemma 3.7.

Let $E_\delta = \{p \in \mathsf{R}^2 \mid \|p\| < \delta\} = \operatorname{int} D_\delta$ and let $\tilde{E}_\delta(\hat{f}) = \hat{f}^{-1}(E_\delta)$.

LEMMA 3.8. *For small $\delta > 0$ the map $\hat{f}|\tilde{E}_\delta \setminus \{0\} : E_\delta \setminus \{0\} \to E_\delta \setminus \{0\}$ is proper.*

PROOF. By Lemma 3.6 there are $C, r, \rho > 0$ such that $\|\hat{f}(p)\| \geq C\,\|p\|^r$ for all $p \in D_\rho$. Assume that δ is so small that $\max\{\delta, \left(\frac{\delta}{C}\right)^{\frac{1}{r}}\} < \rho$. Redefine $\hat{f}$ putting $\hat{f} := \hat{f}|E_\rho$. Then $\tilde{D}_\delta(\hat{f}) \subset D_{\left(\frac{\delta}{C}\right)^{\frac{1}{r}}} \subset E_\rho$, and hence, $\tilde{D}_\delta(\hat{f})$ is compact.

Let $K \subset E_\delta \setminus \{0\}$ be a compact set. Let $\tilde{K} = (\hat{f}|\tilde{E}_\delta \setminus \{0\})^{-1}(K)$ and let (p_n) be a sequence in $\tilde{K}$. Then (p_n) is a sequence in $\tilde{D}_\delta(\hat{f})$, and hence, there is a subsequence $p_{n(k)}$ of p_n and a point $p \in \tilde{D}_\delta(\hat{f})$ such that $p_{n(k)} \to p$ as $k \to \infty$. Then $\hat{f}(p_{n(k)}) \to \hat{f}(p) \in K$, and hence, $p \in \tilde{K}$. It follows that $\tilde{K}$ is compact and that $\hat{f}|\tilde{E}_\delta \setminus \{0\}$ is proper.

PROPOSITION 3.9. *For small $\epsilon > 0$, the restriction $\hat{f}|\tilde{S}_\epsilon(\hat{f}) : \tilde{S}_\epsilon(\hat{f}) \to S_\epsilon$ is stable.*

PROOF. It is enough to show that $\hat{f}|\tilde{S}_\epsilon(\hat{f})$ has only Morse singularities and no singular double points. Corollary 3.4 implies that $\tilde{S}_\epsilon(\hat{f}) \pitchfork \Sigma(\hat{f})$ close to the origin. We also observe that $\Sigma(\hat{f}|\tilde{S}_\epsilon(\hat{f})) \subset \Sigma(\hat{f})$. In fact, $\Sigma(\hat{f}|\tilde{S}_\epsilon(\hat{f})) = \Sigma(\hat{f}) \cap \tilde{S}_\epsilon(\hat{f})$. Let $p \in \Sigma(\hat{f}|\tilde{S}_\epsilon(\hat{f}))$, and let β be a centered chart about p in $\tilde{S}_\epsilon(\hat{f})$, and let π be the projection of R^2 onto the line L perpendicular to $\Delta(\hat{f})$ at $\hat{f}(p)$. The restriction of π to a neighbourhood of $\hat{f}(p)$ in S_ϵ is a chart about $\hat{f}(p)$ in S_ϵ. Let Ψ and Φ be diffeomorphisms of neighbourhoods of p, $\hat{f}(p)$ in U, R^2 respectively such that $\hat{f} = \Phi \circ F \circ \Psi$ where $F(x, y) = (x, y^2)$. Such diffeomorphisms exist because p is a fold point of $\hat{f}$. Now, choose a linear isomorphism $T : L \to \mathsf{R}$ which identifies L with R such that $T(\pi(\hat{f}(p))) = 0$.

Let $\alpha = (\alpha_1, \alpha_2) = \Psi \circ \beta^{-1}$ and let $A = T \circ \pi \circ \Phi$. Then $\hat{f}|\tilde{S}_\epsilon(\hat{f}) \sim_{\mathscr{A}} A \circ F \circ \alpha$. Now we compute

$$(A \circ F \circ \alpha)'(t) = A_x \alpha_1'(t) + 2A_y \alpha_2(t)\alpha_2'(t)$$

and

$$\begin{aligned}(A \circ F \circ \alpha)''(t) = {} & [A_{xx}\alpha_1'(t) + 2A_{xy}\alpha_2(t)\alpha_2'(t)]\alpha_1'(t) + A_x\alpha_1''(t) \\ & + [A_{yx}\alpha_1'(t) + 2A_{yy}\alpha_2(t)\alpha_2'(t)] \cdot 2\alpha_2(t)\alpha_2'(t) \\ & + A_y[2(\alpha_2'(t))^2 + 2\alpha_2(t)\alpha_2''(t)].\end{aligned}$$

Here all the partial derivatives of A are to be taken at $F \circ \alpha(t)$. Since there is no neighbourhood of p in $\tilde{S}_\epsilon(\hat{f})$ restricted to which $\hat{f}|\tilde{S}_\epsilon(\hat{f})$ is injective and since $\tilde{S}_\epsilon(\hat{f}) \pitchfork \Sigma(\hat{f})$, we see from the normal form F of folds that $\alpha_1'(0) = 0$ and $\alpha_2'(0) \neq 0$. We have also chosen $\alpha_2(0) = 0$. The choice of L gives

$A_x(F(\alpha(0)) = 0$. Therefore we must have $A_y(F(\alpha(0))) \neq 0$. We get

$$(A \circ F \circ \alpha)''(0) = 2A_y((F(\alpha(0))) \cdot (\alpha_2'(0))^2 \neq 0.$$

This shows that $A \circ F \circ \alpha$ has a Morse singularity at 0, and hence, $\hat{f}|\tilde{S}_\epsilon(\hat{f})$ has a Morse singularity at p.

3.2. Generic mappings as cones of smooth stable mappings between spheres

In this section we follow the steps in [2] pp. 246–247.

Let $f \in \mathcal{O}_g$ and let $\hat{f} : U \to \mathsf{R}^2$ be a fixed representative of f with U so small that the lemmas of the previous section hold. We simplify notation putting $\tilde{S}_\epsilon := \tilde{S}_\epsilon(\hat{f})$ and similar simplifications for $\tilde{D}_\epsilon(\hat{f})$ and $\tilde{E}_\epsilon(\hat{f})$. Let δ be so small that $\nabla(\theta \circ \hat{f}) \neq 0$ on $\tilde{D}_\delta \setminus \{0\}$ and let $\epsilon, \alpha > 0$ be such that $\epsilon + \alpha < \delta$. Let $\varphi_p(t)$ be the flowline of $\nabla(\theta \circ \hat{f})$ passing through p, and let t_p be such that $\varphi_p(t_p) \in \tilde{S}_\epsilon$. Define maps

$$\begin{aligned}
\phi &: \tilde{E}_{\epsilon+\alpha} - \{0\} \to \tilde{S}_\epsilon,\\
\Phi &: \tilde{E}_{\epsilon+\alpha} - \{0\} \to \tilde{S}_\epsilon \times (0, \epsilon + \alpha),\\
\Psi &: E_{\epsilon+\alpha} - \{0\} \to S_\epsilon \times (0, \epsilon + \alpha)
\end{aligned}$$

by

$$\begin{aligned}
\phi(p) &= \varphi_p(t_p)\\
\Phi(p) &= (\phi(p), \|\hat{f}(p)\|)\\
\Psi(q) &= \left(\epsilon \frac{q}{\|q\|}, \|q\|\right)
\end{aligned}$$

Both Φ and Ψ are certainly diffeomorphisms, and we can define

$$F : \tilde{S}_\epsilon \times (0, \epsilon + \alpha) \to S_\epsilon \times (0, \epsilon + \alpha)$$

by $F = \Psi \circ \hat{f} \circ \Phi^{-1}$. Then $F(\tilde{S}_\epsilon \times \{t\}) \subset S_\epsilon \times \{t\}$ and the following diagram commutes.

$$\begin{array}{ccc}
E_{\epsilon+\alpha} - \{0\} & \xrightarrow{\hat{f}} & E_{\epsilon+\alpha} - \{0\}\\
\Phi \big\downarrow & & \big\downarrow \Psi\\
\tilde{S}_\epsilon \times (0, \epsilon + \alpha) & \xrightarrow{F} & S_\epsilon \times (0, \epsilon + \alpha)
\end{array}$$

Let $f_t : \tilde{S}_\epsilon \to S_\epsilon$ be defined by $F(p, t) = (f_t(p), t)$. Then f_t is a smooth homotopy and $f_\epsilon = \hat{f}|\tilde{S}_\epsilon$. If we let $\pi : \mathsf{R}^2 \to \mathsf{R}$ be the projection onto the first factor, we get

$$\begin{aligned} f_t &= \pi \circ F|\tilde{S}_\epsilon \times \{t\} \\ &= \pi \circ \Psi \circ \hat{f} \circ \Phi^{-1}|\tilde{S}_\epsilon \times \{t\} \\ &= \pi \circ \Psi \circ \hat{f}|\tilde{S}_t \circ \Phi^{-1}|\tilde{S}_\epsilon \times \{t\}. \end{aligned}$$

Thus, f_t is C^∞-equivalent to $\hat{f}|\tilde{S}_t$. It follows from Proposition 3.9 that all $\hat{f}|\tilde{S}_t$ and hence, every f_t is smoothly stable. Hence, there are C^∞ diffeomorphisms

$$h'_t : \tilde{S}_\epsilon \to \tilde{S}_\epsilon$$

and

$$h''_t : S_\epsilon \to S_\epsilon$$

such that $\hat{f}|\tilde{S}_\epsilon \circ h'_t = h''_t \circ f_t$ and we can choose h'_t and h''_t such that $h'_\epsilon = \mathrm{id}$ and $h''_\epsilon = \mathrm{id}$ and the mappings

$$H' : \tilde{S}_\epsilon \times (0, \epsilon + \alpha) \to \tilde{S}_\epsilon \times (0, \epsilon + \alpha)$$

and

$$H'' : S_\epsilon \times (0, \epsilon + \alpha) \to S_\epsilon \times (0, \epsilon + \alpha)$$

defined by $H'(x, t) = (h'_t(x), t)$ and $H''(y, t) = (h''_t(y), t)$ are diffeomorphisms. It follows that $\hat{f}|\tilde{E}_{\epsilon+\alpha} \setminus \{0\} \sim_{\mathscr{A}_\infty} F = (f_t, \mathrm{id}) \sim_{\mathscr{A}_\infty} (\hat{f}|\tilde{S}_\epsilon(\hat{f}), \mathrm{id})$ and that $\hat{f}$ is $\mathscr{A}_0$-equivalent to the cone of the map $\hat{f}|\tilde{S}_\epsilon(\hat{f})$ for small ϵ.

3.3. The main theorem

According to Proposition 3.9, if $f \in \mathscr{O}_{\mathrm{g}}$, then $\hat{f}|\tilde{S}_\epsilon : \tilde{S}_\epsilon(\hat{f}) \to S_\epsilon$ is stable for small ϵ. Also, the homotopy f_t of Section 3.2 is a smooth homotopy of C^∞ stable mappings, and hence, they are all C^∞-equivalent. Therefore, regarding $\hat{f}|\tilde{S}_\epsilon$ as a map between 1-spheres, we can associate a tuple $\mathrm{Ast}(f)$ unambiguously to f by the rule $\mathrm{Ast}(f) = [\mathrm{Ast}(\hat{f}|\tilde{S}_\epsilon)]_E$, the equivalence class of $\mathrm{Ast}(\hat{f}|\tilde{S}_\epsilon)$ under the equivalence relation introduced in Section 2.2. In the same way, we define $\mathrm{Ast}^\#(f) = [\mathrm{Ast}(\hat{f}|\tilde{S}_\epsilon)]_E$ when $\Sigma(f) \neq \{0\}$. It is clear that

$$\mathrm{Ast}^\#(f) = \mathrm{Ast}^\#(g) \Leftrightarrow \mathrm{Ast}(f) = \mathrm{Ast}(g).$$

THEOREM 3.10. *If $f, g \in \mathscr{O}_{\mathrm{g}}$ and $\Sigma(f) \setminus \{0\}, \Sigma(g) \setminus \{0\} \neq \emptyset$, then*

$$f \sim_{\mathscr{A}_0} g \Leftrightarrow \mathrm{Ast}(f) = \mathrm{Ast}(g).$$

PROOF. Assume $\mathrm{Ast}(f) = \mathrm{Ast}(g)$. Choose representatives $\hat{f}$ and $\hat{g}$ for f and g and construct the homotopies f_t and g_t as in Section 3.2. Clearly, for small ϵ and α, $\hat{f}|\tilde{E}_{\epsilon+\alpha}(\hat{f})\setminus\{0\} \sim_{\mathscr{A}_\infty} F \sim_{\mathscr{A}_\infty} (f_\epsilon, \mathrm{id})$ and $\hat{g}|\tilde{E}_{\epsilon+\alpha}(\hat{g})\setminus\{0\} \sim_{\mathscr{A}_\infty}$

$G \sim_{\mathscr{A}_\infty} (g_\epsilon, \mathrm{id})$. Now, by hypothesis and Theorem 2.5, there are suitable homeomorphisms k_ϵ and h_ϵ (which can be chosen to be smooth) such that

$$f_\epsilon = k_\epsilon \circ g_\epsilon \circ h_\epsilon^{-1}.$$

It follows that $F \sim_{\mathscr{A}_\infty} G$, and hence, $f \sim_{\mathscr{A}_0} g$.

Conversely, assume that $f \sim_{\mathscr{A}_0} g$. Then f and g have representatives $\hat{f}$ and $\hat{g}$ which are topologically equivalent to cones of maps of S^1 and there are homeomorphisms $\Sigma(\hat{f}) \approx \Sigma(\hat{g})$, $\Delta(\hat{f}) \approx \Delta(\hat{g})$ and therefore also $\hat{f}^{-1}(\Delta(\hat{f})) \setminus \Sigma(\hat{f}) \approx \hat{g}^{-1}(\Delta(\hat{g})) \setminus \Sigma(\hat{g})$. By Lemma 3.3, Corollary 3.4 and Lemma 3.7, when we pass to the topologically equivalent cones of maps of circles, these sets appear as disjoint curves in source and target intersecting each t-level exactly once. It is clear that this implies that $[\mathrm{Ast}(\hat{f}|\tilde{S}_\epsilon(\hat{f}))]_E = [\mathrm{Ast}(\hat{g}|\tilde{S}_\epsilon(\hat{g}))]_E$ and hence, that $\mathrm{Ast}(f) = \mathrm{Ast}(g)$.

REMARK 3.11. The implication

$$\mathrm{Ast}(f) = \mathrm{Ast}(g) \Rightarrow f \sim_{\mathscr{A}_0} g$$

is also true when $\Sigma(f) \setminus \{0\} = \Sigma(g) \setminus \{0\} = \emptyset$ with the same proof as above.

3.4. Stable perturbations

The notion of stable perturbations of generic smooth map-germs is introduced in [3] and is defined as follows: Let f and $\hat{f}$ be as in Section 3.1 and let δ be so small that both $\hat{f}|\tilde{E}_\delta \setminus \{0\} : \tilde{E}_\delta \setminus \{0\} \to E_\delta \setminus \{0\}$ and $\hat{f}|\tilde{S}_\delta : \tilde{S}_\delta \to S_\delta$ are C^∞ stable. By Proposition 3.9 such δ exist. Let $\tilde{f} : \tilde{E}_\delta \to E_\delta$ be a stable map such that $\{p \in \tilde{E}_\delta \mid \tilde{f}(p) \neq \hat{f}(p)\} \subset \mathrm{int}\, \tilde{E}_\delta$. Such a map $\tilde{f}$ is called a *stable perturbation* of f.

In [3] it is shown that the number $\kappa(\tilde{f})$ of cusps of $\tilde{f}$ has to satisfy the formula

$$\kappa(\tilde{f}) \equiv 1 + \frac{1}{2}\#\{\text{branches of } \Sigma(f) \setminus \{0\}\} + \deg f \mod 2.$$

Proposition 2.11 enables us to reformulate this formula for $\kappa(\tilde{f})$ in terms of the components of Ast f.

PROPOSITION 3.12. *Let $f \in \mathscr{O}_{\mathrm{g}}$ with $\mathrm{Ast}^\#(f) = [x_1, \ldots, x_n]_E$ and let $\tilde{f}$ be a stable perturbation of f. Then the number $\kappa(\tilde{f})$ of cusps of $\tilde{f}$ satisfies*

$$\kappa(\tilde{f}) \equiv 1 + \frac{n}{2} + \frac{1}{n}\left|\sum_{i=1}^{n}(-1)^{i+1}[x_i + 1]\right| \mod 2.$$

PROOF. By Theorem 2.1 of [3],

$$\kappa(\tilde{f}) \equiv 1 + \frac{1}{2}\#\{\text{branches of } \Sigma(f) \setminus \{0\}\} + \deg f.$$

By definition, $n = \#\{\text{branches of } \Sigma(f) \setminus \{0\}\}$ and furthermore, $|\deg \hat{f}|\tilde{S}_\epsilon(\hat{f})| = |\deg f| \equiv \deg f \mod 2$. By Proposition 2.11, $|\deg \hat{f}|\tilde{S}_\epsilon(\hat{f})| = \frac{1}{n}\left|\sum_{i=1}^{n}(-1)^{i+1}[x_i + 1]\right|$ and this finishes the proof.

3.5. *Examples and tables*

When calculating Ast, one has to check that the germ in question has only fold singularities outside the origin. Let $\omega : \mathsf{R}^2 \to \mathsf{R}^2$ be a smooth map and let $p \in \Sigma(\omega)$. It is shown in [1], Section 3, that p is a fold point if and only if

$$D\omega(p)\begin{pmatrix} \frac{\partial}{\partial y}J\omega(p) \\ -\frac{\partial}{\partial x}J\omega(p) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For simplicity, put

$$\nabla_\perp J\omega(p) = \begin{pmatrix} \frac{\partial}{\partial y}J\omega(p) \\ -\frac{\partial}{\partial x}J\omega(p) \end{pmatrix}.$$

EXAMPLE 3.13. Let $\omega(x, y) = (x, y^3 + x^k y)$. We find $J\omega(x, y) = 3y^2 + x^k$, and therefore, $\Sigma(\omega)$ is given by $3y^2 + x^k = 0$. We see that k has to be odd in order for $\Sigma(\omega) \setminus \{0\}$ to be non-empty. Assume that k is odd. It is clear that $\omega^{-1}(0) = \{0\}$.

The branches of $\Sigma(\omega)$ is given by

$$y = \pm\frac{1}{\sqrt{3}}(-x^k)^{\frac{1}{2}}.$$

Let $z(x) = \frac{1}{\sqrt{3}}(-x^k)^{\frac{1}{2}}$. We compute

$$\omega(x, \pm z(x)) = \left(x, \mp\frac{2}{3\sqrt{3}}(-x^k)^{\frac{3}{2}}\right).$$

This shows that ω has no singular double points. Also,

$$D\omega(x, y)\nabla_\perp J\omega(x, y) = \begin{pmatrix} 6y \\ 0 \end{pmatrix}$$

for $(x, y) \in \Sigma(\omega)$. This shows that ω has only fold singularities outside the origin.

To find the branches of $\omega^{-1}(\Delta(\omega)) \setminus \Sigma(\omega)$, let $x < 0$ and consider $f_x(y) = y^3 + x^k y$. We want to solve the equations

$$f_x(y) = f_x(z(x))$$

and

$$f_x(y) = f_x(-z(x)).$$

Since f_x is a polynomial of degree 3 in y and $\pm z(x)$ are local extremal points of f_x, there are $y_1(x) < -z(x)$ with $f_x(y_1(x)) = f_x(z(x))$ and $y_2(x) > z(x)$ with $f_x(y_2(x)) = f_x(-z(x))$. No other solution exist. We need to show that $y_1(x) \to 0$ as $x \to 0$ and $y_2(x) \to 0$ as $x \to 0$. We know that $f(x, \pm z(x)) \to 0$ as $x \to 0$. Therefore, $f_x(y_1(x)) = (y_1(x))^3 + x^k y_1(x) \to 0$ as $x \to 0$ and hence, $y_1(x) \to 0$ as $x \to 0$. The same argument applies to y_2. Altogether, we have proved that

$$[\mathrm{Ast}(\omega)]_E = [(p, s, s, p)]_E.$$

In [4], T. Gaffney presents a table ([4], 9.14) with normal forms of topologically distinct map germs $\mathsf{C}^2 \to \mathsf{C}^2$. Using Theorem 3.10, we are able to reduce this list when we think of it as a list of map germs $\mathsf{R}^2 \to \mathsf{R}^2$. Table 3 is Gaffney's list of germs with the Ast calculated. We see that many of the $\mathscr{A}_0$-equivalence classes in Table 3 are the same in the real case. In the real case, Table 3 reduces to Table 4.

Type		$[\mathrm{Ast}]_E$
(1)	(x, y)	$[(p)]_E$
(2)	(x, y^2)	$[(s, s)]_E$
(3)	$(x, xy + y^3)$	$[(p, s, s, p)]_E$
$(4)_k$	$(x, y^3 + x^k y)$	$[(p, s, s, p)]_E$
(5)	$(x, xy + y^4)$	$[(s, s)]_E$
(6)	$(x, xy + y^5)$	$[(p, s, s, p)]_E$
(7)	$(x, xy + y^6)$	$[(s, s)]_E$
(8)	$(x, xy + y^7)$	$[(p, s, s, p)]_E$
$(9)_{2k+1}$	$(x, xy^2 + y^4 + y^{2k+1})$	$[(s, p, s, s, p, s, p, p)]_E$
(10)	$(x, xy^2 + y^5)$	$[(p, s, s, p, p, s, s, p)]_E$
(11)	$(x, x^2 y + y^4)$	$[(s, s)]_E$
(12)	$(x, xy^2 + y^6 + y^7)$	$[(s, p, s, s, p, s, p, p)]_E$
(13)	$(x, x^2 y + xy^3 + y^5)$	$[(p)]_E$
(14)	$(x, x^3 y + y^4 + x^3 y^2)$	$[(s, s)]_E$

TABLE 3. Gaffney's table

Type		Ast
(1)	(x, y)	$[(p)]_E$
(2)	(x, y^2)	$[(s, s)]_E$
(3)	$(x, xy + y^3)$	$[(p, s, s, p)]_E$
(4)	$(x, xy^2 + y^5)$	$[(p, s, s, p, p, s, s, p)]_E$
(5)	$(x, xy^2 + y^6 + y^7)$	$[(s, p, s, s, p, s, p, p)]_E$

TABLE 4. Reduced table

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