

THE UNIQUENESS PROBLEM FOR MEROMORPHIC FUNCTIONS IN THE UNIT DISC SHARING VALUES AND A SET IN AN ANGULAR DOMAIN

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Abstract

We investigate the uniqueness problem for meromorphic functions in the unit disc sharing four distinct values and one set in an angular domain, and obtain some relations between the Borel points and shared values of meromorphic functions in an angular domain. These results improve the theorem of Mao-Liu [10].

1. Introduction and main results

We use \mathbb{C} to denote the open complex plane, $\widehat{\mathbb{C}} (= \mathbb{C} \cup \{\infty\})$ to denote the extended complex plane, $\mathbb{D} = \{z : |z| < 1\}$ to denote the unit disc, and $X (\subseteq \mathbb{C})$ to denote an angular domain. Let g be a transcendental meromorphic function defined in the whole complex plane \mathbb{C} and not rational. It is assumed that the readers are familiar with the notations of the Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$ and so on, that can be found, for instance, in [5], [19].

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$. Define

$$E_{\mathbb{D}}(S, f) = \bigcup_{a \in S} \{z \in \mathbb{D} \mid f_a(z) = 0, \text{ counting multiplicities}\},$$

$$\overline{E}_{\mathbb{D}}(S, f) = \bigcup_{a \in S} \{z \in \mathbb{D} \mid f_a(z) = 0, \text{ ignoring multiplicities}\},$$

where $f_a(z) = f(z) - a$ if $a \in \mathbb{C}$ and $f_{\infty}(z) = 1/f(z)$.

Let f and g be two non-constant meromorphic functions in \mathbb{C} . If $E_{\mathbb{D}}(S, f) = E_{\mathbb{D}}(S, g)$, we say f and g share the set S *CM* (counting multiplicities) in

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D. If $\overline{E}_D(S, f) = \overline{E}_D(S, g)$, we say f and g share the set S *IM* (ignoring multiplicities) in D . In particular, when $S = \{a\}$, where $a \in \widehat{\mathbb{C}}$, we say f and g share the value a *CM* in D if $E_D(S, f) = E_D(S, g)$, and we say f and g share the value a *IM* in D if $\overline{E}_D(S, f) = \overline{E}_D(S, g)$. Similarly, we use the notations $E_X(S, f)$, $\overline{E}_X(S, f)$, and if $X = \mathbb{C}$, write simply $E(S, f)$, $\overline{E}(S, f)$ and so on (see [18]).

R. Nevanlinna [11] proved the following well known theorems.

THEOREM 1.1 ([11]). *If f and g are two non-constant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 *IM* in $X = \mathbb{C}$, then $f(z) \equiv g(z)$.*

THEOREM 1.2 ([11]). *If f and g are two distinct non-constant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 *CM* in $X = \mathbb{C}$, then f is a Möbius transformation of g , two of the shared values, say a_1 and a_2 are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1993, M. Reinders [12] investigated the uniqueness problem of non-constant meromorphic functions sharing four distinct values *IM* and obtained the following result.

THEOREM 1.3 ([12]). *Let f and g be distinct non-constant meromorphic functions sharing four distinct values a_j ($j = 1, 2, 3, 4$) *IM*. If there exists $a, b \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$ such that $f(z) = a \implies g(z) = b$, then either $f(z)$ and $g(z)$ are the functions $f = L \circ \hat{f} \circ h$ and $g = L \circ \hat{g} \circ h$ or $f = T \circ g$, where L, T are Möbius transformation, h is a non-constant entire function and*

$$\hat{f} = \frac{e^z + 1}{(e^z - 1)^2}, \quad \hat{g} = \frac{(e^z + 1)^2}{8(e^z - 1)}.$$

After their work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (see [18]). It is an interesting topic to investigate the uniqueness with shared values in the remaining part of the complex plane removing an unbounded closed set, see [2], [3], [8]–[10], [15]–[17], [21], [22]. In [21], Zheng studied the uniqueness problem under the condition that five values are shared in some angular domain in \mathbb{C} . Zheng J. H. [22], Cao T. B. and Yi H. X. [3], Xu J. F. and Yi H. X. [17] continued to investigate the uniqueness of meromorphic functions sharing five values and four values in an angular domain, Lin W. C., Mori S. and Tohge K. [8] and Lin W. C., Mori S. and Yi H. X. [9] investigated the uniqueness of meromorphic and entire functions sharing sets in an angular domain. They obtained some important results.

In 2009, Cao T. B. and Yi H. X. [3] investigated the uniqueness problem of two transcendental meromorphic functions sharing five distinct values in an angular domain and obtained the following theorem:

THEOREM 1.4 ([3, Theorem 1.3]). *Let f and g be two transcendental meromorphic functions. Given an angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$, we assume that f and g share five distinct values $a_j \in \widehat{\mathbb{C}}$ ($j = 1, 2, 3, 4, 5$) IM in X . Then $f(z) \equiv g(z)$, provided that*

$$\lim_{r \rightarrow \infty} \frac{S_{\alpha, \beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E),$$

where $S_{\alpha, \beta}(r, f)$ denotes the angular characteristic function of the meromorphic function f .

REMARK 1.1. We refer to Theorem 1.4 as the 5 IM theorem in an angular domain.

In 2009, Zhang [20] proved the following theorems by setting up the relationship between the Nevanlinna angular characteristic function and the Ahlfors angular characteristic function, and applying it to study the uniqueness problems of meromorphic functions in an angular domain.

THEOREM 1.5 ([20]). *Let f, g be two meromorphic functions of finite order in \mathbb{C} , let $a_j \in \widehat{\mathbb{C}}$ ($j = 1, 2, \dots, 5$) be five distinct values, and let $\Delta_\delta = \{z : |\arg z - \theta_0| \leq \delta\}$ ($0 < \delta < \pi$) be an angular domain satisfying*

$$(1) \quad \overline{\lim}_{\varepsilon \rightarrow 0^+} \overline{\lim}_{r \rightarrow +\infty} \frac{\log T(r, \Delta_{\delta-\varepsilon}, f)}{\log r} > \omega,$$

where $\omega = \frac{\pi}{2\delta}$ and $T(r, \Delta_{\delta-\varepsilon}, f)$ denotes the Ahlfors characteristic function of f in $\Delta_{\delta-\varepsilon}$. If f and g share a_j ($j = 1, 2, \dots, 5$) IM in Δ_δ , then $f \equiv g$.

In [4], Fang studied the uniqueness problem of admissible meromorphic functions in the unit disc \mathbb{D} sharing two sets and three sets. It is also an interesting topic to investigate the uniqueness of meromorphic functions in the unit disc \mathbb{D} (see [4], [10], [13]). To state some uniqueness theorems for meromorphic functions in the unit disc \mathbb{D} , we need the following basic notations and definitions of meromorphic functions in \mathbb{D} .

DEFINITION 1.1 ([6]). A meromorphic function f in \mathbb{D} is said to be admissible if and only if

$$\limsup_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = \infty,$$

and non-admissible if and only if

$$\limsup_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} < \infty.$$

Let $f(z)$ be a meromorphic function in the unit disc \mathbb{D} and let $\Delta(\theta_0, \delta)$ denote the domain $\{z : |z| < 1\} \cap \{z : |\arg z - \theta_0| < \delta\}$, where $0 \leq \theta_0 \leq 2\pi$, $0 < \delta < \pi$. We use $n(r, \Delta(\theta_0, \delta), f(z) = a)$ to denote the number of zeros of $f(z) - a$ in $\Delta(\theta_0, \delta) \cap \{z : |z| < r\}$ counting multiplicities.

In 2005, F. Titzhoff [13] investigated the uniqueness problem for admissible functions in the unit disc \mathbb{D} and obtained the five-values theorem in the unit disc \mathbb{D} as follows.

THEOREM 1.6 ([13]). *If two admissible functions f, g share five distinct values, then $f \equiv g$.*

In 2009, Z.-Q. Mao and H.-F. Liu [10] gave a different method to investigate the uniqueness problem of meromorphic functions in the unit disc and obtained the following results.

THEOREM 1.7 ([10]). *Let f, g be two meromorphic functions in \mathbb{D} , $a_j \in \widehat{\mathbb{C}}$ ($j = 1, 2, \dots, 5$) be five distinct values, and $\Delta(\theta_0, \delta)$ ($0 < \delta < \pi$) be an angular domain such that for some $a \in \widehat{\mathbb{C}}$,*

$$(2) \quad \limsup_{r \rightarrow 1^-} \frac{\log n(r, \Delta(\theta_0, \delta/2), f(z) = a)}{\log \frac{1}{1-r}} = \tau > 1.$$

If f and g share a_j ($j = 1, 2, \dots, 5$) IM in $\Delta(\theta_0, \delta)$, then $f(z) \equiv g(z)$.

REMARK 1.2. Let f be a meromorphic function in the unit disc. If for arbitrary small $\varepsilon > 0$, we have

$$\limsup_{r \rightarrow 1^-} \frac{\log n(r, \Delta(\theta_0, \varepsilon), f(z) = a)}{\log \frac{1}{1-r}} := \tau$$

for all but at most two $a \in \widehat{\mathbb{C}}$, then $e^{i\theta_0}$ is called a Borel point of order τ of $f(z)$. In [14], G. Valiron proved that every meromorphic function of finite order ρ in the unit disc must have at least one Borel point of order $\rho + 1$.

It is a natural question to ask whether we can relax the conditions of Theorem 1.7 on the sharing of values in $\Delta(\theta_0, \delta)$?

In this paper, we will answer the above question by studying the uniqueness problem for meromorphic functions in the unit disc \mathbb{D} sharing four distinct values and one set in an angular domain. We obtain the following results.

THEOREM 1.8. *Let f, g be two meromorphic functions in \mathbb{D} , $a_j \in \widehat{\mathbb{C}}$ ($j = 1, 2, 3, 4$) be four distinct values, and $\Delta(\theta_0, \delta)$ ($0 < \delta < \pi$) be an angular domain satisfying (2). We assume that f and g share four distinct values a_j ($j = 1, 2, 3, 4$) IM in $\Delta(\theta_0, \delta)$ and $\overline{E}_{\Delta(\theta_0, \delta)}(S, f) \subset \overline{E}_{\Delta(\theta_0, \delta)}(S, g)$, where $S = \{b_1, \dots, b_m\}$, $m \geq 1$ and $b_1, \dots, b_m \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$. Then f and g share all values CM; thus it follows that either $f \equiv g$ or f is a Möbius transformation of g . Furthermore, if the number of the values in S is odd then $f \equiv g$.*

REMARK 1.3. The special case $m = 1$ of Theorem 1.8 immediately yields Theorem 1.7. In fact, when $m = 1$, set $S = \{a_5\}$. If f, g share a_5 IM, which implies $\overline{E}_{\Delta(\theta_0, \delta)}(S, f) \subset \overline{E}_{\Delta(\theta_0, \delta)}(S, g)$, then by Theorem 1.8, we can get $f \equiv g$.

2. Some Lemmas

To prove our results, we need the following lemmas.

LEMMA 2.1. *Suppose that f is an admissible meromorphic function in \mathbb{D} . Let $P(f) = a_0 f^p + a_1 f^{p-1} + \dots + a_p$ ($a_0 \neq 0$) be a polynomial of f with degree p , where the coefficients a_j ($j = 0, 1, \dots, p$) are constants, and let b_j ($j = 1, 2, \dots, q$) be q ($q \geq p + 1$) distinct finite complex numbers. Then*

$$m\left(r, \frac{P(f) \cdot f'}{(f - b_1)(f - b_2) \dots (f - b_q)}\right) = S(r, f).$$

PROOF. We can get this lemma by using the same argument as in Lemma 4.3 in [18].

LEMMA 2.2. *Let f and g be two distinct admissible meromorphic functions in \mathbb{D} and share four distinct values a_j ($j = 1, 2, 3, 4$) IM in \mathbb{D} . Let*

$$(3) \quad \varphi = \frac{f'g'(f - g)^2}{(f - a_1)(f - a_2)(f - a_3)(f - a_4)(g - a_1)(g - a_2)(g - a_3)(g - a_4)}.$$

Then $T(r, \varphi) = S(r, f) + S(r, g)$ and

- (i) $T(r, f) \leq T(r, g) + S(r, f)$, $T(r, g) \leq T(r, f) + S(r, g)$;
- (ii) $\sum_{j=1}^4 \overline{N}\left(r, \frac{1}{f - a_j}\right) = 2T(r, f) + S(r, f)$, where $S(r) := S(r, f) = S(r, g)$;
- (iii) $\overline{N}\left(r, \frac{1}{f - b}\right) = T(r, f) + S(r, f)$, $\overline{N}\left(r, \frac{1}{g - b}\right) = T(r, g) + S(r, g)$; where $b \neq a_j$ ($j = 1, 2, 3, 4$), and $\overline{N}\left(r, \frac{1}{f - b}\right) = \overline{N}(r, f)$ when $b = \infty$.

PROOF. We can get the conclusions of Lemma 2.2 by using the same argument as in Theorem 4.4 in [18].

LEMMA 2.3. *Suppose that f and g are distinct admissible meromorphic functions in \mathbb{D} . If f and g share four distinct values $a_j \in \widehat{\mathbb{C}}$ ($j = 1, 2, 3, 4$) CM in \mathbb{D} , then f is a Möbius transformation of g , two of the shared values, say a_1 and a_2 , are Picard values in \mathbb{D} , and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

PROOF. Proceeding as in the proof of Lemma 2.8 in [9], we can get the conclusions of Lemma 2.3 easily.

LEMMA 2.4. *Let f, g be two admissible meromorphic functions in \mathbb{D} , $a_j \in \widehat{\mathbb{C}}$ ($j = 1, 2, 3, 4$) be four distinct values, and $S = \{b_1, \dots, b_m\}$, $m \geq 1$ and $b_1, \dots, b_m \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$. We assume that f and g share four distinct values a_j ($j = 1, 2, 3, 4$) IM in \mathbb{D} and $\overline{E}_{\mathbb{D}}(S, f) \subset \overline{E}_{\mathbb{D}}(S, g)$. Then f and g share all values CM in \mathbb{D} , thus it follows that either $f \equiv g$ or f is a Möbius transformation of g . Furthermore, if the number of the values in S is odd, then $f \equiv g$.*

PROOF. Suppose that $f \not\equiv g$ and none of the a_j ($j = 1, 2, 3, 4$) is ∞ . Let φ be the function expressed in Lemma 2.2. Then $\varphi \not\equiv 0$. By Lemma 2.2 (iii), we have

$$(4) \quad m \left(r, \frac{1}{f - b_j} \right) = S(r, f), \quad m \left(r, \frac{1}{g - b_j} \right) = S(r, g),$$

for any $b_j \in S$ ($j = 1, 2, \dots, m$).

Set

$$\varphi_1 := \frac{(g - b_1) \dots (g - b_m)}{(f - b_1) \dots (f - b_m)} \cdot \left(\frac{g'(f - g)}{(g - a_1) \dots (g - a_4)} \right)^m$$

and

$$\varphi_2 := \frac{(f - b_1) \dots (f - b_m)}{(g - b_1) \dots (g - b_m)} \cdot \left(\frac{f'(f - g)}{(f - a_1) \dots (f - a_4)} \right)^m.$$

By Lemma 2.1 and (4), we can get that

$$m \left(r, \frac{1}{f - b_j} \cdot \frac{g'(f - g)(g - b_j)}{(g - a_1) \dots (g - a_4)} \right) = S(r)$$

and

$$m \left(r, \frac{1}{g - b_j} \cdot \frac{f'(f - g)(f - b_j)}{(f - a_1) \dots (f - a_4)} \right) = S(r).$$

From the definitions of φ_1 and φ_2 , we have $m(r, \varphi_j) = S(r)$, $j = 1, 2$. By Lemma 2.2(iii), we see that “almost all” of poles and b_j -points of f and g in

the unit disc D are simple. Since f, g share the four distinct values $a_j, j = 1, 2, 3, 4$ in the unit disc D and $\overline{E}_D(S, f) \subset \overline{E}_D(S, g)$, we can easily get that $N(r, \varphi_1) = S(r)$. Therefore, we have

$$(5) \quad T(r, \varphi_1) = S(r).$$

Since $\varphi_1\varphi_2 \equiv \varphi^m$, we can have

$$(6) \quad T(r, \varphi_2) = S(r).$$

Let $L_D^{pq}(a_j)$ be the set of those a_j -points of f and g in the unit disc D such that the multiplicities of f and g at these points are p and q , respectively. For any $z_0 \in L_D^{pq}(a_1)$, by simple computation, we have

$$\varphi_1(z_0) = \left(q \cdot \frac{f'(z_0) - g'(z_0)}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} \right)^m$$

and

$$\varphi_2(z_0) = \left(p \cdot \frac{f'(z_0) - g'(z_0)}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} \right)^m.$$

Hence

$$(7) \quad \frac{1}{q^m}\varphi_1(z_0) - \frac{1}{p^m}\varphi_2(z_0) = 0.$$

Similarly, we can see that (7) holds for any $z_0 \in L_D^{pq}(a_j), j = 2, 3, 4$.

Now we discuss two cases as follows.

CASE 1. Suppose that $\varphi^{pq} := \frac{1}{q^m}\varphi_1 - \frac{1}{p^m}\varphi_2 \not\equiv 0$, for all positive integers p, q .

For the sake of convenience, we denote by $N^{pq}(r, \frac{1}{f-a_j})$ the counting function of f in the unit disc D with respect to the set $L_D^{pq}(a_j)$, denote by $N^{pq}(r, \frac{1}{f-a_j})$ the corresponding reduced counting function. Thus, we have

$$N\left(r, \frac{1}{f-a_j}\right) = \sum_{p,q=1}^{\infty} N^{pq}\left(r, \frac{1}{f-a_j}\right)$$

and

$$N\left(r, \frac{1}{f-a_j}\right) = \sum_{p,q=1}^{\infty} N^{pq}\left(r, \frac{1}{f-a_j}\right).$$

From the above two equations, (4), (5) and (6), we can see that $T(r, \varphi^{pq}) = S(r, f) + S(r, g)$. And by (7) each zero of $f - a_j$ is a zero of φ^{pq} , so with the

help of $\varphi^{pq} \not\equiv 0$, we can get

$$\begin{aligned} \overline{N}^{pq}\left(r, \frac{1}{f-a_j}\right) &\leq \overline{N}^{pq}\left(r, \frac{1}{\varphi^{pq}}\right) \leq T\left(r, \frac{1}{\varphi^{pq}}\right) \leq T(r, \varphi^{pq}) + O(1) \\ &= S(r, f) + S(r, g) := S(r), \end{aligned}$$

for some p, q . By Lemma 2.2 (ii), we have $T(r, f) + S(r, f) = T(r, g) + S(r, g)$. Thus, from the definition of $S(r)$, we can get $T(r, f) = T(r, g) + S(r)$. Therefore, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f-a_j}\right) &= \sum_{\max(p,q) \geq 5} \overline{N}^{pq}\left(r, \frac{1}{f-a_j}\right) + S(r, f) \\ &\leq \frac{1}{5} \left(\sum_{\max(p,q) \geq 5} N^{pq}\left(r, \frac{1}{f-a_j}\right) \right. \\ &\quad \left. + \sum_{\max(p,q) \geq 5} N^{pq}\left(r, \frac{1}{g-a_j}\right) \right) + S(r, f) \\ &\leq \frac{1}{5} \left(N\left(r, \frac{1}{f-a_j}\right) + N\left(r, \frac{1}{g-a_j}\right) \right) + S(r, f) \\ &\leq \frac{2}{5} T(r, f) + S(r), \end{aligned}$$

for $j = 1, 2, 3, 4$.

By the above inequality and Lemma 2.2 (ii), we can get

$$(8) \quad 2T(r, f) \leq \frac{8}{5}T(r, f) + S(r).$$

Thus, we can get a contradiction.

CASE 2. Suppose that $\varphi^{pq} := \frac{1}{q^m}\varphi_1 - \frac{1}{p^m}\varphi_2 \equiv 0$, for some positive integers p, q . From the definitions of φ_1 and φ_2 , we have

$$(9) \quad \left(\frac{p}{q}\right)^m \cdot \frac{(g-b_1)^2 \dots (g-b_m)^2}{(f-b_1)^2 \dots (f-b_m)^2} \equiv \left(\frac{f'(g-a_1) \dots (g-a_4)}{g'(f-a_1) \dots (f-a_4)}\right)^m.$$

Next we take the two subcases in the following into consideration:

SUBCASE 2.1. Suppose that $p \neq q$. Without loss of generality, we may assume that $p < q$. For some two positive integers p_1 and q_1 , if $z_1 \in L_D^{p_1 q_1}(a_j)$ for some $j \in \{1, 2, 3, 4\}$, then (9) implies that $\frac{p}{q} = \frac{p_1}{q_1}$. Hence $q_1 > p_1 \geq 1$,

and $q_1 \geq 2$ which means that any a_j -points ($j = 1, 2, 3, 4$) of g in the unit disc D are multiple. By Lemma 2.2, we can get

$$\begin{aligned} 2T(r, g) &= \sum_{j=1}^4 \bar{N}\left(r, \frac{1}{g-a_j}\right) + S(r, g) \leq \frac{1}{2} \sum_{j=1}^4 N\left(r, \frac{1}{g-a_j}\right) + S(r, g) \\ &\leq 2T(r, g) + S(r, g), \end{aligned}$$

which leads to the following equations

$$(10) \quad T(r, g) = N\left(r, \frac{1}{g-a_j}\right) + S(r),$$

and

$$(11) \quad N\left(r, \frac{1}{g-a_j}\right) = 2\bar{N}\left(r, \frac{1}{g-a_j}\right) + S(r).$$

From (10) and (11), we can see that “almost all” of a_j -points of g have multiplicity 2, and “almost all” of a_j -points of f are simple in the unit disc D . Without loss of generality, we may assume that f and g attain the values a_3 and a_4 in the unit disc D . Set

$$\phi_1 := \frac{2f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}$$

and

$$\phi_2 := \frac{2f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}.$$

Since ϕ_i ($i = 1, 2$) is analytic at the poles of f and of g and also at those common a_j -points of f and g which have multiplicity 1 with respect to f and multiplicity 2 with respect to g , by Lemma 2.1, we have $T(r, \phi_i) = S(r, f)$, $i = 1, 2$. If $\phi_i \not\equiv 0$, then $N(r, \frac{1}{f-a_4}) \leq N(r, \frac{1}{\phi_1}) = S(r, f)$, which contradicts to equation (10). Then $\phi_1 \equiv 0$. Similarly, we have $\phi_2 \equiv 0$. Therefore, from the definitions of ϕ_1 and ϕ_2 , we have

$$(12) \quad \left(\frac{f-a_4}{f-a_3}\right)^2 \equiv \left(\frac{g-a_4}{g-a_3}\right)^2.$$

Since $f \not\equiv g$, and from (12), we have

$$\frac{f-a_4}{f-a_3} \equiv -\frac{g-a_4}{g-a_3},$$

which implies that f and g share a_3, a_4 *CM* in the unit disc D . Since f and g assume the value a_3 there exist positive integers p_1, q_1 such that $L_D^{p_1 q_1}(a_3) \neq \emptyset$. From the considerations above we get $q_1 > p_1$, contradicting the fact that f and g share a_3 *CM*.

SUBCASE 2.2. Suppose that $p = q$.

In this subcase, (10) becomes

$$\frac{(g - b_1)^2 \dots (g - b_m)^2}{(f - b_1)^2 \dots (f - b_m)^2} \equiv \left(\frac{f'(g - a_1) \dots (g - a_4)}{g'(f - a_1) \dots (f - a_4)} \right)^m.$$

which implies that f and g share the four values a_j ($j = 1, 2, 3, 4$) *CM* in the unit disc D . From the conditions of this lemma and applying Lemma 2.3, g is a Möbius transformation of f . Furthermore, two of the four values, say a_1, a_2 are Picard exceptional values of f and g in the unit disc D . Set

$$\chi_1 := \frac{f'(f - a_4)}{(f - a_1)(f - a_2)(f - a_3)} - \frac{g'(g - a_4)}{(g - a_1)(g - a_2)(g - a_3)}$$

and

$$\chi_2 := \frac{f'(f - a_3)}{(f - a_1)(f - a_2)(f - a_4)} - \frac{g'(g - a_3)}{(g - a_1)(g - a_2)(g - a_4)}.$$

Using the analogous argument of Subcase 2.1 for χ_1, χ_2 , we can get

$$\frac{f - a_3}{f - a_4} \equiv -\frac{g - a_3}{g - a_4}.$$

We define the Möbius transformations T, M and L by

$$T(w) := \frac{w - a_3}{w - a_4}, \quad M(w) := -w \quad \text{and} \quad L := T^{-1} \circ M \circ T.$$

Then we have

$$T \circ f = -T \circ g, \quad \text{hence} \quad g = L \circ f.$$

Obviously a_3 and a_4 are the fixed points of L . Therefore, there exist no fixed points of L in the set S . Let some $b \in S$ be given. Then in view of $b \neq a_1, a_2$ there exists a $z_0 \in \mathbb{C}$ such that $b = f(z_0)$, and from $\overline{E}_D(S, f) \subseteq \overline{E}_D(S, g)$ we obtain

$$L(b) = L(f(z_0)) = g(z_0) \in S.$$

So S is invariant under L . Furthermore, we have $L \circ L = I$ where I denotes the identical transformation. Hence we can conclude that S must contain an even number of values.

Thus, we complete the proof of Lemma 2.4.

LEMMA 2.5 ([9]). *Set*

$$u = u(z) = \frac{z^{\frac{\pi}{\delta}} + 2z^{\frac{\pi}{2\delta}} - 1}{z^{\frac{\pi}{\delta}} - 2z^{\frac{\pi}{2\delta}} - 1}.$$

Then $u(z)$ maps conformally $\{z : |\arg z| < \delta, |z| < 1\}$ onto the unit disc $\{u : |u| < 1\}$, where $0 < \delta < \pi$.

3. Proof of Theorem 1.8

Without loss of generality, we may assume $\theta_0 = 0$. Set

$$(13) \quad u = u(z) = \frac{z^{\frac{\pi}{\delta}} + 2z^{\frac{\pi}{2\delta}} - 1}{z^{\frac{\pi}{\delta}} - 2z^{\frac{\pi}{2\delta}} - 1}.$$

Let $z = z(u)$ denote its inverse function. Then by Lemma 2.5 we know that u maps conformally $\Delta(0, \delta)$ onto the unit disc $\{u : |u| < 1\}$.

Using the same method as in the proof of Theorem 1.2 in [10], we can get two meromorphic functions $f(z(u))$ and $g(z(u))$ in the unit disc $D' = \{u : |u| < 1\}$, and $f(z(u))$ is admissible in the disc $D' = \{u : |u| < 1\}$. For the convenience of the reader, we describe the proof of this lemma as follows.

Set $z_0 = pe^{i\vartheta} \in \Delta(0, \delta)$, by (13) we get

$$(14) \quad 1 - |u(z_0)| = 1 - \sqrt{\frac{A^2 + B^2}{C^2 + D^2}} = \frac{C^2 + D^2 - A^2 - B^2}{C^2 + D^2 + \sqrt{(A^2 + B^2)(C^2 + D^2)}} \\ = \frac{8p^{\frac{\pi}{2\delta}}(1 - p^{\frac{\pi}{\delta}}) \cos \frac{\pi\vartheta}{2\delta}}{C^2 + D^2 + \sqrt{(A^2 + B^2)(C^2 + D^2)}},$$

where

$$A = p^{\frac{\pi}{\delta}} \cos \frac{\pi\vartheta}{\delta} + 2p^{\frac{\pi}{2\delta}} \cos \frac{\pi\vartheta}{2\delta} - 1, \quad B = p^{\frac{\pi}{\delta}} \sin \frac{\pi\vartheta}{\delta} + 2p^{\frac{\pi}{2\delta}} \sin \frac{\pi\vartheta}{2\delta}, \\ C = p^{\frac{\pi}{\delta}} \cos \frac{\pi\vartheta}{\delta} - 2p^{\frac{\pi}{2\delta}} \cos \frac{\pi\vartheta}{2\delta} - 1, \quad D = p^{\frac{\pi}{\delta}} \sin \frac{\pi\vartheta}{\delta} - 2p^{\frac{\pi}{2\delta}} \sin \frac{\pi\vartheta}{2\delta}.$$

Since

$$C^2 + D^2 = p^{\frac{2\pi}{\delta}} + 2p^{\frac{\pi}{\delta}} + 1 + 4p^{\frac{2\pi}{\delta}}(1 - p^{\frac{\pi}{\delta}}) \cos \frac{\pi\vartheta}{2\delta} + 2p^{\frac{\pi}{\delta}} \left(1 - \cos \frac{\pi\vartheta}{\delta}\right),$$

we can get

$$(15) \quad 1 \leq C^2 + D^2 \leq C^2 + D^2 + \sqrt{(A^2 + B^2)(C^2 + D^2)} \leq 2(C^2 + D^2) \leq 20.$$

Since $\lim_{p \rightarrow 1^-} \frac{1-p^{\frac{\pi}{\delta}}}{1-p} = \frac{\pi}{\delta}$, then there exists $b \in \left(\left(\frac{1}{2}\right)^{\frac{2\delta}{\pi}}, 1\right)$ such that for all p satisfying $b < p < 1$, we have

$$(16) \quad \frac{1}{2} < p^{\frac{\pi}{2\delta}} < 1, \quad \frac{\pi}{2\delta}(1-p) < 1 - p^{\frac{\pi}{\delta}} < \frac{3\pi}{2\delta}(1-p).$$

Hence, from (14)–(16), we can get

$$(17) \quad \min \left\{ 1 - |u(pe^{i\vartheta})| : b < p < r, |\vartheta| < \frac{\delta}{2} \right\} > \frac{\pi}{20\delta}(1-r)$$

for all $r \in (b, 1)$.

We will prove that $f(z(u))$ is admissible in the unit disc $\{u : |u| < 1\}$ as follows.

From (2), we get that there exists a sequence $\{r_n\}$ of positive numbers such that $r_n \rightarrow 1$ for $n \rightarrow +\infty$ and such that for τ_1 we have

$$(18) \quad n(r_n, \Delta(0, \delta/2), f(z) = a) > \left(\frac{1}{1-r_n}\right)^{\tau_1}$$

for sufficiently large n and $\tau > \lambda_1 > 1$.

Then from (18) and Theorem 1.3.2 in [7, pp. 16–17], we have

$$(19) \quad \limsup_{t \rightarrow 1} \frac{T(t, f(z(u)))}{\log \frac{1}{1-t}} \geq \limsup_{t'_n \rightarrow 1} \frac{T(t'_n, f(z(u)))}{\log \frac{1}{1-t'_n}} \geq \infty.$$

Since $f(z(u))$ a meromorphic function in $\{u : |u| < 1\}$, from (19), we get that $f(z(u))$ is admissible in the unit disc $\{u : |u| < 1\}$.

From the assumption of Theorem 1.8, we can get that $f(z(u))$ and $g(z(u))$ share a_1, a_2, a_3, a_4 *IM* in D' and $\overline{E}_{D'}(S, f(z(u))) = \overline{E}_{D'}(S, g(z(u)))$. Thus, from Lemma 2.4, we can get that $g(z(u))$ is admissible in D' .

Therefore, by Lemma 2.4, we can get the conclusions of Theorem 1.8.

Thus, we complete the proof of Theorem 1.8.

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