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ON THE REAL RANK OF C^*-ALGEBRAS OF NILPOTENT LOCALLY COMPACT GROUPS

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Abstract

If G is an almost connected, nilpotent, locally compact group then the real rank of the C^*-algebra $C^*(G)$ is given by $\mathrm{RR}(C^*(G)) = \mathrm{rank}(G/[G,G]) = \mathrm{rank}(G_0/[G_0,G_0])$, where G_0 is the connected component of the identity element. In particular, for the continuous Heisenberg group G_3, $\mathrm{RR}(C^*(G_3)) = 2$.

1. Introduction

For a C^*-algebra A, the real rank $\mathrm{RR}(A)$ [5] and the stable rank $\mathrm{sr}(A)$ [21] have been defined as non-commutative analogues of the real and complex dimension of topological spaces. Several authors have computed or estimated the real and stable rank of group C^*-algebras $C^*(G)$ for various classes of locally compact groups G [2], [3], [10], [11], [14], [18], [22], [23], [24], [25], [26], [27], [28], [29]. In generalizing the result of Sudo and Takai [28] for simply connected nilpotent Lie groups, it was shown in [3] that the stable rank of the C^*-algebra of an almost connected, nilpotent group G is given by the formula

$$\mathrm{sr}(C^*(G)) = 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G/[G,G]) \right\rfloor \tag{1.1}$$

and hence that the real rank $\mathrm{RR}(C^*(G))$ satisfies

$$\mathrm{rank}(G/[G,G]) \le \mathrm{RR}(C^*(G)) \le \mathrm{rank}(G/[G,G]) + 1,$$

with equality on the left if the rank of $G/[G,G]$ is odd.

Subsequently, L. G. Brown [4] has made an incisive analysis of the behaviour of real and stable rank in CCR (liminal) C^*-algebras A, partly based on the notion of the topological dimension $\mathrm{top}\dim(A)$ which was introduced in his earlier work with Pedersen [6]. By using these results, we are now able to

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show that $\mathrm{RR}(C^*(G)) = \mathrm{rank}(G/[G, G])$ in all cases. Furthermore, Brown's results allow us to give simultaneously a similar approach to equation (1.1). This enables us to close a gap in [3, Theorem 1.5] arising from the use of [19, Lemma 2] which omits the statement of an implicit dimensional condition. The significance of dimension is illustrated by the recent example due to Kucerovsky and Ng [15] of a non-stable, continuous trace C^*-algebra A, all of whose irreducible representations are infinite dimensional, such that $\dim \hat{A} = \mathrm{sr}(A) = \infty$. Accordingly, in the next section we shall estimate $\mathrm{top}\dim(C^*(G))$ for certain groups G.

The main result on real rank is obtained in Theorem 3.4. As a corollary, we settle a dichotomy for the real rank of the C^*-algebra of the continuous Heisenberg group: the value is 2 rather than 3.

We conclude this section with some definitions and notation. For a discrete, torsion-free, abelian group D, rank D means the torsion-free rank of D (see [13]). That is, rank D is the maximal number of independent elements if this is finite and $\mathrm{rank}\, D = \infty$ otherwise. For a general locally compact group G, let G^c denote the set of all compact elements of G, where an element is called compact if it generates a relatively compact subgroup. If G is a locally compact group with relatively compact conjugacy classes, then G^c is a closed normal subgroup of G and G/G^c is a compact-free, locally compact, abelian group [12, Theorem 3.16]. As such, G/G^c has the form $R^k \times D$ where D is a torsion-free discrete group. The rank of G is then defined to be $k + \mathrm{rank}\, D$. In particular, $\mathrm{rank}\, D < \infty$ whenever G/G^c is compactly generated.

Finally, for any locally compact group G, we will denote by G_0 the connected component of the identity element, by $[G, G]$ the closed commutator subgroup of G and by G_F the subgroup of G consisting of all elements with relatively compact conjugacy classes. The group G is said to be *almost connected* if the quotient group G/G_0 is compact.

2. The topological dimension of some group C^*-algebras

Let A be a type I C^*-algebra. It is well-known that the spectrum $\hat{A}$ is homeomorphic to the primitive ideal space $\mathrm{Prim}(A)$, via the map which sends the unitary equivalence class $[\pi]$ of an irreducible representation π to the primitive ideal $\ker \pi$, and that both spaces are *almost Hausdorff* in the sense that every non-empty closed subset contains a non-empty, relatively open, Hausdorff subset. It follows from [6, 2.2(v) and Remark 2.5(ii)] that the topological dimension of A is given by

$$\mathrm{top}\dim(A) = \sup\{\dim K : K \text{ a compact Hausdorff subset of } \hat{A}\} \in [0, \infty],$$

where $\dim K$ is the covering dimension of K (see [20]).

The next lemma is essentially a special case of [6, Proposition 2.3], but it is convenient to express it in the following form.

LEMMA 2.1. *Let A be a type I C^*-algebra and let $\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \hat{A}$ be an increasing finite sequence of open subsets of $\hat{A}$. Then*

$$\operatorname{top\,dim}(A) = \max_{1\le j\le n} \sup\{\dim K : K \text{ a compact Hausdorff subset of } V_j \backslash V_{j-1}\}.$$

PROOF. For $1 \le j \le n$, let I_j be the closed two-sided ideal of A such that $\hat{I}_j = V_j$ (so that $I_n = A$). Then $\widehat{I_j/I_{j-1}} = V_j \setminus V_{j-1}$ and so by [6, Proposition 2.3]

$$\begin{aligned}\operatorname{top\,dim}(A) &\\ &= \max_{1\le j\le n} \operatorname{top\,dim}(I_j/I_{j-1})\\ &= \max_{1\le j\le n} \sup\{\dim K : K \text{ a compact Hausdorff subset of } V_j \setminus V_{j-1}\}.\end{aligned}$$

PROPOSITION 2.2. *Let G be a connected, simply connected, nilpotent Lie group of dimension d. Then* $\operatorname{top\,dim}(C^*(G)) \le d$.

PROOF. Let $\mathfrak{g}$ be the Lie algebra associated with G and let $k : \mathfrak{g}^* \to \hat{G}$ be the Kirillov map, which is an open, continuous surjection. By [7, Theorem 3.1.14], there is a strictly increasing sequence $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = \mathfrak{g}^*$ of $\mathrm{Ad}^*(G)$-invariant open subsets of $\mathfrak{g}^*$ and, for $1 \le j \le n$, a vector subspace W_j of $\mathfrak{g}^*$ of (linear) dimension m_j and a homeomorphism

$$\psi_j : U_j \setminus U_{j-1} \to \big((U_j \setminus U_{j-1}) \cap W_j\big) \times \mathsf{R}^{d-m_j}$$

such that

(i) the restriction k_j of k to $(U_j \setminus U_{j-1}) \cap W_j$ is a (continuous) bijection onto $k(U_j) \setminus k(U_{j-1})$,

(ii) $k|_{U_j \setminus U_{j-1}} = k_j \circ P_j \circ \psi_j$, where P_j is the projection from $((U_j \setminus U_{j-1}) \cap W_j) \times \mathsf{R}^{d-m_j}$ onto $(U_j \setminus U_{j-1}) \cap W_j$.

Let $V_j = k(U_j)$, an open subset of $\hat{G}$ ($1 \le j \le n$). Note that $V_j \setminus V_{j-1} = k(U_j \setminus U_{j-1})$ since U_{j-1} is $\mathrm{Ad}^*(G)$-invariant.

Temporarily fix j and let E be an open subset of $(U_j \setminus U_{j-1}) \cap W_j$. Then $\psi_j^{-1}(E \times \mathsf{R}^{d-m_j}) = F_j \cap (U_j \setminus U_{j-1})$ for some open subset F_j of U_j. Hence, by (ii) and the fact that U_j and U_{j-1} are $\mathrm{Ad}^*(G)$-invariant,

$$k_j(E) = k(F_j \cap (U_j \setminus U_{j-1})) = k(F_j) \cap (V_j \setminus V_{j-1}),$$

which is open in $V_j \setminus V_{j-1}$ since F_j is open in $\mathfrak{g}^*$. Thus k_j is a homeomorphism (and so its image $V_j \setminus V_{j-1}$ is, in fact, Hausdorff). Let K be a compact, Hausdorff subset of $V_j \setminus V_{j-1}$. Then $k_j^{-1}(K)$ is a compact subset of $(U_j \setminus U_{j-1}) \cap W_j$ and hence is a closed subset of the Hausdorff space W_j. By [20, Chapter 3, Proposition 1.5] and the fact that the linear and covering dimensions of W_j coincide, we have $\dim K \leq \dim(W_j) = m_j$. Since $m_j \leq d$ and j was arbitrary, $\operatorname{top}\dim(C^*(G)) \leq d$ by Lemma 2.1.

In the context of Proposition 2.2, if G is abelian then

$$\operatorname{top}\dim(C^*(G)) = \operatorname{top}\dim(C_0(\mathsf{R}^d)) = d.$$

If G is non-abelian then $m_j \leq d - 2$ $(1 \leq j \leq n-1)$ and, although $m_n = d$, if K is a compact Hausdorff subset of $\hat{G} \setminus V_{n-1}$ then

$$\begin{aligned} \dim K = \dim(k_n^{-1}(K)) &\leq \dim(\mathfrak{g}^* \setminus U_{n-1}) = \dim(G/[G,G])^\wedge \\ &= \operatorname{rank}(G/[G,G]) \leq d - 1. \end{aligned}$$

Thus $\operatorname{top}\dim(C^*(G)) \leq d - 1$, with equality in the case of the Heisenberg group $(d = 3)$.

COROLLARY 2.3. *Let G be a connected, nilpotent Lie group. Then* $\operatorname{top}\dim(C^*(G)) < \infty$.

PROOF. Let H be the simply connected covering group of G. Then $C^*(G)$ is a quotient of $C^*(H)$ and, in particular, any compact Hausdorff subset of $\hat{G}$ is homeomorphic to a compact Hausdorff subset of $\hat{H}$ (see also [6, Proposition 2.4]). Hence

$$\operatorname{top}\dim(C^*(G)) \leq \operatorname{top}\dim(C^*(H)) < \infty,$$

by Proposition 2.2.

We recall from [8, 4.7.12] that a generalised continuous trace C^*-algebra A of finite length has a composition series

$$\{0\} = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = A$$

such that $I_j/I_{j-1} = J(A/I_{j-1})$ for $1 \leq j \leq n$ (where the ideal $J(B)$ of a C^*-algebra B is the closed linear span of the set of positive elements with finite, continuous trace on the spectrum $\hat{B}$). For $1 \leq j \leq n$, write $V_j = \hat{I}_j$, an open subset of $\hat{A}$. Since I_j/I_{j-1} is a continuous trace C^*-algebra, $V_j \setminus V_{j-1}$ is a locally compact Hausdorff space for $1 \leq j \leq n$ (where $V_0 = \emptyset$). Now suppose that G is a group of automorphisms of A. Since $I_1 = J(A)$, I_1 is G-invariant and so V_1 is invariant for the action of G on $\hat{A}$. Since I_1 is G-invariant, there is an induced action of G on A/I_1. Since $I_2/I_1 = J(A/I_1)$, I_2/I_1 is G-invariant. It follows that I_2 is G-invariant and that V_2 is a G-invariant subset of $\hat{A}$. Proceeding by induction, we obtain that I_j and V_j are G-invariant for $1 \leq j \leq n$.

PROPOSITION 2.4. *Let G be a locally compact group containing a closed, normal, second countable subgroup N of finite index such that $C^*(N)$ is a generalised continuous trace C^*-algebra of finite length. Then*

$$\operatorname{top}\dim(C^*(G)) \leq \operatorname{top}\dim(C^*(N)).$$

PROOF. The group G acts on N via the restriction of inner automorphisms. The induced action of G on $\hat{N}$ is given by $g \cdot \tau(n) = \tau(g^{-1}ng)$ $(g \in G, \tau \in \hat{N}, n \in N)$ and hence $g \cdot \tau$ is unitarily equivalent to τ whenever $g \in N$.

Since $C^*(N)$ is CCR [8, 4.7.12(c)] and G/N is finite, the G-orbits

$$G(\tau) := \{g \cdot \tau : g \in G\} \qquad (\tau \in \hat{N})$$

are finite closed sets. Let $\hat{N}/G$ be the quotient space, consisting of all G-orbits. Then the quotient map $q : \hat{N} \to \hat{N}/G$ is continuous and also open since $q^{-1}(q(V))) = \bigcup_{g \in G} g \cdot V$ is open in $\hat{N}$ for every open subset V of $\hat{N}$. If $\pi \in \hat{G}$ then $\operatorname{supp}(\pi|_N) = G(\tau)$ for some $\tau \in \hat{N}$. Conversely, if $\tau \in \hat{N}$ and π is any element of $\operatorname{supp}(\operatorname{ind}_N^G \tau)$, then $\operatorname{supp}(\pi|_N) = G(\tau)$. Thus we have a surjective mapping $r : \hat{G} \to \hat{N}/G$ defined by $r(\pi) = \operatorname{supp}(\pi|_N)$. It follows from the continuity of restricting representations that r is continuous.

Let $\sigma \in \widehat{N}/G$ and choose $\tau \in \widehat{N}$ such that $G(\tau) = \sigma$. We claim that $r^{-1}(\sigma)$ is finite. Note first that $r^{-1}(\sigma) = \operatorname{supp}(\operatorname{ind}_N^G \tau)$. It therefore suffices to show that the commutant of $\operatorname{ind}_N^G \tau$ is finite dimensional. This can be deduced from the work of Mackey and Blattner on induced representations. For the reader's convenience, we indicate the argument. To fix notation, given two representations π and ρ of G, let $\operatorname{Hom}_G(\pi, \rho)$ denote the space of bounded linear operators from $H(\pi)$ into $H(\rho)$ (the Hilbert spaces of π and ρ, respectively) intertwining π and ρ.

Now let $\pi = \operatorname{ind}_N^G \tau$, where τ is as above, and realize $H(\pi)$ as the space of all continuous mappings $\xi : G \to H(\tau)$ satisfying the covariance condition $\xi(xn) = \tau(n^{-1})\xi(x)$ for all $x \in G$ and $n \in N$. Then $\pi(x)$ acts on ξ by $\pi(x)\xi(y) = \xi(x^{-1}y)$, $y \in G$. Fix a finite set X of representatives for the cosets of N in G, and to any $T \in \operatorname{Hom}_N(\tau, \rho|_N)$ associate an operator $\phi(T) : H(\pi) \to H(\rho)$ by setting

$$\phi(T)\xi = \sum_{x \in X} \rho(x)T(\xi(x)), \qquad \xi \in H(\pi).$$

It is then straightforward to verify that this definition does not depend on the choice of X and that the map $T \to \phi(T)$ is a linear isomorphism from $\mathrm{Hom}_N(\tau, \rho|_N)$ onto $\mathrm{Hom}_G(\pi, \rho)$. Now take $\rho = \pi$ and observe that $\mathrm{Hom}_N(\tau, \pi|_N)$ is finite dimensional since $\mathrm{supp}(\pi|_N) = G(\tau)$ is finite. It follows that $\mathrm{Hom}_G(\mathrm{ind}_N^G \tau, \mathrm{ind}_N^G \tau)$ is finite dimensional.

Since $C^*(N)$ is a generalised continuous trace C^*-algebra of finite length, there is a sequence $\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \hat{N}$ of open subsets of $\hat{N}$, which are invariant for the action of G on $\hat{N}$, such that $V_j \setminus V_{j-1}$ is a locally compact Hausdorff space for $1 \leq j \leq n$. Suppose that $\tau_1, \tau_2 \in V_j \setminus V_{j-1}$ are such that $q(\tau_1) \neq q(\tau_2)$. Then the disjoint finite subsets $G(\tau_1)$ and $G(\tau_2)$ can be separated by G-invariant open subsets of $V_j \setminus V_{j-1}$. Since the restriction of q from $V_j \setminus V_{j-1}$ onto $q(V_j \setminus V_{j-1}) = q(V_j) \setminus q(V_{j-1})$ is open, $q(\tau_1)$ and $q(\tau_2)$ can be separated by open subsets of $q(V_j \setminus V_{j-1})$. So $q(V_j \setminus V_{j-1})$ is a locally compact Hausdorff space.

We define a sequence $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = \hat{G}$ of open subsets of $\hat{G}$ by $U_j = r^{-1}(q(V_j))$ $(0 \leq j \leq n)$. Let K be any compact Hausdorff subset of $U_j \setminus U_{j-1}$. Then $r(K)$ is a compact subset of the Hausdorff space $q(V_j \setminus V_{j-1})$. Since N is second countable, $C^*(N)$ is separable and so $\hat{N}$ is second countable. Hence $\hat{N}/G$ and $r(K)$ are second countable and so the compact Hausdorff space $r(K)$ is metrizable. Since $r^{-1}(\sigma)$ is finite for each $\sigma \in \hat{N}/G$ and $r|_K : K \to r(K)$ is a continuous closed surjection, it follows from [20, Chapter 9, Proposition 2.6] that $\dim r(K) \geq \dim K$. On the other hand, since the action of G on $\hat{N}$ factors through the finite group G/N, $C := q^{-1}(r(K))$ is a compact subset of the Hausdorff space $V_j \setminus V_{j-1}$ and $q^{-1}(\sigma)$ is finite for each $\sigma \in \hat{N}/G$. Since C is G-invariant, the restriction of the open mapping q to C is also open and so it follows from [20, Chapter 9, Proposition 2.16] that $\dim C = \dim r(K)$. Thus $\dim K \leq \dim C$. It now follows from Lemma 2.1 that $\operatorname{top\,dim}(C^*(G)) \leq \operatorname{top\,dim}(C^*(N))$.

THEOREM 2.5. *Let G be a locally compact group containing a closed normal subgroup N of finite index such that N is a connected, nilpotent Lie group. Then*

$$\operatorname{top\,dim}(C^*(G)) \leq \operatorname{top\,dim}(C^*(N)) < \infty.$$

PROOF. Since N is a connected Lie group, it is second countable. Furthermore, since N is also nilpotent, $C^*(N)$ is a generalised continuous trace C^*-algebra of finite length by [9]. So by Proposition 2.4 and Corollary 2.3, $\operatorname{top\,dim}(C^*(G)) \leq \operatorname{top\,dim}(C^*(N)) < \infty$.

If G is an almost connected, locally compact group with G_0 nilpotent, then the pro-Lie structure utilised in the proof of Lemma 3.2 below can be combined with the result of Theorem 2.5 to yield the fact that $\operatorname{top\,dim}(C^*(G)) \leq \operatorname{top\,dim}(C^*(G_0))$.

3. Almost connected, nilpotent groups

In the first two results of this section, it suffices to assume that G_0 is nilpotent (rather than G itself).

LEMMA 3.1. *Let G be a locally compact group such that G_0 is a nilpotent Lie group and G/G_0 is finite. Then*

(1) $\mathrm{RR}(C^*(G)) \leq \max\{1, \operatorname{rank}(G_0/[G_0, G_0])\}$,

(2) $\mathrm{sr}(C^*(G)) \leq \max\{2, 1 + \lfloor \frac{1}{2} \operatorname{rank}(G_0/[G_0, G_0]) \rfloor\}$.

PROOF. Let $A = C^*(G)$, a CCR C^*-algebra. As in the proof of [3, Lemma 1.2], A has a closed two-sided ideal I, all of whose irreducible representations are infinite dimensional, such that A/I is isomorphic to $C^*(G/[G_0, G_0])$, all of whose irreducible representations are finite dimensional. Since A is CCR, it follows from [4, Theorem 3.6] that $\mathrm{RR}(A) = \max\{\mathrm{RR}(I), \mathrm{RR}(A/I)\}$ and similarly for the stable rank. Furthermore, by [6, Proposition 2.2] and Theorem 2.5, $\operatorname{top\,dim}(I) \leq \operatorname{top\,dim}(A) < \infty$. Since all of the irreducible representations of I are infinite dimensional, it follows from [4, Theorem 3.10] that $\mathrm{RR}(I) \leq 1$ and $\mathrm{sr}(I) \leq 2$. Thus

$$\mathrm{RR}(C^*(G)) \leq \max\{1, \mathrm{RR}(C^*(G/[G_0, G_0]))\}$$

and

$$\mathrm{sr}(C^*(G)) \leq \max\{2, \mathrm{sr}(C^*(G/[G_0, G_0]))\}.$$

We temporarily write $N = [G_0, G_0]$. Since $[G/N : G_0/N] = [G : G_0] < \infty$ and G_0/N is abelian, G_0/N is contained in $(G/N)_F$ with necessarily finite index (where $(G/N)_F$ is the subgroup of G/N consisting of all elements with relatively compact conjugacy classes). It follows from [2, Lemma 2.8] that $\operatorname{rank}((G/N)_F) = \operatorname{rank}(G_0/N)$. Since G/N is a Moore group [17], it follows from [2, Theorem 3.4] that

$$\mathrm{RR}(C^*(G/N)) \leq \operatorname{rank}((G/N)_F) = \operatorname{rank}(G_0/N)$$

and also

$$\mathrm{sr}(C^*(G/N)) \leq 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(G_0/N) \right\rfloor,$$

as required. (This estimate for $\mathrm{sr}(C^*(G/N))$ can also be obtained from [22, Corollary 3.3].)

LEMMA 3.2. *Let G be an almost connected, locally compact group such that G_0 is nilpotent. Then*

(1) $\mathrm{RR}(C^*(G)) \le \max\{1, \mathrm{rank}(G_0/[G_0, G_0])\}$,

(2) $\mathrm{sr}(C^*(G)) \le \max\{2, 1 + \lfloor \frac{1}{2}\,\mathrm{rank}(G_0/[G_0, G_0]) \rfloor\}$.

PROOF. By [16, Theorem 4.6], G is a projective limit of Lie groups G/K_α, where the K_α are compact normal subgroups of G, and by [3, Lemma 1.1] $\mathrm{RR}(C^*(G)) = \sup_\alpha \mathrm{RR}(C^*(G/K_\alpha))$ and similarly for the stable rank. So let K be any compact normal subgroup of G such that G/K is a Lie group. It suffices to show that

$$\mathrm{RR}(C^*(G/K)) \le \max\{1, \mathrm{rank}(G_0/[G_0, G_0])\}$$

and that $\mathrm{sr}(C^*(G/K)) \le \max\{2, 1 + \lfloor \frac{1}{2}\,\mathrm{rank}(G_0/[G_0, G_0]) \rfloor\}$.

For this, we use some facts from [3, p. 94]. Firstly, $(G/K)_0 = (G_0K)/K$ and this is nilpotent since it is a quotient of G_0. Then $(G/K)/(G/K)_0 = G/G_0K$ and this is finite since it is both discrete and also a quotient of the compact group G/G_0. Finally,

$$\mathrm{rank}((G/K)_0/[(G/K)_0, (G/K)_0]) \le \mathrm{rank}(G_0/[G_0, G_0]).$$

It follows from Lemma 3.1 that

$$\begin{aligned} \mathrm{RR}(C^*(G/K)) &\le \max\{1, \mathrm{rank}((G/K)_0/[(G/K)_0, (G/K)_0])\} \\ &\le \max\{1, \mathrm{rank}(G_0/[G_0, G_0])\} \end{aligned}$$

and a similar argument applies to the stable rank.

The following corollary is an extension of [3, Proposition 2.6].

COROLLARY 3.3. *Let G be a locally compact group such that G_0 is nilpotent and each compact subset of G/G_0 generates a compact subgroup of G/G_0. Then*

(1) $\mathrm{RR}(C^*(G)) \le \max\{1, \mathrm{rank}(G_0/[G_0, G_0])\}$

(2) $\mathrm{sr}(C^*(G)) \le \max\{2, 1 + \lfloor \frac{1}{2}\,\mathrm{rank}(G_0/[G_0, G_0]) \rfloor\}$.

PROOF. Let $\mathscr{H}$ denote the collection of all compactly generated open subgroups H of G. Then $G = \bigcup_{H\in\mathscr{H}} H$ and, for each $H \in \mathscr{H}$, $H_0 = G_0$ and H/G_0 is compact. Since $C^*(G)$ is the inductive limit of the C^*-subalgebras $C^*(H)$, $\mathrm{RR}(C^*(G)) \le \sup_{H\in\mathscr{H}} \mathrm{RR}(C^*(H))$ and $\mathrm{sr}(C^*(G)) \le \sup_{H\in\mathscr{H}} \mathrm{sr}(C^*(H))$ (see [14, Lemma 4.1] and [21, Theorem 5.1]). Statements (1) and (2) now follow by applying Lemma 3.2 to each $H \in \mathscr{H}$.

In the next result, we will use the fact that if G is an almost connected, nilpotent, locally compact group and G_0 is the connected component of the identity element then $\operatorname{rank}(G/[G,G]) = \operatorname{rank}(G_0/[G_0,G_0])$ [3, Lemma 1.4] and furthermore $\operatorname{rank}(G/[G,G]) \geq 2$ if G_0 is not abelian [3, p. 95]. We will also use the fact that if G is an abelian locally compact group then $\mathrm{RR}(C^*(G)) = \dim \hat{G} = \operatorname{rank}(G)$ and $\mathrm{sr}(C^*(G)) = 1 + \lfloor \frac{1}{2} \operatorname{rank}(G) \rfloor$ (see, for example, the discussion in [2, p. 2170]).

THEOREM 3.4. *Let G be an almost connected, nilpotent, locally compact group and let G_0 be the connected component of the identity element. Then*

(1) $$\mathrm{RR}(C^*(G)) = \operatorname{rank}(G/[G,G]) = \operatorname{rank}(G_0/[G_0,G_0]) = \mathrm{RR}(C^*(G_0)) < \infty,$$

(2) $$\mathrm{sr}(C^*(G)) = 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(G/[G,G]) \right\rfloor = 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(G_0/[G_0,G_0]) \right\rfloor = \mathrm{sr}(C^*(G_0)) < \infty.$$

PROOF. Since G is almost connected, it is compactly generated. So $G/[G,G]$ is a compactly generated abelian group and therefore has finite rank.

(1) Suppose firstly that $\operatorname{rank}(G_0/[G_0,G_0]) \geq 1$. Then it follows from Lemma 3.2 that

$$\mathrm{RR}(C^*(G)) \leq \operatorname{rank}(G_0/[G_0,G_0]) = \operatorname{rank}(G/[G,G]) = \mathrm{RR}(C^*(G/[G,G])) \leq \mathrm{RR}(C^*(G)),$$

where the final inequality follows from the fact that $C^*(G/[G,G])$ is a quotient of $C^*(G)$. This establishes the first two equalities of (1), and the final equality follows from replacing G by G_0.

Now suppose that $\operatorname{rank}(G_0/[G_0,G_0]) = 0$ (<2). Then G_0 is abelian and hence, since the rank is zero, it is compact. Since G/G_0 is compact, we obtain that G is compact and hence that $\mathrm{RR}(C^*(G)) = 0 = \mathrm{RR}(C^*(G_0))$, as required.

(2) Suppose firstly that $\operatorname{rank}(G_0/[G_0,G_0]) \geq 2$. Then it follows from Lemma 3.2 that

$$\mathrm{sr}(C^*(G)) \leq 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(G_0/[G_0,G_0]) \right\rfloor = 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(G/[G,G]) \right\rfloor = \mathrm{sr}(C^*(G/[G,G])) \leq \mathrm{sr}(C^*(G)).$$

This establishes the first two equalities of (2), and the final equality follows from replacing G by G_0.

Now suppose that $\mathrm{rank}(G_0/[G_0, G_0]) \leq 1$ (< 2). Then G_0 is abelian and so

$$\mathrm{sr}(C^*(G_0)) = 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G_0) \right\rfloor = 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G/[G, G]) \right\rfloor = 1.$$

So it remains to show that $\mathrm{sr}(C^*(G)) = 1$. Let K be a compact normal subgroup of G such that G/K is a Lie group. Then, as in the proof of Lemma 3.2, it suffices to show that $\mathrm{sr}(C^*(G/K)) = 1$. As observed in the proof of Lemma 3.2, G/G_0K is finite and so G_0K/K is an abelian normal subgroup of G/K with finite index. Thus G/K is a Moore group [17, Theorem 1] and furthermore G_0K/K is contained in $(G/K)_F$ with necessarily finite index. It then follows from [2, Theorem 3.4 and Lemma 2.8] that

$$\begin{aligned} 1 \leq \mathrm{sr}(C^*(G/K)) &\leq 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}((G/K)_F) \right\rfloor \\ &= 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G_0K/K) \right\rfloor \leq 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G_0) \right\rfloor = 1, \end{aligned}$$

where the third inequality holds because G_0K/K is a quotient of the abelian group G_0 (see, for example, [3, p. 91]).

The following corollary concerns the 'threadlike' nilpotent Lie groups G_N ($N \geq 3$), which have been studied by several authors (see [1] and the references cited therein). The group G_3 is the continuous Heisenberg group. The result for the stable rank is already known as a consequence of [28], but the value of the real rank appears to be new since the rank of $G_N/[G_N, G_N]$ is even (cf. [3, Corollary 1.6]).

COROLLARY 3.5. *Let G_N be a threadlike nilpotent Lie group* ($N \geq 3$). *Then* $\mathrm{RR}(C^*(G_N)) = 2$ *and* $\mathrm{sr}(C^*(G_N)) = 2$.

PROOF. Since $G_N/[G_N, G_N] = \mathsf{R}^2$, the result follows from Theorem 3.4.

In view of Lemma 3.2, the question arises as to whether parts (1) and (2) of Theorem 3.4 remain true if only G_0 is assumed to be nilpotent. We show below that, even when G_0 is abelian, in both (1) and (2) no two of the first three numbers need be equal.

EXAMPLE 3.6. Let $G = \mathsf{R}^n \rtimes \mathsf{Z}_2$ where $n \geq 1$ and $-1 \in \mathsf{Z}_2$ acts on R^n by $x \to -x$. Then the irreducible representations of G are either 1− or 2-dimensional and $\mathsf{R}^n = G_0 = [G, G] = G_F$. In particular, $\mathrm{rank}(G/[G, G]) =$

$\mathrm{rank}(\mathsf{Z}_2) = 0$ and $[G : G_F] = 2$. In the following, we apply the results of [2] on the real rank and stable rank of C^*-algebras of Moore groups.

Let $n = 4$. Then $\mathrm{rank}(G_F) = 4$ and it follows from [2, Theorem 3.4] that

$$2 \leq \mathrm{RR}(C^*(G)) \leq \mathrm{RR}(C^*(G_F)) = 4$$

and

$$2 \leq \mathrm{sr}(C^*(G)) \leq \mathrm{sr}(C^*(G_F)) = 3.$$

On the other hand, since $\mathrm{rank}(G_F) = 4$ and $G_F^c = \{0\}$, it follows from [2, Theorem 4.3] that $\mathrm{RR}(C^*(G)) \neq \mathrm{RR}(C^*(G_F))$ (and in fact the proof of [2, Theorem 4.3] shows that $\mathrm{RR}(C^*(G))$ is 2 rather than 3) and it follows from [2, Theorem 4.4] that $\mathrm{sr}(C^*(G)) \neq \mathrm{sr}(C^*(G_F))$. Thus the three numbers $\mathrm{RR}(C^*(G)), \mathrm{RR}(C^*(G_0))$ and $\mathrm{rank}(G/[G,G])$ are distinct and so are the three numbers $\mathrm{sr}(C^*(G))$, $\mathrm{sr}(C^*(G_0))$ and $1 + \left\lfloor \frac{1}{2}\mathrm{rank}(G/[G,G]) \right\rfloor$.

We note in passing that, for the real rank alone, it suffices to take $n = 2$. For then similar arguments show that $\mathrm{RR}(C^*(G)) = 1$ and $\mathrm{RR}(C^*(G_0)) = 2$.

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