RADIAL GROWTH OF HARMONIC FUNCTIONS IN THE UNIT BALL

KJERSTI SOLBERG EIKREM and EUGENIA MALINNIKOVA*

Abstract

Let Ψ_v be the class of harmonic functions in the unit disk or unit ball in R^m which admit a radial majorant $v(r)$. We prove that a function in Ψ_v may grow or decay as fast as v only along a set of radii of measure zero. For the case when v fulfills a doubling condition, we give precise estimates of these exceptional sets in terms of Hausdorff measures.

1. Introduction

Radial behavior of harmonic functions in the unit disk and unit ball in R^m is a classical topic in analysis. In this article we consider harmonic functions bounded a priori by some radial majorant and discuss their radial growth.

It follows from a theorem of N. N. Lusin and I. I. Privalov, see [15], that there exist harmonic functions in the unit disk that tend to infinity along almost each radius. Moreover, a generalization of this result obtained by J.-P. Kahane and Y. Katznelson [11], shows that such functions may be bounded by an arbitrarily slow growing radial majorant.

Let $v(r)$ be a positive increasing continuous function on $[0, 1)$ and assume that $\lim_{r \to 1} v(r) = +\infty$. Let B be the unit ball in R^m, we define

$$(1) \quad \Phi_v^m = \{\, u : \mathsf{B} \to \mathsf{R}, \Delta u = 0, u(x) \leq K v(|x|) \text{ for some } K > 0 \,\},$$

and

$$\Psi_v^m = \{\, u : \mathsf{B} \to \mathsf{R}, \Delta u = 0, |u(x)| \leq K v(|x|) \text{ for some } K > 0 \,\}.$$

Harmonic functions of the class Φ_v^2 with $v(r) = |\log(1 - r)|$ were studied by B. Korenblum in [12]. This class as well as more general classes that correspond to $v(r) = |\log(1 - r)|^s$ appear in connection with the related spaces of analytic functions in the unit disk, see also [16], [3]. Two-sided estimates and classes Ψ_v^2 arise when one considers invertible elements in these spaces of analytic functions as in [2]. Radial growth of harmonic functions in the unit

disk bounded by a multiple of $|\log(1-r)|$ was studied by A. Borichev, Yu. Lyubarskii, E. Malinnikova and P. Thomas in [4], see also [13]. In this article we discuss to what extent the results in [4] remain true for general majorants and higher dimensional spaces.

The main aim of this work is to estimate the size of the set of the radii along which a function from Φ_v^m or Ψ_v^m grows or decays as fast as the majorant $v(r)$. For each function $u \in \Phi_v^m$ we define the following subsets of the unit sphere

$$E^-(u) = \left\{ y \in S : \limsup_{r \to 1-} \frac{u(ry)}{v(r)} < 0 \right\},$$

$$E^+(u) = \left\{ y \in S : \liminf_{r \to 1-} \frac{u(ry)}{v(r)} > 0 \right\}.$$

Our first result is the following:

THEOREM 1.1. *If* $u \in \Psi_v^m$ *then* $\sigma(E^-(u)) = \sigma(E^+(u)) = 0$, *where* σ *is the usual surface measure on the unit sphere in* R^m.

We give more precise estimates of the sets $E^\pm(u)$ when v satisfies the following doubling condition

(2) $v(1 - d/2) \leq Dv(1 - d)$.

The constants K and D will preserve their identities throughout this article.

For every increasing continuous function $\lambda : [0, +\infty) \to [0, +\infty)$ with $\lambda(0) = 0$ we denote by $\mathcal{H}_\lambda$ the corresponding Hausdorff measure.

THEOREM 1.2. *Let* v *satisfy* (2).
(a) *If* $u \in \Phi_v^m$ *and* λ *is a continuous increasing function,* $\lambda(0) = 0$ *and*

$$\lambda(t) = o(t^{m-1}v(1-t)^\alpha) \qquad (t \to 0),$$

for all $\alpha > 0$, *then* $\mathcal{H}_\lambda(E^+(u)) = \mathcal{H}_\lambda(E^-(u)) = 0$.
(b) *For each* $\beta > 0$ *there exists* $u \in \Psi_v^m$ *and an increasing continuous function* λ_β, $\lambda_\beta(0) = 0$ *and* $\lambda_\beta(t) = O(t^{m-1}v(1-t)^\beta)$ $(t \to 0)$, *such that* $\mathcal{H}_{\lambda_\beta}(E^+(u)) > 0$.

From (b) we see that the estimate for E^- in (a) is sharp as well, since $u \in \Psi_v^m$ and $E^-(-u) = E^+(u)$.

In this theorem there is no difference between the size of the sets $E^\pm(u)$ for $u \in \Phi_v^m$ and $u \in \Psi_v^m$. The situation is different for positive harmonic functions as was also noted in [4]. We generalize the result on positive harmonic functions to a wide class of weights.

THEOREM 1.3. *Assume that* $\lambda(t) = t^{m-1}v(1-t)$ *is an increasing continuous function and* $\lambda(0) = 0$. *There exists a positive function* $u \in \Psi_v^m$ *such that* $\mathscr{H}_\lambda(E^+(u)) > 0$.

As the next result shows, the choice of λ for given v in the theorem above is optimal. Moreover, no a priori growth estimate is needed for the estimate of the Hausdorff measure $\mathscr{H}_\lambda$ of the set of radii along which a positive harmonic function grows as $v(r)$. More precisely,

THEOREM 1.4. *Assume that* $\lambda(t) = t^{m-1}v(1-t)$ *is an increasing continuous function and* $\lambda(0) = 0$. *Let u be a positive harmonic function in the unit ball of* $\mathbb{R}^m$, *we define*

$$(3) \qquad F_v^+(u) = \left\{ y \in S : \limsup_{r \to 1-} \frac{u(ry)}{v(r)} > 0 \right\}.$$

Then $F_v^+(u)$ *is a countable union of sets of finite* $\mathscr{H}_\lambda$*-measure.*

The article is organized as follows. We collect some preliminary results on harmonic measure and Hausdorff measures in the next section. Then we prove Theorem 1.1 and Theorem 1.2. For part (a) of Theorem 1.2 our arguments are similar to those in [4]. In higher dimensions they are based on estimates of harmonic measure due to B. E. Dahlberg, [6]. A new approach is used to construct examples of functions with a large set of extremal growth in dimension larger than two in the proof of Theorem 1.2(b).

In the last section we study the radial growth of positive harmonic functions. We prove Theorem 1.4 first, then we characterize boundary measures that correspond to positive functions in Ψ_v^m and describe an example that proves Theorem 1.3. Both constructions (for Theorem 1.2(b) and Theorem 1.3) employ Cantor-type sets on the unit sphere.

2. Preliminaries

2.1. Poisson kernel and some estimates

Let σ be the $(m-1)$-dimensional surface measure on S and denote $\sigma(S) = \gamma_{m-1}$. The Poisson kernel in the m-dimensional unit ball is

$$P(x, \zeta) = \frac{1}{\gamma_{m-1}} \frac{1 - |x|^2}{|x - \zeta|^m}.$$

Assume for simplicity that $x = (1, 0, \ldots, 0)$. Using hyperspherical coordinates for $\zeta \in S$ we have $\zeta = (\cos \phi, \zeta')$, where $|\zeta'| = \sin \phi$. Let

$$\tilde{P}_{m,r}(\phi) = \frac{1}{\gamma_{m-1}} \frac{1 - r^2}{(1 + r^2 - 2r \cos \phi)^{m/2}}$$

and

$$Q_m(r, \phi) = -\partial_\phi \tilde{P}_{m,r}(\phi) = \frac{1}{\gamma_{m-1}} \frac{mr(1 - r^2) \sin \phi}{(1 + r^2 - 2r \cos \phi)^{(m+2)/2}}.$$

Then $P(rx, \zeta) = \tilde{P}_{m,r}(\phi)$.

Let $d(x, \zeta)$ be the geodesic distance between two points x and ζ on S. Then let $B(x, \phi) = \{\zeta \in S : d(x, \zeta) < \phi\}$ be the hyperspherical cap of radius ϕ with center at x. It can be shown that for the $(m - 1)$-dimensional surface measure of the cap

(4) $$C_1 \phi^{m-1} \leq \sigma(B(x, \phi)) \leq C_2 \phi^{m-1},$$

where the constants depend on m.

We will need some estimates for integrals of Q_m.

(i) We have

$$\int_0^{1-r} Q_m(r, \phi) \, d\phi \leq C_3 \int_0^{1-r} \frac{r(1 - r^2)\phi}{((1 - r)^2 + 2r(1 - \cos \phi))^{(m+2)/2}} \, d\phi$$

$$\leq C_3 \int_0^{1-r} \frac{r(1 - r^2)\phi}{(1 - r)^{m+2}} \, d\phi,$$

hence

(5) $$\int_0^{1-r} Q_m(r, \phi) \, d\phi \leq C_4 \frac{1}{(1 - r)^{m-1}}.$$

(ii) For $d > 0$

$$\int_d^\pi Q_m(r, \phi) \, d\phi \leq C_3 \int_d^\pi \frac{r(1 - r^2)\phi}{\left((1 - r)^2 + r\frac{4}{\pi^2}\phi^2\right)^{m/2+1}} \, d\phi$$

$$\leq C_3 \int_d^\pi \frac{r(1 - r^2)\phi}{\left(r\frac{4}{\pi^2}\phi^2\right)^{m/2+1}} \, d\phi,$$

thus

(6) $$\int_d^\pi Q_m(r, \phi) \, d\phi \leq C_5 r^{-m/2} d^{-m} \leq C_6 d^{-m}$$

when $r > \frac{1}{2}$.

(iii) Furthermore, by (4),

$$\int_{1-r}^{\pi} \sigma(B(x, \phi)) Q_m(r, \phi)\, d\phi \leq \int_{1-r}^{\pi} C_2 \phi^{m-1} Q_m(r, \phi)\, d\phi$$

$$\leq C_7 \int_{1-r}^{\pi} \frac{(1 - r^2)\phi^{-2}}{r^{m/2}}\, d\phi$$

$$\leq C_8 r^{-m/2},$$

so for $r > \frac{1}{2}$,

$$\text{(7)} \qquad \int_{1-r}^{\pi} \sigma(B(x, \phi)) Q_m(r, \phi)\, d\phi \leq C_9.$$

2.2. Harmonic measure in Lipschitz domains

A bounded domain $\Omega \in \mathsf{R}^m$ is a Lipschitz domain if there is a constant C such that to each point $q \in \partial\Omega$ there corresponds a coordinate system $(\xi, \eta), \xi \in \mathsf{R}^{m-1}, \eta \in \mathsf{R}$, and a function φ such that $|\varphi(\xi_1) - \varphi(\xi_2)| \leq C|\eta_1 - \eta_2|$ for which $\Omega \cap V = \{(\xi, \eta) : \varphi(\xi) < \eta\}$ for some neighborhood V of q. The smallest such constant C is called the Lipschitz constant.

Let S be the unit sphere in R^m. For $\zeta \in S$ and $a < 1$ we use the standard notation $\Gamma_\zeta^a = \mathrm{conv}(\zeta, a\mathsf{B})$ for the convex hull of ζ and the m-dimensional ball of radius a. Given a compact set $F \in S$ we consider the cone-domain $G = G(F, a) = \bigcup_{\zeta \in F} \Gamma_\zeta^a$. It is a Lipschitz domain, and the Lipschitz constant of $G(F, a)$ depends on a only. Given a Lipschitz domain Ω, a subset $A \subset \partial\Omega$ and a point $z \in \Omega$, we denote by $\omega(z, A, \Omega)$ the harmonic measure of A at the point z.

A celebrated result by B. E. Dahlberg [6] says that on the boundary of a Lipschitz domain the harmonic measure and the surface measure are mutually absolutely continuous. We need a quantitative form of this result for cone-domains and refer the reader to [6], [10] and [1, Chapter 4.2].

THEOREM A. *Let $a > 0$, then there exist α and C that depend on a and m only such that for any cone-domain $G = G(F, a)$ in the unit ball of R^m and any $A \subset Q \subset \partial G$ the following inequality holds*

$$\frac{\omega(0, A, G)}{\omega(0, Q, G)} \geq C \left(\frac{\eta(A)}{\eta(Q)} \right)^\alpha,$$

where Q is a ball on ∂G and η is the surface measure on ∂G.

2.3. *Hausdorff measures*

We will refer to generalized Hausdorff measures in R^m as they are defined for example in [14, p. 59]. Let h be an increasing continuous function on $[0, +\infty)$, $h(0) = 0$, then for any $E \subset \mathsf{R}^m$

$$\mathscr{H}_h(E) = \liminf_{\delta \to 0} \left\{ \sum_j h(d_j) : E \subset \bigcup_j F_j, \, d_j = \mathrm{diam}(F_j) < \delta \right\}.$$

We assume in addition that $h(t/2) \geq ch(t)$ for some $c > 0$. Then the Hausdorff measure is equivalent to the so-called net measure $N_h(E)$, defined with F_j being half-open dyadic cubes with sides parallel to the coordinate axis, in the following sense: $\mathscr{H}_h(E) \leq N_h(E) \leq A(c, m)\mathscr{H}_h(E)$, see [14, p. 76]. Further, the following property holds, if $f : \mathsf{R}^k \to \mathsf{R}^m$ is a Lipschitz map and $E \subset \mathsf{R}^k$, then $\mathscr{H}_h(f(E)) \leq L\mathscr{H}_h(E)$, where L depends on the Lipschitz constant of f and on c. The proofs follow readily from the definitions.

We will use Cantor-type sets having the following structure:

$$C = \bigcap_s C_s, \qquad C_s \supset C_{s+1}, \qquad C_0 = [0, 1],$$

where each set C_s is a union of N_s segments $\{I_j^{(s)}\}_j$ of the same length l_s. For each such segment the intersection $C_{s+1} \cap I_j^{(s)}$ is a union of k_s non-overlapping segments of length l_{s+1}. We assume, of course, that

(i) $l_s \searrow 0$ as $s \to \infty$,   (ii) $k_s l_{s+1} < l_s$,   and   (iii) $N_s = k_0 k_1 \ldots k_{s-1}$.

The next result is Theorem 3 in [4].

LEMMA A. *Let* $\lambda : [0, 1) \to [0, +\infty)$ *be a continuous increasing function with* $\lambda(0) = 0$*, such that for some* $a > 0$ *and* $s > s_0$

$$(8) \qquad \frac{\lambda(l)}{l} \geq a \frac{\lambda(l_{s+1})}{l_{s+1}} \qquad for \ any \quad l \in [l_{s+1}, l_s).$$

Then

$$(9) \qquad \liminf_{s \to \infty} N_s \lambda(l_s) \geq \mathscr{H}_\lambda(C) \geq \frac{a}{2} \liminf_{s \to \infty} N_s \lambda(l_s).$$

Two slightly more delicate results that we need, give estimates of the Hausdorff measure of (symmetric) Cantor sets and cylinder sets in higher dimensions. Note also that we are not interested in the exact value of the Hausdorff measure but only in its positivity.

LEMMA B (Hatano, [7]). *Let* h *be a continuous increasing function with* $h(0) = 0$*. Let* $\{k_q\}_{q=1}^\infty$ *be a sequence of positive integers and* $\{l_q\}_{q=0}^\infty$*,* $l_0 = 1$

be a sequence of positive numbers that satisfy $k_{q+1}l_{q+1} < l_q$. The general-
ized symmetric Cantor set E in $\mathbb{R}^m$ defined by the sequences $\{k_q\}$ and $\{l_q\}$ is
constructed in the following way: Let $C_0 = [0, 1]$, C_1 is obtained from C_0 by
removing $k_1 - 1$ open intervals of equal lengths such that remaining k_1 closed
intervals are of length l_1. Then, to get C_2, $k_2 - 1$ open intervals are removed
from each interval of C_1 such that remaining intervals are of length l_2, etc.
Define $C = \bigcap_n C_n$ and $E = C \times C \times \cdots \times C \subset [0, 1]^m$.
*   Then $\mathscr{H}_h(E) > 0$ if and only if $\liminf_{q \to \infty}(k_1 \ldots k_q)^m h(l_q) > 0$.*

The measure used in [7] is not the classical Hausdorff measure but one
defined using coverings by all open cubes. As we mentioned above, under
our condition on h the two measures are equivalent (up to a multiplicative
constant).

The next statement is intuitively clear but we were not able to find a precise
reference, so we outline a short proof.

LEMMA 2.1. *Let $h(t) = t^{k-1}v(t)$, where v is an increasing continuous*
function on $[0, +\infty)$ and $v(0) = 0$. Assume also that $v(t/2) \geq cv(t)$ for some
$c > 0$. If $F \subset [0, 1]$ is compact, $\mathscr{H}_v(F) > 0$, and $E = F \times [0, 1]^{k-1} \subset \mathbb{R}^k$,
then $\mathscr{H}_h(E) > 0$.

PROOF. We will use that $\mathscr{H}_v$ is equivalent to N_v and $\mathscr{H}_h$ is equivalent to N_h.
Assume that $N_h(E) = 0$, then for any $\epsilon > 0$ and $\delta > 0$ there exists a *finite*
family of half-open dyadic cubes $\{Q_\alpha\}$ with sides $l_\alpha = 2^{-n_\alpha} < \delta$ that covers
$E = F \times [0, 1]^{k-1}$ and such that $\sum_\alpha h(l_\alpha) < \epsilon$. Indeed we can find an infinite
family for which $\sum h(l_\alpha) < 2^{-k}\epsilon$, then for each cube Q in this family, take an
open cube that contains Q, has side length which is twice that of Q and can be
covered by 2^k half-open dyadic cubes of the same size as Q. Then we choose
a finite sub-cover of the compact set E.

Let $n = \min_\alpha n_\alpha$ and $N = \max_\alpha n_\alpha$, we divide $[0, 1]$ into dyadic intervals
I_j of length 2^{-n}, $[0, 1] = \bigcup_j I_j$. For each j consider cubes $K_{j,s} = I_j \times J_s$, $1 \leq$
$s \leq 2^{n(k-1)}$, where J_s is a dyadic subcube of $[0, 1]^{k-1}$ with side length 2^{-n}.
Now for each s let

$$d_{j,s} = \sum_{\alpha : Q_\alpha \subset K_{j,s}} h(l_\alpha).$$

Choose $t = t(j)$ such that $d_{j,t} = \min_s d_{j,s}$ and replace the covering $\{Q_\alpha\}$ by a
new one $\{Q_\beta\}$ such that $\sum_\beta h(l_\beta) \leq \sum_\alpha h(l_\alpha)$, and for each j the cubes $\{Q_\beta\}$
contained in $K_{j,s}$ can be obtained from the cubes $\{Q_\alpha\}$ contained in $K_{j,t(j)}$
by translation. If for the new family $\min n_\beta > n$, we repeat the procedure.
If not, we get some chains of cubes $K_{j,1}, \ldots, K_{j,2^{(k-1)n}}$ in the new family and
repeat the procedure on the complement of these chains. Anyway the size of

the smallest cubes is always at least 2^{-N} and after finitely many steps we find a family of intervals I_γ of length $l_\gamma < \delta$ that covers F and

$$\sum_\gamma v(l_\gamma) = \sum_\gamma h(l_\gamma) l_\gamma^{-(k-1)} < \epsilon.$$

Thus $\mathscr{H}_v(F) = 0$.

3. Sets of extremal growth or decay

3.1. Lebesgue measure of sets of extremal growth

In this subsection we first estimate the Lebesgue measure of the sets $E^\pm(u)$ for arbitrary $u \in \Phi_v^m$.

PROPOSITION 3.1. *Suppose that $u \in \Phi_v^m$, then $\sigma(E^-(u)) = 0$.*

PROOF. We have $E^-(u) = \bigcup_n F_n = \bigcup_n \{\zeta \in S : u(r\zeta) \le -\frac{1}{n} v(r), r \ge 1 - \frac{1}{n}\}$. Assume $\sigma(E^-(u)) > 0$. Then $\sigma(F_n) > 0$ for some n, and F_n is a compact subset of S. Let $G = \bigcup_{\zeta \in F_n} \Gamma_\zeta^a$ and $G_\alpha = G \cap \alpha B$ for $\alpha < 1$. We have $\partial G = F_n \cup L$, where $L = \partial G \cap B$.

We will estimate $u(0)$ using harmonic measure in the domain G_α. First, it follows from Dahlberg's theorem that $\omega(0, F_n, G) = c > 0$. Now let $L_\alpha = \partial G \cap \alpha B = L \cap \alpha B$ and let $p_\alpha(A)$ be the radial projection of a set A onto αS, where $0 < \alpha \le 1$. Then

$$\partial G_\alpha = L_\alpha \cup \alpha F_n \cup p_\alpha(L \setminus L_\alpha).$$

Choose $s > 1 - \frac{1}{n}$ such that

$$\omega(0, L \setminus L_s, G) < \frac{c}{3n} \qquad \text{and} \qquad \sigma(p_1(L \setminus L_s)) < \frac{\gamma_{m-1} c}{3n}.$$

Let $s < t < 1$. Then, since $G_t \subset G$,

$$\omega(0, L_t \setminus L_s, G_t) \le \omega(0, L_t \setminus L_s, G) \le \omega(0, L \setminus L_s, G) < \frac{c}{3n}.$$

Further,

$$\omega(0, p_t(L \setminus L_t), G_t) \le \omega(0, p_t(L \setminus L_t), t B) = \frac{1}{\gamma_{m-1}} \sigma(p_1(L \setminus L_t)) < \frac{c}{3n}.$$

Finally, we want to estimate $\omega(0, t F_n, G_t)$. Note that $t G \subset G_t$, then

$$\omega(0, t F_n, G_t) \ge \omega(0, t F_n, t G) = \omega(0, F_n, G) = c.$$

Now we apply the estimates for the function u, which is harmonic in G_t, using that $\partial G_t = L_s \cup (L_t \setminus L_s) \cup t F_n \cup p_t (L \setminus L_t)$.

$$u(0) \leq v(s)\omega(0, L_s, G) + v(t)\frac{2c}{3n} - \frac{v(t)}{n}c \leq v(s) - \frac{v(t)c}{3n}.$$

When t goes to 1 we get a contradiction, since $v(t) \to \infty$.

Thus the estimate from above for u, $u(x) \leq Kv(|x|)$, implies that the set of the radii along which u decays at least as fast as $-v$ has zero measure. Theorem 1.1 formulated in the introduction follows.

To deal with the set $E^+(u)$ for $u \in \Phi(v)$, we assume that the function v fulfills (2). The proof follows the argument from [4].

LEMMA 3.2. *Let $u \in \Phi_v^m$ where v satisfies (2) and assume $u(x) > cv(|x|)$ for some $x \in \mathsf{B}$. Then there exists $\tau = \tau(K, D, c) > 0$ such that $u(x') > c/2v(|x|)$ whenever $|x - x'| < \tau (1 - |x|)$, $|x'| = |x|$.*

The same statement holds if we write $<$ in both inequalities and assume that $c < 0$.

PROOF. Let $y, y' \in \mathsf{B}$. Assume that $|y| = |y'|$ and $|y - y'| < \tau_1 (1 - |y|)$, where $\tau_1 < 1$, then for any $\zeta \in S$

$$|y' - \zeta| \geq |y - \zeta| - |y - y'| > |y - \zeta| - \tau_1|y - \zeta| = (1 - \tau_1)|y - \zeta|.$$

Thus $|y - \zeta|^m (1 - \tau_1)^m < |y' - \zeta|^m$ and

(10) $$P(y, \zeta) > (1 - \tau_1)^m P(y', \zeta).$$

Let $r = |x|$, $R = (1 + r)/2$ and denote $q = q(\tau_1) = (1 - \tau_1)^m$. We apply (10) with $y = \frac{x}{R}$, $y' = \frac{x'}{R}$ and $|y - y'| < \tau_1 (1 - |y|)$. Then

$$u(x) = \int_S u(R\zeta)P\left(\frac{x}{R}, \zeta\right) d\sigma(\zeta)$$

$$= qu(x') + \int_S u(R\zeta)\left(P\left(\frac{x}{R}, \zeta\right) - qP\left(\frac{x'}{R}, \zeta\right)\right) d\sigma(\zeta)$$

$$\leq qu(x') + \int_S Kv(R)\left(P\left(\frac{x}{R}, \zeta\right) - qP\left(\frac{x'}{R}, \zeta\right)\right) d\sigma(\zeta)$$

$$= qu(x') + (1 - q)Kv(R) \leq qu(x') + (1 - q)KDv(r).$$

If τ_1 is such that $c - (1-q)KD \geq \frac{c}{2}q$ and $|x - x'| < \frac{\tau_1}{2}(1-r) < \tau_1\left(1 - \frac{r}{R}\right)$, then $u(x') > \frac{c}{2}v(r)$. To complete the proof it suffices to choose $\tau(K, D, c) = \tau_1/2$.

For the second case when $c < 0$, we use the inequality

$$u(x') \leq qu(x) + (1 - q)KDv(r) < (qc + (1 - q)KD) v(r)$$

and choose τ_1 such that $qc + (1 - q) KD \leq \frac{c}{2}$.

COROLLARY 3.3. *If $u \in \Phi_v^m$ where v satisfies (2), then $\sigma(E^+(u)) = 0$.*

PROOF. Note that by Lemma 3.2 u is bounded from below in Γ_ζ^a for any $\zeta \in E^+(u)$ and some $a = a(\zeta)$. Then by results of L. Carleson [5] (see also [9], [6]), u has a finite non-tangential limit at almost each point of $E^+(u)$. Applying the lemma once again, we see that the non-tangential limit at $\zeta \in E^+(u)$ is infinite. Thus $\sigma(E^+(u)) = 0$.

3.2. *Estimates of Hausdorff measures*

For weights that satisfy the doubling condition we can give more precise estimates of the size of the exceptional sets. We now prove Theorem 1.2(a) formulated in the introduction.

PROOF OF THEOREM 1.2(a). We start with $E^+(u)$. It is enough to prove the statement for each set

$$E_n = \left\{ \zeta \in S : u(r\zeta) \geq \frac{1}{n} v(r), r \geq 1 - \frac{1}{n} \right\}.$$

By Lemma 3.2, there exists a such that $u(x) \geq \frac{1}{2n} v(|x|)$ for any $x \in \Gamma_\zeta^a$ where $|x| > 1 - \frac{1}{n}$ and $\zeta \in E^+(u)$.

Let $G = \bigcup_{\zeta \in E_n} \Gamma_\zeta^a$ and $G_t = G \cap t\mathbf{B}$. Clearly we may assume that $u \geq c_0$ on G for some $c_0 < 0$. Let $b = b(a)$ be such that

$$\partial G_t \cap tS = t E_n^{b(1-t)} = \{t\zeta \in tS : |\zeta - \zeta_0| < b(1 - t) \text{ where } \zeta_0 \in E_n\}.$$

Then by harmonic measure estimate for G_t when $t > 1 - \frac{1}{n}$, we obtain

$$u(0) \geq c_0 + \omega(0, t E_n^{b(1-t)}, G_t) \frac{v(t)}{2n}.$$

By Theorem A there exists C and $\gamma > 0$ that depend only on a such that

$$\omega(0, A, G_t) \geq C\sigma(A)^\gamma,$$

here σ is the $(m - 1)$-dimensional surface measure on tS. This implies

$$\sigma(E_n^{b(1-t)})^\gamma \leq \frac{C_1}{v(t)}$$

where $C_1 = C_1(n, u, a)$. So for all $\epsilon > 0$ small enough we get by applying (2)

$$\sigma(E_n^\epsilon) \leq C_2 \left(v \left(1 - \frac{\epsilon}{b} \right) \right)^{-1/\gamma} \leq C_3 \left(v(1 - \epsilon) \right)^{-1/\gamma} .$$

We cover E_n by a finite collection of balls $\{B_j : j \in J\}$ of radius $\frac{\epsilon}{5}$ and centers at points in E_n. By the Vitali covering lemma (see for example [8, p. 2]) there exists a subcollection $J' \subseteq J$ where $\{B_j : j \in J'\}$ are disjoint and $\bigcup_{j \in J} B_j \subseteq \bigcup_{j \in J'} 5B_j$, and we also have $\bigcup_{j \in J'} 5B_j \subseteq E_n^\epsilon$. Then E_n can be covered by N_ϵ balls $\{5B_j : j \in J'\}$ of radius ϵ, where

$$\epsilon^{m-1} N_\epsilon \leq 5^{m-1} \sigma(E_n^\epsilon),$$

thus

$$N_\epsilon \leq 5^{m-1} \epsilon^{-m+1} C_3 \left(v(1 - \epsilon) \right)^{-1/\gamma} .$$

Then

$$\mathscr{H}_\lambda(E_n) \leq \liminf_{\epsilon \to 0} N_\epsilon \lambda(\epsilon) \leq \liminf_{\epsilon \to 0} 5^{m-1} \epsilon^{-m+1} C_3 \left(v(1 - \epsilon) \right)^{-1/\gamma} \lambda(\epsilon).$$

Since $\lambda(t) = o(t^{m-1}(v(1-t))^w)$, $(t \to 0)$, for any $w > 0$, we get $\mathscr{H}_\lambda(E_n) = 0$.

The proof for $\mathscr{H}_\lambda(E^-(u))$ is similar; we then use the second part of Lemma 3.2.

REMARK 3.4. If $v(t) = (1 - t)^{-\gamma}$ for some $\gamma > 0$ and $u \in \Phi_v^m$, then the theorem above implies in particular that $\mathscr{H}_\lambda(E^+(u)) = 0$ and $\mathscr{H}_\lambda(E^-(u)) = 0$ when $\lambda(t) = t^{m-1} \log \frac{1}{t}$. On the other hand, we will show in section 3.4 that for any $\epsilon > 0$ there exists $u \in \Phi_v^m$ such that

$$\dim E^+(u) > m - 1 - \epsilon.$$

3.3. Auxiliary functions

We now begin to prove Theorem 1.2(b). First we construct auxiliary functions u_k in B that resemble $\text{Im}(z^{2^k})$ in the unit disk.

For each positive integer k let S_k and T_k be subsets of the interval $[0, 2\pi)$ defined by

$$S_k = \bigcup_{j=0}^{2^k - 1} [2j\pi 2^{-k}, (2j + 1)\pi 2^{-k}),$$

$$T_k = \bigcup_{j=0}^{2^k - 1} [(2j + 1/4)\pi 2^{-k}, (2j + 3/4)\pi 2^{-k}].$$

Then on the unit sphere S in R^m we define

$$E_k = \{\eta \in S, \eta = (t \cos \phi, t \sin \phi, \eta_3, \ldots, \eta_m), t \geq 0, \phi \in S_k\},$$

and

$$F_k = \{\eta \in S, \eta = (t \cos \phi, t \sin \phi, \eta_3, \ldots, \eta_m), t \geq 3/4, \phi \in T_k\}.$$

Let $f_k = 1$ on E_k and $f_k = -1$ on $S \setminus E_k$. Further, let $u_k = P * f_k$ be the corresponding harmonic function in the unit ball B.

LEMMA 3.5. *The function u_k has the following properties*

(a) $|u_k| \leq 1$ *on* B;

(b) $u_k(r\eta) \geq 0$ *when* $\eta \in E_k$;

(c) *For each $d \in \mathsf{N}$ there exists $c_{d,m}$ such that* $|u_k(x)| \leq c_{d,m} 2^{-kd} (1 - |x|)^{-d}$;

(d) *There exists $a_m > 0$ such that $u_k(r\eta) > 1/4$ when $r > 1 - a_m 2^{-k}$ and $\eta \in F_k$.*

PROOF. By the maximum principle (a) follows immediately. Note further that $f_k(x_1, x_2, \ldots, x_m) = -f_k(x_1, -x_2, \ldots, x_m)$, and thus

$$u_k(x) = \frac{1}{\gamma_{m-1}} \int_S \frac{1 - |x|^2}{|x - y|^m} f_k(y) \, dy$$

satisfies $u_k(x_1, x_2, \ldots, x_m) = -u_k(x_1, -x_2, \ldots, x_m)$. In particular,

(11) $$u_k(x_1, 0, \ldots, x_m) = 0.$$

Let $\alpha_k = \pi 2^{-k}$ and

$$A_k = \begin{bmatrix} \cos \alpha_k & -\sin \alpha_k & 0 & \ldots & 0 \\ \sin \alpha_k & \cos \alpha_k & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{m-2} & \\ 0 & 0 & & & \end{bmatrix},$$

where I_{m-2} is the identity matrix. Then A_k is an orthogonal matrix and the corresponding transformation of R^m maps the unit sphere to itself, moreover $A_k(E_k) = S \setminus E_k$. Then

$$f_k(A_k x) = -f_k(x) \qquad \text{and} \qquad u_k((A_k)^{-1} x) = -u_k(x).$$

Now, taking into account (11), we get

$$u_k(s \cos l\alpha_k, s \sin l\alpha_k, x_3, \ldots, x_m) = 0$$

for any $l = 0, 1, \ldots, 2^{k+1} - 1$. Fix l and consider the set

$$G_{k,l} = \{x \in \mathsf{B}, x = (s\cos\phi, s\sin\phi, x_3, \ldots, x_m), \phi \in (l\alpha_k, (l+1)\alpha_k)\}.$$

The boundary of $G_{k,l}$ consists of a part of the unit sphere and of subsets of the hyperplanes

$$\{(\sin l\alpha_k)x_1 - (\cos l\alpha_k)x_2 = 0\}$$

and

$$\{(\sin(l+1)\alpha_k)x_1 - (\cos(l+1)\alpha_k)x_2 = 0\}.$$

On both subsets of the hyperplanes $u_k = 0$, and on the corresponding part of the sphere all boundary values of u_k equal 1 if l is even and -1 if l is odd. Anyway, u_k does not change sign in $G_{k,l}$ and (b) follows.

To prove (c) assume first that $d = 1$. We write

$$u_k(x) = \frac{1}{\gamma_{m-1}} \int_{E_k} \frac{1 - |x|^2}{|x - y|^m} dy - \frac{1}{\gamma_{m-1}} \int_{S \setminus E_k} \frac{1 - |x|^2}{|x - y|^m} dy$$

$$= \frac{1}{\gamma_{m-1}} \int_{E_k} \left(\frac{1 - |x|^2}{|x - y|^m} - \frac{1 - |x|^2}{|x - A_k y|^m} \right) dy.$$

We want to estimate the difference under the integral sign. Note that

$$\max_{y \in \mathsf{B}} |y - A_k y| = 2\sin\alpha_k/2 < \alpha_k$$

and assume that $1 - |x| > \alpha_k$, then

$$\left| \frac{1}{|x - y|^m} - \frac{1}{|x - A_k y|^m} \right| \leq \frac{m|y - A_k y|(|x - A_k y| + \alpha_k)^{m-1}}{|x - y|^m |x - A_k y|^m}$$

$$\leq \frac{m\alpha_k 2^{m-1}}{(1 - |x|)|x - y|^m}.$$

We obtain $|u_k(x)| \leq c_m \pi 2^{-k}(1 - |x|)^{-1}$ when $1 - |x| > \alpha_k$, otherwise the inequality follows from (a).

In general, for $d \geq 2$, we define $V_x(y) = |x - y|^{-m}$ and notice that

$$\int_{E_k} V_x(A_k^j y) \, dy = \int_{E_k} V_x(y) \, dy, \qquad \text{when } j \text{ is even, and}$$

$$\int_{E_k} V_x(A_k^j y) \, dy = \int_{S \setminus E_k} V_x(y) \, dy, \quad \text{when } j \text{ is odd.}$$

Further, we have $\sum_{j=0}^{d}(-1)^j \binom{d}{j} = (1-1)^d = 0$ and $\sum_{j=0}^{d} \binom{d}{j} = (1+1)^d = 2^d$. Thus

$$u_k(x) = \frac{1-|x|^2}{\gamma_{m-1}} \int_{E_k} V_x(y)\, dy - \frac{1-|x|^2}{\gamma_{m-1}} \int_{S \setminus E_k} V_x(y)\, dy$$

$$= \frac{1-|x|^2}{\gamma_{m-1}} 2^{-d+1} \int_{E_k} \sum_{j=0}^{d} (-1)^j \binom{d}{j} V_x(A_k^j y)\, dy.$$

To estimate the sum under the integral sign let

$$y = (y_0 \cos \psi,\, y_0 \sin \psi,\, y_1) \in \mathsf{R} \times \mathsf{R} \times \mathsf{R}^{m-2}.$$

We have $A_k^j y = (y_0 \cos(\psi + j\alpha_k),\, y_0 \sin(\psi + j\alpha_k),\, y_1)$ and $V_x(y) = h_{x,y_0,y_1}(\psi)$. Then we write the Taylor polynomial of order $d-1$ for h_{x,y_0,y_1} at ψ,

$$V_x(A_k^j y) = h_{x,y_0,y_1}(\psi + j\alpha_k)$$

$$= \sum_{l=0}^{d-1} \frac{h_{x,y_0,y_1}^{(l)}(\psi)}{l!} (j\alpha_k)^l + \frac{h_{x,y_0,y_1}^{(d)}(\psi + \beta_{jk})}{d!} (j\alpha_k)^d.$$

A straightforward estimate of the derivatives of h shows that

$$|h_{x,y_0,y_1}^{(d)}(\phi)| \leq C|x - (y_0 \cos \phi,\, y_0 \sin \phi,\, y_1)|^{-m-d}$$

$$\leq C(|x-y| - y_0|\phi - \psi|)^{-m-d},$$

where C depends only on m and d. Applying the difference relation

$$\sum_{j=0}^{d} (-1)^j \binom{d}{j} j^l = 0,$$

when $l < d$ (see for example [18, p. 42]), we get

$$\left| \sum_{j=0}^{d} (-1)^j \binom{d}{j} V_x(A_k^j y) \right| \leq C 2^{-dk}(|x-y| - d\alpha_k)^{-m-d}.$$

Now for $1 - |x| > 2d\alpha_k$ and $|y| = 1$ we obtain

$$\left| \sum_{j=0}^{d} (-1)^j \binom{d}{j} V_x(A_k^j y) \right| \leq C 2^{m+d} 2^{-dk} |x-y|^{-m} (1-|x|)^{-d},$$

and the integration over E_k gives

$$u_k(x) \leq c_{d,m} 2^{-dk} (1 - |x|)^{-d}.$$

Note also that for $1 - |x| < 2d\alpha_k$ the inequality above (with a large enough constant) follows from $|u_k| \leq 1$, so (c) holds.

Finally, we prove (d). Let $\eta \in F_k$ and $x = r\eta$. It is easy to check that $B(\eta, 2^{-k-1}) \subset E_k$. A direct calculation shows that for a_m small enough

$$\frac{1}{\gamma_{m-1}} \int_{S \setminus B(\eta, 2^{-k-1})} \frac{1 - r^2}{|r\eta - y|^m} \, dy < \frac{3}{8},$$

when $r > 1 - a_m 2^{-k}$. Thus $u_k(x) > 1 - 2\frac{3}{8} = \frac{1}{4}$.

It will be more convenient to use functions like $\operatorname{Re}(z^{2^k})$, so we define

$$h_k(x) = u_k(A_{k+1} x); \qquad B_k = \bigcup_{j=0}^{2^k - 1} [(2j - 1/4)\pi 2^{-k}, (2j + 1/4)\pi 2^{-k}].$$

It is easy to check that (d) implies $h_k(r\eta) > 1/4$ when

$$(12) \quad \eta \in H_k = \left\{ y \in S, y = (t \cos\phi, t \sin\phi, y_3, \ldots, y_m), t \geq \tfrac{3}{4}, \phi \in B_k \right\}$$

and $r > 1 - a_m 2^{-k}$.

3.4. Construction of $u \in \Psi_v^m$ with a large set of radial growth

Now we can prove Theorem 1.2(b).

PROOF OF THEOREM 1.2(b). First we construct $v_\beta(t) = O(tv(1 - t)^\beta)$. The assumption on v implies

$$\frac{d}{2} v \left(1 - \frac{d}{2}\right)^\alpha \leq \frac{d}{2} D^\alpha v(1 - d)^\alpha < \frac{3}{4} dv(1 - d)^\alpha$$

when $\alpha \leq \alpha_0$.

For simplicity we define a new function g such that $v(r) = g\left(\frac{1}{1-r}\right)$. We will keep this notation throughout the paper. Then (2) is equivalent to

$$(13) \qquad\qquad g(2x) \leq Dg(x).$$

We choose $\alpha \leq \min\{\alpha_0, \beta\}$, and define v_β by

$$v_\beta(\pi 2^{-n}) = \pi 2^{-n} v(1 - 2^{-n})^\alpha = \pi 2^{-n} g(2^n)^\alpha, \qquad n \geq 2,$$

and ν_β is linear on $[\pi 2^{-n-1}, \pi 2^{-n}]$. Then $\lim_{t \to 0} \nu_\beta(t) = 0$ and ν_β is continuous and increasing. For $t \in [\pi 2^{-n-1}, \pi 2^{-n})$ we have

$$\frac{\nu_\beta(t)}{tg(\frac{1}{t})^\beta} \leq \frac{\pi 2^{-n} g(2^n)^\alpha}{\pi 2^{-n-1} g(2^n/\pi)^\beta} = 2\frac{g(2^n)^\alpha}{g(2^n/\pi)^\beta}$$

$$\leq 2\frac{g(2^n)^\alpha}{g(2^{n-2})^\beta} \leq 2D^{2\beta}\frac{g(2^n)^\alpha}{g(2^n)^\beta} \leq 2D^{2\beta}$$

when $n \geq n_0$, so $\nu_\beta(t) = O(tv(1-t)^\beta)$ when $t \to 0$. We also define a new function $\lambda_\beta(t) = t^{m-2}\nu_\beta(t)$.

Fix $A_1 > 1$ and define $b_1 = 1$,

$$b_{n+1} = \min\{l : g(2^l) > A_1 g(2^{b_n})\}, \qquad n = 2, 3, \ldots.$$

By assumption, $g(2^{b_{n+1}}) \leq Dg(2^{b_{n+1}-1})$, and by the way the b_n's are defined, $g(2^{b_{n+1}-1}) \leq A_1 g(2^{b_n})$. Then

$$(14) \qquad\qquad g(2^{b_{n+1}}) \leq DA_1 g(2^{b_n}) = A_2 g(2^{b_n}).$$

Let

$$u(x) = \sum_{n=1}^{\infty} g(2^{b_n}) h_{b_n}(x).$$

We want to check that u converges uniformly on compact subsets of $\mathbb{B}$ and $u \in \Phi_v^m$. Since g fulfills (13), there exists γ such that

$$(15) \qquad\qquad \frac{g(2^{l_2})}{g(2^{l_1})} \leq 2^{\gamma(l_2-l_1)}$$

for $l_1, l_2 \in \mathbb{N}$, just let $\gamma = \log_2 D$. Choose $d > \gamma$ and note that (15) implies

$$(16) \qquad\qquad \frac{g(2^{b_{n+1}})2^{-b_{n+1}d}}{g(2^{b_n})2^{-b_n d}} \leq 2^{-(d-\gamma)(b_{n+1}-b_n)} \leq 2^{-(d-\gamma)},$$

when $n > n_0$. Assume that $1 - 2^{-b_N} < |x| < 1 - 2^{-b_{N+1}}$, then by Lemma 3.5(a) and (c),

$$|u(x)| \leq \sum_{n=1}^{N} g(2^{b_n}) + c_{d,m} \sum_{n=N+1}^{\infty} g(2^{b_n}) 2^{-b_n d} (1-|x|)^{-d}.$$

The first sum is bounded by $C_1 g(2^{b_N})$, and for N large enough (16) implies that the second sum is bounded by $C_2 g(2^{b_{N+1}}) 2^{-b_{N+1}d}(1-|x|)^{-d} \leq C_2 g(2^{b_{N+1}})$.

Then by (14)

$$|u(x)| \leq C_3 g(2^{b_{N+1}}) \leq C_4 g(2^{b_N}) \leq C_4 g\left(\frac{1}{1-|x|}\right).$$

Finally, we show that $F = \bigcap_n H_{b_n} \subset E^+(u)$, where H_k are defined by (12). Let $x = |x|\eta$, $\eta \in F \subset S$, and $1 - a_m 2^{-b_N} < |x| \leq 1 - a_m 2^{-b_{N+1}}$, where a_m is as in Lemma 3.5; we may assume also that $x_1^2 + x_2^2 > 1/4$. Then by (b) and (d) in Lemma 3.5 (see also the definition of h_k above), we obtain

$$u(x) = \sum_{n=1}^{\infty} g(2^{b_n}) h_{b_n}(x) \geq g(2^{b_N}) h_{b_N}(x) \geq \frac{1}{4} g(2^{b_N}) \geq C_5 g\left(\frac{1}{1-|x|}\right).$$

Let $C = \bigcap_n B_{b_n} \subset [0, 2\pi)$ and $C_j = \bigcap_{n=1}^{j} B_{b_n}$. Then C_j is a union of N_j intervals of length $l_j = \frac{\pi}{4} 2^{-b_j}$, where some of the intervals are next to each other, and C is a set as in Lemma A. Intervals of length l_j are called intervals from j-th generation. Each of them contains k_{j+1} intervals from the next generation. It is easy to show that $k_{j+1} = 1$ if $b_{j+1} - b_j = 1$, and $k_{j+1} = \frac{1}{4} 2^{b_{j+1} - b_j}$ if $b_{j+1} - b_j > 1$. So $k_{j+1} \geq \frac{1}{4} 2^{b_{j+1} - b_j}$ and $N_j \geq (\frac{1}{4})^j 2^{b_j}$.

Let $0 < l \leq \frac{\pi}{4}$ and pick t and j in N such that $l_j \geq \frac{\pi}{4} 2^{-t} \geq l \geq \frac{\pi}{4} 2^{-t-1} \geq l_{j+1}$. Then

$$\frac{\nu_\beta(l)}{l} \geq \frac{\nu_\beta(\frac{\pi}{4} 2^{-t-1})}{\frac{\pi}{4} 2^{-t}} = \frac{\pi 2^{-t-3} g(2^{t+3})^\alpha}{\pi 2^{-t-2}} \geq \frac{1}{2} g(2^{b_j+3})^\alpha \geq \frac{1}{2} g(2^{b_j})^\alpha$$

$$\geq \frac{1}{2A_2^\alpha} g(2^{b_{j+1}})^\alpha \geq \frac{1}{2A_2^\alpha D^{2\alpha}} g\left(2^{b_{j+1}} 4\right)^\alpha = \frac{1}{2A_2^\alpha D^{2\alpha}} \frac{\nu_\beta(l_{j+1})}{l_{j+1}}.$$

Lemma A with ν_β defined as above and $a = \frac{1}{2A_2^\alpha D^{2\alpha}}$ now yields

$$\mathcal{H}_{\nu_\beta}(C) \geq \frac{a}{2} \liminf_{j\to\infty} N_j \nu_\beta(l_j) \geq \frac{a}{2} \liminf_{j\to\infty} \left(\frac{1}{4}\right)^j 2^{b_j} \nu_\beta\left(\frac{\pi}{4} 2^{-b_j}\right)$$

$$= \frac{a\pi}{8} \liminf_{j\to\infty} \left(\frac{1}{4}\right)^j g(2^{b_j+2})^\alpha \geq \frac{a\pi}{8} \liminf_{j\to\infty} \left(\frac{1}{4}\right)^j A_1^{j\alpha} g(2^{b_0})^\alpha.$$

By choosing $A_1^\alpha > 4$ we obtain $\mathcal{H}_{\nu_\beta}(C) = \infty$.

Then by Lemma 2.1 for $\lambda_\beta(t) = t^{m-2} \nu_\beta(t)$ and the remark on the behavior of the Hausdorff measure under the Lipschitz map, we have

$$\mathcal{H}_{\lambda_\beta}(E^+(u)) \geq \mathcal{H}_{\lambda_\beta}(F) > 0.$$

4. Positive harmonic functions

4.1. Proof of the Theorem 1.4

We will now consider extremal growth on subsets of radii of the unit ball in R^m for positive functions. Let v be a positive increasing continuous function on $[0, 1)$ and assume $\lambda(t) = t^{m-1}v(1 - t)$ is increasing. Let u be a positive harmonic function on B and let $F_v^+(u)$ be defined by (3). For positive $u \in \Psi_v^m$, clearly $E^+(u) \subset F_v^+(u)$. Theorem 1.4 is a generalization of Theorem 2 in [4], where the result is proved for $v(r) = \log(\frac{1}{1-r})$ and $m = 2$. Note that we no longer assume that $u \in \Psi_v^m$.

The proof of Theorem 1.4 is similar to the one in [4], but the proof of Lemma 4.1 is new.

Let

$$F_n = \left\{ \zeta \in S : \limsup_{r \to 1} \frac{u(r\zeta)}{v(r)} \geq \frac{2}{n} \right\}.$$

It suffices to prove that $\mathscr{H}_\lambda(F_n) < \infty$ for all n.

Clearly $u = P * \mu$ for some positive Borel measure μ on S. Let $h : S \to [0, \pi]$ be given by $h(\cos \phi, \zeta') = \phi$ and define a measure on $[0, \pi]$ by $\nu = h_* \mu$, which means that $\nu(A) = \mu(h^{-1}(A))$ for any measurable set $A \subset [0, \pi]$. The formula

$$\int_0^\pi f(\psi) \, d\nu(\psi) = \int_0^\pi f'(\phi)\nu((\phi, \pi]) \, d\phi$$

is valid for $f \in C^1[0, \pi]$ that is non-decreasing and fulfills $f(0) = 0$ and $f(t) > 0$ for $t > 0$ (see for example [17, p. 84]). By using it with $f(\phi) = \tilde{P}_{m,r}(0) - \tilde{P}_{m,r}(\phi)$, we get the following integration by parts on S

$$\int_S P(rx, \zeta) \, d\mu(\zeta) = \int_S \tilde{P}_{m,r}(h(\zeta)) \, d\mu(\zeta) = \int_0^\pi \tilde{P}_{m,r}(\phi) \, d\nu$$

$$= -\int_0^\pi (\tilde{P}_{m,r}(0) - \tilde{P}_{m,r}(\phi)) \, d\nu + \tilde{P}_{m,r}(0)\nu([0, \pi])$$

$$= -\int_0^\pi Q_m(r, \phi)\nu((\phi, \pi]) \, d\phi + \tilde{P}_{m,r}(0)\mu(S)$$

$$= \tilde{P}_{m,r}(\pi)\mu(S) + \int_0^\pi Q_m(r, \pi)\nu((0, \phi]) \, d\phi,$$

thus

(17) $\quad \displaystyle\int_S P(rx, \zeta) \, d\mu(\zeta) = \tilde{P}_{m,r}(\pi)\mu(S) + \int_0^\pi Q_m(r, \phi)\mu(\bar{B}(x, \phi)) \, d\phi.$

We need the following lemma:

LEMMA 4.1. *For each n there exists $k = k(m, n) > 0$ such that for any $x \in F_n$ there is a decreasing sequence $\{\Delta_j\}$, $\Delta_j \to 0$ as $j \to \infty$, which satisfies*

$$\mu(B(x, \Delta_j)) \geq k\sigma(B(x, \Delta_j))v(1 - \Delta_j). \tag{18}$$

Suppose this lemma is already proved. Let K be a compact subset of F_n and let $B_j = B(x_j, a_j)$, where $x_j \in K$ and $a_j < \epsilon$. For each $\epsilon > 0$ we can cover K with a finite collection of such balls $\{B_j : j \in J\}$ which satisfy $\mu(B_j) \geq k\sigma(B_j)v(1 - a_j)$. By the Vitali covering lemma (see for example [8, p. 2]) there exists a subcollection $J' \subseteq J$ where $\{B_j : j \in J'\}$ are disjoint and $\bigcup_{j \in J} B_j \subseteq \bigcup_{j \in J'} 5B_j$. Using (4) and Lemma 4.1 we obtain

$$\sum_{j \in J'} \lambda(5a_j) \leq C \sum_{j \in J'} \sigma(B(x_j, 5a_j))v(1 - a_j) \leq C5^{m-1} \sum_{j \in J'} \sigma(B_j)v(1 - a_j)$$

$$\leq C\frac{5^{m-1}}{k} \sum_j \mu(B_j) \leq C\frac{5^{m-1}}{k}\mu(S),$$

which yields $\mathscr{H}_\lambda(K) \leq C\frac{5^{m-1}}{k}\mu(S)$. Thus $\mathscr{H}_\lambda(F_n) \leq C\frac{5^{m-1}}{k}\mu(S) < \infty$.

PROOF OF LEMMA 4.1. Assume that $x = (1, 0, \ldots, 0)$. Then by (17) and (6),

$$u(rx) = \int_S P(rx, \zeta)\, d\mu(\zeta) \leq \mu(S) + \int_0^\pi \mu(B(x, \phi))Q_m(r, \phi)\, d\phi$$

$$\leq \mu(S) + \int_0^d \mu(B(x, \phi))Q_m(r, \phi)\, d\phi + \mu(S)\int_d^\pi Q_m(r, \phi)\, d\phi$$

$$\leq \mu(S) + \int_0^d \mu(B(x, \phi))Q_m(r, \phi)\, d\phi + \mu(S)C_6 d^{-m}$$

Let $k < [(C_2C_4 + C_9)3n]^{-1}$, where the constants are from (4), (5) and (7). For $x \in F_n$ there exists a sequence $\{r_j\}_1^\infty$ such that $r_j \nearrow 1$ and $u(r_j) > \frac{1}{n}v(r_j)$. We may assume that $r_j > \frac{1}{2}$. Now choose $d_j = d_j(r_j, v) \geq 1 - r_j$ such that

$$Cd_j^{-m} < \frac{1}{3n}v(r_j)$$

and $d_j \to 0$ when $j \to \infty$. For $j > j_0$, $\mu(S) < \frac{1}{3n}v(r_j)$. Then

$$\int_0^{d_j} \mu(B(x, \phi))Q_m(r_j, \phi)\, d\phi > \frac{1}{3n}v(r_j).$$

We claim that this implies that for any j there exists $\Delta_j \in (0, d_j)$ such that

$$\mu(B(x, \Delta_j)) \geq k\sigma(B(x, \Delta_j))v(1 - \Delta_j),$$

and the lemma follows. If not, there exists j such that

$$\mu(B(x, \phi)) < k\sigma(B(x, \phi))v(1 - \phi)$$

for any $\phi \in (0, d_j)$. Using (4) and the fact that $t^{m-1}v(1 - t)$ is increasing, and then applying (5) and (7), we obtain

$$\int_0^{d_j} \mu(B(x, \phi))Q_m(r_j, \phi) \, d\phi$$

$$< k \int_0^{d_j} \sigma(B(x, \phi))v(1 - \phi)Q_m(r_j, \phi) \, d\phi$$

$$\leq kC_2(1 - r_j)^{m-1}v(r_j) \int_0^{1-r_j} Q_m(r_j, \phi) \, d\phi$$

$$+ kv(r_j) \int_{1-r_j}^{\pi} \sigma(B(x, \phi))Q_m(r_j, \phi) \, d\phi$$

$$\leq k\left(C_2(1 - r_j)^{m-1}v(r_j)C_4\frac{1}{(1 - r_j)^{m-1}} + C_9v(r_j)\right) < \frac{1}{3n}v(r_j),$$

and we have a contradiction.

COROLLARY 4.2. *If $v(r) = \left(\frac{1}{1-r}\right)^{\gamma}$ for $0 \leq \gamma < m - 1$, then for any positive harmonic function u, the Hausdorff dimension of $F_v^+(u)$ is less than or equal to $m - 1 - \gamma$.*

We will show in section 4.3 that this estimate is sharp, i.e. there exists u such that

$$\dim F_v^+(u) = m - 1 - \gamma.$$

4.2. *Measures that correspond to positive functions in Ψ_v^m*

Let

$$\Theta_v^m = \{u : \mathsf{B} \to \mathsf{R}, \Delta u = 0, 0 < u(x) \leq Kv(|x|)\}.$$

We want to characterize all functions in Θ_v^m by their corresponding measure on S.

PROPOSITION 4.3. *Suppose v satisfies (2) and let $u(x) = \int_S P(x, \zeta) \, d\mu(\zeta)$ where μ is a positive Borel measure on S. Then $u \in \Theta_v^m$ if and only if*

(19) $$\mu(B(x, \phi)) \leq C\sigma(B(x, \phi))g\left(\frac{\pi}{\phi}\right)$$

for each ball $B(x, \phi) \subset S$.

PROOF. Assume $u \in \Theta_v^m$ and let $x \in S$. Then

$$v\left(1 - \frac{\phi}{\pi}\right) \geq \frac{1}{K} u\left(\left(1 - \frac{\phi}{\pi}\right)x\right) = \frac{1}{K\gamma_{m-1}} \int_S \frac{1 - \left(1 - \frac{\phi}{\pi}\right)^2}{\left|x\left(1 - \frac{\phi}{\pi}\right) - \zeta\right|^m} \, d\mu(\zeta)$$

$$\geq \frac{1}{K\pi\gamma_{m-1}} \int_{B(x,\phi)} \frac{2\phi - \frac{\phi^2}{\pi}}{(2\phi)^m} \, d\mu(\zeta) \geq \frac{1}{K\pi\gamma_{m-1}} \frac{\mu(B(x,\phi))}{2^m \phi^{m-1}}$$

$$\geq C\mu(B(x,\phi)) \frac{1}{\sigma(B(x,\phi))},$$

thus $\mu(B(x,\phi)) \leq C\sigma(B(x,\phi))g\left(\frac{\pi}{\phi}\right)$.

Conversely, suppose that (19) is fulfilled. Without loss of generality we may assume that $x = (1, 0, \ldots, 0)$. Then by (17),

$$u(rx) = \int_S P(rx, \zeta) \, d\mu(\zeta) \leq \mu(S) + \int_0^\pi \mu(B(x,\phi)) Q_m(r, \phi) \, d\phi$$

$$\leq \mu(S) + \int_0^{1-r} \mu(B(x,\phi)) Q_m(r, \phi) \, d\phi$$

$$+ \int_{1-r}^\pi \mu(B(x,\phi)) Q_m(r, \phi) \, d\phi$$

$$\leq \mu(S) + \mu(B(x, 1-r)) \int_0^{1-r} Q_m(r, \phi) \, d\phi$$

$$+ C \int_{1-r}^\pi \sigma(B(x,\phi))g\left(\frac{\pi}{\phi}\right) Q_m(r, \phi) \, d\phi$$

$$\leq \mu(S) + C\sigma(B(x, 1-r))g\left(\frac{\pi}{1-r}\right) \int_0^{1-r} Q_m(r, \phi) \, d\phi$$

$$+ Cg\left(\frac{\pi}{1-r}\right) \int_{1-r}^\pi \sigma(B(x,\phi)) Q_m(r, \phi) \, d\phi.$$

Furthermore, by (5) and (7), $u(rx) \leq \tilde{C}g\left(\frac{\pi}{1-r}\right) \leq \tilde{C}_1 v(r)$.

4.3. Proof of Theorem 1.3

First, note that v satisfies (2), in fact

$$v\left(1 - \frac{t}{2}\right) = \frac{2^{m-1}\lambda\left(\frac{t}{2}\right)}{t^{m-1}} < \frac{2^{m-1}\lambda(t)}{t^{m-1}} = 2^{m-1}v(1-t).$$

Consider the set $A = [0, \pi] \times \ldots \times [0, \pi] \times [0, 2\pi]$ and the hyperspherical coordinates on S, i.e., we consider the function $f : A \to S$ defined by

$$f(\phi_1, \ldots, \phi_{m-1}) = (\cos \phi_1, \sin \phi_1 \cos \phi_2, \ldots, \sin \phi_1 \ldots \sin \phi_{m-1}).$$

This function is bilipschitz on $[1, 2]^{m-1}$. We will use a Cantor-type construction to get a set $C \subset [1, 2]^{m-1} \subset A$. We first construct a set in $[1, 2]$.

Let $F_0 = [1, 2]$. Define d_k as

$$d_k = \min\{n \in \mathsf{N} : g(2^n) \geq 2^{(m-1)k}\}, \qquad k = 2, 3, \ldots$$

Then $d_{k+1} > d_k$ for all k. By (13) we also have

(20) $$\frac{g(2^{d_k})}{g(2^{d_{k-1}})} \leq \frac{Dg(2^{d_k-1})}{g(2^{d_{k-1}})} \leq \frac{D2^{(m-1)k}}{2^{(m-1)(k-1)}} = D2^{m-1} = \delta_m.$$

We construct by induction sets $F_k \subset F_{k-1}$ such that F_k consists of $n_k = 2^{d_k-k}$ closed intervals of length 2^{-d_k} each. To obtain F_k we divide each of the intervals of F_{k-1} into $2^{d_k-d_{k-1}}$ equal subintervals and choose each second of them for F_k. Now let $C_k = F_k \times \cdots \times F_k \subset [1, 2]^{m-1}$. The number of squares in C_k is $N_k = 2^{(d_k-k)(m-1)}$. Let also $C = \cap C_k$.

Let ν_k be the measures defined by $d\nu_k = 2^{(m-1)k}\chi(C_k)dy$ on $[1, 2]^{m-1}$, where $\chi(C_k)$ is the characteristic function of C_k. We also define the measures $\mu_k = f_*\nu_k$ on S. Denote $G_k = f(C_k)$ and $G = f(C)$, clearly $G = \bigcap G_k$.

LEMMA 4.4. *The sequence $\{\mu_k\}$ converges $*$-weakly to a measure μ and $u = P * \mu \in \Theta_v^m$.*

PROOF. The $*$-weak convergence of $\{\mu_k\}$ follows from the $*$-weak convergence of $\{\nu_k\}$, which we will prove now. Note that $\nu_k(C_0) = 1$ for each k. Let $\{J_i\}_{i=1}^{N_k}$ be the squares of C_k. For each square J_i the limit $\nu_k(J_i)$ as $k \to \infty$ exists because all values $\nu_k(J_i)$ are the same when $k > s$. For squares in $S \setminus C_k$ the limit will be 0. Now each continuous function on $[1, 2]^{m-1}$ can be uniformly approximated by linear combinations of characteristic functions of small squares. Thus for each continuous function f on $[1, 2]^{m-1}$ there exists

$$\lim_{k \to \infty} \int_{[1,2]^{m-1}} f \, d\nu_k$$

and ν_k converge weakly to some positive measure ν.

By Proposition 4.3 it suffices to check that

$$\mu(B(x, r)) \leq C\sigma(B(x, r))g\left(\frac{\pi}{r}\right)$$

for each ball $B(x, r) \subset S$, in order to prove that $u = P * \mu \in \Theta_v^m$. This is true if a similar estimate is true for v.

Let $y \in [1, 2]^{m-1}$ and let $B_e(y, r)$ be a Euclidean ball. Choose s such that $2^{-d_s} < r \le 2^{-d_{s-1}}$. Now take a square $Q \supset B_e(y, r)$ that is a union of dyadic cubes with side lengths 2^{-d_s}, and let the side length of Q be $2^{-d_s}l$ for some $l \in \mathsf{N}$ such that $2^{-d_s}l < 4r$. Then $|Q| < A|B_e(y, r)|$ where $A = A(m)$. By using (20), we obtain

$$v_k(B_e(y, r)) \le v_k(Q) = v_s(Q) \le 2^{(m-1)s}|Q| < A|B_e(y, r)|g(2^{d_s})$$

$$\le A\delta_m|B_e(y, r)|g(2^{d_{s-1}}) \le A\delta_m|B_e(y, r)|g\left(\frac{1}{r}\right),$$

which is the desired inequality.

To finish the proof of Theorem 1.3 we will show that $G \subset E^+(v)$ and $\mathscr{H}_\lambda(G) > 0$. We have

$$u(rx) = \int_S P(rx, \zeta) \, d\mu(\zeta) \ge \int_0^\pi \mu(B(x, \phi)) Q_m(r, \phi) \, d\phi.$$

Let $x \in \bigcap G_k = G$ and $r \in [0, 1)$. Choose k_0 such that $2^{(m-1)(k_0-1)} \le g\left(\frac{1}{1-r}\right) < 2^{(m-1)k_0}$. Then it is not difficult to see that $\mu(B(x, \phi)) \ge c2^{(m-1)k_0}\phi^{m-1}$ for $\phi < 1 - r$ and

$$u(rx) \ge \int_0^{1-r} \mu(B(x, \phi)) Q_m(rx, \phi) d\phi$$

$$\ge c2^{(m-1)k_0} \int_0^{1-r} \phi^{m-1} Q_m(rx, \phi) d\phi \ge c_1 2^{(m-1)k_0} \ge c_1 g\left(\frac{1}{1-r}\right),$$

when $r > r_0$. Thus $G = \bigcap G_k \subset E^+(u)$.

Finally we use Lemma B to estimate $\mathscr{H}_\lambda(C)$. We have

$$\liminf_{k \to \infty} N_k\lambda(l_k) = \liminf_{k \to \infty} 2^{d_k-k} 2^{-d_k} g\left(2^{d_k}\right) \ge \liminf_{k \to \infty} \pi 2^{-k+1} 2^k = c > 0.$$

Then $\mathscr{H}_\lambda(C) > 0$ and $\mathscr{H}_\lambda(G) \ge c_1 \mathscr{H}_\lambda(C) > 0$ since $C = f^{-1}(G)$ and f^{-1} is Lipschitz. We use also that

$$\lambda\left(\frac{t}{2}\right) = \left(\frac{t}{2}\right)^{m-1} v\left(1 - \frac{t}{2}\right) \ge t^{m-1} v(1 - t) 2^{-m+1} = 2^{-m+1}\lambda(t),$$

see Section 2.3.

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DEPARTMENT OF MATHEMATICAL SCIENCES
NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
NO-7491 TRONDHEIM
NORWAY
E-mail: kjerstei@math.ntnu.no
         eugenia@math.ntnu.no