MATH. SCAND. 110 (2012), 297–319

THE PIMSNER-VOICULESCU SEQUENCE FOR COACTIONS OF COMPACT LIE GROUPS

MAGNUS GOFFENG

Abstract

The Pimsner-Voiculescu sequence is generalized to a Pimsner-Voiculescu tower describing the KK-category equivariant with respect to coactions of a compact Lie group satisfying the Hodgkin condition. A dual Pimsner-Voiculescu tower is used to show that coactions of a compact Hodgkin-Lie group satisfy the Baum-Connes property.

Introduction

When G is a second countable, locally compact group and A is a separable C^*-algebra with a continuous G-action, the Baum-Connes conjecture states that the K-theory of the reduced crossed product $A \rtimes_r G$ can be calculated by means of geometric and representation theoretical properties of G and A, see more in [4]. To be more precise, the Baum-Connes conjecture states that the assembly mapping $\mu_A : K_*^G(\mathscr{E}G; A) \to K_*(A \rtimes_r G)$ is an isomorphism. The space $\mathscr{E}G$ is the universal proper G-space and $K_*^G(\mathscr{E}G; A)$ is the proper equivariant K-homology with coefficients in A. There are known counterexamples when μ_A is not an isomorphism, so it is more natural to speak of groups having the Baum-Connes property. In [10], the equivariant K-homology with coefficients in A was proved to be the left derived functor of $F(A) = K_*(A \rtimes_r G)$ and the assembly mapping being the natural transformation from $\mathsf{L}F$ to F. The approach to the Baum-Connes property using triangulated categories can be generalized to discrete quantum groups, see [9], which indicates that geometric techniques such as universal proper G-spaces can be generalized to discrete quantum groups.

The generalization of the Baum-Connes property to quantum groups has been studied in for instance [11] and [17]. The case studied in [11] is that of quantum group actions of the dual of a compact Lie group which correspond to coactions of the Lie group. In [11] duals of compact Lie groups were shown to satisfy the strong Baum-Connes property, i.e., the embedding of

Received 29 October 2010, in final form 4 January 2011.

the triangulated category generated by proper coactions, the C^*-algebras that are Baaj-Skandalis dual to trivial G-actions, into the KK-category equivariant with respect to coactions is essentially surjective. In this paper we construct an analogue of the Pimsner-Voiculescu sequence for coactions of a compact Hodgkin-Lie group G that describes how the KK-category equivariant with respect to coactions of G is built up from the C^*-algebras with coactions of G which are proper in the sense of [11].

The starting point is to express the Pimsner-Voiculescu sequence for Z-actions in terms of a property of the representation ring of a rank one torus. Using the Universal Coefficient Theorem, the Pimsner-Voiculescu sequence can be constructed from a Koszul complex

$$0 \longrightarrow R(T) \xrightarrow{\alpha} R(T) \longrightarrow 0,$$

where α is defined as multiplication by $1-t$ under the isomorphism $R(T) \cong \mathsf{Z}[t, t^{-1}]$. When A has a coaction of T, i.e., a Z-action, the tensor product over $R(T)$ between this Koszul complex and $K_*^T(A \rtimes_r \mathsf{Z})$ gives the Pimsner-Voiculescu sequence. In the generalization to higher rank, when T is a torus of rank n we consider the Koszul complex

$$\begin{aligned} 0 \longrightarrow \textstyle\bigwedge^n R(T)^n \longrightarrow \textstyle\bigwedge^{n-1} R(T)^n \longrightarrow \cdots \\ \longrightarrow \textstyle\bigwedge^2 R(T)^n \longrightarrow R(T)^n \longrightarrow R(T) \longrightarrow 0. \end{aligned}$$

The boundary mappings in this complex are defined from interior multiplication with the element $\sum(1-t_i)e_i^* \in \mathrm{Hom}_{R(T)}(R(T)^n, R(T))$. If G is a compact Hodgkin-Lie group with maximal torus T, the representation ring $R(T)$ is a free $R(G)$-module by [15], so the generalization from a torus to compact Hodgkin-Lie groups goes in a straightforward fashion. Just as when the rank is 1, the Koszul complex above can be used to produce sequence of distinguished triangles which is the analogue of a Pimsner-Voiculescu sequence for the K-theory of crossed products by coactions of G.

We will give a geometric description of a sequence of distinguished triangles in the KK-category equivariant with respect to coactions of G that corresponds to the above Koszul complex under the Universal Coefficient Theorem. As for the Pimsner-Voiculescu sequence for Z we will obtain a projective resolution of the crossed product by a coaction in the sense of triangulated categories rather than exact sequences. Using suitable tensor products we produce in Theorem 3.4 a sequence of distinguished triangles in the KK-category equivariant with respect to coactions of G that we call the generalized Pimsner-Voiculescu tower for A:

$$
\begin{tikzcd}
\mathsf{C}^w \otimes A \arrow[rr] & & \Sigma^n D_{n-1}(A) \arrow[rr] \arrow[dl] & & \Sigma^n D_{n-2}(A) \arrow[r] \arrow[dl] & \cdots \arrow[dl, "\circ" marking] \\
& \Sigma \mathsf{C}^{wn} \otimes A \arrow[ul, "\circ" marking] & & \Sigma^2 \mathsf{C}^{wk_{n-1}} \otimes A \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] & \cdots \arrow[l, "\circ" marking] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \Sigma^n D_2(A) \arrow[rr] \arrow[dl] & & \Sigma^n D_1(A) \arrow[rr] \arrow[dl] & & t(A \rtimes_r \hat{G}) \arrow[dl] \\
\cdots & & \Sigma^{n-1} \mathsf{C}^{wk_2} \otimes A \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] & & \Sigma^n \mathsf{C}^w \otimes A \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] &
\end{tikzcd}
$$

Here $t(A \rtimes_r \hat{G})$ denotes the C^*-algebra $A \rtimes_r \hat{G}$ equipped with the trivial $\hat{G}$-action and the terms $D_i(A)$ can be explicitly described as braided tensor products. Taking K-theory of the lower row will give a complex similar to the Koszul complex that in a sense forms a projective resolution of the K-theory of $A \rtimes \hat{G}$. The dual Pimsner-Voiculescu gives a more precise description of the results of [11] by a sequence of distinguished triangles in KK^G that describes the crossed product $A \rtimes_r \hat{G}$ in terms of G–C^*-algebras with trivial G-action, thus giving a direct route to the strong Baum-Connes property of $\hat{G}$.

The paper is organized as follows; the first section consists of a review of KK-theory of actions and coactions. In particular we gather some known results about the braided tensor product and the Drinfeld double which plays a mayor role in constructing the dual Pimsner-Voiculescu tower. The main references of this section are [1], [2], [3], [7], [10], [12] and [16]. In the second section a geometric construction of the Pimsner-Voiculescu sequence for Z-actions is presented and generalized to higher rank via a Koszul complex. In the third section the restriction functor for coactions is used to generalize the Pimsner-Voiculescu sequence to coactions of compact Hodgkin-Lie groups G. As an example of this we calculate the K-theory of some compact homogeneous spaces. By similar methods, a dual Pimsner-Voivulescu tower is constructed in KK^G, following the ideas of [10]. At the end of the paper we discuss some possible generalizations to duals of Woronowicz deformations.

ACKNOWLEDGMENTS: The author would like to thank Ryszard Nest for posing the question on how to explicitly construct the crossed product of a $\hat{G}$–C^*-algebra from trivial actions and for much inspiration in the writing process. The author also wishes to thank the referee for several valuable suggestions.

1. Actions and coactions of compact groups

The standard approach to equivariant K-theory is to introduce equivariant KK-theory. If A and B are two separable C^*-algebras with a continuous action of a locally compact group G, the equivariant KK-group $KK^G(A, B)$ is defined as the set of homotopy classes of G-equivariant $A-B$-Kasparov modules which forms an abelian group under direct sum. The KK-groups can be equipped with a product such that if C is a third separable C^*-algebra with a continuous G-action there is an additive pairing called the Kasparov product

$$KK^G(A, B) \times KK^G(B, C) \longrightarrow KK^G(A, C).$$

Following the standard construction, we let KK^G denote the additive category of all separable C^*-algebras with a continuous G-action and a morphism in KK^G from A to B is an element of $KK^G(A, B)$. The composition of two KK^G-morphisms is defined to be their Kasparov product. The group $KK^G(\mathsf{C}, A)$ coincides with the equivariant K-theory of A. In particular, if G is compact $KK^G(\mathsf{C}, \mathsf{C}) = R(G)$, the representation ring of G. The action of $R(G)$ on equivariant K-theory generalizes to an $R(G)$-module structure on the bivariant groups $KK^G(A, B)$.

The category KK^G can be equipped with a triangulated structure with a mapping cone coming from the mapping cone construction of a $*$-homomorphism. The triangulated structure on KK^G is universal in the sense that any homotopy invariant, stable, split-exact functor on the category of C^*-algebras with a continuous G-action defines a homological functor on KK^G. The construction of the triangulated structure and its universality are thoroughly explained in [10]. Let us just recall the basics of the construction of the triangulated structure on KK^G. The suspension ΣA of a G–C^*-algebra is defined by $C_0(\mathsf{R}) \otimes A$. By Bott periodicity $\Sigma^2 \cong \mathrm{id}$. A distinguished triangle in KK^G is a triangle isomorphic to one of the form

$$\begin{tikzcd} C(f) \arrow[rr] & & A \arrow[dl, "f"] \\ & B, \arrow[ul] & \end{tikzcd}$$

where $C(f)$ is the mapping cone of the equivariant $*$-homomorphism $f : A \to B$. In particular, if $f : A \to B$ is a surjection and admits an equivariant completely positive splitting the natural mapping $\ker(f) \to C(f)$ defines an equivariant KK-isomorphism, so under suitable assumptions a distinguished triangle is isomorphic to a short exact sequence.

How to construct KK-theory of coactions of groups is easiest seen in the simpler case when G is an abelian group. If A is a C^*-algebra equipped with an action α of the abelian group G, the crossed product $A \rtimes_r G$ carries a natural action of the Pontryagin dual $\hat{G}$. This action is called the dual action of $\hat{G}$. Since abelian groups are exact, the crossed product by an abelian group defines a triangulated functor $KK^G \to KK^{\hat{G}}$. The crossed product by the dual action is described by Takesaki-Takai duality which states that there is an equivariant isomorphism

$$A \rtimes_r G \rtimes_r \hat{G} \cong A \otimes \mathcal{K}(L^2(G)),$$

where $A \rtimes_r G \rtimes_r \hat{G}$ is equipped with the dual action of G and the G-action on $A \otimes \mathcal{K}(L^2(G))$ is defined as $\alpha \otimes \mathrm{Ad}$. Takesaki-Takai duality implies that the crossed product defines a triangulated equivalence $KK^G \to KK^{\hat{G}}$.

An action α of a group G on A defines a $*$-homomorphism $\Delta_\alpha : A \to \mathcal{M}(A \otimes C_0(G))$ by letting $\Delta_\alpha(a)$ be the function $g \mapsto \alpha_g(a)$. When G is abelian there is a natural isomorphism $C_0(\hat{G}) \cong C_r^*(G)$ and a $\hat{G}$-action corresponds to a non-degenerate $*$-homomorphism $\Delta_A : A \to \mathcal{M}(A \otimes_{\min} C_r^*(G))$ satisfying certain conditions. The first instance of a coaction of a group G is on $C_r^*(G)$. Using the universal property of $C_r^*(G)$, one can construct a non-degenerate mapping $\Delta : C_r^*(G) \to \mathcal{M}(C_r^*(G) \otimes_{\min} C_r^*(G))$ called the comultiplication and is induced from the diagonal homomorphism $G \to G \times G$. Clearly, the mapping Δ satisfies:

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$

so we say that Δ is coassociative. Since $\Delta_{21} = \Delta$ the comultiplication Δ is cocommutative, so if we interpret $C_r^*(G)$ as the functions on a reduced locally compact quantum group $\hat{G}$ then $\hat{G}$ can be thought of as abelian, see more in [7]. With the abelian setting as motivation, we say that a separable C^*-algebra A has a coaction of the locally compact second countable group G if there is non-degenerate $*$-homomorphism $\Delta_A : A \to \mathcal{M}(A \otimes_{\min} C_r^*(G))$ satisfying the two conditions that $\Delta_A(A) \cdot 1_A \otimes_{\min} C_r^*(G)$ is a dense subspace of $A \otimes_{\min} C_r^*(G)$ and that Δ_A is coassociative in the sense that

$$(\Delta_A \otimes \mathrm{id}_{C_r^*(G)})\Delta_A = (\mathrm{id}_A \otimes \Delta)\Delta_A. \tag{1}$$

A separable C^*-algebra equipped with a coaction of G will be called a $\hat{G}$–C^*-algebra. Sometimes we will abuse the notation and call a coaction of G a $\hat{G}$-action. An example of a coaction is the dual coaction on C^*-algebras of the form $A = B \rtimes_r G$, for some G–C^*-algebra B. When G is discrete we can decompose $B \rtimes_r G$ by means of the dense subspace $\bigoplus_{g \in G} B\lambda_g$ and the dual coaction is defined by $\Delta_A(b\lambda_g) := b\lambda_g \otimes \lambda_g$. In the general setting, the construction of the dual coaction goes analogously and we refer the reader to [1].

Much of the theory for group actions also hold for group coactions, the crossed product will as for abelian groups be a stepping stone back and forth between actions and coactions. In [1], the KK-theory equivariant with respect to a bi-C^*-algebras and the corresponding Kasparov product was constructed. In [12] it was proved that the KK-theory equivariant with respect to a locally compact quantum group has a triangulated structure defined in the same fashion as for a group.

Let us explain the setting of [1] more explicitly in the case of coactions of a group. An $A-B$-Hilbert bimodule $\mathscr{E}$ is called $\hat{G}$-equivariant if there is a coaction $\delta_{\mathscr{E}} : \mathscr{E} \to \mathscr{L}_{B\otimes_{\min} C_r^*(G)}(B \otimes_{\min} C_r^*(G), \mathscr{E} \otimes C_r^*(G))$ satisfying a coassociativity condition similar to (1) and $\delta_{\mathscr{E}}$ should commute with the A-action and B-action in the obvious ways. By Proposition 2.4 of [1], the coaction $\delta_{\mathscr{E}}$ is uniquely determined by a unitary $V_{\mathscr{E}} \in \mathscr{L}(\mathscr{E} \otimes_{\Delta_B} (B \otimes_{\min} C_r^*(G)), \mathscr{E} \otimes C_r^*(G))$ via the equation $\delta_{\mathscr{E}}(x)y = V_{\mathscr{E}}(x \otimes_{\Delta_B} y)$ for $x \in \mathscr{E}$ and $y \in B \otimes_{\min} C_r^*(G)$. A $\hat{G}$-equivariant $A-B$-Kasparov module is an $A-B$-Kasparov module $(\mathscr{E}, F)$ such that $\mathscr{E}$ is a $\hat{G}$-equivariant $A-B$-Hilbert module and the operator F commutes with the unitary $V_{\mathscr{E}}$ up to a compact operator. The group $KK^{\hat{G}}(A, B)$ is defined as the homotopy classes of $\hat{G}$-equivariant $A-B$-Kasparov modules. The additive category $KK^{\hat{G}}$ is defined by taking the objects to be separable $\hat{G}$–C^*-algebras and the group of morphisms from A to B is $KK^{\hat{G}}(A, B)$. The composition in $KK^{\hat{G}}$ is Kasparov product of $\hat{G}$-equivariant Kasparov modules.

To a closed subgroup H of G, the restriction of a G-action to H defines a restriction functor $\mathrm{Res}^G_H : KK^G \to KK^H$ and its right adjoint is the induction functor $\mathrm{Ind}^G_H : KK^H \to KK^G$. However the restriction goes in the other direction for coactions. When H is a closed subgroup of G, there is a non-degenerate embedding $C^*(H) \subseteq \mathscr{M}(C^*(G))$ so a coaction of H can be restricted to a coaction of G. This construction defines a triangulated functor $\mathrm{Res}^{\hat{H}}_{\hat{G}} : KK^{\hat{H}} \to KK^{\hat{G}}$.

The crossed product $B \mapsto B \rtimes_r G$ sends a G–C^*-algebra to a $\hat{G}$–C^*-algebra and if G is exact the crossed product induces a triangulated functor $KK^G \to KK^{\hat{G}}$. In order to construct a duality similar to Takesaki-Takai duality one introduces the crossed product by a coaction. If A is a $\hat{G}$–C^*-algebra we define

$$A \rtimes_r \hat{G} := [\Delta_A(A) \cdot 1_A \otimes C_0(G)] \subseteq \mathscr{M}(A \otimes \mathscr{K}(L^2(G))).$$

It follows from Lemma 7.2 of [2] that $A \rtimes_r \hat{G}$ forms a C^*-algebra. For a thorough introduction to crossed products by coactions see [13]. The C^*-algebra $A \rtimes_r \hat{G}$ carries a continuous G-action defined in the dense subspace

$\Delta_A(A) \cdot 1_A \otimes C_0(G)$ by

$$g.(\Delta_A(a) \cdot 1_A \otimes f) := \Delta_A(a) \cdot 1_A \otimes g.f.$$

Similarly to the abelian setting, Takesaki-Takai duality holds so there are equivariant isomorphisms $B \rtimes_r G \rtimes_r \hat{G} \cong B \otimes \mathscr{K}(L^2(G))$ and $A \rtimes_r \hat{G} \rtimes_r G \cong A \otimes \mathscr{K}(L^2(G))$ which ensures that the crossed product defines an equivalence of triangulated categories known as Baaj-Skandalis duality.

The tensor product on the category of G–C^*-algebras is well defined. If A and B have actions α respectively β of G the tensor product $A \otimes_{\min} B$ can be equipped with the action $\alpha \otimes \beta : G \to \mathrm{Aut}(A \otimes_{\min} B)$. However, for a non-abelian group G the construction of a tensor product of $\hat{G}$–C^*-algebras can not be done by just taking tensor products of the C^*-algebras. The tensor product relevant for $\hat{G}$–C^*-algebras is the braided tensor product over $\hat{G}$ which requires one further structure. Suppose that A is a $\hat{G}$-algebra with a continuous G-action α. If the action α satisfies that

$$\Delta_A \circ \alpha_g = (\alpha_g \otimes \mathrm{Ad}(g))\Delta_A \tag{2}$$

we say that A is a Yetter-Drinfeld algebra. An example of a Yetter-Drinfeld algebra is $C_r^*(G)$ with G-action defined by the adjoint action $G \to \mathrm{Aut}(G)$. It is much easier to construct a Yetter-Drinfeld algebra from a G–C^*-algebra, if A is a G–C^*-algebra we can in a functorial way define a coaction of G on A by setting $\Delta_A(a) := a \otimes 1$. When A is a Yetter-Drinfeld algebra, the C^*-algebra $A \rtimes_r \hat{G}$ is also a Yetter-Drinfeld algebra since the morphism Δ_A is covariant with respect to the G-action and Δ_A extends to a coaction of G on $A \rtimes_r \hat{G}$, see more in [12]. This construction is functorial and the crossed product can be seen as a functor on the category of Yetter-Drinfeld algebras.

When A is a Yetter-Drinfeld algebra and B is a $\hat{G}$–C^*-algebra we define the mappings

$$\iota_A : A \longrightarrow \mathscr{M}(A \otimes_{\min} B \otimes \mathscr{K}(L^2(G))), \qquad \iota(a) := \Delta_\alpha(a)_{13}$$
$$\iota_B : B \longrightarrow \mathscr{M}(A \otimes_{\min} B \otimes \mathscr{K}(L^2(G))), \qquad \iota(b) := \Delta_B(b)_{23}.$$

Following [12], the braided tensor product $A \boxtimes_{\hat{G}} B$ is defined as the closed linear span of $\iota_A(A) \cdot \iota_B(B)$. By Proposition 8.3 of [16], $A \boxtimes_{\hat{G}} B$ forms a $*$-subalgebra of $\mathscr{M}(A \otimes_{\min} B \otimes \mathscr{K}(L^2(G)))$ so the braided tensor product is a C^*-algebra. The coaction of G on $A \boxtimes_{\hat{G}} B$ is defined by

$$\Delta_A \boxtimes_{\hat{G}} \Delta_B(\iota_A(a) \cdot \iota_B(b)) := (\iota_A \otimes \mathrm{id})(\Delta_A(a)) \cdot (\iota_B \otimes \mathrm{id})(\Delta_B(b)).$$

Observe that since $C_r^*(G)$ is cocommutative, the adjoint $\hat{G}$-action is trivial and a similar construction of a braided tensor product over G between G–

C^*-algebras with trivial $\hat{G}$-actions coincides with the usual tensor product. In general, the braided tensor product over G does not need to coincide with the usual tensor product. By Lemma 3.5 of [12] there is a G-equivariant isomorphism

$$(A \boxtimes_{\hat{G}} B) \rtimes_r \hat{G} \cong (A \rtimes_r \hat{G}) \boxtimes_{\hat{G}} B \tag{3}$$

where the G-coaction on the right hand side is the trivial one on B. More generally, this identity holds for any quantum group and in particular also for braided tensor products over G. We will prove this statement in special case of braided tensor products over a compact group G with $C(G)$ below in Lemma 3.3.

If we interpret the structure of a Yetter-Drinfeld algebra as two actions of the quantum groups G and $\hat{G}$ satisfying a certain cocycle relation, the cocycle defines a quantum group by means of a double crossed product such that Yetter-Drinfeld algebras are precisely the C^*-algebras with an action of this double crossed product. The right quantum group to look at is the Drinfeld double $D(G)$. Using the notations of quantum groups, the algebra of functions on $D(G)$ is $C_0(G, C_r^*(G)) = C_0(G) \otimes C_r^*(G)$ with the obvious action and coaction of G. The action and coaction define a comultiplication

$$\Delta_{D(G)} : C_0(D(G)) \longrightarrow \mathscr{M}(C_0(D(G)) \otimes C_0(D(G)))$$

by $\Delta_{D(G)} := \sigma_{23}\,\mathrm{Ad}(W_{23})(\Delta_{C_0(G)} \otimes \Delta_{C_r^*(G)})$ where $W \in \mathscr{B}(L^2(G) \otimes L^2(G))$ is the multiplicative unitary of G defined by $Wf(g_1, g_2) = f(g_1, g_1g_2)$. The comultiplication $\Delta_{D(G)}$ makes $D(G)$ into a quantum group by Theorem 5.3 of [3]. A Yetter-Drinfeld algebra A with the action α and coaction Δ_A correspond to a $D(G) - C^*$-algebra by defining the $D(G)$-coaction

$$\Delta_A^{D(G)} := (\Delta_\alpha \otimes \mathrm{id})\Delta_A : A \longrightarrow \mathscr{M}(A \otimes_{\min} C_0(D(G))),$$

see more in Proposition 3.2 of [12]. Therefore we can consider the braided tensor product as a tensor product between $D(G) - C^*$-algebras and $\hat{G}$–C^*-algebras. The braided tensor product induces a biadditive functor

$$\boxtimes_{\hat{G}} : KK^{D(G)} \times KK^{\hat{G}} \longrightarrow KK^{\hat{G}}.$$

Much of the theory of coactions can be done without introducing any quantum groups, but in order to construct the Pimsner-Voiculescu sequence for coactions of compact Hodgkin-Lie groups we will need the braided tensor product as a biadditive functor between KK-categories.

2. The Pimsner-Voiculescu sequence from the viewpoint of representation rings

In this section we will study the Pimsner-Voiculescu sequence for Z and generalize to a Pimsner-Voiculescu tower for Z^n. We will use representation theory to calculate all the mappings explicitly. These calculations will in a surprisingly straightforward way give a natural route to a Pimsner-Voiculescu tower for coactions of compact Lie groups.

Consider the evaluation mapping $l : C_0(\mathsf{R}) \to C_0(\mathsf{Z})$. This mapping fits into a Z-equivariant short exact sequence

$$0 \longrightarrow \Sigma C_0(\mathsf{Z}) \longrightarrow C_0(\mathsf{R}) \xrightarrow{l} C_0(\mathsf{Z}) \longrightarrow 0. \tag{4}$$

The Z-equivariant Dirac operator $D\!\!\!/$ on R defines a Z-equivariant odd unbounded K-homology class, thus an element $[D\!\!\!/] \in KK^{\mathsf{Z}}(C_0(\mathsf{R}), \Sigma\mathsf{C})$. While R is the universal proper Z-space the element $[D\!\!\!/]$ is the Dirac element of Z and the strong Baum-Connes property of Z implies that $[D\!\!\!/]$ is a KK^{Z}-isomorphism. The exact sequence (4) induces a distinguished triangle in KK^{Z} which after using the isomorphism $C_0(\mathsf{R}) \cong \Sigma\mathsf{C}$ and rotation 4 steps to the left becomes

$$\begin{tikzcd} C_0(\mathsf{Z}) \arrow[rr] & & C_0(\mathsf{Z}) \arrow[dl] \\ & \mathsf{C}. \arrow[ul, "\circ" marking] & \end{tikzcd} \tag{5}$$

In a certain sense, the distinguished triangle (5) captures the entire behavior of the Pimsner-Voiculescu sequence. If A is a $\mathsf{Z} - C^*$-algebra we can apply Baaj-Skandalis duality to (5) and tensor with $A \rtimes_r \mathsf{Z}$. If we apply Baaj-Skandalis duality again, we obtain a distinguished triangle in KK^{Z}:

$$\begin{tikzcd} A \arrow[rr] & & A \arrow[dl] \\ & A \rtimes_r \mathsf{Z}, \arrow[ul, "\circ" marking] & \end{tikzcd}$$

where $A \rtimes_r \mathsf{Z}$ is given the trivial Z-action. Taking K-theory of this distinguished triangle gives back the classical Pimsner-Voiculescu sequence due to the following proposition:

PROPOSITION 2.1. *When T is a torus of rank 1 and the element $\kappa \in KK^T(\mathsf{C}, \mathsf{C})$ is defined using the isomorphisms $KK^T(\mathsf{C}, \mathsf{C}) \cong \mathrm{Hom}_{R(T)}(R(T), R(T))$ and $R(T) \cong \mathsf{Z}[t, t^{-1}]$ as*

$$\kappa f(t, t^{-1}) = (1 - t) f(t, t^{-1}),$$

the KK-morphism κ is Baaj-Skandalis dual to the KK-morphism $C_0(\mathsf{Z}) \to C_0(\mathsf{Z})$ defined by (4).

Observe that the K-theory of the exact sequence (4) is described from the exact sequence:

$$0 \longrightarrow R(T) \xrightarrow{1-t} R(T) \longrightarrow \mathsf{Z} \longrightarrow 0,$$

by Proposition 2.1. The first terms in this exact sequence is the Koszul complex defined by $1 - t \in \mathrm{Hom}_{R(T)}(R(T), R(T))$ and Z is the cohomology of the Koszul complex.

Proof. Let $\kappa_0 \in \mathrm{Hom}_{R(T)}(R(T), R(T))$ denote the Baaj-Skandalis dual of the KK-morphism induced from (4). It follows directly from the construction that the mapping $R(T) \to \mathsf{Z}$ induced from $\Sigma C_0(\mathsf{Z}) \to C_0(\mathsf{R})$ is the augmentation mapping $\mathsf{Z}[t, t^{-1}] \to \mathsf{Z}$ onto the generator of $K_1(C_0(\mathsf{R}))$. Therefore the image of κ_0 is the ideal generated by either $1 + t$ of $1 - t$ so κ_0 is of the form $u \cdot (1 \pm t)$ for some unit $u \in \mathsf{Z}[t, t^{-1}]$. The sign and $u = 1$ is found by either a direct calculation or by considering the Pimsner-Voiculescu sequence for $C_0(\mathsf{Z})$.

We will return to the Koszul complexes later on. First we will construct a geometric interpretation of the higher rank situation. Assume that T is a torus of rank n and consider the semi-open unit cube $I = [0, 1[^n \subseteq \mathsf{R}^n$. For $i = 1, \ldots, n$ we define $\tilde{X}_i$ as the set of open $i-1$-dimensional faces of I. The union satisfies

$$\bigcup_{i=1}^{n} \tilde{X}_i = \partial I \cap I.$$

We let k_i, for $i = 1, 2, \ldots n$, denote the integers $k_i := \binom{n}{i-1}$. The set $\tilde{X}_i$ has k_i connected components so if we choose a homeomorphism $]0, 1[\cong \mathsf{R}$ there are homeomorphisms

$$\tilde{X}_i \cong \coprod_{j=1}^{k_i} \mathsf{R}^{i-1} \qquad \text{for} \quad i = 1, 2, \ldots, n, \tag{6}$$

where we interpret R^0 as the one-point space. We take X_i to be the Z^n-translates of $\bigcup_{j \leq i} \tilde{X}_j$ and define $Y_i := \mathsf{R}^n \setminus X_i$ for $1 = 1, 2, \ldots, n$ and $Y_0 := \mathsf{R}^n$.

Proposition 2.2. *For $i = 1, 2, \ldots, n$ there are Z^n-equivariant isomorphisms*

$$C_0(Y_{i-1})/C_0(Y_i) \cong \mathsf{C}^{k_i} \otimes \Sigma^{i-1} C_0(\mathsf{Z}^n).$$

PROOF. By equation (6) there is a Z^n-equivariant homeomorphism

$$Y_{i-1} \setminus Y_i \cong \coprod_{m \in \mathsf{Z}^n} \Bigl(\coprod_{j=1}^{k_i} \mathsf{R}^{i-1} \Bigr),$$

where Z^n acts by translation on the first disjoint union. Therefore

$$\begin{aligned} C_0(Y_{i-1})/C_0(Y_i) &\cong C_0(Y_{i-1} \setminus Y_i) \cong C_0\Bigl(\coprod_{m \in \mathsf{Z}^n} \Bigl(\coprod_{j=1}^{k_i} \mathsf{R}^{i-1} \Bigr) \Bigr) \\ &\cong \mathsf{C}^{k_i} \otimes C_0(\mathsf{Z}^n \times \mathsf{R}^{i-1}) \equiv \mathsf{C}^{k_i} \otimes \Sigma^{i-1} C_0(\mathsf{Z}^n). \end{aligned}$$

Consider the classifying space R^n for proper actions of Z^n. Since Z^n has the strong Baum-Connes property, the Dirac element $[\not{D}]$ induces a KK^{Z^n}-isomorphism $C_0(\mathsf{R}^n) \cong \Sigma^n \mathsf{C}$. An alternative approach to constructing this isomorphism is the Julg theorem which implies that for any $T - C^*$-algebra A there is an isomorphism $K_*^T(A) \cong K_*(A \rtimes_r T)$. Therefore $K_*^T(\Sigma^n \mathsf{C} \rtimes \mathsf{Z}^n) \cong K_*^T(C_0(\mathsf{R}^n) \rtimes \mathsf{Z}^n)$ and the statement follows from the Universal Coefficient Theorem for the compact Hodgkin-Lie group T, see more in [14].

For $i = 1, 2, \dots, n$, Proposition 2.2 implies that there is a Z^n-equivariant short exact sequence

$$0 \longrightarrow C_0(Y_i) \longrightarrow C_0(Y_{i-1}) \longrightarrow \mathsf{C}^{k_i} \otimes \Sigma^{i-1} C_0(\mathsf{Z}^n) \longrightarrow 0. \tag{7}$$

We will by $\kappa_i \in KK^{\mathsf{Z}^n}(\mathsf{C}^{k_i} \otimes C_0(\mathsf{Z}^n), \mathsf{C}^{k_{i+1}} \otimes C_0(\mathsf{Z}^n))$ denote the Z^n-equivariant KK-morphism defined in such a way that the extension class defined by (7) composed with the restriction mapping $C_0(Y_i) \to \mathsf{C}^{k_{i+1}} \otimes \Sigma^i C_0(\mathsf{Z}^n)$ coincides with $\Sigma^{i-1}\kappa_i$. Notice that $Y_n = \mathsf{Z}^n \times]0,1[^n$ and $Y_0 = \mathsf{R}^n$ so we have that $C_0(Y_n) = \Sigma^n C_0(\mathsf{Z}^n)$ and $C_0(Y_0) = C_0(\mathsf{R}^n)$, the latter being KK^{Z^n}-isomorphic to $\Sigma^n \mathsf{C}$. Thus we get a sequence of distinguished triangles in KK^{Z^n}:

(8)

$$\begin{tikzcd}[column sep=small]
\Sigma^n C_0(\mathsf{Z}^n) \arrow[rr] & & C_0(Y_{n-1}) \arrow[rr] \arrow[dl] & & \cdots \arrow[dl] & \\
& \mathsf{C}^n \otimes \Sigma^{n-1} C_0(\mathsf{Z}^n) \arrow[ul, "\circ" marking, "\Sigma^n \kappa_n"'] & & \mathsf{C}^{k_{n-1}} \otimes \Sigma^{n-2} C_0(\mathsf{Z}^n) \arrow[ll, "\circ" marking, "\Sigma^{n-1}\kappa_{n-1}"'] \arrow[ul, "\circ" marking] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] & & C_0(Y_2) \arrow[rr] \arrow[dl] & & C_0(Y_1) \arrow[rr] \arrow[dl] & & \Sigma^n \mathsf{C} \arrow[dl] \\
& \cdots & & \mathsf{C}^n \otimes \Sigma C_0(\mathsf{Z}^n) \arrow[ll, "\circ" marking, "\Sigma^2\kappa_2"'] \arrow[ul, "\circ" marking] & & C_0(\mathsf{Z}^n) \arrow[ll, "\circ" marking, "\Sigma\kappa_1"'] \arrow[ul, "\circ" marking] &
\end{tikzcd}$$

A sequence of distinguished triangles of this type will be called a *tower*. The tower (8) in $KK^{\mathbb{Z}^n}$ is the higher rank analogue of the distinguished triangle (5). The tower (8) can be generalized to contain any coefficient ring.

To find a better description of the morphisms κ_i let us recall the notion of a Koszul complex. Let R denote a commutative ring and E an R-module. For simplicity we will assume that E is free and finitely generated, let us say of rank N. For an element $v \in \mathrm{Hom}_R(E, R)$, the Koszul complex of E with respect to v is the complex

$$0 \longrightarrow \wedge^N E \xrightarrow{\partial_1} \wedge^{N-1} E \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_{N-2}} \wedge^2 E \xrightarrow{\partial_{N-1}} E \xrightarrow{v} R \longrightarrow 0,$$

where each ∂_k is defined as interior multiplication by v. Since we have assumed E to be free, we may write $v = \sum \nu_i \mathrm{e}_i^*$ for some $\nu_1, \nu_2, \ldots, \nu_N \in R$ and the dual basis e_i^* of a basis e_i, $i = 1, 2, \ldots, N$ of E. If the sequence $\nu_1, \nu_2, \ldots, \nu_N$ is a regular sequence the Koszul complex is exact except at R. The cohomology of the Koszul complex is in this case $R/v(E)$ at R. See more in [5].

The Koszul complex of interest to us is constructed from the module $E := R(T)^n$ over the representation ring of the torus T which has the following form:

$$R(T) \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

Observe that Baaj-Skandalis duality and the Universal Coefficient Theorem implies that

$$\begin{aligned} KK^{\mathbb{Z}^n}(\mathbb{C}^{k_i} \otimes C_0(\mathbb{Z}^n), \mathbb{C}^{k_{i+1}} \otimes C_0(\mathbb{Z}^n)) &\cong KK^T(\mathbb{C}^{k_i}, \mathbb{C}^{k_{i+1}}) \\ &\cong \mathrm{Hom}_{R(T)}(R(T)^{k_i}, R(T)^{k_{i+1}}). \end{aligned}$$

We have that $R(T)^{k_i} \cong \wedge^{n-i+1} E$ so the lower row in (8) have the right ranks for coinciding with a Koszul complex. Let $f_i \in \mathrm{Hom}_{R(T)}(\wedge^{n-i+1} E, \wedge^{n-i} E)$ denote the image of κ_i under the isomorphisms above. To simplify notations, we will by $(e_i)_{i=1}^n$ denote the $R(T)$-basis of E coming from the isomorphism $E \cong R(T) \otimes_{\mathbb{Z}} \mathbb{Z}^n$ and by $(e_i^*)_{i=1}^n$ denote the dual basis.

THEOREM 2.3. *Under the isomorphisms $R(T)^{k_i} \cong \wedge^{n-i+1} E$ the mappings f_i coincide with interior multiplication by the element $v := \sum_{i=1}^n (1 - t_i)\mathrm{e}_i^*$. Therefore the sequence*

$$0 \longrightarrow \wedge^n E \xrightarrow{f_1} \wedge^{n-1} E \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} \wedge^2 E \xrightarrow{f_{n-1}} E \xrightarrow{f_n} R(T) \longrightarrow 0$$

defines a complex isomorphic to the Koszul complex of E whose cohomology at $R(T)$ is $\mathbb{Z}$.

PROOF. While both f_i and the mapping defined by interior multiplication by v are $R(T)$-linear it is sufficient to prove that $f_i(u) = v\neg u$ for elements of the form $u = \mathrm{e}_{m_1} \wedge \cdots \wedge \mathrm{e}_{m_{n-i+1}} \in \bigwedge^{n-i+1} E$, where $m_1, \ldots, m_{n-i+1} \in \{1, 2, \ldots, n\}$. Let $(m_p)_{p=n-i+1}^n$ be an enumeration of all $j = 1, 2, \ldots, n$ such that $j \notin (m_p)_{p=1}^{n-i+1}$. If we view Z^n as a subset of R^n we can define $X_u \subseteq \tilde{X}_i$ as the open face in R^n spanned by the vectors $\mathrm{e}_{m_{n-i+1}}, \mathrm{e}_{m_{n-i+2}}, \ldots, \mathrm{e}_{m_n}$.

Under the isomorphism $\bigwedge^{n-i+1} E \cong K_{i-1}(\mathsf{C}^{k_i} \otimes \Sigma^{i-1} C_0(\mathsf{Z}^n))$ the element u corresponds to a K-theory class on $\tilde{X}_i$ which is trivial except on the face X_u. Therefore there exists sequences of numbers $(a_j)_{j=1}^{n-i+1}, (b_j)_{j=1}^{n-i+1} \subseteq \mathsf{Z}$ such that

$$f_i(u) = \sum_{j=1}^{n-i+1} (a_j + b_j t_j)\mathrm{e}_{m_j} \neg u.$$

If $j = 1, 2, \ldots, n-i+1$, we will let $X_{u,j}$ denote the open face spanned by the vectors $\mathrm{e}_{m_j}, \mathrm{e}_{m_{n-i+1}}, \mathrm{e}_{m_{n-i+2}}, \ldots, \mathrm{e}_{m_n}$. It follows from restricting to $X_{u,j}$ that $a_j = 1$ since Bott periodicity implies that the index mapping $K_{i-1}(C_0(X_u)) \to K_i(C_0(X_{u,j}))$ is an isomorphism. In a similar fashion it follows that $b_i = -1$.

While $v(E)$ is the ideal generated by the regular sequence $1 - t_1, 1 - t_2, \ldots, 1 - t_n$, the cohomology of the Koszul complex is $R(T)/v(E) = \mathsf{Z}$ and the quotient mapping $R(T) \to \mathsf{Z}$ coincides with the augmentation mapping.

Consider the tower Baaj-Skandalis dual to (8). Given $A, B \in KK^T$ we can apply the homological functor $KK^T(A, -\otimes_{\min} B)$ to this tower. This functor is only homological on the bootstrap category if B is not exact, but all objects in the tower Baaj-Skandalis dual to (8) are in the bootstrap category. The lowest row of the corresponding tower in the category of $R(T)$-modules is a Koszul complex:

$$(9)\quad 0 \longrightarrow \bigwedge^n \mathsf{Z}^n \otimes KK_*^T(A, B) \xrightarrow{v_A\neg} \bigwedge^{n-1} \mathsf{Z}^n \otimes KK_*^T(A, B) \xrightarrow{v_A\neg} \ldots$$
$$\xrightarrow{v_A\neg} \mathsf{Z}^n \otimes KK_*^T(A, B) \xrightarrow{v_A\neg} KK_*^T(A, B) \longrightarrow 0$$

where

$$v_A := \sum_{i=1}^n (1 - \beta_{i*})\mathrm{e}_i^* \in \mathrm{Hom}_{R(T)}(KK_*^T(A, B)^n, KK_*^T(A, B))$$

and $(\beta_i)_{i=1}^n$ are the commuting equivariant automorphisms of A that are Baaj-Skandalis to the Z^n-action on $B \rtimes_r T$. The cohomology of this Koszul complex can be calculated from $KK_*^T(A, B)$. We will return to this subject in the next section in the more general case of Hodgkin-Lie groups and explain this procedure further.

3. The generalized Pimsner-Voiculescu-towers

As mentioned in the introduction, the representation ring $R(T)$ is free over $R(G)$ when G is a Hodgkin-Lie group, so the step to coactions of a compact Hodgkin-Lie group will not be too large. We will throughout this section assume that G is a compact Hodgkin-Lie group of rank n with maximal torus T. Recall that a group satisfies the Hodgkin condition if it is connected and the fundamental group is torsion-free.

The embedding $T \subseteq G$ induces a restriction functor $KK^{\hat{T}} \to KK^{\hat{G}}$. Using the isomorphism $\hat{T} \cong \mathsf{Z}^n$, the tower (8) can be restricted to a $KK^{\hat{G}}$-tower:

$$\begin{tikzcd}
\Sigma^n C^*(T) \arrow[rr] & & C_0(Y_{n-1}) \arrow[dl] \arrow[rr] & & \cdots \arrow[dl] & \\
& \mathsf{C}^n \otimes \Sigma^{n-1} C^*(T) \arrow[ul, "\Sigma^n \kappa_n"] & & \mathsf{C}^{k_{n-1}} \otimes \Sigma^{n-2} C^*(T) \arrow[ul] \arrow[ll, "\Sigma^{n-1}\kappa_{n-1}"'] & & \cdots \arrow[ul] \arrow[ll]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_0(Y_2) \arrow[dl] \arrow[rr] & & C_0(Y_1) \arrow[dl] \arrow[rr] & & C_0(\mathsf{R}^n) \arrow[dl] \\
\cdots & & \mathsf{C}^n \otimes \Sigma C^*(T) \arrow[ul] \arrow[ll, "\Sigma^2\kappa_2"'] & & C^*(T) \arrow[ul] \arrow[ll, "\Sigma\kappa_1"'] &
\end{tikzcd}$$

In order to work with this $KK^{\hat{G}}$-tower we need to describe the terms $C^*(T)$ in the second row.

LEMMA 3.1. *If G is a compact Hodgkin-Lie group with Weyl group of order w there is an isomorphism*

$$C^*(T) \cong \mathsf{C}^w \otimes C^*(G) \qquad \textit{in} \quad KK^{\hat{G}}.$$

Observe that the condition on G to be a Hodgkin group is equivalent to $\hat{G}$ being a torsion-free quantum group in the sense of Meyer, see [9]. The torsion-free quantum groups are the only non-classical discrete quantum groups for which there is a general formulation of the Baum-Connes property in terms of triangulated categories. In [11], coactions of compact non-Hodgkin Lie groups were considered and the "torsion" turned out to be the torsion elements of $H^2(G, S^1)$. The less precise statement $C(G/T) \cong \mathsf{C}^k$ in KK^G for some k is stated and proved in [11]. An explicit calculation that $k = |W|$ can be found in [15]. We will review the conceptually important part of the proof of a Proposition in [11] which proves Lemma 3.1 aside from the calculation of k.

PROOF. By [15], the representation ring $R(T)$ is free of rank w over the representation ring $R(G)$ if $\pi_1(G)$ is torsion-free. If we let $\mathscr{S}$ denote the localizing subcategory of KK^G generated by C and $C(G/T)$, Lemma 11 of [10] states that for $A \in \mathscr{S}$ the natural homomorphism

$$R(T) \otimes_{R(G)} KK^G(A, \mathsf{C}) \longrightarrow KK^T(A, \mathsf{C})$$

is an isomorphism. Thus the representable functor on $\mathscr{S}$

$$A \longrightarrow KK^G(A, \mathsf{C}^w) \cong R(T) \otimes_{R(G)} KK^G(A, \mathsf{C})$$

coincides with the representable functor

$$A \longrightarrow KK^G(A, C(G/T)) \cong KK^T(A, \mathsf{C}).$$

The last isomorphism exists as a consequence of the fact that the induction functor Ind_T^G is the right adjoint of the restriction functor from G to T. So the Yoneda lemma implies that $C(G/T) \cong \mathsf{C}^w$ in $\mathscr{S}$ and therefore in KK^G. Applying Baaj-Skandalis duality it follows that there is an equivariant KK-isomorphism $C^*(T) \cong \mathsf{C}^w \otimes C^*(G)$.

Using Lemma 3.1 the tower (8) takes the form:

(10)
$$\begin{tikzcd}
\Sigma^n \mathsf{C}^w \otimes C^*(G) \arrow[rr] & & C_0(Y_{n-1}) \arrow[dl] \arrow[r] & \cdots \\
& \Sigma^{n-1}\mathsf{C}^{wn} \otimes C^*(G) \arrow[ul, "\circ" marking] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_0(Y_2) \arrow[dl] \arrow[rr] & & C_0(Y_1) \arrow[dl] \arrow[rr] & & \Sigma^n \mathsf{C} \arrow[dl] \\
\cdots & & \Sigma \mathsf{C}^{wn} \otimes C^*(G) \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] & & \mathsf{C}^w \otimes C^*(G) \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] &
\end{tikzcd}$$

We will call this $KK^{\hat{G}}$-tower the fundamental G–PV-tower. The dual fundamental G–PV-tower is defined to be the KK^G-tower which is Baaj-Skandalis dual to the fundamental G–PV-tower:

(11)
$$\begin{tikzcd}
\Sigma^n \mathsf{C}^w \arrow[rr] & & D_{n-1} \arrow[dl] \arrow[rr] & & \cdots \arrow[dl] \\
& \Sigma^{n-1}\mathsf{C}^{wn} \arrow[ul, "\circ" marking] & & \Sigma^{n-2}\mathsf{C}^{wk_{n-1}} \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] & \cdots \arrow[l, "\circ" marking] \arrow[u, "\circ" marking]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & D_2 \arrow[dl] \arrow[rr] & & D_1 \arrow[dl] \arrow[rr] & & \Sigma^n C(G) \arrow[dl] \\
\cdots & & \Sigma \mathsf{C}^{wn} \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] & & \mathsf{C}^w \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] &
\end{tikzcd}$$

where $D_i := C_0(Y_i) \rtimes_r \hat{G}$.

As mentioned above, if A is a G–C^*-algebra, the trivial coaction of G on A makes A into a Yetter-Drinfeld algebra. This follows from that $C(G)$ is commutative so we can extend a G-action via the $D(G)$-equivariant $*$-monomorphism $C(G) \to \mathcal{M}(C_0(D(G)))$. Clearly, a G-equivariant mapping is equivariant in this new $D(G)$-action. Furthermore, since mapping cones does not depend on the action, the trivial extension of a G-action to a $D(G)$-action is functorial and respects mapping cones. The following proposition follows from universality.

PROPOSITION 3.2. *If G is a locally compact group, the functor mapping a G–C^*-algebra to a G-Yetter-Drinfeld algebra with trivial $\hat{G}$-action defines a triangulated functor $KK^G \to KK^{D(G)}$.*

Using the triangulated functor of Proposition 3.2, we may consider the tower (11) as a tower in $KK^{D(G)}$. Applying a crossed product by G we obtain that also the tower (10) is a tower in $KK^{D(G)}$. For a C^*-algebra B we will use the notation $t(B)$ for the $\hat{G}$–C^*-algebra with trivial coaction, or in the context of G–C^*-algebras $t(B)$ will denote the G–C^*-algebra with trivial action. Let us state and prove the corresponding version of (3) in a simple case of a braided tensor product over G with $C(G)$, a more general proof can be found in [12].

LEMMA 3.3. *When B has a continuous G-action, there is a $\hat{G}$-equivariant Morita equivalence*

$$(C(G) \otimes B) \rtimes_r G \sim_M t(B).$$

PROOF. By Baaj-Skandalis duality, it suffices to prove that there is a $\hat{G}$-equivariant isomorphism $(C(G)\otimes B)\rtimes_r G \cong (C(G)\rtimes_r G)\otimes t(B)$. Denote the G-action on B by β and define the equivariant mapping $\varphi_0 : L^1(G, C(G, B)) \to (C(G) \rtimes_r G) \otimes t(B)$ by setting

$$\varphi_0(f)(g_1, g_2) := \beta_{g_1^{-1}} f(g_1, g_2).$$

The linear mapping φ_0 is a $*$-homomorphism when $L^1(G, C(G, B))$ is equipped with the convolution twisted by the G-action on $C(G) \otimes B$. It is straightforward to verify that φ_0 is bounded in C^*-norm so we can define $\varphi : (C(G)\otimes B) \rtimes_r G \to (C(G) \rtimes_r G) \otimes B$ by continuity. The $*$-homomorphism φ is an equivariant isomorphism since an inverse can be constructed by extending

$$\varphi^{-1}(f \otimes b)(g_1, g_2) := f(g_1, g_2)\beta_{g_1}(b)$$

to a $*$-homomorphism $\varphi^{-1} : (C(G) \rtimes_r G) \otimes t(B) \to (C(G) \otimes B) \rtimes_r G$.

Theorem 3.4 (The Pimsner-Voiculescu tower). *Let G be a compact Hodgkin-Lie group of rank n and Weyl group of order w. For any separable $\hat{G}$–C^*-algebra A there is a $KK^{\hat{G}}$-tower*

$$\begin{tikzcd}[column sep=small]
\mathsf{C}^w \otimes A \arrow[rr] & & \Sigma^n D_{n-1}(A) \arrow[rr] \arrow[dl] & & \Sigma^n D_{n-2}(A) \arrow[rr] \arrow[dl] & & \cdots \\
& \Sigma \mathsf{C}^{wn} \otimes A \arrow[ul, "\circ" marking] & & \Sigma^2 \mathsf{C}^{wk_{n-1}} \otimes A \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] &
\end{tikzcd} \tag{12}$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] & & \Sigma^n D_2(A) \arrow[rr] \arrow[dl] & & \Sigma^n D_1(A) \arrow[rr] \arrow[dl] & & t(A \rtimes_r \hat{G}) \arrow[dl] \\
& \cdots & & \Sigma^{n-1} \mathsf{C}^{wk_2} \otimes A \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] & & \Sigma^n \mathsf{C}^w \otimes A \arrow[ll, "\circ" marking] \arrow[ul, "\circ" marking] &
\end{tikzcd}$$

where $D_i(A) := (C_0(Y_i) \otimes \mathscr{K}(L^2(G))) \boxtimes_G (A \rtimes_r \hat{G})$ and is equipped with the $\hat{G}$-action induced from the diagonal $\hat{G}$-action on $C_0(Y_i) \otimes \mathscr{K}(L^2(G))$.

Observe that the $D(G)$-actions on the C^*-algebras $C_0(Y_i) \otimes \mathscr{K}(L^2(G))$ is defined to come from those on their Baaj-Skandalis duals $C_0(Y_i) \rtimes_r \hat{G}$, which are $D(G) - C^*$-algebras in the dual G-actions on the crossed products and the trivial $\hat{G}$-actions. So in general, $D_i(A)$ is not the tensor product of $C_0(Y_i) \otimes \mathscr{K}(L^2(G))$ and $A \rtimes_r \hat{G}$.

Proof. By Lemma 3.3 the $\hat{G}$–C^*-algebra A admits the equivariant Morita equivalence:

$$(C(G) \otimes (A \rtimes_r \hat{G})) \rtimes_r G \sim_M t(A \rtimes_r \hat{G}). \tag{13}$$

Furthermore, the isomorphism of equation (3) holds for braided tensor products over G so while the $\hat{G}$-actions on $D_i = C_0(Y_i) \rtimes_r \hat{G}$ are trivial there are equivariant isomorphisms

$$\begin{aligned} (D_i \otimes (A \rtimes_r \hat{G})) \rtimes_r G &\cong ((C_0(Y_i) \rtimes_r \hat{G}) \boxtimes_G (A \rtimes_r \hat{G})) \rtimes_r G \\ &\cong (C_0(Y_i) \otimes \mathscr{K}(L^2(G))) \boxtimes_G (A \rtimes_r \hat{G}). \end{aligned} \tag{14}$$

Thus if we tensor the dual fundamental G–PV-tower (11) by the G–C^*-algebra $A \rtimes_r \hat{G}$ we obtain a new KK^G-tower which becomes the Pimsner-Voiculescu tower of A after applying Baaj-Skandalis duality, using the Morita equivalence (13) and the isomorphisms (14).

The Pimsner-Voiculescu tower (12) is the generalization of the resolution in (9) to compact Hodgkin-Lie groups. Applying the cohomological functor $KK(-, B)$ to the Pimsner-Voiculescu tower we obtain a similar resolution of

$KK_*(A \rtimes_r \hat{G}, B)$ in terms of $KK_*(A, B)$ as in (9). Similarly, the homological functor $KK(B, -)$ applied to the Pimsner-Voiculescu tower gives a resolution of $KK(B, A \rtimes \hat{G})$ in terms of $KK(B, A)$. Observe that since A has a $\hat{G}$-action, the groups $KK(\mathbb{C}^w \otimes A, B)$ and $KK(B, \mathbb{C}^w \otimes A)$ will always have an $R(G)$-module structure and since $R(T)$ is free over $R(G)$ also an $R(T)$-module structure.

As an example of this, we will use the Pimsner-Voiculescu tower to calculate the K-theory of the homogeneous space G/H when $H \subseteq G$ is a Lie subgroup. More generally, this technique can be used to calculate $K_*(A \rtimes_r \hat{G})$ for any $\hat{G}$–C^*-algebra A when one knows $K_*(A)$ and its $R(G)$-module structure coming from the Julg isomorphism $K_*(A) \cong K_*^G(A \rtimes_r \hat{G})$. To calculate $K^*(G/H)$, consider the C^*-algebra $A := C^*(H)$ equipped with the $\hat{G}$-action induced from the natural mapping $C^*(H) \to \mathcal{M}(C^*(G))$. Green's imprimitivity theorem implies that $C^*(H) \rtimes \hat{G}$ is KK-equivalent with $C(G/H)$. Thus, if we take the K-theory of the Pimsner-Voiculescu tower of $C^*(H)$ we obtain a tower of abelian groups of the form

$$\begin{tikzcd}
R(T) \otimes_{R(G)} R(H) \arrow[rr] & & K_{*-n}(D_{n-1}(C^*(H))) \arrow[r] \arrow[dl] & \cdots \\
& \Sigma R(T)^n \otimes_{R(G)} R(H) \arrow[ul, "\Sigma v \otimes 1"] & & \cdots \arrow[ul] \arrow[ll, "\Sigma v \otimes 1"'] \\
\cdots \arrow[r] & K_{*-n}(D_1(C^*(H))) \arrow[rr] \arrow[dl] & & K^*(G/H) \arrow[dl] \\
\cdots & & \Sigma^n R(T) \otimes_{R(G)} R(H) \arrow[ul] \arrow[ll, "\Sigma v \otimes 1"] &
\end{tikzcd} \tag{15}$$

We use Σ to denote degree shift in the category of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Here we have used that $R(T)$ is a free $R(G)$-module of rank w so $K_*(\mathbb{C}^w \otimes C^*(H)) \cong R(T) \otimes_{R(G)} R(H)$. Thus the lowest row is the tensor product of $R(H)$ with the Koszul complex of $R(T)$ that is associated with the regular sequence $1 - t_1, 1 - t_2, \ldots, 1 - t_n$ under the isomorphism $R(T) \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$.

If we restrict our attention to simple compact Lie groups we can perform an explicit calculation of all the groups in (15). Assume that $G = G_n$ is a simple compact Hodgkin-Lie group in the classical A, B, C- or D-series of rank n and assume that $H = G_k \subseteq G_n$ is a simple simply connected compact Lie group in the same classical serie being of rank $k < n$. We may take a maximal torus $T_n \subseteq G_n$ such that $T_k := T_n \cap G_k$ is a maximal torus in G_k. In this case we may consider $R(T_k)$ as an ideal in $R(T_n)$ and $R(T_n) \otimes_{R(G_n)} R(G_k) \cong R(T_k)$ as $R(T_n)$-modules. Under the isomorphisms $R(T_k) \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_k^{\pm 1}]$ and

$R(T_n) \cong \mathsf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, the Koszul vector v is identified with $\sum_{i=1}^{k}(1-t_i)\mathrm{e}_i^* \in \mathrm{Hom}(R(T_k)^n, R(T_k))$. Thus we arrive at the tower

$$\begin{tikzcd}
R(T_k) \arrow[rr] & & K_{*-n}(D_{n-1}(C^*(G_k)))) \arrow[rr] \arrow[dl] & & \cdots \\
& \Sigma\mathsf{Z}^n \otimes_{\mathsf{Z}} R(T_k) \arrow[ul, "\Sigma\partial_n", "\circ" description] & & \bigwedge^2 \mathsf{Z}^n \otimes_{\mathsf{Z}} R(T_k) \arrow[ll, "\Sigma\partial_{n-1}"', "\circ" description] \arrow[ul, "\circ" description] & & \cdots \arrow[ll, "\circ" description]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & K_{*-n}(D_1(C^*(G_k))) \arrow[rr] \arrow[dl] & & K^*(G_n/G_k) \arrow[dl] \\
\cdots & & \Sigma^n \bigwedge^n \mathsf{Z}^n \otimes_{\mathsf{Z}} R(T_k) \arrow[ll, "\Sigma\partial_1"', "\circ" description] \arrow[ul, "\circ" description] &
\end{tikzcd}$$

Let us use the notation E^* for the complex $\bigwedge^{n-*} \mathsf{Z}^n \otimes R(T_k)$ equipped with the Koszul differential from the vector $\sum_{i=1}^{k}(1-t_i)\mathrm{e}_i^*$ which we as above denote by $\partial_l : E^{l-1} \to E^l$. After some simpler calculations in this Koszul complex we arrive at the conclusion that

$$K_{*-n}(D_l(C^*(G_k))) \cong \ker(\partial_{l+1}) \oplus \bigoplus_{j=l+2}^{n+1} \Sigma^{n-j} H^j(E^*).$$

Hence we obtain the isomorphism $K^*(G_n/G_k) \cong \bigoplus_{j=1}^{n+1} \Sigma^{n-j} H^j(E^*)$. These cohomology groups are calculated in Corollary 17.10 of [5] and $H^j(E^*)$ is a free group of rank $k(j) := (n-k)!/(n-j)!(n-j-k)!$ if $0 \le j \le n-k$ and 0 otherwise. Therefore

$$K^*(G_n/G_k) \cong \bigoplus_{j=0}^{n-k} \Sigma^{n-j} \mathsf{Z}^{k(j)} = \mathsf{Z}^{2^{n-k-1}} \oplus \Sigma\mathsf{Z}^{2^{n-k-1}}.$$

THEOREM 3.5 (The dual Pimsner-Voiculescu tower). *Under the assumptions of Theorem 3.4 there is a KK^G-tower*

(16)
$$\begin{tikzcd}
\mathsf{C}^w \otimes t(A) \arrow[rr] & & \Sigma^n \tilde{D}_{n-1}(A) \arrow[rr] \arrow[dl] & & \cdots \\
& \Sigma\mathsf{C}^{wn} \otimes t(A) \arrow[ul, "\circ" description] & & \Sigma^2\mathsf{C}^{wk_{n-1}} \otimes t(A) \arrow[ll, "\circ" description] \arrow[ul, "\circ" description] & & \cdots \arrow[ll, "\circ" description] \\
\cdots \arrow[r] & \Sigma^n \tilde{D}_2(A) \arrow[rr] \arrow[dl] & & \Sigma^n \tilde{D}_1(A) \arrow[rr] \arrow[dl] & & A \rtimes_r \hat{G} \arrow[dl] \\
\cdots & & \Sigma^{n-1}\mathsf{C}^{wk_2} \otimes t(A) \arrow[ll, "\circ" description] \arrow[ul, "\circ" description] & & \Sigma^n\mathsf{C}^w \otimes t(A) \arrow[ll, "\circ" description] \arrow[ul, "\circ" description] &
\end{tikzcd}$$

where $\tilde{D}_i(A) := D_i \boxtimes_{\hat{G}} A$.

For a homological functor $F : KK^{\hat{G}} \to Ab$, the dual Pimsner-Voiculescu tower of A allows us to calculate $F(A)$ in terms of the objects $F(C_r^*(G) \otimes t(A))$. As we shall see below, $\hat{G}$–C^*-algebras of the form $C_r^*(G) \otimes t(A)$ behaves similarly to *proper* actions. Compare this result to Theorem 4.4 of [8].

PROOF. Consider the braided tensor product by $\Sigma^n A$ and the tower (10):

$$\begin{tikzcd}
\mathsf{C}^w \otimes C^*(G) \boxtimes_{\hat{G}} A \arrow[rr] & & \Sigma^n C_0(Y_{n-1}) \boxtimes_{\hat{G}} A \arrow[r] \arrow[dl] & \cdots \\
& \Sigma \mathsf{C}^{wn} \otimes C^*(G) \boxtimes_{\hat{G}} A \arrow[ul, "\circ"] & & \cdots \arrow[ll, "\circ"] \arrow[ul, "\circ"]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma^n C_0(Y_1) \boxtimes_{\hat{G}} A \arrow[rr] \arrow[dl] & & A \arrow[dl] \\
\cdots & & \Sigma^n \mathsf{C}^w \otimes C^*(G) \boxtimes_{\hat{G}} A \arrow[ll, "\circ"] \arrow[ul, "\circ"] &
\end{tikzcd}$$

Taking crossed product between this tower and $\hat{G}$ implies the Theorem since the following equivariant Morita equivalences follows from (3)

$$(C^*(G) \boxtimes_{\hat{G}} A) \rtimes_r \hat{G} \sim_M t(A)$$

and

$$(C_0(Y_i) \boxtimes_{\hat{G}} A) \rtimes_r \hat{G} \sim_M (C_0(Y_i) \rtimes_r \hat{G}) \boxtimes_{\hat{G}} A = D_i \boxtimes_{\hat{G}} A.$$

One of the main motivations behind this paper was to give a precise description of the Baum-Connes property of duals of Hodgkin-Lie groups. The Baum-Connes property for coactions of compact Lie groups was given meaning to and was proved to hold in [11]. More generally, this fits into the program of generalizing the Baum-Connes property to quantum groups. So far, it is not known what a suitable property the Baum-Connes property should be for a general locally compact quantum group. For discrete quantum groups which are torsion-free, in the sense of [9], there is a formulation and as mentioned above duals of compact Hodgkin-Lie groups are torsion-free.

The problem that arises when one tries to define the Baum-Connes assembly mapping for a quantum group is that there is no natural notion of a proper action and there are in general too many quantum homogeneous spaces. It is much easier to generalize certain notions of free actions than proper actions of a quantum group by just saying that an action of a discrete quantum group Γ on a C^*-algebra A is *truly* free if there is a C^*-algebra A_0 and an equivariant

$*$-isomorphism $A \cong A_0 \otimes_{\min} C_0(\Gamma)$ with Γ only acting on the second leg. In the case of a group, there are many free actions that are not truly free but this stronger notion of a free action will suffice for our purposes.

Restricting one's attention to generalizing the Baum-Connes property of the simpler class of torsion-free discrete groups to the quantum setting, when proper actions are free, Meyer introduced a class of quantum groups known as torsion-free in [9]. Following [9], we say that a discrete quantum group Γ is torsion-free if every coaction of the compact quantum group $\hat{\Gamma}$ on a finite-dimensional C^*-algebra is Morita equivalent to a trivial coaction on a direct sum of C:s. This fact implies that any finite-dimensional projective representation of the dual compact quantum group is equivalent to a representation. If Γ is a discrete group, coactions of the dual compact quantum group on finite-dimensional C^*-algebras that are not Morita equivalent to a trivial coaction on a direct sum of C:s correspond to finite subgroups so a discrete group is torsion-free if and only if it is torsion-free in the sense of [9].

For a torsion-free quantum group a proper action should correspond to a free action. Under Baaj-Skandalis duality, a truly free $\Gamma - C^*$-algebra corresponds to a trivial $\hat{\Gamma}$-action. Let $\mathscr{C}\mathscr{I}_{\hat{\Gamma}}$ denote the image of $t : KK \to KK^{\hat{\Gamma}}$. The triangulated category $\langle \mathscr{C}\mathscr{I}_{\hat{\Gamma}} \rangle$ is defined as the localizing subcategory generated by $\mathscr{C}\mathscr{I}_{\hat{\Gamma}}$. Following the formulation of [9], Γ is said to satisfy the strong Baum-Connes property if the embedding of triangulated categories $\langle \mathscr{C}\mathscr{I}_{\hat{\Gamma}} \rangle \to KK^{\hat{\Gamma}}$ is essentially surjective. The strong Baum-Connes property of Γ is equivalent to that any $\Gamma - C^*$-algebra is in the localizing category generated by all truly free actions. So regardless of what notion of a proper action we choose, the strong Baum-Connes conjecture will imply that the localizing category generated by all such proper actions will be KK^{Γ}. The quantum group is said to satisfy the Baum-Connes property if the same statement holds after localizing with respect to the kernel of equivariant K-theory.

In [11] the Baum-Connes property was formulated in the slightly more general setting of duals of compact Lie groups. The finite-dimensional projective representations of a compact Lie group G correspond to the torsion classes of $H^2(G, S^1)$, which can be thought of as the torsion of $\hat{G}$. When G is Hodgkin, $H^2(G, S^1)$ is torsion-free so $\hat{G}$ is torsion-free. In this case a "proper" action is an object of the additive category generated by $\hat{G}$-algebras that are Baaj-Skandalis dual to $A_0 \otimes \mathsf{C}_\omega$, with C_ω denoting the endomorphisms of a projective representation ω and A_0 having trivial G-action. So the substitute in the setting of [11] for proper actions is the category of tensor products between Baaj-Skandalis duals of coactions on finite-dimensional C^*-algebras and trivial actions, just as the truly free actions form a substitute for proper actions of torsion-free quantum groups. The Baum-Connes property of coactions of a compact Hodgkin-Lie group is a direct consequence of Theorem 3.5.

The method of proof of Proposition 2.1 of [11] can be used to generalize both Theorem 3.4 and Theorem 3.5 to arbitrary compact Lie group.

Finally, let us mention a promising generalization of Theorem 3.5 to Woronowicz deformations. It was proved in [12] that the compact quantum group $SU_q(2)$ satisfies that $C(SU_q(2)/T)$ is $KK^{D(SU_q(2))}$-isomorphic to $\mathbb{C}^2$ for $q \in]0,1[$. So if we apply the induction functor $\mathrm{Ind}_T^{SU_q(2)} : KK^T \to KK^{SU_q(2)}$ to the distinguished triangle Baaj-Skandalis dual to (5) and use the isomorphism of Nest-Voigt we arrive at the distinguished triangle in $KK^{D(SU_q(2))}$:

$$\begin{tikzcd} \mathbb{C}^2 \arrow[rr] & & \mathbb{C}^2 \arrow[dl] \\ & C(SU_q(2)). \arrow[ul, "\circ" marking] & \end{tikzcd}$$

Using the technique from the proof of Theorem 3.5 any $A \in KK^{\widehat{SU_q(2)}}$ fits into a distinguished triangle

$$\begin{tikzcd} \mathbb{C}^2 \otimes t(A) \arrow[rr] & & \mathbb{C}^2 \otimes t(A) \arrow[dl] \\ & A \rtimes \widehat{SU_q(2)}. \arrow[ul, "\circ" marking] & \end{tikzcd}$$

This distinguished triangle gives an alternative proof of the strong Baum-Connes property for $\widehat{SU_q(2)}$, a result first proved in [17]. The interesting part about this proof is that it only relies on the isomorphism $C(G_q/T) \cong \mathbb{C}^w$ in $KK^{D(G_q)}$. So if such an isomorphism exists for a simply connected semi-simple compact Lie group G, the strong Baum-Connes conjecture holds for $\hat{G}_q$, the quantum dual of the Woronowicz deformation of G. To formulate the Baum-Connes property for $\hat{G}_q$ we must of course know that it is torsion-free, a statement proved in [17] for $G = SU(2)$ and the general case was proved in [6]. Another striking application of such an isomorphism is that the method above for calculating K-theory of homogeneous spaces can be generalized to classical quantum homogeneous spaces of the Woronowicz deformations.

REFERENCES

1. Baaj, S., and Skandalis, G., *C^*-algèbres de Hopf et théorie de Kasparov équivariante*, K-theory 2 (1989), 683–721.
2. Baaj, S., and Skandalis, G., *Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres*, Ann. Sci. École Norm. Sup. (4) 26 (1993), 425–488.

3. Baaj, S., Vaes, S., *Double crossed products of locally compact quantum groups*, J. Inst. Math. Jussieu 4 (2005), 135–173.
4. Baum, P., Connes, A., and Higson, N., *Classifying space for proper actions and K-theory of group C^*-algebras*, pp. 240–291 in: C^*-algebras: 1943–1993, Proc. San Antonio 1993, Contemp. Math. 167, Amer. Math. Soc., Providence, RI 1994.
5. Eisenbud, D., *Commutative Algebra. With a View toward Algebraic Geometry*, Graduate Texts in Math. 150, Springer, New York 1995.
6. Goffeng, M., *A remark on twists and the notion of torsion-free discrete quantum groups*, arXiv:1004.0072, to appear in Algebr. Represent. Theory, DOI: 10.1007/s10468-011-9266-6.
7. Kustermans, J., and Vaes, S., *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. (4) 33 (2000), 837–934.
8. Meyer, R., and Nest, R., *Homological algebra in bivariant K-theory and other triangulated categories I*, pp. 236–289 in: Triangulated Categories, London Math. Soc. Lecture Note 375, Cambridge Univ. Press, Cambridge 2010.
9. Meyer, R., *Homological algebra in bivariant K-theory and other triangulated categories II*, Tbil. Math. J. 1 (2008), 165–210.
10. Meyer, R., and Nest, R., *The Baum-Connes conjecture via localisation of categories*, Topology 45 (2006), 209–259.
11. Meyer, R., and Nest, R., *An analogue of the Baum-Connes isomorphism for coactions of compact groups*, Math. Scand. 100 (2007), 301–316.
12. Nest, R., and Voigt, C., *Equivariant Poincaré duality for quantum group actions*, J. Funct. Anal. 258 (2010), 1466–1503.
13. Raeburn, I., *On crossed products by coactions and their representation theory*, Proc. London Math. Soc. (3) 64 (1992), 625–652.
14. Rosenberg, J., and Schochet, C., *The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory*, Mem. Amer. Math. Soc. 62:348 (1986).
15. Steinberg, R., *On a theorem of Pittie*, Topology 14 (1975), 173–177.
16. Vaes, S., *A new approach to induction and imprimitivity results*, J. Funct. Anal. 229 (2005), 317–374.
17. C. Voigt, *The Baum-Connes conjecture for free orthogonal quantum groups*, Adv. Math. 227 (2011), 1873–1913.

MATHEMATICAL SCIENCES
CHALMERS UNIVERSITY OF TECHNOLOGY AND THE UNIVERSITY OF GOTHENBURG
SE-412 96 GÖTEBORG
SWEDEN
E-mail: goffeng@chalmers.se