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# MINIMIZING ROOTS OF MAPS INTO THE TWO-SPHERE

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## Abstract

This article is a study of the root theory for maps from two-dimensional CW-complexes into the 2-sphere. Given such a map $f : K \to S^2$ we define two integers $\zeta(f)$ and $\zeta(K, d_f)$, which are upper bounds for the minimal number of roots of $f$, denote be $\mu(f)$. The number $\zeta(f)$ is only defined when $f$ is a cellular map and $\zeta(K, d_f)$ is defined when $K$ is homotopy equivalent to the 2-sphere. When these two numbers are defined, we have the inequality $\mu(f) \leq \zeta(K, d_f) \leq \zeta(f)$, where $d_f$ is the so-called homological degree of $f$. We use these results to present two very interesting examples of maps from 2-complexes homotopy equivalent to the sphere into the sphere.


## 1. Introduction

The purpose of this article is to present some results and, in particular, two interesting examples involving the root problem for maps from finite and connected two-dimensional CW-complexes into the 2-sphere. The root problem consists in determining the minimal number of roots of maps in a determined homotopy class, namely, given a map $f : K \to S^2$, where $K$ is a finite and connected 2-complex, and fixed a point $a \in S^2$, the *minimum number of roots of $f$* is defined by

$$\mu(f) = \min\{ \#\varphi^{-1}(a) \text{ such that } \varphi \text{ is homotopic to } f \}.$$

where # denotes cardinality. Since the range of $f$ is a manifold it is easy to prove that the minimum number of roots is independent of the selected point $a \in S^2$. Moreover, by Theorems 2.10 and 2.12 of [5], $\mu(f)$ is really a number, that is, it is finite. When $\mu(f) = 0$ we say that the map $f$ is *root free*, and since the range is the 2-sphere, this occurs if and only if $f$ is homotopic to a constant map. If $f$ is not root free, we say also that $f$ is *strongly surjectivity*. If this is the case, we would like to determine exactly the minimum number of roots $\mu(f)$, or at least to make a good estimate of it. In the Nielsen root theory

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one defines $N(f)$, the *Nielsen root number* of $f$, which is a lower bound for $\mu(f)$.

By using the concept of *vector-degree* of a cellular map $f : K \to S^2$ presented in [3], we define, in Section 2 of this article, a number $\zeta(f)$ satisfying the inequality $\mu(f) \leq \zeta(f)$. We also present some conditions for the identity $\mu(f) = \zeta(f)$ to occur.

In Section 3, we study roots of maps from a 2-complex homotopy equivalent to the 2-sphere into the 2-sphere. Here, contrary to what occurs for self-maps of the 2-sphere, a strongly surjectivity map may have minimum root number strictly greater than 1. (It is well known that a strongly surjectivity self-map of $S^2$ is homotopic to a map with a single root). We define the number $\zeta(K, d_f)$, which is an upper bound for $\mu(f)$. Its definition uses the generator of the infinite cyclic group $H_2(K)$ and the so-called *homological degree* $d_f$ of the map $f$. When both the numbers $\zeta(f)$ and $\zeta(K, d_f)$ are defined, we have the inequality

$$\mu(f) \leq \zeta(K, d_f) \leq \zeta(f).$$

In Section 4 we present two interesting examples. The questions presented in the first example (Example 4.1) are completely answered by using the latter inequality and some simple arguments. It is a more complete version of Example 4 given by D. L. Gonçalves in [6]. The second example (Example 4.2) is more complicated and requires a deeper study. It provides an example in which $\mu(f) < \zeta(K, d_f) = \zeta(f)$.

Throughout the text, we use the capital letter $K$ to denote a finite and connected two-dimensional CW-complex. We also simplify two-dimensional CW-complex to 2-complex. Given such a 2-complex $K$, we write $\#^2(K)$ to denote the number of cells of dimension two. We also simplify $f$ is a continuous map to $f$ is a map.

## 2. The number $\zeta(f)$

Let $K$ be a finite and connected 2-complex and let $f : K \to S^2$ be a cellular map. We consider the sphere $S^2$ with its minimal cellular decomposition with a 0-cell $e^0_*$ and a 2-cell $e^2_*$, henceforth adopted, $S^2 = e^0_* \cup e^2_*$. Next, we define the *vector-degree* and the *cellular factorization* of $f$ as in [3].

Suppose that $K$ has $m = \#^2(K)$ two-dimensional cells, say $e^2_1, \ldots, e^2_m$. Let $K^1$ be the 1-skeleton of $K$. We define $\omega : K \to K/K^1$ to be the natural quotient map which collapses $K^1$ to a single point $e^0$. Then $K/K^1$ is isomorphic to the bouquet $\bigvee_{i=1}^m S^2_i$ of $m$ two-dimensional spheres with its minimal cellular decomposition. Since all of $K^1$ is mapped by $f$ onto $e^0_*$, there exists a unique map $\bar{f} : K/K^1 \to S^2$ such that $f = \bar{f} \circ \omega$. Consider the homomorphism $\bar{f}^* : H^2(S^2) \to H^2(K/K^1)$ induced by $f$ on second integral cohomology groups. Let $\varrho$ be a fixed generator of the infinite cyclic group $H^2(S^2)$ and let $\varrho_1, \ldots, \varrho_m$ be fixed generators of the abelian free group $H^2(K/K^1)$ of rank $m$. There exists integer numbers $d_1^f, \ldots, d_m^f$ such that

$$\bar{f}^*(\varrho) = d_1^f \varrho_1 + \cdots + d_m^f \varrho_m.$$

The integer vector-column $\overrightarrow{\deg}(f) = (d_1^f, \ldots, d_m^f)^{\mathrm{T}}$, where the superscript T indicates transposition of matrices, is called the *vector-degree* of $f$. It is invariant by homotopy relative to the 1-skeleton $K^1$, but it is not a homotopy invariant. As in [3], a geometric interpretation of the vector-degree can be given in the following way: The map $\bar{f}$ can be seen as a map from the bouquet $\bigvee_{i=1}^m S_i^2$ into $S^2$. Then, for each $1 \leq i \leq m$, the restriction map $\bar{f}_i = \bar{f}|_{S_i^2} : S_i^2 \to S^2$ has degree $\deg(\bar{f}_i) = d_i^f$. The maps $\bar{f}_1, \ldots, \bar{f}_m$ are called the *cellular factorization* of $f$.

If $X$ is a 2-complex, then since $S^2$ is simply connected, the based homotopy classes $[X, S^2]_*$ of based maps from $X$ into $S^2$ and the homotopy classes $[X, S^2]$ of maps from $X$ into $S^2$ are in one-to-one correspondence. Furthermore, by theorems of Hopf (see [2] and [7]), given a map $\varphi : X \to S^2$, the assignment $\varphi \mapsto \varphi^*(\varrho)$ sets up an one-to-one correspondence between the homotopy classes of the maps $\varphi : X \to S^2$ and the elements of the integral cohomology group $H^2(X)$. This implies that, for a cellular map $f : K \to S^2$, the condition $\overrightarrow{\deg}(f) = 0$ implies that $\bar{f}$ is homotopic to a constant map, which in turn implies that also $f$ is homotopic to a constant map. Now, for each $1 \leq i \leq m$, the map $\bar{f}_i$ is homotopic to a constant map if and only if $d_i^f = 0$. If the degree $d_i^f \neq 0$, then every map homotopic to $\bar{f}_i$ is surjective and $\mu(\bar{f}_i) = 1$. Moreover, in this case, there exists a cellular map $\bar{g}_i$ based homotopic to $\bar{f}_i$ such that $\bar{g}_i^{-1}(-e_*^0)$ is a single point situated in the interior of the 2-cell corresponding to $\omega(e_i^2)$, where $-e_*^0$ is the antipodal point of $e_*^0$. Since a based homotopy $\{\bar{h}_t\} : K/K^1 \to S^2$ can be lifted to a homotopy $\{h_t\} : K \to S^2$, it proves that the number $\mu(f)$ is less than or equal to the number of non-zero coordinates of the vector-degree $\overrightarrow{\deg}(f)$.

To facilitate future references, we define $\zeta(f) =$ *number of non-zero coordinates of* $\overrightarrow{\deg}(f)$. We have proved the following lemma:

LEMMA 2.1. *For every cellular map $f : K \to S^2$ we have $\mu(f) \leq \zeta(f)$.*

In order to see that the identity $\mu(f) = \zeta(f)$ may be false, let $D^2$ be the (closed) 2-disc with the minimal cellular decomposition $e_1^0 \cup e_1^1 \cup e_1^2$ and let $f : D^2 \to S^2$ be the canonical map which identifies the boundary of the disc at a single point, which we have chosen to be the 0-cell $e_*^0$ of $S^2$. We have $\mu(f) = 0$ but $\zeta(f) = 1$.

Lemma 2.1 implies that for each (not necessarily cellular) map $f : K \to S^2$, we have $\mu(f) \leq \#^2(K)$. In fact, such a map has a cellular approximation $f_{\text{cel}} : K \to S^2$ for which we have $\mu(f) = \mu(f_{\text{cel}}) \leq \zeta(f_{\text{cel}}) \leq \#^2(K)$. Actually, something stronger can be shown, namely, we can prove that the number $\mu(f)$ is less than or equal to the number of 2-cells of any subcomplex $L$ such that $K'$ collapses to $L$, where $K'$ is any cellular subdivision of $K$.

The following proposition presents conditions for the identity $\mu(f) = \zeta(f)$ to occur. Following it we present an example showing that the assumption in this proposition is not superfluous.

PROPOSITION 2.2. *Let $f : K \to S^2$ be a map. Suppose that $f$ has a cellular approximation $\varphi : K \to S^2$ such that, for some point $a \in S^2$, $\mu(f) = \#\varphi^{-1}(a)$ with $\varphi^{-1}(a) \subset K - K^1$. Then $\mu(f) = \zeta(\varphi)$.*

PROOF. Since $\varphi$ is a cellular approximation of $f$, we have $\mu(f) = \mu(\varphi) \leq \zeta(\varphi)$, where the inequality comes from Lemma 2.1. We will prove now that $\mu(\varphi) \geq \zeta(\varphi)$. Let $\bar{\varphi} : K/K^1 \to S^2$ be defined such that $\varphi = \bar{\varphi} \circ \omega$. Suppose that the equation $\varphi(x) = a$ has no roots in some 2-cell $e_j^2$ of $K$. Then the equation $\bar{\varphi}(y) = a$ has no roots in the 2-cell corresponding to the image $\omega(e_j^2)$. This implies that $d_j^{\varphi} = 0$. It follows that the equation $\varphi(x) = a$ has at least $\zeta(\varphi)$ roots, each one of them belong to the interior of a different 2-cell of $K$. Therefore, since $\mu(\varphi) = \mu(f) = \#\varphi^{-1}(a)$, we conclude that $\mu(\varphi) \geq \zeta(\varphi)$.

In Definition 5.3 of [4], a map $f$ satisfying the assumptions in the previous proposition is said to be of *type* $\nabla_3$. By Proposition 5.5 of [4], every map between closed surfaces is of type $\nabla_3$, therefore, so is a map from a closed surface into the 2-sphere. We use this result in the proof of the next proposition.

Let $f : K \to S^2$ be a map and suppose that $\varphi : K \to S^2$ is a (not necessarily cellular) map homotopic to $f$ such that $\mu(f) = \#\varphi^{-1}(a)$ for some $a \in S^2$, with $\varphi^{-1}(a) \subset K - K^1$. Up to composition of $\varphi$ with a self-homeomorphism of $S^2$ homotopic to the identity map, we can consider $a \neq e_*^0$. Moreover, it is clear that $a \notin \varphi(K^1)$. Let $V$ be a closed neighborhood of $a$ in $S^2$ homeomorphic to a closed 2-disc and not intersecting the set $\{e_*^0\} \cup \varphi(K^1)$. Let $\vartheta : V \to S^2$ be the quotient map which identifies the boundary of $V$ to $e_*^0$. Let $h : S^2 \to S^2$ be the map defined so that $h|_V = \vartheta$ and $h(S^2 - V) = e_*^0$. It is easy to see that $h$ is homotopic to the identity map and, moreover, the map $\psi : K \to S^2$ defined by the composition $\psi = h \circ \varphi$ is a cellular approximation of $f$ satisfying $\mu(f) = \#\psi^{-1}(a)$ with $\psi^{-1}(a) \subset K - K^1$. Thus, by Proposition 2.2, $\mu(f) = \zeta(\psi)$.

However, in general, it may occur that the pre-image $\varphi^{-1}(a)$ is not contained in $K - K^1$, whatever the map $\varphi : K \to S^2$ homotopic to $f$ with $\mu(f) = \#\varphi^{-1}(a)$. In this case, in general, we can not conclude the equality of Proposition 2.2. We present now an example to illustrate this fact: Let

$K = S_1^2 \vee S_2^2$ be the bouquet of two 2-spheres with the minimal cellular decomposition with one 0-cell $e^0$ and two 2-cell $e_1^2$ and $e_2^2$. Let $f : K \to S^2$ be a map with restricted to each one of the spheres is homotopic to the identity map. Consider the surface $S^2$ with its minimal cellular decomposition $S^2 = e_*^0 \cup e_*^2$. Then, certainly, there exists a cellular map $\varphi : K \to S^2$ homotopic to $f$ such that $\varphi^{-1}(e_*^0) = \{e^0\}$. Thus, $\#\varphi^{-1}(e_*^0) = 1 = \mu(f)$. Now, it is clear that each map $g$ homotopic to $f$, restricted to each $S_i^2$, $i = 1, 2$, is surjective. So, for all these maps $g$, the equation $g(x) = a$ has at least one root in each $S_i^2$, $i = 1, 2$, for any point $a \in S^2$. Therefore, if a root $x_0$ of $g(x) = a$ belongs to the interior of a 2-cell of $K$, then the equation $g(x) = a$ should have a second root, which must be in the closure of the other 2-cell of $K$. But in this case $\#g^{-1}(a) \geq 2$ and thus $\#g^{-1}(a) \neq \mu(f)$. This shows that the number of roots of any map homotopic to $f$ whose roots are not in $K^1$ is strictly greater than $\mu(f)$.

PROPOSITION 2.3. *Every map $f : M \to S^2$ from a closed surface into $S^2$ has a cellular approximation $\varphi$ such that $\#\varphi^{-1}(a) = \mu(f) = \zeta(\varphi)$, for some $a \in S^2$.*

PROOF. By Proposition 2.2 and the considerations made after its proof, it is enough to prove that there exists a map $\psi : M \to S^2$ homotopic to $f$ with $\mu(f) = \#\psi^{-1}(a)$ and $\psi^{-1}(a) \subset M - M^1$, for some $a \in S^2$, where $M^1$ is the 1-skeleton of $M$. (Here we consider the closed surface $M$ with its minimal cellular decomposition). But this is Proposition 5.5 of [4].

Before concluding this section, we wish to observe that the number $\zeta(f)$ is defined for every cellular map $f : K \to S^2$. However, this number is not a homotopy invariant (it is only invariant by homotopies relative to the 1-skeleton). It may occur that we have $f, g : K \to S^2$ two cellular maps, with $f$ homotopic to $g$, but $\zeta(f) \neq \zeta(g)$. Therefore, given a (not necessarily cellular) map $f : K \to S^2$, each cellular approximation $f_{\text{cel}}$ of $f$ may provides a different number $\zeta(f_{\text{cel}})$. Thus, we can not define the number $\zeta(f)$ for non-cellular maps. When $f$ is not cellular, we can guarantee only that

$$\mu(f) \leq \min\{\, \zeta(f_{\text{cel}}) : f_{\text{cel}} \text{ is a cellular approximation of } f \,\}$$

In the next section we will define a new upper bound for $\mu(f)$, which is defined not only for cellular maps. However, another restriction is necessary, namely, we will suppose that the 2-complex $K$ is homotopy equivalent to the 2-sphere.

## 3. Maps from complexes homotopy equivalent to the 2-sphere

We study in this section the special case of maps $f : K \to S^2$ where $K$ is a 2-complex homotopy equivalent to $S^2$. In this case, $K$ is connected and simply connected and the infinite cyclic group $H_2(K)$ has a generator of the form

$$\zeta_K = a_1 e_1^2 + \cdots + a_m e_m^2,$$

where $m = \#^2(K)$ and $e_i^2$, for $1 \leq i \leq m$, are exactly the 2-cells of $K$. We define the *homological degree* of a map $f : K \to S^2$ to be the integer number $d_f$ which satisfies the equation

$$f_*(\zeta_K) = d_f \cdot \zeta_{S^2},$$

where $f_* : H_2(K) \to H_2(S^2)$ is the homomorphism induced by $f$ on second homology groups and $\zeta_{S^2}$ is the fundamental class of $S^2$. When we consider a map $g : S^2 \to K$, its homological degree is defined similarly to be the integer number $d_g$ such that $g_*(\zeta_{S^2}) = d_g \cdot \zeta_K$.

It is very important to note that two maps from $K$ into $S^2$ are homotopic if and only if its homological degree are equal. A proof for this result is straightforward.

If we assume that $f : K \to S^2$ is a cellular map and we consider $\zeta_K$ as the integer $m$-vector $(a_1, \ldots, a_m)$, then the homological degree of $f$ is equal to the product $\zeta_K \cdot \overrightarrow{\deg}(f)$, that is

$$d_f = a_1 d_1^f + \cdots + a_m d_m^f.$$

Since, moreover, there are maps $K \to S^2$ with arbitrary homological degrees, we have the following: For each integer $d$, there are integers $d_1, \ldots, d_m$ such that

$$d = a_1 d_1 + \cdots + a_m d_m.$$

If $(d_1, \ldots, d_m)$ is an $m$-tuple of integers, we define $\int(d_1, \ldots, d_m) =$ *number of non-zero coordinates of the vector* $(d_1, \ldots, d_m)$. Finally, we define

$$\zeta(K, d) = \min\left\{ \int (d_1, \ldots, d_m) \ \text{ such that } \ d = a_1 d_1 + \cdots + a_m d_m \right\}.$$

Obviously, for each cellular map $f : K \to S^2$ with homological degree $d_f$, we have the inequality $\zeta(K, d_f) \leq \zeta(f)$. The reader may extract from Example 4.2 below some examples where we have the strict inequality $\zeta(K, d_f) < \zeta(f)$. We shall return to comment on that.

We will now state and prove the main result of this section.

THEOREM 3.1. *Let $K$ be a 2-complex homotopy equivalent to the 2-sphere and let $f : K \to S^2$ be a map with homological degree $d_f$. Then $\mu(f) \leq \zeta(K, d_f)$.*

PROOF. Certainly, $f$ has a cellular approximation $f_{cel}$ and the homological degree of $f_{cel}$ is equal to $d_f$. So we can assume that $f$ itself is a cellular map.

Let $(d_1, \ldots, d_m)$ be an $m$-uple of integers satisfying $\zeta(K, d_f) = \int(d_1, \ldots, d_m)$. Let $\bar{\varphi} : K/K^1 \to S^2$ be a cellular map such that $\bar{\varphi}^*(\varrho) = d_1\varrho_1 + \cdots + d_m\varrho_m$, whose restriction $\bar{\varphi}_i$ is the constant map at $e^0_*$ if $d_i = 0$. (We use the notation of the previous section). Such map lifts through $\omega : K \to K/K^1$ to a cellular map $\varphi : K \to S^2$ such that

$$d_\varphi = a_1d_1 + \cdots + a_md_m = d \qquad \text{and} \qquad \zeta(\varphi) = \int(d_1, \ldots, d_m).$$

By Lemma 2.1, $\mu(\varphi) \leq \zeta(\varphi) = \int(d_1, \ldots, d_m) = \zeta(K, d_f)$. Now, since the homological degree of $\varphi$ is equal to $d_f$, it follows that $\varphi$ is homotopic to $f$. Therefore $\mu(f) = \mu(\varphi) \leq \zeta(K, d_f)$.

As an immediately consequence of this theorem, we have the following one:

COROLLARY 3.2. *Under the assumption of Theorem 3.1, we have:*

1. $\mu(f) = 0$ *if and only if* $\zeta(K, d_f) = 0$*;*
2. $\mu(f) = 1$ *if* $\zeta(K, d_f) = 1$.

PROOF. The first part follows from the fact that a map $f : K \to S^2$ is root free if and only if it is homotopic to a constant map, what occurs if and only if $d_f = 0$. The second statement is an immediate consequence of the inequality $\mu(f) = \mu(\varphi) \leq \zeta(K, d_f)$ and the first statement.

In the end of the next section, we will extract from Example 4.2, examples of maps for which the converse of the second statement of the previous corollary is not true, that is, $\mu(f) = 1$ does not imply $\zeta(K, d_f) = 1$.

## 4. Two interesting examples

In order to show the applicability of Theorem 3.1, we answer completely the questions posed by D. L. Gonçalves in Example 4 of [6]. There the author did some estimates and indicated his belief about the values of $\mu(f)$ for certain maps $f : K \to S^2$, for a special 2-complex $K$ homotopy equivalent to $S^2$.

EXAMPLE 4.1. Let $K_i$, $i = 1, 2$, be the two 2-complexes obtained from $S^1$ by attaching a 2-cell by the maps $\xi_i : S^1 \to S^1$, $i = 1, 2$, of degrees 2 and 3, respectively. ($K_1$ is just the two-dimensional projective space). Take

$$K = \frac{K_1 \sqcup (S^1 \times [0, 1]) \sqcup K_2}{\sim}$$

where we identify the 1-skeleton $S^1 \subset K_1$ with $S^1 \times 0$ and the 1-skeleton $S^1 \subset K_2$ with $S^1 \times 1$. In order to prove that $K$ is homotopy equivalent to the sphere $S^2$, we first prove that $K$ is simply connected. Let $A$ be the open subset of $K$ corresponding to the image of the set $K_1 \sqcup (S^1 \times [0, 3/4))$ by the identification, and let $B$ be the open subset of $K$ corresponding to image the of the set $(S^1 \times (1/4, 1]) \sqcup K_2$ by the identification. Then $A \cap B$ is the open subset of $K$ corresponding to $S^1 \times (1/4, 3/4)$. It is easy to see that $A$, $B$ and $A \cap B$ are path-connected subsets of $K$ and, moreover,

$$\pi_1(A) = \langle \alpha \mid \alpha^2 \rangle, \qquad \pi_1(B) = \langle \beta \mid \beta^3 \rangle \qquad \text{and} \qquad \pi_1(A \cap B) = \langle \sigma \mid \cdot \rangle.$$

By the van Kampen Theorem we have $\pi_1(K) = \langle \alpha, \beta \mid \alpha^2, \beta^3, \alpha = \beta \rangle = \mathbb{1}$.

We will now show that $H_2(K) \approx \mathsf{Z}$ and determine a generator $\zeta_K$ for this group. Consider the natural cellular decomposition of $K$ as shown in Figure 1.

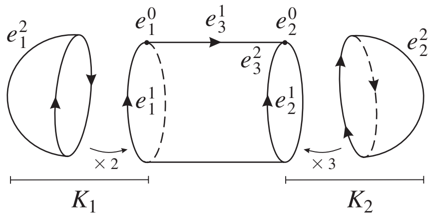


FIGURE 1. A 2-complex $K$ homotopy equivalent to the 2-sphere.

The cellular chain complex of $K$,

$$0 \to \langle e_1^2, e_2^2, e_3^2 \rangle \xrightarrow{\partial_2} \langle e_1^1, e_2^1, e_3^1 \rangle \xrightarrow{\partial_1} \langle e_1^0, e_2^0 \rangle \to 0$$

is such that

$$\partial_2(e_1^2) = 2e_1^1, \qquad \partial_2(e_2^2) = 3e_2^1 \qquad \text{and} \qquad \partial_2(e_3^2) = e_1^1 + e_2^1.$$

Hence, $\partial_2(\alpha e_1^2 + \beta e_2^2 + \gamma e_3^2) = (2\alpha + \gamma)e_1^1 + (3\beta + \gamma)e_2^1$. Thus, the cellular chain $\alpha e_1^2 + \beta e_2^2 + \gamma e_3^2$ belongs to $\ker(\partial_2)$ if and only if $2\alpha + \gamma = 0 = 3\beta + \gamma$. Therefore

$$H_2(K) \approx \ker(\partial_2) = \langle \zeta_K \rangle \approx \mathsf{Z} \qquad \text{where} \qquad \zeta_K = 3e_1^2 + 2e_2^2 - 6e_3^2.$$

We have thus shown that $K$ is a connected and simply connected 2-complex with $H_2(K) \approx \mathsf{Z}$. We will conclude that $K$ is homotopy equivalent to $S^2$. In fact, by the Hurewicz Isomorphism Theorem (Theorem 3.2, page 78 of [8]), we have $\pi_2(K) \approx H_2(K) \approx \mathsf{Z}$. Thus, there exists a (based) map $F$ :

$S^2 \to K$ representing the generator of $\pi_2(K)$ and, in this case, the induced homomorphisms $F_\# : \pi_1(S^2) \to \pi_1(K)$ and $F_\# : \pi_2(S^2) \to \pi_2(K)$ are both isomorphisms. By the Whitehead Theorem (Theorem 2.12, page 74 of [8]), $F$ is a homotopy equivalence.

We will now discuss the minimizing of the roots of an arbitrary cellular map $f : K \to S^2$, that is, we will determine the number $\mu(f)$. Let us remember that the requirement that $f$ is a cellular map is not a restriction, since every map from a complex into $S^2 = e^0_* \cup e^2_*$ has a cellular approximation and the minimum number of roots is a homotopy invariant.

By what we saw in the previous section, each cellular map $f : K \to S^2$ has a homological degree $d_f$ satisfying the equation $d_f = 3d_1^f + 2d_2^f - 6d_3^f$. We know that $\mu(f) = 0 \Leftrightarrow d_f = 0$. Consider then $d_f \neq 0$. We can assume, without loss of generality, $d_f > 0$. We have:

$$\zeta(K, d_f) = \min\left\{ \int (d_1, d_2, d_3) : d_f = 3d_1 + 2d_2 - 6d_3 \right\}.$$

The complete characterization of $\mu(f)$ in terms of $d_f$ is as follows:

**1.** *If $d_f = 1$ then $\mu(f) = 2$.*

In fact, $\zeta(K, d_f) = 2 = \int(1, -1, 0)$ and this show that $\mu(f) \leq 2$. On the other hand, if $1 = 3d_1 + 2d_2 - 6d_3$ with $d_3 \neq 0$ then also $d_1, d_2 \neq 0$ and $\int(d_1, d_2, d_3) = 3$. It follows that any map of homological degree equal to 1 (and thus any map homotopic to $f$) has at least a root in $K_1$ and a root in $K_2$. Thus $\mu(f) \geq 2$, since $K_1 \cap K_2 = \emptyset$.

**2.** *If $d_f$ is relatively prime to 6, then $\mu(f) = 2$.*

In fact, any integer relatively prime to 6 is of the form $3d_1 + 2d_2$ with $d_1, d_2 \neq 0$. Therefore $\zeta(K, d_f) = 2$ and $\mu(f) \leq 2$. Now, the same arguments as above can be applied to prove that $\mu(f) \geq 2$. This shows that $\mu(f) = 2$ and the roots are located one in $K_1$ and one in $K_2$.

**3.** *If $d_f$ is relatively prime to 2, then $d_f$ is odd and we have two cases:*

(a) *If $d_f \equiv 0 \mod 3$, then $\mu(f) = 1$.*

In this case $d_f$ is of the form $3d_1$ and, since $d_f$ is odd, it is not a multiple of 6. Then $\mu(f) = \zeta(K, d_f) = 1$, with the root located in $K_1$.

(b) *If $d_f \equiv 1 \mod 3$ or $d_f \equiv 2 \mod 3$, then $\mu(f) = 2$.*

In this case $d_f$ is of the form $3d_1 + 2d_2$ with $d_1, d_2 \neq 0$ or of the form $3d_1 + 2d_2 - 6d_3$ with $d_1, d_2, d_3 \neq 0$. It follows that $\mu(f) \leq \zeta(K, d_f) = 2$. On the other hand, again the same arguments of the item **1** show that $\mu(f) \geq 2$. Therefore $\mu(f) = 2$ and the roots are one in $K_1$ and one in $K_2$.

**4.** *If $d_f$ is relatively prime to 3, then we have the two following cases:*

(a) *If $d_f$ is even, then $\mu(f) = 1$.*

In this case $d_f$ is of the form $2d_2$ or of the form $2d_2 - 6d_3$ with $d_2, d_3 \neq 0$ and $d_f$ is not multiple of 6. Then $\mu(f) = \zeta(K, d_f) = 1$ and the root is located in $K_2$.

(b) *If $d_f$ is odd, then $\mu(f) = 2$.*

In this case or $d_f \equiv 1 \mod 3$ or $d_f \equiv 2 \mod 3$ and the result is identical to the case (b) of the previous item.

**5.** *If 6 divides $d_f$, then $\mu(f) = 1$.*

In this case $d_f$ can be of the form $3d_1$ or $2d_2$ or $-6d_3$, among other possibilities involving two or three terms, and it is evident that $\zeta(K, d_f) = 1$. Moreover, the root can be located anywhere in $K$.

Now, we present a more complicated example. We consider the 2-complex of the previous example and delete the cells $e_3^2$ and $e_3^1$ and identifies the 1-cells $e_1^1$ and $e_2^1$ to obtain a new 2-complex homotopy equivalent to the first. Then, we study the minimum number of roots of maps from this new 2-complex into the 2-sphere.

EXAMPLE 4.2. Let $\mathsf{RP}^2$ be the projective plane constructed by attaching a 2-cell $e_1^2$, through its boundary, into a 1-sphere by a map $S^1 \to S^1$ of degree 2. Also, let $\mathsf{P}_3^2$ be the pseudo-projective plane of degree 3, obtained by attaching a 2-cell $e_2^2$ into a 1-sphere, through its boundary, by a map $S^1 \to S^1$ of degree 3. Now, let $K$ be the 2-complex obtained by identifying the sphere $S^1$ corresponding to the 1-skeleton of $\mathsf{RP}^2$ with the sphere $S^1$ corresponding to the 1-skeleton of $\mathsf{P}_3^2$. The complex $K$ and its natural cellular decomposition induced by this construction are illustrated in Figure 2.

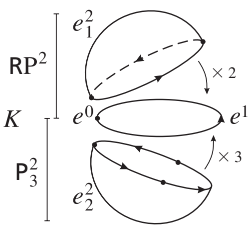


FIGURE 2. A 2-complex $K$ homotopy equivalent to the 2-sphere.

It is easy to check that $K$ is homotopy equivalent to the 2-sphere $S^2$ and $\zeta_K = 3e_1^2 + 2e_2^2$ represents a generator of the homology group $H_2(K) \approx \mathsf{Z}$.

We know that a map $f : K \to S^2$ is root free if and only if $d_f = 0$. We will prove now that:

*If $f : K \to S^2$ has nonzero homological degree, then $\mu(f) = 1$.*

Note that it is sufficient to prove that this result is true when $d_f = 1$. In fact: Suppose that $f : K \to S^2$ has homological degree $d_f = 1$ and minimum root number $\mu(f) = 1$. Then, let $\varphi : K \to S^2$ be a map homotopic to $f$ and let $a \in S^2$ be a point such that $\#\varphi^{-1}(a) = 1$, say $\varphi^{-1}(a) = \{b\}$. Given an integer $l \neq 0$, there is a map $g_l : S^2 \to S^2$ of degree $l$ such that $g_l^{-1}(a) = \{a\}$. Let $\varphi_l : K \to S^2$ be the map $\varphi_l = g_l \circ \varphi$. Then, the homological degree of $\varphi_l$ is equal to $l$ and, moreover, $\varphi_l^{-1}(a) = \{b\}$. Now, if $f' : K \to S^2$ is any map with homological degree $l$, then $f'$ is homotopic to $\varphi_l$ and so $\mu(f') = 1$.

Knowing this, we will prove that:

*There exists $f : K \to S^2$ with homological degree $d_f = 1$ such that $\mu(f) = 1$.*

The proof of this statement is quite long and, to facilitate the understanding, it will be divided into several steps.

At first, let $x_0 \in K^1$ be the point of the 1-skeleton $K^1$ of $K$ corresponding to the antipodal point of the 0-cell $e^0$.

STEP 1: We will define a map $\bar{f}_1 : \mathsf{RP}^2 \to S^2$ whose single root is the point $x_0$.

Note that $x_0$ is a point of the projective plane $\mathsf{RP}^2$ which belongs to its 1-skeleton. Let $x_1 \in \mathsf{RP}^2$ be a point in the interior of the 2-cell $e_1^2$ "near" $x_0$. Then, there is an open subset $V$ of $\mathsf{RP}^2$, homeomorphic to the open 2-ball, containing $x_0$ and $x_1$, and such that the closure of $K^1 \cap V$ in $\mathsf{RP}^2$ is a closed arc $\sigma$ which does not contain $e^0$. Obviously, $x_0$ is a point in the interior of $\sigma$. Now, Figure 3 illustrates precisely the construction of a homeomorphism $h : \mathsf{RP}^2 \to \mathsf{RP}^2$ homotopic to the identity map, such that $h|_{\mathsf{RP}^2 - V}$ is the identity map and $h(x_0) = x_1$, where $V = V_1 \cup V_2$.

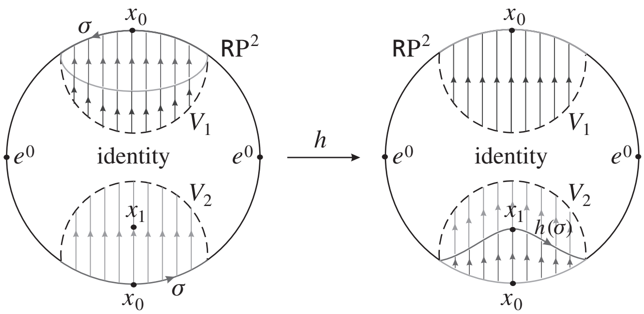


FIGURE 3. A specific homeomorphism of the projective plane.

Let $\omega_1 : \mathsf{RP}^2 \to S^2$ be the canonical quotient map which collapses the whole 1-skeleton of $\mathsf{RP}^2$ onto the 0-cell $e_*^0$ of $S^2$ and identifies the interior of the 2-cell

$e_1^2$ to the interior of the 2-cell $e_*^2$ of $S^2$. Let $a$ be the image of $x_1$ by $\omega_1$, that is, $a = \omega_1(x_1)$. Then $a$ is a point of the interior of the 2-cell $e_*^2$ of $S^2$ and we have $\omega_1^{-1}(a) = \{x_1\}$. Now, define the map $\bar{f}_1 : \mathsf{RP}^2 \to S^2$ to be the composition $\bar{f}_1 = \omega_1 \circ h$. Then $\bar{f}_1$ is a continuous (but not cellular) map homotopic to $\omega_1$ and such that $\bar{f}_1^{-1}(a) = \{x_0\}$.

Note that $\bar{f}_1$ carries the whole closed subset $K^1 - V$ of $K^1$ onto $e_*^0$ and the image of $\sigma$ by $\bar{f}_1$ is a simple closed path $\bar{f}_1(\sigma)$ in the sphere $S^2$, with base point $e_*^0$, containing the point $a$. See Figure 4.

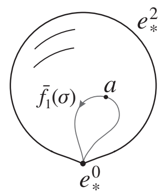


FIGURE 4. The simple closed path $\bar{f}_1(\sigma)$ in $S^2$.

STEP 2: We will define a map $\bar{f}_2 : \mathsf{P}_3^2 \to S^2$, which agrees with $\bar{f}_1$ on $K^1$, whose single root with respect to $a$ is the point $x_0$.

Let $\omega_2 : \mathsf{RP}^2 \to S^2$ be the canonical quotient map which collapses the whole 1-skeleton of $\mathsf{P}_3^2$ onto the 0-cell $e_*^0$ of $S^2$ and identifies the interior of the 2-cell $e_2^2$ to the interior of the 2-cell $e_*^2$ of $S^2$.

Let $x_2 = \omega_2^{-1}(a)$. (Remember that $a = \omega_1(x_1) = \bar{f}_1(x_0)$). Then $x_2 \in \mathsf{P}_3^2$ is a point in the interior of the 2-cell $e_2^2$ "near" $x_0$. Certainly, there exists an open subset $U$ of $\mathsf{P}_3^2$, containing the points $x_0$ and $x_2$, such that the closure of the intersection $U \cap K^1$ is exactly the closed arc $\sigma$ and, moreover, $U - \sigma$ is a disjoint union of three open subsets of $\mathsf{P}_3^2$, all homeomorphic to the open 2-ball. Figure 5 illustrates the closure of $U = U_1 \cup U_2 \cup U_3$ in $\mathsf{P}_3^2$.

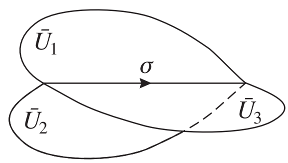


FIGURE 5. Closure of the open neighborhood $U$ of $x_0$ and $x_2$ in $\mathsf{P}_3^2$.

Now, Figure 6 illustrates precisely the construction of a map $g : \mathsf{P}_3^2 \to \mathsf{P}_3^2$, homotopic to the identity map, such that $g|_{\mathsf{P}_3^2 - U}$ is the identity map and $g(x_0) = x_2$.

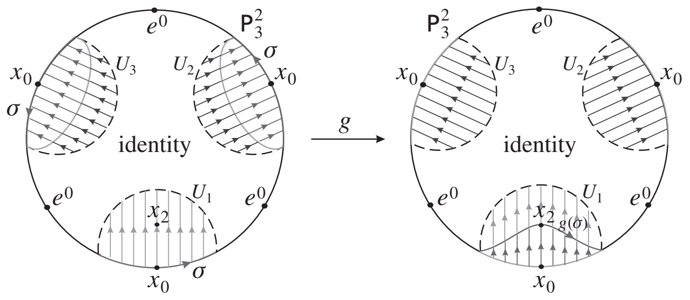


FIGURE 6. A specific self-map of the pseudo-projective plane $\mathsf{P}_3^2$.

It is clear that there is a cellular map $\omega_2' : \mathsf{P}_3^2 \to S^2$, homotopic to $\omega_2$, carrying the interior of the 2-cell $e_2^2$ of $\mathsf{P}_3^2$ homeomorphically onto the 2-cell $e_*^2$ of $S^2$ and such that $(\omega_2')^{-1}(a) = \{x_2\}$ and the closed path, corresponding to the image $\omega_2'(g(\sigma))$, coincides with $\bar{f}_1(\sigma)$.

Let $\bar{f}_2 : \mathsf{P}_3^2 \to S^2$ be the map defined by the composition $\bar{f}_2 = \omega_2' \circ g$. Then $\bar{f}_2$ is a continuous (but not cellular) map, homotopic to $\omega_2$ and such that $\bar{f}_2^{-1}(a) = x_0$.

To finalize this step, we note that, by construction, $\bar{f}_1|_{K^1} = \bar{f}_2|_{K^1}$.

STEP 3: We will define maps $f_1 : \mathsf{RP}^2 \to S^2$ and $f_2 : \mathsf{P}_3^2 \to S^2$ using $\bar{f}_1$ and $\bar{f}_2$.

Let $\gamma \subset S^2$ be the path (loop) $\bar{f}_1(\sigma)$, which coincides with the path (loop) $\bar{f}_2(\sigma)$, by Step 2. Let $\varphi_1 : S^2 \to S^2$ be a cellular map of degree 1 such that $\varphi_1^{-1}(a) = \{a\}$ and $\varphi_1(\gamma)$ is the "geodesic arc" $\gamma'$ connecting $e_*^0$ to $a$. Similarly, let $\varphi_2 : S^2 \to S^2$ be a cellular map of degree $-1$ such that $\varphi_2^{-1}(a) = \{a\}$ and $\varphi_2(\gamma)$ is also the "geodesic arc" $\gamma'$. See Figure 7.

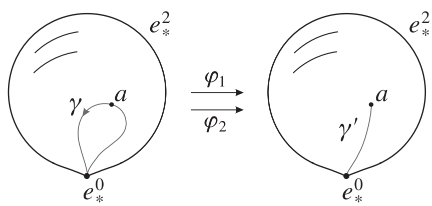


FIGURE 7. Specific self-maps of $S^2$ of degrees 1 and $-1$.

Define maps $f_1 : \mathsf{RP}^2 \to S^2$ and $f_2 : \mathsf{P}_3^2 \to S^2$ by the compositions $f_1 = \varphi_1 \circ \bar{f}_1$ and $f_2 = \varphi_2 \circ \bar{f}_2$. By construction, we have

$$f_1^{-1}(a) = \{x_0\} = f_2^{-1}(a).$$

STEP 4: Definition of $f$.

Note that the maps $f_1$ and $f_2$ coincide on $K^1$, that is, $f_1(x) = f_2(x)$ for all $x \in K^1$. Moreover, $K^1$ is a closed subset of $K = \mathsf{RP}^2 \cup_{K^1} \mathsf{P}^2_3$. Then, we can define a continuous map $f : K \to S^2$ such that $f|_{\mathsf{RP}^2} = f_1$ and $f|_{\mathsf{P}^2_3} = f_2$. Such a map $f$ satisfies

$$f^{-1}(a) = \{x_0\}.$$

Now, we will prove that the homological degree of $f$ is equal to 1, that is, $d_f = 1$. Since $f$ is not a cellular map, it is not easy to establish this assertion directly. However, $f$ has a natural cellular approximation. We will construct it: Remember that $\omega_1$ is a cellular approximation of $\bar{f}_1$ and $\omega_2$ is a cellular approximation of $\bar{f}_2$. We define $\varphi : K \to S^2$ as follows: Given $x \in K$, we chose an index $i(x) \in \{1, 2\}$ such that $x \in e^2_{i(x)}$. Then, we define $\varphi(x) = \varphi_{i(x)}(\omega_{i(x)}(x))$. Since each $\omega_i$ and $\varphi_i$, $i = 1, 2$, is cellular, the map $\varphi$ is well defined and is also cellular. Moreover, it is easy to see that $\varphi$ is a cellular approximation of $f$ and the maps $\varphi_1$ and $\varphi_2$ are a cellular factorization of $\varphi$. It follows that

$$d_f = d_\varphi = 3\deg(\varphi_1) + 2\deg(\varphi_2) = 3 - 2 = 1.$$

By the reasons presented in the beginning of the example, this completes what we wanted to demonstrate. Moreover, it is proved that, given a map $\psi : K \to S^2$ of nonzero homological degree, there exists a map homotopic to it, necessarily not cellular, having a single root, which necessarily belongs to $K^1$ and can be chosen to be any point of this subcomplex.

Let $K$ be the 2-complex of this latter example. Then, we have proved that any strongly surjectivity map $f : K \to S^2$ may be homotoped to a map with a single root, that is, $\mu(f) = 1$. Now, every such a map having homological degree $d_f = 2a + 3b$ with $ab \neq 0$ has number $\zeta(K, d_f) = 2$. This proves that, in fact, the converse of the second statement of Corollary 3.2 is not true.

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