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ON POLARS OF BLASCHKE-MINKOWSKI HOMOMORPHISMS

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Abstract

In this paper we establish Minkowski, Brunn-Minkowski, and Aleksandrov-Fenchel type inequalities for the volume difference of polars of Blaschke-Minkowski homomorphisms.

1. Introduction and statement of main results

The well-known classical Minkowski inequality and Brunn-Minkowski inequality can be stated as follows:

If K and L are convex bodies in R^n, then (see, e.g., [19])

$$V_1(K, L)^n \geq V(K)^{n-1} V(L),$$

and

$$V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}.$$

In each case, equality holds if and only if K and L are homothetic. Here, $+$ is usual Minkowski sum and $V_1(K, L)$ denotes the mixed volume of the convex bodies K and L defined by

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)\, dS(K, u),$$

where $h(L, u) = \max\{u \cdot x : x \in L\}$ is the support function of L and $S(K, u)$ is the surface area measure of K (see, e.g., [19]).

Let K and L be star bodies in R^n, then the dual Minkowski inequality and the dual Brunn-Minkowski inequality state that (see [15]).

$$\tilde{V}_1(K, L)^n \leq V(K)^{n-1} V(L),$$

and

$$V(K \mathbin{\tilde{+}} L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}.$$

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In each case, equality holds if and only if K and L are dilates. Here, $\tilde{+}$ is radial sum and $\tilde{V}_1(K, L)$ denotes the dual mixed volume of the star bodies K and L, defined by

$$\tilde{V}_1(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-1} \rho(L, u)\, dS(u),$$

where $\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}$ is the radial function of K and $S(u)$ is the spherical Lebesgue measure (see [4]).

In 2004 Leng [11] defined the volume difference function of compact domains D and K, where $D \subseteq K$, by

$$D_V(K, D) = V(K) - V(D).$$

The following Minkowski and Brunn-Minkowski type inequalities for volume difference functions were also established by Leng [11].

THEOREM A. *If K, L, D and D' are compact domains, $D \subseteq K$, $D' \subseteq L$, and D' is a homothetic copy of D, then*

$$(V_1(K, L) - V_1(D, D'))^n \geq (V(K) - V(D))^{n-1}(V(L) - V(D')), \tag{1.1}$$

and

$$(V(K + L) - V(D + D'))^{1/n} \geq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}. \tag{1.2}$$

In each case, equality holds if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

Recently, Lv [18] introduced the *dual volume difference function* for star bodies and established the following dual Minkowski and Brunn-Minkowski type inequalities for them:

THEOREM B. *If K, L, D and D' are star bodies in R^n, and $D \subseteq K$, $D' \subseteq L$, and L is a dilation of K, then*

$$(\tilde{V}_1(K, L) - (\tilde{V}_1(D, D'))^n \geq (V(K) - V(D))^{n-1}(V(L) - V(D')), \tag{1.3}$$

with equality if and only if D and D' are dilates and $(K, D)) = \mu(L, D')$, where μ is a constant, and

$$(V(K \tilde{+} L) - (V(D \tilde{+} D'))^{1/n} \geq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}, \tag{1.4}$$

with equality if and only if D and D' are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

In fact, more general versions on these types of inequalities were proved in [11] and [18], respectively. Moreover, inequalities for p-quermassintegral difference functions were established in [31].

Let $\mathscr{K}^n$ denote the space of convex bodies in R^n, i.e. compact, convex subsets of R^n with non-empty interior. The topology on $\mathscr{K}^n$ is induced by the Hausdorff metric.

DEFINITION 1.1 ([20]). A map $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ is called Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) Φ is continuous.

(b) For all $K, L \in \mathscr{K}^n$,

$$\Phi(K \ddot{+} L) = \Phi(K) + \Phi(L),$$

where $\ddot{+}$ denotes the Blaschke sum of the convex bodies K and L.

(c) For all $K, L \in \mathscr{K}^n$ and every $\vartheta \in SO(n)$,

$$\Phi(\vartheta K) = \vartheta \Phi(K),$$

where $SO(n)$ is the group of rotations in n dimensions.

Blaschke-Minkowski homomorphism is an important notion in the theory of convex body valued valuations (see, e.g., [1], [5], [8], [10], [12]–[14], [17], [21], [23]–[25], [30]). Their natural dual, radial Blaschke-Minkowski homomorphism, was introduced by Schuster [20] and further investigated to be meaningful (see [22]).

Let $\Phi(K_1, \ldots, K_{n-1})$ denote mixed Blaschke-Minkowski homomorphisms of convex bodies $K_1, \ldots, K_{n-1}$ (see Section 2). The convex body $\Phi(K_1, \ldots, K_{n-1})$ contains the origin in its interior, as was shown in [20]–[22].

If K is a convex body that contains the origin in its interior, the polar body of K is defined by

$$K^* := \{x \in \mathsf{R}^n \mid x \cdot y \le 1, y \in K\}.$$

Thus, the polar body $(\Phi(K_1, \ldots, K_{n-1}))^*$, in particular, $(\Phi K)^*$ is well defined. We will simply write $\Phi_i^*(K_1, \ldots, K_{n-1})$ and $\Phi^* K$ rather than $(\Phi(K_1, \ldots, K_{n-1}))^*$ and $(\Phi K)^*$. If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, then write $\Phi_i^* K$ for $\Phi^*(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$, and write $\Phi_i^*(K, L)$ for the mixed $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i})$. We write $\Phi_0^* K$ as $\Phi^* K$.

In 2006, Schuster [20] established the following Minkowski, Brunn-Minkowski, and Aleksandrov-Fenchel type inequalities for Blaschke-Minkowski homomorphisms.

THEOREM C. *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. If K, L are convex bodies in $\mathbb{R}^n$, then*

$$V(\Phi_1^*(K, L))^{n-1} \leq V(\Phi^* K)^{n-2} V(\Phi^* L), \tag{1.5}$$

with equality if and only if K and L are homothetic.

THEOREM D. *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. If K, L are convex bodies in $\mathbb{R}^n$, then*

$$V(\Phi^*(K + L))^{-1/n(n-1)} \geq V(\Phi^* K)^{-1/n(n-1)} + V(\Phi^* L)^{-1/n(n-1)}. \tag{1.6}$$

with equality if and only if K and L are homothetic.

THEOREM E. *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. If K_i $(1 \leq i \leq n-1)$ are convex bodies in $\mathbb{R}^n$, and $1 \leq r \leq n-1$, then*

$$V(\Phi^*(K_1, \ldots, K_{n-1}))^r \leq \prod_{j=1}^{r} \Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}). \tag{1.7}$$

Motivated by the work of Leng and Lv, we give the following definition:

DEFINITION 1.2. The volume difference function for polar Blaschke-Minkowski homomorphism of convex bodies K and D, $D_V(\Phi^* K, \Phi^* D)$, is defined by

$$D_V(\Phi^* K, \Phi^* D) = V(\Phi^* K) - V(\Phi^* D).$$

The aim of this paper is to establish the following Minkowski, Brunn-Minkowski, and Aleksandrov-Fenchel type inequalities for volume difference of polars of Blaschke-Minkowski homomorphisms.

THEOREM C′. *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. If $D, D', K, L \in \mathscr{K}^n$, $V(\Phi^*(D)) \leq V(\Phi^*(K))$ and $V(\Phi^*(D')) \leq V(\Phi^*(L))$, and L is a homothetic copy of K, then*

$$\begin{aligned}[V(\Phi_1^*(K, L)) - V(\Phi_1^*(D, D'))]^{n-1} \\ \geq [V(\Phi^* K) - V(\Phi^* D)]^{n-2}[V(\Phi^* L) - V(\Phi^* D')],\end{aligned} \tag{1.8}$$

with equality if and only if D and D' are homothetic and $(V(\Phi^* K), V(\Phi^* L)) = \mu(V(\Phi^* D), V(\Phi^* D'))$, where μ is a constant.

Theorem C′ just is a special case of Theorem 4.3 established in Section 4.

THEOREM D′. *Let* $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ *be an even Blaschke-Minkowski homomorphism. If* $D, D', K, L \in \mathcal{K}^n$, $V(\Phi^*(D)) \le V(\Phi^*(K))$ *and* $V(\Phi^*(D')) \le V(\Phi^*(L))$, *and* L *is a homothetic copy of* K, *then*

$$\begin{aligned}(1.9)\quad & [V(\Phi^*(K+L)) - V(\Phi^*(D+D'))]^{-1/n(n-1)} \\ & \le [V(\Phi^*K) - V(\Phi^*D)]^{-1/n(n-1)} + [V(\Phi^*L) - V(\Phi^*D')]^{-1/n(n-1)},\end{aligned}$$

with equality if and only if D *and* D' *are homothetic and* $(V(\Phi^*K), V(\Phi^*L)) = \mu(V(\Phi^*D), V(\Phi^*D'))$, *where* μ *is a constant.*

Theorem D′ just is a special case of Theorem 4.1 established in Section 4.

THEOREM E′. *Let* $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ *be an even Blaschke-Minkowski homomorphism. If* K_i *and* D_i $(1 \le i \le n-1)$ *are convex bodies in* R^n,

$$\begin{aligned} & V(\Phi^*(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_{n-1})) \\ & \qquad \ge V(\Phi^*(\underbrace{D_j, \dots, D_j}_{r}, D_{r+1}, \dots, D_{n-1})),\end{aligned}$$

and K_j $(j = 1, \dots, r)$ *be homothetic copies of each other, then*

$$\begin{aligned}(1.10)\quad & [V(\Phi^*(K_1, \dots, K_{n-1})) - V(\Phi^*(D_1, \dots, D_{n-1}))]^r \\ & \ge \prod_{j=1}^{r} D_V\big(\Phi^*(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_{n-1}), \\ & \qquad \Phi^*(\underbrace{D_j, \dots, D_j}_{r}, D_{r+1}, \dots, D_{n-1})\big).\end{aligned}$$

2. Definitions and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathsf{R}^n (n > 2)$. Let $\mathcal{K}^n$ denote the set of all convex bodies (compact, convex subsets with nonempty interiors) in R^n. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. If $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, then write $v(K_1^u, \dots, K_{n-1}^u)$ for the mixed volume of the figures $K_1^u, \dots, K_{n-1}^u$ in the space E_u. If $K_1 = \cdots = K_{n-1} = K$, then write $v(K^u)$ for $v(K^u, \dots, K^u)$.

We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K,\cdot): S^{n-1} \to \mathrm{R}$, denote the support function of $K \in \mathscr{K}^n$; i.e. for $u \in S^{n-1}$

$$h(K,u) = \max\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product u and x in R^n.

Let δ denote the Hausdorff metric on $\mathscr{K}^n$, i.e., for $K, L \in \mathscr{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

2.1. Mixed volumes

If $K_i \in \mathscr{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial in λ_i given by

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (2.1), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is usually written $V_i(K,L)$. If $L = B$, then $V_i(K,B)$ is the i-th projection measure (Quermassintegral) of K and is written as $W_i(K)$.

2.2. Projection bodies and mixed projection bodies

If $K \in \mathscr{K}^n$, then the projection body of convex body K will be denoted as $\mathbf{\Pi} K$ and whose support function is defined by

$$h(\mathbf{\Pi} K, u) = v(K^u), \qquad u \in S^{n-1}. \tag{2.2}$$

If $K_1, \ldots, K_r \in \mathscr{K}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, then the projection body of the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathscr{K}^n$ can be written as a symmetric homogeneous polynomial of degree $(n-1)$ in the λ_i ([17]):

$$\mathbf{\Pi}(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \lambda_{i_1} \ldots \lambda_{i_{n-1}} \mathbf{\Pi}_{i_1 \cdots i_{n-1}}, \tag{2.3}$$

where the sum is a Minkowski sum taken over all $(n-1)$-tuples $(i_1, \ldots, i_{n-1})$ of positive integers not exceeding r. The body $\mathbf{\Pi}_{i_1 \ldots i_{n-1}}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_{n-1}}$, and is uniquely determined by (2.3), it is called *the mixed projection bodies* of $K_{i_1}, \ldots, K_{i_{n-1}}$, and is written as $\mathbf{\Pi}(K_i, \ldots, K_{i_{n-1}})$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{\Pi}(K_{i_1}, \ldots, K_{i_{n-1}})$

will be written as $\mathbf{\Pi}_i(K, L)$. If $L = B$, then $\mathbf{\Pi}_i(K, L)$ is denoted $\mathbf{\Pi}_i K$ and when $i = 0$, $\mathbf{\Pi}_i K$ is denoted $\mathbf{\Pi} K$.

The support function of mixed projection bodies of $K_1, \ldots, K_{n-1}$ given by

$$h(\mathbf{\Pi}(K_1, \ldots, K_{n-1}), u) = v(K_1^u, \ldots, K_{n-1}^u). \tag{2.4}$$

2.3. Mixed Blaschke-Minkowski homomorphisms

There is a continuous operator (see [20])

$$\Phi : \underbrace{\mathscr{K}^n \times \cdots \times \mathscr{K}^n}_{n-1} \to \mathscr{K}^n,$$

symmetric in its arguments such that, for $K_1, \ldots, K_r$ and $\lambda_1, \ldots, \lambda_r \geq 0$,

$$\Phi(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \ldots \lambda_{n-1} \Phi(K_{i_1}, \ldots, K_{i_{n-1}}).$$

Clearly, above the continuous operator generalizes the notion of Blaschke-Minkowski homomorphism. We call

$$\Phi : \underbrace{\mathscr{K}^n \times \cdots \times \mathscr{K}^n}_{n-1} \to \mathscr{K}^n$$

the mixed Blaschke-Minkowski homomorphism induced by Φ. The mixed Blaschke-Minkowski homomorphisms were first studied in more detail in [20]. If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, we write $\Phi_i K$ for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$ and call Φ_i the mixed Blaschke-Minkowski homomorphism of order i. For $0 \leq i \leq n$, we write $\Phi_i(K, L)$ for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i})$. We write $\Phi_0 K$ as ΦK.

3. Auxiliary Results

The following results will be required to prove our main theorems.

LEMMA 3.1 ([20]). *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. If $K, L \in \mathscr{K}^n$, and $0 \leq j \leq n-3$, then*

$$\begin{aligned} V(\Phi_j^*(K+L))^{-1/(n-1)(n-1-j)} \\ \geq V(\Phi_j^* K)^{-1/(n-1)(n-1-j)} + V(\Phi_j^* L)^{-1/(n-1)(n-1-j)}, \end{aligned} \tag{3.1}$$

with equality if and only if K and L are homothetic.

LEMMA 3.2 ([2], p. 38, Reversed Bellman's inequality). *Let $a = \{a_1, \dots, a_n\}$ and $b = \{b_1, \dots, b_n\}$ be two series of positive real numbers and $p < 0$ (or $0 < p < 1$) such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \geq \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{1/p}, \tag{3.2}$$

with equality if and only if $a = \upsilon b$ where υ is a constant.

The inequality is reversed for $p > 1$.

LEMMA 3.3 ([20]). *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. If $K, L \in \mathscr{K}^n$ and $0 \leq j \leq n-2$, then*

$$V(\Phi_j^*(K, L))^{1/(n-1)} \leq V(\Phi^* K)^{n-j-1} + V(\Phi^* L)^j, \tag{3.3}$$

with equality if and only if K and L are homothetic.

LEMMA 3.4 ([31]). *If $a, b, c, d > 0$, $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta = 1$. Let $a > b$ and $c > d$, then*

$$a^\alpha c^\beta - b^\alpha d^\beta \geq (a-b)^\alpha (c-d)^\beta, \tag{3.4}$$

with equality if and only if $a/b = c/d$.

LEMMA 3.5 ([20]). *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. If $K_1, \dots, K_1 \in \mathscr{K}^n$, and $1 \leq r \leq n-1$, then*

$$V(\Phi^*(K_1, \dots, K_{n-1}))^r \leq \prod_{j=1}^r V(\Phi^*(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_n)). \tag{3.5}$$

LEMMA 3.6 ([2], p. 26). *If $c_i > 0$, $b_i > 0$, $c_i > b_i$, $i = 1, \dots, n$, then*

$$\left(\prod_{i=1}^n (c_i - b_i)\right)^{1/n} \leq \left(\prod_{i=1}^n c_i\right)^{1/n} - \left(\prod_{i=1}^n b_i\right)^{1/n}, \tag{3.6}$$

with equality if and only if $c_1/b_1 = c_2/b_2 = \cdots = c_n/b_n$.

4. Inequalities for volume differences of polar Blaschke-Minkowski homomorphisms

4.1. Brunn-Minkowski-type inequalities

In the following we establish the Brunn-Minkowski inequality for volume differences of Blaschke-Minkowski homomorphisms stated in the introduction.

In fact, Theorem D′ is just the special case $j = 0$ of

THEOREM 4.1. *Let $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ be an even Blaschke-Minkowski homomorphism. Let D, D', K and L be convex bodies in R^n, $V(\Phi_j^* D) \le V(\Phi_j^* K)$ and $V(\Phi_j^* D') \le V(\Phi_j^* L)$, and let L be a homothetic copy of K. If $0 \le j < n-1$, then*

$$[V(\Phi_j^*(K+L)) - \Phi_j^*(D+D')]^{-1/n(n-j-1)} \le [V(\Phi_j^* K) - V(\Phi_j^* D)]^{-1/n(n-j-1)} + [V(\Phi_j^* L) - V(\Phi_j^* D')]^{-1/n(n-j-1)}, \tag{4.1}$$

with equality if and only if D and D' are homothetic and $(V(\Phi^* K), V(\Phi^* L)) = \mu(V(\Phi^* D), V(\Phi^* D'))$, where μ is a constant.

PROOF. By Lemma 3.1, we have

$$V(\Phi_j^*(D+D'))^{-1/(n-1)(n-j-1)} \ge V(\Phi_j^* D)^{-1/(n-i)(n-j-1)} + V(\Phi_j^* D')^{-1/(n-i)(n-j-1)}, \tag{4.2}$$

with equality if and only if D and D' are homothetic. Since L is a homothetic copy of K, note that

$$V(\Phi_j^*(K+L))^{-1/(n-1)(n-j-1)} = V(\Phi_j^* K)^{-1/(n-i)(n-j-1)} + V(\Phi_j^* L)^{-1/(n-i)(n-j-1)}. \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$\begin{aligned} D_V\big(\Phi_j^*(K+L), \Phi_j^*(D+D')\big)^{-1/n(n-j-1)} &\le \Big\{\big[V(\Phi_j^* K)^{-1/n(n-j-1)} + V(\Phi_j^* L)^{-1/(n-i)(n-j-1)}\big]^{-n(n-j-1)} \\ &- \big[V(\Phi_j^* D)^{-1/n(n-j-1)} + V(\Phi_j^* D')^{-1/n(n-j-1)}\big]^{-n(n-j-1)}\Big\}^{-1/n(n-j-1)}, \end{aligned} \tag{4.4}$$

with equality if and only if D and D' are homothetic.

From (4.4) and an application of Bellman's inequality, Lemma 3.2, we thus obtain the desired inequality

$$D_V(\Phi_j^*(K+L), \Phi_j^*(D+D'))^{-1/n(n-j-1)} \\ \leq (V(\Phi_j^* K) - V(\Phi_j^* D))^{-1/n(n-j-1)} + (V(\Phi_j^* L) - V(\Phi_j^* D'))^{-1/n(n-j-1)}.$$

By the equality conditions of inequalities (4.4) and (3.2), equality holds in (4.1) if and only if D and D' are homothetic and $(V(\Phi_j^* K), V(\Phi_j^* L)) = \mu(V(\Phi_j^* D), V(\Phi_j^* D'))$, where μ is a constant.

Since the projection body operator $\mathbf{\Pi} : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke-Minkowski homomorphism, we have

COROLLARY 4.2. *Let D, D', K and L be convex bodies in R^n, $K \subseteq D$, $L \subseteq D'$ and let L be a homothetic copy of K. If $0 \leq j < n-1$, then*

$$D_V(\mathbf{\Pi}_j^*(K+L), \mathbf{\Pi}_j^*(D+D'))^{-1/n(n-j-1)} \\ \leq (V(\mathbf{\Pi}_j^* K) - V(\mathbf{\Pi}_j^* D))^{-1/n(n-j-1)} \\ + (V(\mathbf{\Pi}_j^* L) - V(\mathbf{\Pi}_j^* D'))^{-1/n(n-j-1)}, \tag{4.5}$$

with equality if and only if D and D' are homothetic and $(V(\mathbf{\Pi}_j^* K), V(\mathbf{\Pi}_j^* L)) = \mu(V(\mathbf{\Pi}_j^* D), V(\mathbf{\Pi}_j^* D'))$, where μ is a constant.

4.2. Minkowski-type inequalities

In the following we establish the Minkowski inequality for volume differences of Blaschke-Minkowski homomorphisms stated in the introduction.

In fact, Theorem C$'$ is just the special case $j = 1$ of

THEOREM 4.3. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke-Minkowski homomorphism. Let D, D', K and L be convex bodies in R^n, $V(\Phi^*(D)) \leq V(\Phi^*(K))$ and $V(\Phi^*(D')) \leq V(\Phi^*(L))$, and let L is a dilated copy of K. If $1 \leq j < n-1$, then*

$$D_V(\Phi_j^*(K, L), \Phi_j^*(D, D')) \\ \geq (V(\Phi^* K) - V(\Phi^* D))^{(n-j-1)/(n-1)} (V(\Phi^* L) - V(\Phi^* D'))^{j/(n-1)}, \tag{4.6}$$

with equality if and only if D and D' are homothetic and $(V(\Phi^* K), V(\Phi^* L)) = \mu(V(\Phi^* D), V(\Phi^* D'))$, where μ is a constant.

PROOF. By Lemma 3.3, we have

$$V(\Phi_j^*(D, D'))^{n-1} \leq V(\Phi^* D)^{n-j-1} V(\Phi^* D')^j,$$

with equality if and only if D and D' are homothetic. Since L is a homothetic copy of K, note that

$$V(\Phi_j^*(K, L))^{n-1} = V(\Phi^* K)^{n-j-1} V(\Phi^* L)^j.$$

Therefore, in view of $\frac{n-j-1}{n-1} + \frac{j}{n-1} = 1$ by Lemma 3.4, we obtain

$$\begin{aligned} D_V(\Phi_j^*(K, L), \Phi_j^*(D, D')) &\geq V(\Phi^* K)^{(n-j-1)/(n-1)} V(\Phi^* L)^{j/(n-1)} \\ &\quad - V(\Phi^* D)^{(n-j-1)/(n-1)} V(\Phi^* D')^{j/(n-1)} \\ &\geq (V(\Phi^* K) - V(\Phi^* D))^{(n-j-1)/(n-1)} (V(\Phi^* L) - V(\Phi^* D'))^{j/(n-1)}. \end{aligned}$$

By the equality conditions of Lemma 3.3 and (3.4), equality holds if and only if D and D' are homothetic and $(V(\Phi^* K), V(\Phi^* L)) = \mu(V(\Phi^* D), V(\Phi^* D'))$, where μ is a constant.

If we take the projection body operator $\mathbf{\Pi}$ as the Blaschke-Minkowski homomorphism in Theorem 4.3, we have the following

COROLLARY 4.4. *Let D, D', K and L be convex bodies in R^n, $K \subseteq D$, $L \subseteq D'$, and let L be a homothetic copy of K. If $1 \leq j < n-1$, then*

$$\begin{aligned} D_V(\mathbf{\Pi}_j^*(K, L), \mathbf{\Pi}_j^*(D, D')) \\ \geq (V(\mathbf{\Pi}^* K) - V(\mathbf{\Pi}^* D))^{(n-j-1)/(n-1)} (V(\mathbf{\Pi}^* L) - V(\mathbf{\Pi}^* D'))^{j/(n-1)}, \end{aligned}$$

with equality if and only if D and D' are homothetic and $(V(\mathbf{\Pi}^* K), V(\mathbf{\Pi}^* L)) = \mu(V(\mathbf{\Pi}^* D), V(\Phi^* D'))$, where μ is a constant.

4.3. Aleksandrov-Fenchel-type inequalities

The Aleksandrov-Fenchel inequality for volume differences of polar mixed Blaschke-Minkowski homomorphisms stated in the introduction will be established as follows:

THEOREM 4.5. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke-Minkowski homomorphism. If K_i and D_i $(1 \leq i \leq n-1)$ are convex bodies in R^n,*

$$\begin{aligned} V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) \\ \geq V(\Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})), \end{aligned}$$

and K_j $(j=1,\dots,r)$ be homothetic copies of each other, then

$$[V(\Phi^*(K_1,\dots,K_{n-1}))-V(\Phi^*(D_1,\dots,D_{n-1}))]^r \geq \prod_{j=1}^{r} D_V\big(\Phi^*(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_{n-1}), \Phi^*(\underbrace{D_j,\dots,D_j}_{r},D_{r+1},\dots,D_{n-1})\big). \tag{4.7}$$

PROOF. By Lemma 3.5, we have

$$V(\Phi^*(D_1,\dots,D_{n-1}))^r \leq \prod_{j=1}^{r} V(\Phi^*(\underbrace{D_j,\dots,D_j}_{r},D_{r+1},\dots,D_{n-1})).$$

Suppose $K_j(j=1,\dots,r$ are homothetic copies of each other, we have

$$V(\Phi^*(K_1,\dots,K_{n-1}))^r = \prod_{j=1}^{r} V(\Phi^*(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_{n-1})).$$

Hence

$$V(\Phi^*(K_1,\dots,K_{n-1}))-V(\Phi^*(D_1,\dots,D_{n-1})) \geq \left(\prod_{j=1}^{r} V(\Phi^*(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_{n-1}))\right)^{1/r} - \left(\prod_{j=1}^{r} V(\Phi^*(\underbrace{D_j,\dots,D_j}_{r},D_{r+1},\dots,D_{n-1}))\right)^{1/r}, \tag{4.8}$$

with equality if and only if $D_1,\dots,D_r$ are homothetic.

By using Lemma 3.6 in (4.8), we obtain

$$D_V(\Phi^*(K_1,\dots,K_{n-1}),\Phi^*(D_1,\dots,D_{n-1})) \geq \left(\prod_{j=1}^{r}\big[V(\Phi^*(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_{n-1})) - V(\Phi^*(\underbrace{D_j,\dots,D_j}_{r},D_{r+1},\dots,D_{n-1}))\big]\right)^{1/r}$$

$$= \prod_{j=1}^{r} D_V\big(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}),$$
$$\Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})\big)^{1/r}.$$

If we take the projection body operator $\mathbf{\Pi}$ as the Blaschke-Minkowski homomorphism in Theorem 4.5, we have

COROLLARY 4.6. *If K_i and D_i, $1 \le i \le n-1$, are convex bodies in R^n, $K_i \subseteq D_i$ and K_j ($j = 1, \ldots, r, 1 \le r \le n-1$) be homothetic copies of each other, then*

$$(4.9)\quad (V(\mathbf{\Pi}^*(K_1, \ldots, K_{n-1})) - V(\mathbf{\Pi}^*(D_1, \ldots, D_{n-1})))^r$$
$$\ge \prod_{j=1}^{r} D_V\big(\mathbf{\Pi}^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n),$$
$$\mathbf{\Pi}^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_n)\big).$$

Moreover, Zhao [32] defined the volume sum function of polar projection bodies of convex D and K, by

$$S_V(\mathbf{\Pi}^*K, \mathbf{\Pi}^*D) = V(\mathbf{\Pi}^*K) + V(\mathbf{\Pi}^*D).$$

We finally remark that inequalities for the sum function of polar of mixed projection bodies were established in [32], inequalities for L_p-intersection bodies were established in [3], [6], [7], [26], [28]–[29] and [33], and for L_p-mixed intersection bodies in [28].

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