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QUASI-DIAGONAL FLOWS II

A. KISHIMOTO

Abstract

Two similar notions defined for flows, quasi-diagonality and pseudo-diagonality, are shown to be equivalent; so approximately inner flows on a quasi-diagonal C^*-algebra are quasi-diagonal (not just pseudo-diagonal). We define a notion of MF flow which is weaker than quasi-diagonality and study equivalent conditions following Blackadar and Kirchberg's results on MF algebras and we characterize the dual flow of such on the crossed product as a dual MF flow. In the same spirit we introduce a notion of NF flow and show that NF flows are MF flows on nuclear C^*-algebras, or equivalently, quasi-diagonal flows on nuclear C^*-algebras. We also introduce a notion of strong quasi-diagonality (in parallel with strong quasi-diagonality versus quasi-diagonality for C^*-algebras), whose examples contain AF flows.

1. Introduction

We mean by a *flow* a strongly continuous one-parameter automorphism group of a C^*-algebra. We refer to [4], [14] for some background on flows. We are particularly interested in approximately inner flows since they have close relevance to applications to physics and were a cause for C^*-algebras to have been introduced. But we are still trying to understand the situations surrounding approximately inner flows (see, e.g., [3], [8]).

We have defined two similar notions for flows on C^*-algebras: pseudo-diagonality and quasi-diagonality, in [11], which are naturally derived from the notion of quasi-diagonality for C^*-algebras (e.g., [15], [16]). But as we shall see in this note they are in fact equivalent. Thus quasi-diagonality holds for approximately inner flows on quasi-diagonal C^*-algebras. For example if α is an approximately inner flow on an AF algebra A then there is a covariant representation (π, U) of (A, α) such that π is faithful and $(\pi(A), U)$ is quasi-diagonal, i.e., $\|[E_n, \pi(x)]\| \to 0$ for $x \in A$ and $\sup\{\|[E_n, U_t]\| \mid -1 \le t \le 1\} \to 0$ for some increasing sequence (E_n) of finite-rank projections on $\mathcal{H}_\pi$ with $\lim_n E_n = 1$. Note also that for *any* covariant representation (π, U) there is an increasing sequence (E_n) of finite-rank projections on $\mathcal{H}_\pi$ with $\lim_n E_n = 1$ and a sequence (V_n) of unitary flows such that $V_{n,t}E_n = V_{n,t}$ and $\|[E_n, \pi(x)]\| \to 0$ and $\|E_n\pi\alpha_t(x)E_n - V_{n,t}E_n\pi(x)E_nV_{n,t}^*\| \to 0$ uniformly in t on every compact subset of R for any $x \in A$. If a covariant representation

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(π, U) induces a faithful representation of the crossed product then $(\pi(A), U)$ is quasi-diagonal by Voiculescu's theorem (Theorem 3.1 of [11]). Thus an approximately inner flow on an AF algebra can be approximated by flows on finite-dimensional C^*-algebras in a sense.

We have also noted in [11] we could define a notion of MF flows when the C^*-algebra is separable, which is derived from pseudo-diagonality, following the notion of MF algebras introduced and studied by Blackadar and Kirchberg [1]. We will examine this notion closely following [1]. See Theorem 3.10 for equivalent conditions.

Let us be specific about the definition of MF flows. Let M_n be the C^*-algebra of $n \times n$ matrices. Any flow on M_n is given as $t \mapsto \mathrm{Ad}\, e^{ith}$ with $h = h^* \in M_n$. Let (k_n) be a sequence of natural numbers and let $B = \prod_{n=1}^{\infty} M_{k_n}$ be the C^*-algebra consisting of bounded sequences (x_n) with $x_n \in M_{k_n}$. Let β_n be a flow on M_{k_n} and let $\beta_t = \prod \beta_{n,t}$, $t \in \mathsf{R}$ as automorphisms of B. Since $t \mapsto \beta_t$ is not continuous on B in general, we let B_β be the maximal C^*-subalgebra of B on which $t \mapsto \beta_t$ is continuous. Thus β restricts to a flow on B_β. Let $I = \bigoplus_{n=1}^{\infty} M_{k_n}$ be the C^*-algebra consisting of sequences converging to zero, which is an ideal of B contained in B_β and is left invariant under β. We denote by the same symbol β the flow on B_β/I induced from β on B_β.

When α is a flow on a separable C^*-algebra A we consider the following conditions:

(1) There is an isomorphism ϕ of A into B_β/I such that $\phi\alpha_t = \beta_t\phi$ (for some $B = \prod_{n=1}^{\infty} M_{k_n}$ and $\beta = \prod_{n=1}^{\infty} \beta_n$).

(2) There is a completely positive (CP) contraction ϕ of A into B_β such that $Q\phi$ is an isomorphism and $Q\phi\alpha_t = \beta_t Q\phi$, where Q is the quotient map of B_β onto B_β/I.

(3) There is an isomorphism ϕ of A into B_β such that $\phi\alpha_t = \beta_t\phi$.

We will call α an *MF flow* if it satisfies the first condition. The second condition on α is equivalent to α's being quasi-diagonal (by Theorem 2.3), which is stronger than the first in general (if A is not nuclear). We will call α an *RF flow* if it satisfies the third condition. In this case A is residually finite-dimensional as a C^*-algebra. This is stronger than the second because, if $Q\phi$ is not an injection or $\phi(A) \cap I$ is non-zero, there is another (B, β), where B may be obtained by repeating an infinite copies of each M_{k_n} from the original B, and an isomorphism ψ of A into this new B_β such that $Q\psi$ is an isomorphism and $Q\psi\alpha_t = \beta_t Q\psi$. We note in 3.5 that we may replace all (M_{k_n}, β_n) by a single $(\mathscr{K}, \mathrm{Ad}\,\lambda)$ in the definition of MF flows, where $\mathscr{K}$ is the C^*-algebra of compact operators on $L^2(\mathsf{R})$ and λ is the unitary flow defined by $(\lambda_t\xi)(s) = \xi(s-t)$. We also note in 3.12 that an MF flow is obtained as a quotient of an RF flow. As in the case of pseudo-diagonal flows, if α is an MF flow on a unital C^*-algebra, it has KMS states for all inverse temperatures as shown in 3.14. This is what motivates us to introduce MF flows. We shall also introduce a notion of dual MF flows; α is a *dual MF flow* on A if (A, α) is realized in $\big((\prod M_{k_n} \otimes C_0(\mathsf{R}))_\gamma / \bigoplus M_{k_n} \otimes C_0(\mathsf{R}), \gamma\big)$ for some (k_n) where $\gamma = \prod \gamma_n$ and γ_n is the flow induced from translations on R. It follows in 3.19 that α is an MF flow (resp. a dual MF flow) if and only if $\hat{\alpha}$ is a dual MF flow (resp. a MF flow) on the crossed product $A \times_\alpha \mathsf{R}$.

We will also define a notion of *NF flows* following [1] and study some equivalent conditions in Theorem 4.7. It will turn out that an NF flow is an MF flow on a nuclear C^*-algebra as expected and has a characterization in terms of CP contractions through finite-dimensional C^*-algebras as follows: There is a sequence of flows (B_n, β_n) with B_n finite-dimensional and CP contractions $\sigma_n : A \to B_n$ and $\tau_n : B_n \to A$ such that $\tau_n \sigma_n \to \mathrm{id}$, $\|\sigma_n(xy) - \sigma_n(x)\sigma_n(y)\| \to 0$ for all $x, y \in A$, and $\|\sigma_n \alpha_t - \beta_{n,t}\sigma_n\| \to 0$ uniformly in t on every compact subset of R. By the way quasi-diagonality is characterized without τ_n in the above condition replacing $\tau_n\sigma_n \to \mathrm{id}$ by $\|\sigma_n(x)\| \to \|x\|$, $x \in A$ (see Theorem 1.5 of [11]).

We will also define a notion of *strongly quasi-diagonal flows*, which is naturally stronger than quasi-diagonality, and note that such a flow on a separable C^*-algebra is obtained as the limit of a *canonical* increasing sequence of RF flows after a cocycle perturbation (see 5.9 for details). An AF flow is strongly quasi-diagonal (see 5.6), where an AF flow is defined as the limit of an increasing sequence of FD flows (i.e., flows on finite-dimensional algebras).

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2. Quasi-diagonal flows

First we note the following result, which we should have noticed before.

PROPOSITION 2.1. *Pseudo-digonality and quasi-diagonaliy for flows are equivalent.*

PROOF. We shall show that the condition (2) of Theorem 1.6 of [11] implies the condition (2) of Theorem 1.5 of [11]. The converse is trivial.

Let α be a pseudo-diagonal flow on A. Hence α satisfies the following condition: For any finite subset $\mathscr{F}$ of A, $T > 0$, and $\delta > 0$ there is a finite-dimensional C^*-algebra B, a flow β on B and a CP map ϕ of A into B such that $\|\phi\| \leq 1$, $\|\phi(x)\| \geq (1-\delta)\|x\|$ for $x \in \mathscr{F}$ and $\|\phi(x)\phi(y) - \phi(xy)\| \leq \delta\|x\|\|y\|$ for $x, y \in \mathscr{F}$, and $\|\beta_t\phi(x) - \phi\alpha_t(x)\| \leq \epsilon\|x\|$ for $x \in \mathscr{F}$ and $t \in [-T, T]$. This is slightly different from the condition (2) of 1.6 of [11] but they are equivalent as we can see easily. Especially we have allowed T to be arbitrarily large instead of fixing it to be 1.

Let $\epsilon > 0$ be smaller than 1. We define a CP map ψ of A into B by

$$\psi = \frac{\epsilon}{2}\int e^{-\epsilon|t|}\beta_{-t}\phi\alpha_t\,dt.$$

For $x \in \mathscr{F}$ we compute

$$\begin{aligned}\|\psi(x) - \phi(x)\| &\leq \frac{\epsilon}{2}\int e^{-\epsilon|t|}\|\beta_{-t}\phi\alpha_t(x) - \phi(x)\|\,dt \\ &\leq \frac{\epsilon}{2}\int_{-T}^{T} e^{-\epsilon|t|}\delta\|x\|\,dt + \epsilon\|x\|\int_{|t|\geq T} e^{-\epsilon|t|}\,dt \\ &\leq (\delta + 2e^{-\epsilon T})\|x\|.\end{aligned}$$

Thus if we set $\delta = \epsilon/2$ and $T = \epsilon^{-1}\log(4/\epsilon)$, we obtain that $\|\psi(x) - \phi(x)\| \leq \epsilon\|x\|$, $x \in \mathscr{F}$. Hence we have that $\|\psi(x)\| \geq (1-2\epsilon)\|x\|$ for $x \in \mathscr{F}$ and $\|\psi(x)\psi(y) - \psi(xy)\| \leq 3\epsilon\|x\|\|y\| + \|\phi(x)\phi(y) - \phi(xy)\| \leq 4\epsilon\|x\|\|y\|$ for $x, y \in \mathscr{F}$.

Since

$$\beta_{-t}\psi\alpha_t = \frac{\epsilon}{2}\int e^{-\epsilon|s-t|}\beta_{-s}\phi\alpha_s\,ds$$

and $|s| \leq |s-t| + |t|$ and $|s-t| \leq |s| + |t|$, we obtain that $\|\beta_{-t}\psi\alpha_t - \psi\| \leq e^{\epsilon|t|} - 1$ or $\|\beta_t\psi - \psi\alpha_t\| \leq e^{\epsilon} - 1$ for $t \in [-1, 1]$. Thus the condition (2) of Theorem 1.5 of [11] is satisfied with ψ in place of ϕ starting with a smaller ϵ.

REMARK 2.2. Suppose that α is an approximately inner flow on a quasi-diagonal C^*-algebra A. Then $(\pi(A), U)$ is pseudo-diagonal for any covariant representation (π, U) of (A, α) (see the proof of Proposition 2.17 of [11]). It follows from the above proof that for any covariant representation (π, U) there is a covariant representation (ρ, W) such that $\operatorname{Ker}\rho = \operatorname{Ker}\pi$ and $(\rho(A), W)$ is quasi-diagonal.

THEOREM 2.3. *Let α be a flow on a separable C^*-algebra A. Then the following conditions are equivalent.*

(1) *α is quasi-diagonal.*
(2) *α is pseudo-diagonal.*
(3) *There is a CP contraction ϕ of A into $\left(\prod_{n=1}^{\infty} M_{k_n}\right)_\beta$ such that $Q\phi$ is an isomorphism and $Q\phi\alpha_t = \beta_t Q\phi$ with $\beta = \prod_{n=1}^{\infty}\beta_n$ for some*

(k_n) and some (β_n), where Q is the quotient map of $\left(\prod M_{k_n}\right)_\beta$ onto $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$.

PROOF. We have already shown that the first two conditions are equivalent.

Suppose α is pseudo-diagonal. Let (x_n) be a dense sequence in A. We choose M_{k_n} and a flow β_n on M_{k_n} and a CP contraction $\phi_n : A \to M_{k_n}$ such that $\|\phi_n(x_k)\| \geq (1 - 1/n)\|x_k\|$, $\|\phi_n(x_k)\phi_n(x_\ell) - \phi(x_k x_\ell)\| \leq 1/n$, and $\|\beta_{n,t}\phi(x_k) - \phi_n\alpha_t(x_k)\| < 1/n$, $t \in [-1, 1]$ for all $k, \ell \leq n$. (As easily shown we may assume the target algebra for ϕ_n is a full matrix algebra.) We define a CP contraction ϕ of A into $\prod_{n=1}^\infty M_{k_n}$ by $\phi(x) = (\phi_n(x))_n$. Then one can show that $Q\phi$ is an isomorphism and $Q\phi\alpha_t = \beta_t Q\phi$. One can also show that $t \mapsto \beta_t\phi(x) = (\beta_{n,t}\phi_n(x))_n$ is continuous since $\beta_t\phi(x) - \phi\alpha_t(x) \in \bigoplus M_{k_n}$. That is, we have that $\phi(A) \subset \left(\prod M_{k_n}\right)_\beta$.

Suppose (3). Let ϕ be a CP contraction of A into $\left(\prod M_{k_n}\right)_\beta$ as given there. Let ϕ_n denote the component of ϕ mapping A into M_{k_n}. Then for any finite subset $\mathscr{F}$ of $A \setminus \{0\}$ and $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that for $\phi^{(n)} = \prod_{k=n}^\infty \phi_k$ the conditions $\|\phi^{(n)}(x)\phi^{(n)}(y) - \phi^{(n)}(xy)\| \leq \epsilon\|x\|\|y\|$, and $\|\beta_t\phi^{(n)}(x) - \phi^{(n)}\alpha_t(x)\| \leq \epsilon\|x\|$, $t \in [-1, 1]$ are satisfied for all $x, y \in \mathscr{F}$. We then find $m > n$ such that $\prod_{k=n}^m \phi_k$ instead of $\phi^{(n)}$ still satisfies the above conditions together with $\left\|\prod_{k=n}^m \phi_k(x)\right\| \geq (1 - \epsilon)\|x\|$ for $x \in \mathscr{F}$. This implies that α is pseudo-diagonal.

3. MF flows and dual MF flows

DEFINITION 3.1. Let (k_n) be a sequence of positive integers and let β_n be a flow on M_{k_n}. Let $\beta_t = \prod_{n=1}^\infty \beta_{n,t}$ which forms a (non-continuous) flow on $\prod_{n=1}^\infty M_{k_n}$. Let $(\prod_{n=1}^\infty M_{k_n})_\beta$ denote the maximal C^*-subalgebra of $\prod_{n=1}^\infty M_{k_n}$ on which β is continuous. A flow α on a separable C^*-algebra A is called an *MF flow* if there is an embedding of A into $\left(\prod_{n=1}^\infty M_{k_n}\right)_\beta / \bigoplus_{n=1}^\infty M_{k_n}$ for some (k_n) and (β_n) such that $\beta_t\phi = \phi\alpha_t$.

We first state a technical lemma.

LEMMA 3.2. *There is a constant $C > 0$ satisfying: Let α be a flow on a C^*-algebra A. If $e \in A$ is a projection such that $\max_{|t|\leq 1} \|\alpha_t(e) - e\| = \delta$ is sufficiently small, there is an α-cocycle u in A (or in $A + \mathbb{C}1$ if $A \not\ni 1$) such that $\mathrm{Ad}\, u_t\alpha_t(e) = e$ and $\max_{|t|\leq 1} \|u_t - 1\| \leq C\delta^{1/2}$.*

PROOF. Let δ_α denote the generator of α. If $e \in D(\delta_\alpha)$ then $e\delta_\alpha(e)e = (1-e)\delta_\alpha(e)(1-e) = 0$. Thus $[ih, e] = -\delta_\alpha(e)$ for $h = i(\delta_\alpha(e)e - e\delta_\alpha(e)) = i(1-e)\delta_\alpha(e)e + ie\delta_\alpha(e)(1-e)$, which is a self-adjoint element of A of norm less than or equal to $\|\delta_\alpha(e)\|$. Thus the differentiable α-cocycle u defined by $du_t/dt = u_t\alpha_t(ih)$ satisfies the conditions that $\mathrm{Ad}\, u_t\alpha_t(e) = e$ and that $\max_{|t|\leq 1} \|u_t - 1\| \leq \|\delta_\alpha(e)\|$.

If we only assume that $\max_{|t|\leq 1} \|\alpha_t(e) - e\|$ is small, we have to resort to the above situation. Namely we find a projection $e' \in A$ such that $\|\delta_\alpha(e')\|$ is small and e' is close to e. Then finding a unitary $w \approx 1$ such that $w^*ew = e'$ and an α-cocycle v such that $\mathrm{Ad}\, v_t\alpha_t(e') = e'$ and $\|v_t - 1\| \approx 0$, $t \in [-1, 1]$, we would obtain the desired α-cocycle $t \mapsto wv_t\alpha_t(w^*)$.

The following arguments are standard and mostly found in [4], but we shall give out some details (see the proof of Proposition 1.3 of [11]).

Let $e \in A$ be a projection and let $\delta = \max_{|t|\leq 1} \|\alpha_t(e) - 1\| > 0$.

Let g be a non-negative C^∞-function on R such that g has compact support and $\int g(t)\, dt = 1$. We define

$$q = \int \delta^{1/2} g(\delta^{1/2}t)\alpha_t(e)\, dt,$$

which satisfies that $0 \leq q \leq 1$. Since $\|\alpha_t(e) - e\| \leq \delta(1 + |t|)$, we deduce that

$$\|q - e\| \leq \int \delta^{1/2} g(\delta^{1/2}t)\|\alpha_t(e) - e\|\, dt \leq \delta + C_1\delta^{1/2} \leq (1 + C_1)\delta^{1/2},$$

where $C_1 = \int g(t)|t|\, dt$. We assume that $(1 + C_1)\delta^{1/2} < 1/8$, which insures that $\mathrm{Sp}(q) \subset [0, 1/8] \cup [7/8, 1]$. Note that $q \in D(\delta_\alpha)$ and

$$\|\delta_\alpha(q)\| = \left\| -\int \delta g'(\delta^{1/2}t)\alpha_t(e)\, dt \right\| \leq C_2\delta^{1/2},$$

where $C_2 = \int |g'(t)|\, dt$.

Let f be a non-negative C^∞-function on R such that $\mathrm{supp}(f) \subset [1/2, 3/2]$ and $f(t) = 1$ on $[7/8, 1]$. Define $\hat{f}$ by $\hat{f}(p) = (2\pi)^{-1} \int e^{-ipt} f(t)\, dt$ and set $C_3 = \int |t\hat{f}(t)|\, dt$. We define

$$e' = f(q) = \int \hat{f}(t)e^{itq}\, dt,$$

which is a projection such that $\|e' - q\| \leq (1 + C_1)\delta^{1/2}$. By Theorem 3.2.32 of [4] it follows that $e' \in D(\delta_\alpha)$ and

$$\|\delta_\alpha(e')\| \leq C_3\|\delta_\alpha(q)\| \leq C_2C_3\delta^{1/2}.$$

Hence there is an α-cocycle v such that $\mathrm{Ad}\, v_t\alpha_t(e') = e'$ and $\max_{|t|\leq 1} \|v_t - 1\| \leq \|\delta_\alpha(e')\| \leq C_2C_3\delta^{1/2}$.

Note that $\|e - e'\| \leq \|e - q\| + \|q - e'\| \leq 2(1 + C_1)\delta^{1/2} \leq 1/4$. Since $\|ee' + (1 - e)(1 - e') - 1\| \leq 2\|e - e'\| \leq 1/2$, the unitary w obtained by the polar decomposition of $ee' + (1-e)(1-e')$ satisfies that $\|w-1\| \le 4\|e-e'\|$. Since $we'w^* = e$, we conclude that the α-cocycle $u : t \mapsto wv_t\alpha_t(w)^*$ satisfies that $\mathrm{Ad}\, u_t\alpha_t(e) = e$. Note that if $|t| \le 1$, then $\|u_t - 1\| \le 2\|w-1\| + \|v_t - 1\| \le (16 + 16C_1 + C_2C_3)\delta^{1/2}$. Thus if $\delta < 8^{-2}(1+C_1)^{-2}$ then we obtain the desired cocycle u for the constant $C = 16 + 16C_1 + C_2C_3$.

LEMMA 3.3. *Let α be an MF flow on a unital separable C^*-algebra A. Then there is a unital embedding ϕ of A into $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$ such that $\phi\alpha_t = \beta_t\phi$ with $\beta = \prod \beta_n$ for some sequence (k_n) in N and (β_n).*

PROOF. Suppose that A is embedded into $\left(\prod_{n=1}^{\infty} M_{k_n}\right)_\beta / \bigoplus_{n=1}^{\infty} M_{k_n}$ as in the definition. Let $(p_n) \in \prod M_{k_n}$ be a representative of the unit of A. We may suppose that $p_n^* = p_n$. Since $\|p_n^2 - p_n\| \to 0$ we may also suppose that each p_n is a projection by functional calculus. Since $\|\beta_{n,t}(p_n) - p_n\|$ converges to zero uniformly in $t \in [-1, 1]$, there is a sequence $(u_{n,t})$ of cocycles by Lemma 3.2 such that $u_{n,t}$ is a β_n-cocycle in M_{k_n}, $\mathrm{Ad}\, u_{n,t}\beta_{n,t}(p_n) = p_n$, and $\|u_{n,t} - 1\| \to 0$ uniformly in $t \in [-1, 1]$. Thus we can replace M_{k_n} by $p_nM_{k_n}p_n$ and β_n by $\mathrm{Ad}\, u_{n,t}\beta_{n,t} | p_nM_{k_n}p_n$ and obtain the desired unital embedding.

Let $\mathcal{K} = \mathcal{K}(L^2(\mathsf{R}))$, the compact operators on $L^2(\mathsf{R})$, and define a unitary flow λ on $L^2(\mathsf{R})$ by $(\lambda_t\xi)(s) = \xi(s-t)$, $\xi \in L^2(\mathsf{R})$. We denote by $\mathrm{Ad}\,\lambda$ the flow on $\mathcal{K}$ defined by $t \mapsto \mathrm{Ad}\,\lambda_t$. The following proposition shows that there is a *universal flow* (on a non-separable C^*-algebra) for MF flows in the sense that the flow is MF if and only if it is realized as a subflow of the universal one.

The following is a technical lemma about almost commuting pairs of self-adjoint operators, one compact and the other possibly unbounded (cf. [12]).

LEMMA 3.4. *For every $\epsilon > 0$ there is a $\nu > 0$ satisfying the following condition: Let $a \in (\mathcal{K}(\mathcal{H}))_{sa}$ and H a self-adjoint operator (which may be unbounded) on $\mathcal{H}$ such that $\|a\| \le 1$ and $\|[a, H]\| < \nu$. Then there is an $a_1 \in (\mathcal{K}(\mathcal{H}))_{sa}$ and a self-adjoint operator H_1 on $\mathcal{H}$ such that a_1 is of finite rank, $\|a - a_1\| < \epsilon$, $\|H - H_1\| < \epsilon$, $H - H_1 \in \mathcal{K}(\mathcal{H})$, and $[a_1, H_1] = 0$.*

PROOF. This follows from Theorem 3.1 of [2], where this is stated as a result valid for a and H on an arbitrary finite-dimensional space $\mathcal{H}$ without depending on the dimensionality.

PROPOSITION 3.5. *Let α be a flow on a separable C^*-algebra. Then the following conditions are equivalent.*

(1) *α is an MF flow.*
(2) *(A, α) can be embedded into $\left(\left(\prod \mathcal{K}_n\right)_\gamma / \bigoplus \mathcal{K}_n, \gamma\right)$, where $\mathcal{K}_n = \mathcal{K}$ and $\gamma = \prod \mathrm{Ad}\,\lambda$.*

PROOF. Suppose (1), i.e., suppose that (A, α) can be embedded into

$$\Big(\prod M_{k_n}\Big)_\beta \Big/ \bigoplus M_{k_n}$$

with $\beta = \prod \beta_n$ for some (k_n) and (β_n). Let v_n be a unitary flow in M_{k_n} such that $\beta_{n,t} = \operatorname{Ad} v_{n,t}$. Then, since the spectrum of λ is R, by using the Weyl-von Neumann theorem one can obtain a sequence of λ-cocycles u_n in $\mathcal{K} + \mathsf{C}1$ and a sequence of finite-rank projections $e_n \in \mathcal{K}$ such that $u_{n,t} - 1$ is compact, $\|u_{n,t} - 1\| \to 0$ uniformly in $t \in [-1, 1]$ as $n \to \infty$, $\operatorname{Ad}(u_{n,t}\lambda_t)(e_n) = e_n$, and the spectrum of $t \mapsto u_{n,t}\lambda_t e_n$ is equal to that of v_n with multiplicity included. Then there is an embedding of M_{k_n} into $e_n\mathcal{K}_n e_n \subset \mathcal{K}_n$ such that v_n is mapped to $u_n\lambda e_n$. Thus one can embed $\big((\prod M_{k_n})_\beta, \beta\big)$ into $\big((\prod \mathcal{K}_n)_\sigma, \sigma\big)$ with $\sigma = \prod(u_n\lambda)$. Since $u_{n,t} \to 1$ uniformly in t on any bounded set of R and $u_{n,t} - 1 \in \mathcal{K}_n$, one derives that $\prod u_{n,t} \in \bigoplus \mathcal{K}_n + \mathsf{C}1$; thus σ and $\gamma = \prod \lambda$ are equal on the quotient $\prod \mathcal{K}_n \big/ \bigoplus \mathcal{K}_n$. Thus (A, α) can be embedded into $\big((\prod \mathcal{K}_n)_\gamma \big/ \bigoplus \mathcal{K}_n, \gamma\big)$.

Suppose (2). If A is unital, this follows from the proof of Lemma 3.3. Suppose that A is not unital. Let (p_k) be an approximate identity for A and let $(p_{k,n})_n \in \big(\prod \mathcal{K}_n\big)_\gamma$ be a sequence representing p_k with $0 \le p_{k,n} \le 1$. Let f be a smooth non-negative function on R such that $\int f(t)\,dt = 1$ and $\int |f'(t)|\,dt$ is small. Note that $\big(\int f(t) \operatorname{Ad}\lambda_t(p_{k,n})\,dt\big)_n$ represents $\alpha_f(p_k) = \int f(t)\alpha_t(p_k)\,dt$ and that $\|\delta_\alpha(\alpha_f(p_k))\| \le \int |f'(t)|\,dt$ etc., where δ_α is the generator of α. By using these facts we obtain a sequence (e_k) in A with $0 \le e_k \le 1$ and $(e_{k,n})_n \in \big(\prod \mathcal{K}_n\big)_\gamma$ representing e_k with $0 \le e_{k,n} \le 1$ such that $\|e_k x - x\| \to 0$ for any $x \in A$, $\|\delta_\alpha(e_k)\| \to 0$ as $k \to \infty$ and $\|[H, e_{k,n}]\| \to 0$ uniformly in n as $k \to \infty$, where H is the self-adjoint generator of λ.

Let (x_k) be a dense sequence in the unit ball of A_{sa} and let $(x_{k,n})_n$ be a sequence of self-adjoint elements in the unit ball of $\big(\prod \mathcal{K}_n\big)_\gamma$ representing x_k. Let $n \in \mathsf{N}$. We choose $\nu > 0$ for $\epsilon = 2^{-n}$ as in Lemma 3.4. We choose $k \in \mathsf{N}$ such that $\|e_k x_i - x_i\| < \epsilon$ for any $i \le n$ and $\|[H, e_{k,m}]\| < \nu$ for all $m \in \mathsf{N}$. We choose $M_n \in \mathsf{N}$ such that $\|e_{k,m}x_{i,m} - x_{i,m}\| < \epsilon$ for all $m \ge M_n$. Then by Lemma 3.4 we choose a self-adjoint H_m on $L^2(\mathsf{R})$ and a finite-rank self-adjoint operator $e'_{k,m}$ for $m \ge M_n$ such that $[H_m, e'_{k,m}] = 0$, $H_m - H$ is compact, $\|H_m - H\| < \epsilon$, and $\|e_{k,m} - e'_{k,m}\| < \epsilon$. Let P_m be the support projection of $e'_{k,m}$. Then P_m is a finite-rank projection commuting with H_m and satisfies that $\|P_m x_{i,m} - x_{i,m}\| \le 2\epsilon + \|P_m e'_{k,m} x_{i,m} - x_{i,m}\| = 2\epsilon + \|e'_{k,m} x_{i,m} - x_{i,m}\| \le 4\epsilon = 2^{-n+2}$ for $i \le n$. We may suppose that (M_n) is strictly increasing and we make such a choice for $M_n \le m < M_{n+1}$ and set $B_m = P_m \mathcal{K} P_m$ and $\beta_{m,t} = \operatorname{Ad} e^{itH_m} | B_m$. Then it follows that $(P_m x_{i,m} P_m)$ is equal to $(x_{i,m})$ modulo $\bigoplus \mathcal{K}_m$ for all i and $\gamma'_t = \prod \operatorname{Ad} e^{itH_m} = \operatorname{Ad} u_t\gamma_t$ for some γ-cocycle u with $u_t - 1 \in$

$\bigoplus \mathcal{K}_m$. Hence (A, α) can be embedded into $\left(\prod B_m\right)_\beta / \bigoplus B_m$ equipped with $\beta = \prod \beta_m$ which is embedded into $\left(\prod \mathcal{K}_m\right)_{\gamma'} / \bigoplus \mathcal{K}_m = \left(\prod \mathcal{K}_m\right)_\gamma / \bigoplus \mathcal{K}_m$ equipped with γ such that the composition is the original embedding of (A, α). This completes the proof.

REMARK 3.6. In the above proposition the property we needed for λ is that its spectrum contains arbitrarily long intervals of R.

PROPOSITION 3.7. *The class of MF flows on a separable C^*-algebra is closed under cocycle perturbations.*

PROOF. Let α be an MF flow on A and let u be an α-cocycle. Let ϕ be an embedding of A into $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$ such that $\phi\alpha_t = \beta_t\phi$ for some (k_n) and $\beta = \prod \beta_n$.

If A is unital, then u_t belongs to A and we may assume that ϕ is unital. By Lemma 1.1 of [7] it follows that u is given as $wu_t^{(h)}\alpha_t(w^*)$, where w is a unitary and $u^{(h)}$ is the differentiable α-cocycle defined by $du_t^{(h)}/dt|_{t=0} = ih$ with $h = h^* \in A$. Then we find a β-cocycle v in $\left(\prod M_{k_n}\right)_\beta$, by lifting w and h to a unitary and a self-adjoint element respectively, such that $v_t = \prod v_{n,t}$ maps to $\phi(u_t)$ under the quotient map. Hence we obtain that $\phi \operatorname{Ad} u_t\alpha_t = \beta_t'\phi$ with $\beta_t' = \prod \operatorname{Ad} v_{n,t}\beta_{n,t}$ (regarded as a flow on the quotient).

If A is not unital and u is an α-cocycle in the multiplier algebra $M(A)$ of A, we approximate u by α-cocycles in $A + \mathsf{C}1$ [9]. More precisely let (x_i) be a dense sequence in A and let $(u^{(n)})$ be a sequence of α-cocycles in $A + \mathsf{C}1$ such that

$$\|(u_t - u_t^{(n)})x_i\| \leq 2^{-n}\|x_i\|, \qquad t \in [-1, 1]$$

for $i = 1, 2, \ldots, n$. We extend ϕ to a CP map from $A + \mathsf{C}1$ into $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$ by setting $\phi(1) = 1$. We then lift each $\phi(u^{(n)})$ to an β-cocycle $v^{(n)}$ in $\left(\prod M_{k_n}\right)_\beta$ as stated above. We also fix a lifting $y_i \in \left(\prod M_{k_n}\right)_\beta$ of each $\phi(x_i)$. We then have for $i \leq n$

$$\|Q((v_t^{(n)} - v_t^{(n+1)})y_i)\| \leq (2^{-n} + 2^{-n-1})\|x_i\|, \qquad t \in [-1, 1],$$

where Q is the quotient map onto $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$. Hence one can choose a sequence (K_n) of integers such that $K_0 = 0$, $K_n > K_{n-1}$, and

$$\sup_{k \geq K_n} \|(v_{k,t}^{(n)} - v_{k,t}^{(n+1)})y_{i,k}\| \leq 2^{-n+1}\|x_i\|, \qquad t \in [-1, 1]$$

for $i \leq n$. We define a β-cocycle $w \in \left(\prod M_{k_n}\right)_\beta$ by $w_{k,t} = v_{k,t}^{(n)}$ for $K_n \leq k < K_{n+1}$. If $m > n$ and $K_m \leq k < K_{m+1}$ then the norm of the k'th coordinate of $(w_t - v_t^{(n)})y_i$ is

$$\|v_{k,t}^{(m)} y_{i,k} - v_{k,t}^{(n)} y_{i,k}\| \leq 2^{-n+2}\|x_i\|$$

for $i \le n$. Hence it follows that $\|Q(w_t)\phi(x_i) - \phi(u_t^{(n)}x_i)\| = \|Q(w_t y_i - v_t^{(n)} y_i)\| \le 2^{-n+2}\|x_i\|$ for $i \le n$, which implies that $\|Q(w_t)\phi(x_i) - \phi(u_t x_i)\| \le 2^{-n+3}\|x_i\|$. Since n is arbitrary, we can conclude that $Q(w_t)\phi(a) = \phi(u_t a)$ for any $a \in A$. We replace the flow β_n on M_{k_n} by $t \mapsto \mathrm{Ad}\, w_{n,t}\beta_{n,t}$. Then it follows that $\phi\, \mathrm{Ad}\, u_t \alpha_t = \beta_t \phi$.

A $*$-linear generalized inductive system of flows is a sequence of flows (A_n, α_n) together with $*$-linear maps $\phi_{m,n} : A_m \to A_n$ for $m < n$ with $\phi_{m,n}\phi_{k,m} = \phi_{k,n}$ for all $k < m < n$ such that for all k and all $x, y \in A_k$ and $\epsilon > 0$ there is an $K > k$ such that for all $n > m \ge K$ and $t \in [-1, 1]$

(1) $\|\phi_{m,n}(\phi_{k,m}(x)\phi_{k,m}(y)) - \phi_{k,n}(x)\phi_{k,n}(y)\| < \epsilon$,

(2) $\|\phi_{m,n}\alpha_{m,t}\phi_{k,m}(x) - \alpha_{n,t}\phi_{k,n}(x)\| < \epsilon$,

(3) $\sup_{r>m} \|\phi_{mr}|L(\{\alpha_{m,t}\phi_{km}(x) \,|\, |t| < \delta\})\| < \infty$,

for some $\delta > 0$, where $L(S)$ is the linear span of S.

This notion and the following consequences are adapted from Section 2 of [1]. The above condition 3 replaces $\sup_{r>k} \|\phi_{k,r}(x)\| < \infty$ there.

For such a system one defines the inductive limit C^*-algebra A and the flow α on A, which may be realized as follows. Let $\prod_{n=1}^\infty A_n$ be the full C^*-direct product of the A_n's and let $\beta_t = \prod_{n=1}^\infty \alpha_{n,t}$. Let $\bigoplus_{n=1}^\infty A_n$ be the C^*-direct sum, the ideal of $\prod_{n=1}^\infty A_n$ consisting of sequences converging to zero in norm. Define a map ϕ_m of A_m into $\prod A_n$ by $\phi_m(x)_n = \phi_{m,n}(x)$ for $n \ge m$ and 0 for $n < m$. Since $\phi_m(x) - \phi_n\phi_{m,n}(x) \in \bigoplus A_n$ one can define a $*$-linear map ϕ of $\bigcup A_n$ into $\prod A_n / \bigoplus A_n$ by $\phi|A_m = Q\phi_m$, where Q is the quotient map of $\prod M_{k_n}$ onto $\prod M_{k_n} / \bigoplus M_{k_n}$. Since $\phi(x)\phi(y) = Q(\phi_m(x)\phi_m(y))$ is the limit of $\phi(\phi_{m,n}(x)\phi_{m,n}(y))$ as $n \to \infty$ for $x, y \in A_m$, ϕ extends to an isomorphism of the inductive limit A of the system (A_n, ϕ_{mn}) into $\prod A_n / \bigoplus A_n$. Now we could identify the inductive limit A with the closure of $\phi\big(\bigcup A_n\big)$.

Since $Q\beta_t\phi_m(x) = Q((\alpha_{n,t}\phi_{m,n}(x))_n)$ is the limit of $\phi\alpha_{n,t}\phi_{m,n}(x)$ as $n \to \infty$ for $x \in A_m$, β_t induces an automorphism of A which we denote by α_t.

We shall show that $t \mapsto \alpha_t\phi(x)$ is continuous for $x \in A_m$. Let $\epsilon > 0$. Then there is $M > m$ such that for $n > \ell \ge M$ and $t \in [-1, 1]$ we have that $\|\phi_{\ell,n}\alpha_{\ell,t}\phi_{m,\ell}(x) - \alpha_{n,t}\phi_{m,n}(x)\| < \epsilon$. Hence $\|\alpha_t\phi(x) - \phi(x)\| \le \|\phi(\alpha_{\ell,t}\phi_{m,\ell}(x) - \phi_{m,\ell}(x))\| + \epsilon$. Fixing $\ell \ge M$ there is a $1 > \delta > 0$ such that if $|t| < \delta$ then $\|\alpha_{\ell,t}\phi_{m,\ell}(x) - \phi_{m,\ell}(x)\| < \epsilon$. Hence we obtain that if $|t| < \delta$ then $\|\alpha_t\phi(x) - \phi(x)\| < 2\epsilon$. Thus α is a (continuous) flow. Note that α is realized as the restriction of $\beta = \prod \alpha_n$.

Let $\big(\prod A_n\big)_\beta$ be the maximal C^*-subalgebra of $\prod A_n$ on which β is continuous. We note that the image of ϕ_m is contained in $\big(\prod A_n\big)_\beta$. Suppose that there are a sequence (t_i) in R, a sequence (n_i) in N, $x \in A_m$, and a $\delta > 0$

such that $\lim_i t_i = 0$ and $\|\alpha_{n_i,t_i}\phi_{m,n_i}(x) - \phi_{m,n_i}(x)\| > \delta$. Since each α_n is continuous we must have that $n_i \to \infty$. Note that there is $\ell > m$ such that $\|\phi_{\ell,n}\alpha_{\ell,t}\phi_{m,\ell}(x) - \alpha_{n,t}\phi_{m,n}(x)\| < \delta/2$ for $t \in [-1, 1]$ and

$$\sup_{r>\ell} \|\phi_{\ell,r}|L(\{\alpha_{\ell,t}\phi_{m,\ell}(x) \mid |t| < s\}\| < \infty$$

for some $s > 0$. Since $\|\phi_{\ell,n_i}(\alpha_{\ell,t_i}\phi_{m,\ell}(x) - \phi_{m,\ell}(x))\| > \delta/2$ for $n_i > \ell$, this contradicts that $t \mapsto \alpha_{\ell,t}$ is continuous. Hence one concludes that ϕ embeds A into $\left(\prod A_n\right)_\beta / \bigoplus A_n$.

LEMMA 3.8. *Suppose that (A, α) can be embedded into $\left(\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}, \beta\right)$ with $\beta = \prod \beta_n$. Then there exist a (separable) C^*-algebra B on a separable Hilbert space $\mathscr{H}$ and a unitary flow U on $\mathscr{H}$ such that B includes $\mathscr{K}(\mathscr{H})$, $t \mapsto \operatorname{Ad} U_t(x)$ defines a flow on B, there is an isomorphism ϕ of $B/\mathscr{K}(\mathscr{H})$ onto A such that $\phi Q \operatorname{Ad} U_t(x) = \alpha_t \phi Q(x)$ for $x \in B$, and (B, U) is quasi-diagonal, where Q is the quotient map of B onto $B/\mathscr{K}(\mathscr{H})$. Conversely if there is such (B, U) then (A, α) can be embedded into $\left(\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}, \beta\right)$ for some (k_n) and (β_n).*

PROOF. Let $\mathscr{H}$ be an infinite-dimensional separable Hilbert space and let (E_n) be a sequence of projections on $\mathscr{H}$ such that $E_n\mathscr{H}$ is k_n-dimensional, $E_m E_n = 0$ for $m \neq n$, and $\sum_{n=1}^\infty E_n = 1$. Let σ be a map of A into $\left(\prod M_{k_n}\right)_\beta$ such that $Q'\sigma$ is the given embedding of A into $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$, where Q' is the quotient map of $\prod M_{k_n}$ onto $\prod M_{k_n} / \bigoplus M_{k_n}$. We identify $E_n\mathscr{B}(\mathscr{H})E_n$ with M_{k_n} and denote by ι the embedding of $\prod M_{k_n}$ into $\mathscr{B}(\mathscr{H})$ by $\iota(x) = \sum_{n=1}^\infty x_n$ for $x = (x_n)_n$. Note that ι induces the embedding of $\prod M_{k_n} / \bigoplus M_{k_n}$ into $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ since $\iota\left(\prod M_{k_n}\right) \cap \mathscr{K}(\mathscr{H}) = \iota\left(\bigoplus M_{k_n}\right)$. We let $\psi = \iota\sigma$, which is a map of A into $\mathscr{B}(\mathscr{H})$.

Let $B = \psi(A) + \mathscr{K}(\mathscr{H})$, which is a quasi-diagonal C^*-algebra such that $Q\psi$ is an isomorphism of A onto $B/\mathscr{K}(\mathscr{H})$. Thus ϕ is obtained as the inverse of $Q\psi$.

Let U_n be a unitary flow in $M_{k_n} = E_n\mathscr{B}(\mathscr{H})E_n$ such that $\operatorname{Ad} U_{n,t} = \beta_{n,t}$ and let $U_t = \iota((U_{n,t})_n)$ which is a unitary flow in $\mathscr{B}(\mathscr{H})$ such that $t \mapsto U_t$ is strongly continuous. Note that $t \mapsto \operatorname{Ad} U_t(x)$ is norm-continuous for $x \in B$. Then we have for $x \in A$ that $Q \operatorname{Ad} U_t \psi(x) = Q\iota\beta_t\sigma(x) = Q\psi\alpha_t(x)$, where we use $Q'\beta_t\sigma = Q'\sigma\alpha_t$ and $Q\iota = 0$ on $\operatorname{Ker} Q'$. Since $\phi = (Q\psi)^{-1}$, we obtain that $\phi Q \operatorname{Ad} U_t \psi(x) = \alpha_t(x)$. For $y = \psi(x) + c$ with $c \in \mathscr{K}(\mathscr{H})$ we obtain that $\phi Q \operatorname{Ad} U_t(y) = \phi \operatorname{Ad} U_t \psi(x) = \alpha_t(x) = \alpha_t \phi Q(y)$. This concludes the proof of the first part.

Conversely if there is such a (B, U) then there is an increasing sequence (P_n) of finite-rank projections on $\mathscr{H}$ such that $\lim_n P_n = 1$, $\|[P_n, U_t]\| \to 0$

uniformly in $t \in [-1, 1]$, and $\|[P_n, b]\| \to 0$ for $b \in B$. We may suppose that $[P_n, U_t] = 0$ by perturbing of U by compacts and passing to a subsequence of (P_n). Set $E_n = P_n - P_{n-1}$ and $k_n = \text{rank}\, E_n$ with $P_0 = 0$. Identifying $E_n \mathscr{B}(\mathscr{H}) E_n$ with M_{k_n} we define a map $\phi : B \to \prod M_{k_n}$ by $\phi(x) = (E_n x E_n)_n$. This drops to a $*$-homomorphism of B into $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$, intertwining α with β, whose kernel is exactly $\mathscr{K}(\mathscr{H})$.

A continuous field of flows over $\mathsf{N} \cup \{\infty\}$ is a continuous field of C^*-algebras A_n, $n \in \mathsf{N} \cup \{\infty\}$ and flows α_n on A_n such that if $n \mapsto x_n$ is a continuous field so is $n \mapsto \alpha_{n,t}(x_n)$ for all $t \in \mathsf{R}$. Since $\|x_n - \alpha_{n,t}(x_n)\|$ converges to $\|x_\infty - \alpha_{\infty,t}(x_\infty)\|$ as $n \to \infty$ in N, it follows that $t \mapsto \alpha_{n,t}(x_n)$ is continuous uniformly in $n \in \mathsf{N} \cup \{\infty\}$. Hence if $n \mapsto x_n$ is a continuous field then so is $n \mapsto \int f(t) \alpha_{n,t}(x_n)\, dt$ for $f \in L^1(\mathsf{R})$. Note also that the flow $\alpha = \prod_{n=1}^\infty \alpha_n \times \alpha_\infty$ defined on the C^*-algebra generated by the continuous fields is strongly continuous.

We will present a version of Proposition 2.2.3 of [1] by borrowing the terminology there; a finite product $\prod_{n=r}^s (A_n, \alpha_n)$ for $1 \le r \le s < \infty$ is called a *segment* of $\prod_{n=1}^\infty (A_n, \alpha_n)$ and two segments are *disjoint* if their intersection is zero when they are naturally regarded as subsystems of $\prod_{n=1}^\infty (A_n, \alpha_n)$.

LEMMA 3.9. *Let α_n be a flow on a separable C^*-algebra A_n and $\beta = \prod_{n=1}^\infty \alpha_n$. Let (A, α) be a flow with A separable. Then the following are equivalent:*

(1) *(A, α) can be embedded into $\left(\left(\prod A_n\right)_\beta / \bigoplus A_n, \beta\right)$.*

(2) *There is a continuous field of flows (B_n, β_n) over $\mathsf{N} \cup \{\infty\}$ such that (B_n, β_n) is a segment of $\prod (A_n, \alpha_n)$ for $n \in \mathsf{N}$ with disjoint segments for different n and such that $(B_\infty, \beta_\infty) \cong (A, \alpha)$.*

(3) *(A, α) can be embedded into $\left(\left(\prod B_n\right)_\gamma / \bigoplus B_n, \gamma\right)$, where (B_n, β_n) is a segment of $\prod (A_n, \alpha_n)$ for $n \in \mathsf{N}$ with disjoint segments for different n and $\gamma = \prod \beta_n$, such that $\|x\| = \lim_n \|x_n\|$ holds for every $x \in A$ and sequence (x_n) representing x.*

PROOF. We follow the proof of Proposition 2.2.3 of [1].

We shall prove (1) ⇒ (2) as follows: Let (x_i) be a dense sequence in A with $(x_{in})_n \in \left(\prod A_n\right)_\beta$ representing x_i and let (t_j) be an enumeration of the rationals. For i, j and $n \in \mathsf{N} \cup \{\infty\}$ we set $y_{i,j}(n) = \alpha_{n,t_i}(x_{j,n}) \in A_n$ for $n \in \mathsf{N}$ and $y_{i,j}(\infty) = \alpha_{t_i}(x_j) \in A_\infty = A$. Let P be the set of all polynomials in non-commuting variables $Y_{i,j}$, $i, j \in \mathsf{N}$ and their formal adjoints $Y_{i,j}^*$, $i, j \in \mathsf{N}$ with coefficients in $\mathsf{Q} + i\mathsf{Q}$. Since P is countable, let (f_i) be a fixed enumeration of P. For $n \in \mathsf{N} \cup \{\infty\}$ we set $f_i(n)$ to be the element in A_n obtained from f_i substituting $Y_{i,j} = y_{i,j}(n)$ for all i, j.

There are disjoint segments $[r_m, s_m]$ in N such that for $i = 1, 2, \ldots, m$

$$\left| \|f_i(\infty)\| - \left\| \prod_{n=r_m}^{s_m} f_i(n) \right\| \right| < 1/m.$$

Set $B(m) = \prod_{n=r_m}^{s_m} A_n$ and $\beta_{m,t} = \prod_{n=r_m}^{s_m} \alpha_{n,t}$. We set $F_i(m) = \prod_{n=r_m}^{s_m} f_i(n) \in B(m)$ and $F_i(\infty) = f_i(\infty)$. Then the function $n \mapsto \|F_i(n)\|$ is continuous on $\mathsf{N} \cup \{\infty\}$ and the set of F_i's, together with the sequences converging to zero, forms a $*$-algebra $\mathscr{A}$ over $\mathsf{Q} + i\mathsf{Q}$ invariant under $\prod_{m=1}^{\infty} \beta_{m,t} \times \alpha_t$, $t \in \mathsf{Q}$. Since $(x_{in})_n \in \left(\prod A_n\right)_\beta$, $m \mapsto \beta_{m,t_j} F_i(m) - F_i(m) \in \mathscr{A}$ converges to zero uniformly in $m \in \mathsf{N}$ as $t_j \to 0$. Hence the closure of $\mathscr{A}$ is a C^*-algebra invariant under $\prod \beta_m \times \alpha$ on which $t \mapsto \prod \beta_{m,t} \times \alpha_t$ is continuous. Thus the continuous fields are invariant under the flow.

For the other implications see the proof of Proposition 2.2.3 in [1].

The following result will be proved by mimicking the proof of Theorem 3.2.2 of [1].

THEOREM 3.10. *Let α be a flow on a separable C^*-algebra A. Then the following conditions are equivalent:*

1. (A, α) *is obtained as the inductive limit of a $*$-linear generalized inductive system of flows on finite-dimensional C^*-algebras.*
2. α *is an MF flow.*
3. *There is an essential quasi-diagonal extension B of A by the compact operators $\mathscr{K}$ and a unitary flow $U \in M(\mathscr{K})$ such that* $\mathrm{Ad}\, U_t(B) = B$ *for $t \in \mathsf{R}$, $t \mapsto$* $\mathrm{Ad}\, U_t(x)$ *is norm-continuous for $x \in B$, (B, U) is quasi-diagonal and Q* $\mathrm{Ad}\, U_t = \alpha_t Q$ *where Q is the quotient map of B onto A.*
4. *There is a continuous field of flows (A_n, α_n) over $\mathsf{N} \cup \{\infty\}$ such that A_n is finite-dimensional for $n \in \mathsf{N}$ and $(A_\infty, \alpha_\infty) \cong (A, \alpha)$.*
5. *There is a continuous field of flows (A_n, α_n) over $\mathsf{N} \cup \{\infty\}$ such that $A_n \cong M_{k_n}$ for some k_n for each $n \in \mathsf{N}$ and $(A_\infty, \alpha_\infty) \cong (A, \alpha)$.*

PROOF. We proved (1) $\Rightarrow$ (2) before Lemma 3.8 and (2) $\Leftrightarrow$ (3) in Lemma 3.8 and (2) $\Leftrightarrow$ (4) in Lemma 3.9. (5) $\Rightarrow$ (4) is trivial and (4) $\Rightarrow$ (5) is easy since the fibres at any isolated points may be enlarged.

It remains to show (2) $\Rightarrow$ (1). Suppose that (A, α) is embedded into

$$\left(\prod M_{k_n}\right)_\beta \Big/ \bigoplus M_{k_n}$$

with $\beta = \prod \beta_n$ for some (M_{k_n}, β_n). For $x \in A$ let $\mathrm{Sp}_\alpha(x)$ denote the α-spectrum of x and let $A^\alpha(F) = \{x \in A \mid \mathrm{Sp}_\alpha(x) \subset F\}$ for a closed set

F of R. Let $A_C = \bigcup_{n=1}^{\infty} A^\alpha[-n, n]$, which is a dense $*$-subalgebra of A. Similarly let $\left(\prod M_{k_n}\right)_C = \left(\left(\prod M_{k_n}\right)_\beta\right)_C = \bigcup_{n=1}^{\infty}\left(\prod M_{k_n}\right)^\beta[-n, n]$, where $\left(\prod M_{k_n}\right)^\beta[-n, n] = \left(\left(\prod M_{k_n}\right)_\beta\right)^\beta[-n, n]$. For each $x \in A_C$ there is a $(x_n) \in \left(\prod M_{k_n}\right)_\beta$ representing x. If $f \in L^1(\mathrm{R})$ has Fourier transform with compact support and is 1 on the α-spectrum of x then we have that $\int f(t)\alpha_t(x)\,dt = x$, which implies that $\int f(t)\beta_t((x_n))\,dt$ also represents x. In this way we deduce that $A^\alpha[-n, n]$ is embedded into $\left(\prod M_{k_n}\right)^\beta[-n-1, n+1]/\bigoplus M_{k_n}$. Note that $A^\alpha[-n, n]$ etc. are self-adjoint. We choose a $*$-linear map σ of A_C into $\left(\prod M_{k_n}\right)_C$ such that $Q\sigma = \mathrm{id}$ on A_C and $\sigma(A^\alpha[-n, n]) \subset \left(\prod M_{k_n}\right)^\beta[-n-1, n+1]$. We also choose a dense sequence (x_n) in A_C.

We shall define finite-dimensional C^*-algebras A_n with flows α_n on A_n and $*$-linear maps $\gamma_n : A_n \to A_C \subset A$ and $\delta_n : A \to A_{n+1}$ such that the sequence (A_n, α_n) of flows with maps $\phi_{n,n+1} \equiv \phi_n = \delta_n\gamma_n : A_n \to A_{n+1}$ is a $*$-linear generalized inductive system of flows with the desired properties, appearing as the upper sequence of the commutative diagram:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \longrightarrow & \dots \\
{\scriptstyle\gamma_1}\downarrow & \nearrow{\scriptstyle\delta_1} & {\scriptstyle\gamma_2}\downarrow & \nearrow{\scriptstyle\delta_2} & \downarrow & & \\
A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \dots
\end{array}$$

In particular our system will satisfy the following conditions:

$$\|\phi_{n+1}(xy) - \phi_{n+1}(x)\phi_{n+1}(y)\| \leq 2^{-n}\|x\|\|y\|$$

for all $x, y \in \phi_n(A_n) \subset A_{n+1}$ and

$$\|\phi_{n+1}\alpha_{n+1,t}(x) - \alpha_{n+2,t}\phi_{n+1}(x)\| \leq 2^{-n}\|x\|$$

for all $x \in \phi_n(A_n)$ and $t \in [-1, 1]$, which is enough to imply that the system has the desired properties together with the condition $\sup_{n>k}\|\phi_{k,n}\| < \infty$ for $k \in \mathrm{N}$.

On the other hand the lower sequence of copies of (A, α) of the above commutative diagram with maps $\gamma_{n+1}\delta_n : A \to A$ defines (A, α), which follows from: $(\gamma_n(A_n))$ is increasing with dense union in A, $\gamma_{n+1}\delta_n(x) = x$ for $x \in \gamma_n(A_n)$, and

$$\|\gamma_{n+1}\delta_n\alpha_t(x) - \alpha_t\gamma_{n+1}\delta_n(x)\| \leq 2^{-n+1}\|x\|$$

for $x \in \gamma_n(A_n)$ and $t \in [-1, 1]$. We shall require the intertwining properties for δ_n and γ_n with α_n and α, which will imply that both the upper and lower sequences define the same object, i.e., (A, α).

In the course of the inductive construction below we shall define a finite-dimensional $*$-subspace V_n of $A_C \subset A$ depending on γ_n which is a vital ingredient for constructing A_{n+1}, δ_n and then γ_{n+1} such that (V_n) forms an increasing sequence with dense union in A. In particular the norms of δ_n and γ_{n+1} will be almost dominated by $(\dim V_n)^{1/2}$ and V_n will equal $\gamma_{n+1}(A_{n+1})$. To obtain the above inequalities we shall require the following properties for γ_n and δ_n with $n \geq 2$. The first two are discussed in the proof of Theorem 3.2.2 of [1] and the second two are new being concerned with the flows:

$$\|\delta_n(xy) - \delta_n(x)\delta_n(y)\| \leq 2^{-n-1}(\dim V_n)^{-1/2}\|x\|\|y\|$$

for all $x, y \in \gamma_n(A_n)$,

$$\gamma_{n+1}\delta_n(x) = x$$

for all $x \in \gamma_n(A_n) \cdot \gamma_n(A_n)$ or $x \in \gamma_n(A_n)$,

$$\|\alpha_{n+1,t}\delta_n(x) - \delta_n\alpha_t(x)\| \leq 2^{-n}\|x\|$$

for $x \in \gamma_n(A_n)$, and

$$\|\alpha_t\gamma_{n+1}(x) - \gamma_{n+1}\alpha_{n+1,t}(x)\| \leq 2^{-n-1}\|x\|$$

for $x \in \phi_n(A_n)$.

We set $A_1 = \mathbf{C}$ and $\gamma_1 : A_1 \to A_C$ be an arbitrary $*$-linear map. Suppose that

$$A_2, \delta_1, \gamma_2, A_3, \delta_2, \gamma_3, \ldots, A_n, \delta_{n-1}, \gamma_n$$

are constructed so that $\gamma_k(A_{k-1}) \subset \gamma_k(A_k)$ and $x_{k-1} \in \gamma_k(A_k)$ for $k \leq n$ as well as the above inequalities, where (x_n) was chosen as a dense sequence in A_C. We shall define A_{n+1}, and $\delta_n : A \to A_{n+1}$, and $\gamma_{n+1} : A_{n+1} \to A_C$.

Let d be the dimension of A_n. Let $E \in \mathbf{N}$ be such that $\mathrm{Sp}_\alpha(x) \subset [-E, E]$ and $\mathrm{Sp}_\beta\, \sigma(x) \subset [-E, E]$ for all $x \in \gamma_n(A_n)$, which exists by the assumption on γ_n and σ. We choose $N \in \mathbf{N}$ such that $E\sqrt{d(2N+1) + d^2 + 2}/N < 2^{-n-3}$. Let V_n be the $*$-subspace of A generated by $\gamma_n(A_n) \cdot \gamma_n(A_n), x_n, x_n^*$ and $\alpha_{k/N}(\gamma_n(A_n))$ with $k = 0, \pm 1, \pm 2, \ldots, \pm N$. Note that $V_n \subset A_C$ and the $\dim(V_n) \leq d(2N+1) + d^2 + 2$. Note also that $\alpha_t(x)$ with $x \in \gamma_n(A_n)$, $t \in [-1, 1]$ is *almost* contained in V_n; more precisely, there is a $y \in V_n$ such that $\|\alpha_t(x) - y\| \leq (E/N)\|x\|$. This follows by setting $y = \alpha_{k/N}(x)$ for some k due to the estimate: $\|\alpha_s(x) - \alpha_t(x)\| \leq E|s-t|\|x\|$, which is derived from $\mathrm{Sp}_\alpha(x) \subset [-E, E]$.

Since $Q\sigma = \mathrm{id}$ and $Q\beta_t\sigma = \alpha_t$ we will then choose $r_n < s_n$ such that the linear map $\rho_n : A \to A_{n+1} \equiv \prod_{i=r_n}^{s_n} M_{k_i}$ defined by $x \mapsto \prod_{i=r_n}^{s_n} \sigma_i(x)$ satisfies the following conditions: $\rho_n|V_n$ is almost isometric and $\rho_n|\gamma_n(A_n)$ is almost multiplicative and $\alpha_{n+1,t}\rho_n \equiv \prod_{i=r_n}^{s_n} \beta_{i,t}\rho_n$ is nearly equal to $\rho_n\alpha_t$ on $\gamma_{n+1}(V_n)$, i.e., for any prescribed $\epsilon > 0$,

$$\|\rho_n|V_n\| < 1+\epsilon,$$

$$\|(\rho_n|V_n)^{-1}\| < 1+\epsilon,$$

$$\|\rho_n(x)\rho_n(y) - \rho_n(xy)\| \le \epsilon\|x\|\|y\|, \quad x, y \in \gamma_n(A_n),$$

$$\|\alpha_{n+1,k/N}\rho_n(x) - \rho_n\alpha_{k/N}(x)\| \le \epsilon\|x\|, \quad x \in \gamma_n(A_n), \quad k = 0, \pm 1, \ldots, \pm N.$$

Let P_n be a projection from A onto V_n such that $\|P_n\| \le \sqrt{\dim V_n}$ (see 1.14 of [13]; we need this stronger estimate rather than $\|P_n\| \le \dim V_n$). We set $\delta_n = \rho_n P_n : A \to A_{n+1}$. Let R_n be a projection from A_{n+1} onto $\delta_n(A) = \rho_n(V_n)$ such that $\|R_n\| \le \sqrt{\dim V_n}$ and set $\gamma_{n+1} = (\rho_n|V_n)^{-1}R_n : A_{n+1} \to A_C$. Then it is immediate that $\gamma_{n+1}\delta_n|V_n = \mathrm{id}$. We set

$$\epsilon = 2^{-n-3}(\dim V_n)^{-1/2},$$

which assures the first inequalities on δ_n.

Note that $\gamma_{n+1}(A_{n+1}) = (\rho_n|V_n)^{-1}(\rho_n(V_n)) = V_n$, which implies that $\gamma_{n+1}(A_{n+1}) \supset \gamma_n(A_n)$ and $\gamma_{n+1}(A_{n+1}) \ni x_n$.

We have defined $\phi_n = \delta_n\gamma_n : A_n \to A_{n+1}$. Since

$$\gamma_n\delta_{n-1} = (\rho_{n-1}|V_{n-1})^{-1}R_{n-1}\rho_{n-1}P_{n-1} = P_{n-1}$$

is a projection onto V_{n-1} and the range of γ_m is V_{m-1}, we obtain that $\phi_{m,n} = \phi_{n-1}\phi_{n-2}\ldots\phi_m = \delta_{n-1}\gamma_m = \rho_{n-1}(\rho_{m-1}|V_{m-1})^{-1}R_{m-1}$, i.e., $\|\phi_{m,n}\| < 4\sqrt{\dim V_{m-1}}$ for all $n > m$.

Let us repeat here the proof from [1] for ϕ_{n+1} being approximately multiplicative. For $x, y \in A_n$, since $\phi_{n,n+2} = \delta_{n+1}\gamma_n$, $\|\phi_{n+1}(\phi_n(x)\phi_n(y)) - \phi_{n,n+2}(x)\phi_{n,n+2}(y)\|$ is less than or equal to

$$\|\delta_{n+1}\{\gamma_{n+1}(\delta_n\gamma_n(x)\delta_n\gamma_n(y)) - \gamma_n(x)\gamma_n(y)\}\| \\ + \|\delta_{n+1}(\gamma_n(x)\gamma_n(y)) - \delta_{n+1}\gamma_n(x)\delta_{n+1}\gamma_n(y)\|.$$

Substituting $\gamma_n(x)\gamma_n(y) = \gamma_{n+1}\delta_n(\gamma_n(x)\gamma_n(y))$ the first term is less than or equal to

$$\|\delta_{n+1}\gamma_{n+1}\|\|\delta_n\gamma_n(x)\delta_n\gamma_n(y) - \delta_n(\gamma_n(x)\gamma_n(y))\|$$

which is roughly smaller than $2^{-n-1}\|\gamma_n(x)\|\|\gamma_n(y)\|$. The second term is roughly smaller than $2^{-n-2}(\dim V_n)^{-1/2}\|\gamma_n(x)\|\|\gamma_n(y)\|$. Thus one can estimate that

$$\|\phi_{n+1}(\phi_n(x)\phi_n(y)) - \phi_{n,n+2}(x)\phi_{n,n+2}(y)\| \le 2^{-n}\|\phi_n(x)\|\|\phi_n(y)\|.$$

Now we come to the proof of the intertwining properties of δ_n and γ_{n+1} with $\alpha_t, \alpha_{n+1,t}$.

Let $x \in \gamma_n(A_n)$. For $t \in [-1,1]$ we want to estimate $\|\delta_n\alpha_t(x)-\alpha_{n+1,t}\delta_n(x)\|$. First assume that $t = k/N$ with $k \in [-N, N]$. Since $\alpha_t(x), x \in V_n$ we have $\|\delta_n\alpha_t(x) - \alpha_{n+1,t}\delta_n(x)\| = \|\rho_n\alpha_t(x) - \alpha_{n+1,t}\rho_n(x)\| \le \epsilon\|x\|$. If $t \in [-1, 1]$ in general there is k/N such that $|t - k/N| < 1/N$. Since

$$\begin{aligned}\|\delta_n\alpha_t(x) - \delta_n\alpha_{k/N}(x)\| &\le \|\delta_n\|\|\alpha_t(x) - \alpha_{k/N}(x)\| \\ &\le (1+\epsilon)\sqrt{\dim V_n}EN^{-1}\|x\|\end{aligned}$$

and

$$\|\alpha_{n+1,t}\delta_n(x) - \alpha_{n+1,k/N}\delta_n(x)\| \le E/N\|\rho_n(x)\| \le (1+\epsilon)EN^{-1}\|x\|$$

and $\sqrt{\dim V_n}EN^{-1} < 2^{-n-3}$, we obtain that

$$\|\delta_n\alpha_t(x) - \alpha_{n+1,t}\delta_n(x)\| \le (\epsilon + 2^{-n-2}(1+\epsilon))\|x\| \le 2^{-n}\|x\|.$$

Let $x \in V_n$ and $t = k/N$. Since $\gamma_{n+1}\rho_n(x) = x$ and $\gamma_{n+1}\rho_n\alpha_t(x) = \alpha_t(x)$, we have

$$\|\alpha_t\gamma_{n+1}\rho_n(x) - \gamma_{n+1}\alpha_{n+1,t}\rho_n(x)\| = \|\gamma_{n+1}(\rho_n\alpha_t(x) - \alpha_{n+1,t}\rho_n(x))\|$$

which is less than or equal to $\epsilon\|\gamma_{n+1}\|\|x\| \le (1+\epsilon)^2\epsilon\sqrt{\dim V_n}\|\rho_n(x)\|$. If $t \in [-1, 1]$ in general there is k such that $|t - k/N| < 1/N$. Since

$$\|\alpha_t\gamma_{n+1}\rho_n(x) - \alpha_{k/N}\gamma_{n+1}\rho_n(x)\| = \|\alpha_t(x) - \alpha_{k/N}(x)\|$$

is less than or equal to $(1+\epsilon)EN^{-1}\|\rho_n(x)\| \le (1+\epsilon)2^{-n-3}\|\rho_n(x)\|$ and

$$\|\gamma_{n+1}\alpha_{n+1,t}\rho_n(x) - \gamma_{n+1}\alpha_{n+1,k/N}\rho_n(x)\|$$

is less than or equal to $\|\gamma_{n+1}\|EN^{-1}\|\rho_n(x)\| \le (1+\epsilon)\sqrt{\dim V_n}EN^{-1}\|\rho_n(x)\|$, we obtain that

$$\begin{aligned}&\|\alpha_t\gamma_{n+1}\rho_n(x) - \gamma_{n+1}\alpha_{n+1,t}\rho_n(x)\| \\ &\qquad\le \left(2(1+\epsilon)2^{-n-3} + (1+\epsilon)^2\epsilon\sqrt{\dim V_n}\right)\|\rho_n(x)\|,\end{aligned}$$

which is less than or equal to $2^{-n-1}\|\rho_n(x)\|$. This completes the proof.

REMARK 3.11. In the above proof (2) $\Rightarrow$ (1) of the theorem we have chosen a lifting σ of $A \subset \left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$ such that $\sigma(A_C) \subset \left(\prod M_{k_n}\right)_C$ and constructed V_n in A_C. (We actually defined σ only on A_C.) We could have chosen a σ such that $\sigma(D(\delta_\alpha)) \subset D(\delta_\beta)$, where δ_β is the generator of β

(on $\left(\prod M_{k_n}\right)_\beta$), and constructed V_n in $D(\delta_\alpha)$. Because what we needed was Lipschitz continuity of $t \mapsto \alpha_t(x)$ and $t \mapsto \beta_t\sigma(x)$ of $x \in \gamma_n(A_n) = V_{n-1}$.

COROLLARY 3.12. *Let α be an MF flow on a separable C^*-algebra A. Then there is an RF flow (B, β) and a β-invariant ideal I such that the quotient of (B, β) by I is isomorphic to (A, α).*

PROOF. Let B be the C^*-algebra generated by the continuous fields as in Condition (5) of Theorem 3.10 applied to (A, α), which has the flow β determined by α_n, $n \in \mathsf{N} \cup \{\infty\}$. Let π_n be the representation of B which picks up the fiber M_{k_n} at $n \in \mathsf{N}$. Then the family π_n, $n \in \mathsf{N}$ is faithful and each π_n is β-covariant, i.e., β is an RF flow. Let I be the ideal of B generated by the fields $n \mapsto a_n$ with $a_\infty = 0$. (Note that $I = \bigoplus_{n=1}^\infty M_{k_n}$ is β-invaraint.) Then the quotient of (B, β) by I is isomorphic to (A, α).

The following is about KMS states.

PROPOSITION 3.13. *Let $B = \prod_{n=1}^\infty M_{k_n}$ and $I = \bigoplus M_{k_n}$ for some (k_n) and $\beta_t = \prod \beta_{n,t}$. The flow β on B_β/I has KMS states for all inverse temperatures.*

PROOF. Fix an inverse temperature. Then each β_n has a unique KMS state ω_n on M_{k_n}. Let $\mathcal{U}$ be an ultra filter on N and define a state ω on B_β by $\omega((x_n)) = \lim_{n\to\mathcal{U}} \omega_n(x_n)$, which is a KMS state and satisfies that $\omega|I = 0$. Thus we may regard ω as a state of B_β/I.

COROLLARY 3.14. *Let α be an MF flow on a unital separable C^*-algebra. Then α has KMS states for all inverse temperatures.*

PROOF. There is a unital embedding of (A, α) into $\left(\prod M_{k_n}\right)_\beta / \bigoplus M_{k_n}$ by 3.3. Hence this follows from the previous proposition.

From now on we are concerned with the dual object of MF flows.

LEMMA 3.15. *If there is a continuous field of flows (B_n, β_n) over $\mathsf{N} \cup \{\infty\}$ then there is a continuous field of flows $(B_n \times_{\beta_n} \mathsf{R}, \hat{\beta}_n)$ over $\mathsf{N} \cup \{\infty\}$ such that if $n \mapsto x_n$ is a continuous field for the former and $f \in L^1(\mathsf{R})$ then $n \mapsto x_n\lambda_n(f)$ is a continuous field for the latter, where λ_n is the natural embedding of $L^1(\mathsf{R})$ into $M(B_n \times_{\beta_n} \mathsf{R})$.*

PROOF. Let $x_i \in B_\infty$ and $f_i \in L^1(\mathsf{R})$ for $i = 1, 2, \ldots, k$. Let $(x_{i,n})$ be a continuous field with $x_{i,\infty} = x_i$. We shall show that $\left\|\sum_{i=1}^k x_{i,n}\lambda_n(f_i)\right\|$ converges to $\left\|\sum_{i=1}^k x_i\lambda(f_i)\right\|$ as $n \to \infty$, where $\lambda = \lambda_\infty$. Since $\hat{\beta}_{n,p}(x_{in}\lambda_n(f_i)) = x_{in}\lambda_n(\chi_p f_i)$ with $\chi_p(t) = e^{ipt}$, this suffices to conclude the proof.

Let $\rho\left(\sum_{i=1}^k x_i\lambda(f_i)\right) = \limsup_n \left\|\sum_{i=1}^k x_{in}\lambda_n(f_i)\right\|$. Since

$$\rho\left(\sum_{i=1}^k x_i\lambda(f_i)\right) \le \int \left\|\sum x_i f_i(t)\right\| dt,$$

ρ is well-defined on the L^1-closure of the linear span of $x\lambda(f)$, $x \in B_\infty$, $f \in L^1(\mathbb{R})$. Since $\left(\sum x_{in}\lambda_n(f_i)\right)^*\left(\sum x_{in}\lambda_n(f_i)\right) = \sum_{i,j}\int \beta_{n,-t}(x_{jn}^* x_{in})\bar{f}_j(t)$ $\lambda_n(t)^*\,dt\lambda(f_i)$ can be approximated in L^1 norm by $\sum_{i,j}\sum_\ell \beta_{n,-t_\ell}(x_{jn}^* x_{in})$ $\lambda(f_j\Delta_\ell)^*\lambda(f_i)$ uniformly in n, where Δ_ℓ's are non-negative functions supported around $-t_\ell$ such that $\left(\sum_\ell \Delta_\ell\right)f_j \approx f_j$ in L^1 norm for all j, one can conclude that

$$\rho\Big(\Big(\sum x_i\lambda(f)\Big)^*\Big(\sum x_i\lambda(f_i)\Big)\Big) = \rho\Big(\sum x_i\lambda(f_i)\Big)^2.$$

Hence ρ is a C^*-semi-norm. Since $\rho\hat{\beta}_{\infty,p} = \rho$, if ρ is not a norm it vanishes on the ideal generated by a non-zero ideal of B_∞. If x is a non-zero element of that ideal and (x_n) is a continuous field with $x_\infty = x$, then it should follow that $\lim_n \|x_n\lambda_n(f)\| = 0$ for any $f \in L^1(\mathbb{R})$. Since $t \mapsto \beta_{n,t}(x_n)$ is continuous uniformly in n we may suppose that the β_n-spectrum of x_n is contained in $(-1,1)$ for all n. If $\hat{f}$ is 1 on $[0,1]$ one deduces $\|x_n\lambda(f)\| \geq \|x_n\|/3$, which contradicts that $x \neq 0$. (Assuming $B_n \times_{\beta_n} \mathbb{R}$ is faithfully represented, let E be the spectral measure of $t \mapsto \lambda_t$ and set $P_i = E(i-1,i]$. Since $x_n = \sum_i P_{i+1}xP_i + \sum_i P_i x_n P_i + \sum_i P_{i-1}x_n P_i$ one deduces that one of the three terms has at least norm $\|x_n\|/3$. Note that the norm of the first term is $\sup\|P_{i+1}x_nP_i\| = \|P_1x_nP_0\| \leq \|x_nP_0\|$ using the fact that the norm is invariant under the dual flow. With similar formulas for other terms one reaches the conclusion.) Thus one can conclude that ρ is the C^*-norm on $B_\infty \times_{\beta_\infty} \mathbb{R}$. Since the same arguments apply to any subsequence one concludes that $\lim_n\left\|\sum x_{in}\lambda(f_i)\right\| = \left\|\sum x_i\lambda(f_i)\right\|$.

DEFINITION 3.16. Let (k_n) be a sequence of positive integers and let γ_n be the flow on $M_{k_n} \otimes C_0(\mathbb{R})$ induced from translations, i.e., $(\gamma_{n,t}f)(s) = f(s-t)$ for $f \in M_{k_n} \otimes C_0(\mathbb{R}) = C_0(\mathbb{R}, M_{k_n})$. A flow α on a separable C^*-algebra is called a *dual MF flow* if there is such a sequence (k_n) and an embedding of (A,α) into $\left(\prod_{n=1}^\infty M_{k_n} \otimes C_0(\mathbb{R})\right)_\gamma / \bigoplus M_{k_n} \otimes C_0(\mathbb{R})$ equipped with $\gamma = \prod \gamma_n$.

PROPOSITION 3.17. *The class of dual MF flows on a separable C^*-algebra is closed under cocycle perturbations.*

PROOF. This is proved in the same way as Proposition 3.7 once we notice the following: Any γ-cocycle u in $M(M_k \otimes C_0(\mathbb{R}))$ is a coboundary. In fact if we set $w(s) = u_s(s)$ for such a γ-cocycle u then $w \in M(M_k \otimes C_0(\mathbb{R}))$ and $w\gamma_t(w^*)(s) = w(s)w(s-t)^* = u_s(s)u_{s-t}(s-t)^* = u_t(s)$.

We provide some details. Let (A,α) be a dual MF flow and ϕ an embedding of (A,α) into $\left(\prod_{n=1}^\infty M_{k_n} \otimes C_0(\mathbb{R})\right)_\gamma / \bigoplus M_{k_n} \otimes C_0(\mathbb{R})$. Note that A is non-unital (see 3.21 below) and let u be an α-cocycle in $M(A)$. If (x_i) is a dense sequence in A there is a sequence $(u^{(n)})$ of α-cocycles in $A + \mathsf{C}1$ [9] such that

$$\|(u_t - u_t^{(n)})x_i\| \leq 2^{-n}\|x_i\|, \qquad t \in [-1, 1], \ \ i = 1, 2, \ldots, n.$$

There are self-adjoint $h_n, b_n \in A + \mathsf{C}1$ such that $u_t^{(n)} = e^{ib_n} u_t^{(h_n)} \alpha_t(e^{-ib_n})$ (see Lemma 1.1 of [7]). By lifting $\tilde{\phi}(h_n), \tilde{\phi}(b_n)$ to self-adjoint elements in

$$\left(\prod_{n=1}^{\infty} M_{k_n}\right)_{\gamma} \otimes C_0(\mathsf{R}) + \mathsf{C}1,$$

where $\tilde{\phi}$ is the unitization of ϕ, we obtain a γ-cocycle $v^{(n)}$ in $\left(\prod_{n=1}^{\infty} M_{k_n}\right)_{\gamma} \otimes C_0(\mathsf{R}) + \mathsf{C}1$ such that $Q(v_t^{(n)}) = u_t^{(n)}$, where Q is the quotient map. We write $v_t^{(n)} = (v_{k,t}^{(n)})$, where $v_k^{(n)}$ is a γ_n-cocycle in $M_{k_n} \otimes C_0(\mathsf{R}) + \mathsf{C}1$. By patching up these $v_k^{(n)}$ we can construct a γ-cocycle w in $\prod_{n=1}^{\infty}(M_{k_n} \otimes C_0(\mathsf{R}) + \mathsf{C}1)$ such that $Q(w_t)\phi(a) = \phi(u_t a)$ for all $a \in A$ (see the proof of 3.6 for details). Then we conclude that $(\mathrm{Ad}\, w_t\gamma_t)^{-}\phi(a) = \phi(\mathrm{Ad}\, u_t\alpha_t(a))$, $a \in A$, where $(\mathrm{Ad}\, w_t\gamma_t)^{-}$ is the flow on the quotient induced by $\mathrm{Ad}\, w_t\gamma_t$. Since w_t is given as $U\gamma_t(U)^*$ with a unitary U in $\prod_{n=1}^{\infty} M(M_{k_n} \otimes C_0(\mathsf{R}))$, it follows that $(\gamma_t)^{-}\,\mathrm{Ad}\, Q(U^*)\phi(a) = \mathrm{Ad}\, Q(U^*)\phi(\mathrm{Ad}\, u_t\alpha_t(a))$, $a \in A$. Thus the embedding $\mathrm{Ad}\, Q(U^*)\phi$ intertwines $\mathrm{Ad}\, u_t\alpha_t$ with γ_t concluding the proof that $\mathrm{Ad}\, u\alpha$ is a dual MF flow.

LEMMA 3.18. *Let α be a flow on a separable C^*-algebra A.*

(1) *If α is an MF flow, then $\hat{\alpha}$ is a dual MF flow on $A \times_\alpha \mathsf{R}$.*

(2) *If α is a dual MF flow, then $\hat{\alpha}$ is an MF flow on $A \times_\alpha \mathsf{R}$.*

PROOF. By Theorem 3.10 if α is an MF flow then there is a continuous field of flows (A_n, α_n) over $\mathsf{N} \cup \{\infty\}$ such that $A_n = M_{k_n}$ for $n \in \mathsf{N}$ and $(A_\infty, \alpha_\infty) \cong (A, \alpha)$. Hence by Lemma 3.15 there is a continuous field of flows $(A_n \times_{\alpha_n} \mathsf{R}, \hat{\alpha}_n)$. Since $(A_n \times_{\alpha_n} \mathsf{R}, \hat{\alpha}_n) \cong (M_{k_n} \otimes C_0(\mathsf{R}), \gamma_n)$ one concludes that $(A \times_\alpha \mathsf{R}, \hat{\alpha})$ is a dual MF flow, where γ_n is induced from translations.

If α is a dual MF flow then there is a continuous field of flows (B_n, γ_n) over $\mathsf{N} \cup \{\infty\}$ such that $B_n = M_{k_n} \otimes C_0(\mathsf{R})$ and γ_n is induced from translations for $n \in \mathsf{N}$ and $(B_\infty, \gamma_\infty) \cong (A, \alpha)$. Then by Lemma 3.15 we obtain a continuous field of flows $(B_n \times_{\gamma_n} \mathsf{R}, \hat{\gamma}_n)$. Note that $B_n \times_{\gamma_n} \mathsf{R} \cong M_{k_n} \otimes \mathcal{K}$ and $\hat{\gamma}_n = \mathrm{id} \otimes \mathrm{Ad}\,\lambda$ for $n \in \mathsf{N}$. Hence by Proposition 3.5 (and the remark after that) we conclude that $(A \times_\alpha \mathsf{R}, \hat{\alpha})$ is MF.

PROPOSITION 3.19. *Let α be a flow on a separable C^*-algebra. Then α is an MF flow* (*resp. a dual MF flow*) *if and only if $\hat{\alpha}$ is a dual MF flow* (*resp. an MF flow*).

PROOF. The "only if" part is shown in the above lemma. Suppose that $\hat{\alpha}$ is a dual MF flow. Then $\hat{\hat{\alpha}}$ is an MF flow by the above lemma, i.e., we conclude that $\alpha \otimes \mathrm{Ad}\,\lambda$ is an MF flow on $A \otimes \mathcal{K}$ by the Takesaki-Takai duality. Hence $\alpha \otimes \mathrm{id}$ is also an MF flow on $A \otimes \mathcal{K}$ by 3.7; thus α is because $A \otimes e$ is an $\alpha \otimes \mathrm{id}$-invariant C^*-subalgebra of $A \otimes \mathcal{K}$, where e is a minimal projection in $\mathcal{K}$.

Suppose that $\hat{\alpha}$ is an MF flow. Then $\hat{\hat{\alpha}} = \alpha \otimes \mathrm{Ad}\,\lambda$ is a dual MF flow on $A \otimes \mathcal{K}$. Then one concludes that α is a dual MF flow just as above.

PROPOSITION 3.20. *Let α be a flow on a separable C^*-algebra A. Then the following conditions are equivalent.*

(1) *α is a dual MF flow.*

(2) *(A, α) can be embedded into $\left(\left(\prod_{n=1}^{\infty} \mathcal{K}_n \otimes C_0(\mathbf{R})\right)_{\gamma} / \bigoplus \mathcal{K}_n \otimes C_0(\mathbf{R}), \gamma\right)$, where $\mathcal{K}_n = \mathcal{K}$, $\gamma = \prod \gamma_n$, and γ_n is the flow induced by translations.*

PROOF. (1) $\Rightarrow$ (2) is easy. Suppose (2). Then one derives that $(A \times_{\alpha} \mathbf{R}, \hat{\alpha})$ satisfies the condition (2) in Proposition 3.5 since the crossed product of $C_0(\mathbf{R})$ by translations is $\mathcal{K}$. Hence $\hat{\alpha}$ is an MF flow. Thus α is a dual MF flow.

REMARK 3.21. If α is a dual MF flow on A, then A has no non-zero projections because $\mathcal{K} \otimes C_0(\mathbf{R})$ has no non-zero projections. In particular A has no unit. If α is a dual MF flow then no $\alpha_t \neq \mathrm{id}$ is approximately inner (i.e., no sequence of unitaries in $A + \mathbf{C}1$ approximates α_t by adjoint action).

Here we give some examples. The flow γ on $\mathcal{K} \otimes C_0(\mathbf{R})$ induced by translations is not a MF flow (see Example 2.10 in [11]) but of course it is a dual MF flow. The flow $\mathrm{Ad}\,\lambda$ on $\mathcal{K}$ is an MF flow but not a dual MF flow. (By the duality given in 3.19 these two statements are equivalent, giving another proof of Example 2.10 quoted above.) The identity flow on $\mathcal{K} \otimes C_0(\mathbf{R})$ is both an MF flow and a dual MF flow. (It is quasi-diagonal. To see that it is a dual MF flow define an isomorphism ϕ of $\mathcal{K} \otimes C_0(\mathbf{R})$ into $\prod_{n=1}^{\infty} \mathcal{K}_n \otimes C_0(\mathbf{R})$ by $\phi(f) = (f_1, f_2, \ldots)$ with $f_n(t) = f(t/n)$ for $f \in \mathcal{K}_n \otimes C_0(\mathbf{R}) \cong C_0(\mathbf{R}, \mathcal{K})$. Then ϕ embeds $(\mathcal{K} \otimes C_0(\mathbf{R}), \mathrm{id})$ into $\left(\prod \mathcal{K}_n \otimes C_0(\mathbf{R}) / \bigoplus \mathcal{K}_n \otimes C_0(\mathbf{R}), \gamma\right)$. From this it follows that $\mathrm{id} \otimes \mathrm{id} \otimes \gamma$ on $\mathcal{K} \otimes C_0(\mathbf{R}) \otimes C_0(\mathbf{R})$ is both an MF flow and a dual MF flow.

4. NF flows

The condition in the following lemma is a flow version of (vi) of Theorem 5.2.2 of [1].

LEMMA 4.1. *Let A be a nuclear C^*-algebra and α a quasi-diagonal flow on A. Then for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a flow β on a finite-dimensional C^*-algebra B and completely positive contractions $\sigma : A \to B$*

and $\tau : B \to A$ *such that*

$$\begin{aligned} \|x - \tau\sigma(x)\| &< \epsilon, & x &\in \mathcal{F}, \\ \|\sigma(xy) - \sigma(x)\sigma(y)\| &< \epsilon, & x, y &\in \mathcal{F}, \\ \|\sigma\alpha_t - \beta_t\sigma\| &< \epsilon, & t &\in [-1, 1]. \end{aligned}$$

PROOF. Since A is nuclear and quasi-diagonal, for any finite subset $\mathcal{F}$ and $\epsilon > 0$ there is a triple (B, σ, τ) which satisfies the first two conditions in the lemma (see (iv) of Theorem 5.2.2 of [1]). Though this σ has nothing to do with α, one can approximate σ by a CP contraction $\sigma' : A \to B$ which is α-covariant, i.e., the representation of A induced by σ' is α-covariant. More specifically we take a large $\gamma > 0$ such that

$$\frac{\gamma}{2}\int e^{-\gamma|t|}\|\alpha_t(x) - x\|\,dt \approx 0, \qquad x \in \mathcal{F} \cup (\mathcal{F}\cdot\mathcal{F})$$

where $\mathcal{F}\cdot\mathcal{F} = \{xy \mid x, y \in \mathcal{F}\}$ and set

$$\sigma' = \frac{\gamma}{2}\int e^{-\gamma|t|}\sigma\alpha_t\,dt.$$

Then it follows that σ' is a CP contraction of A into B such that $\|\sigma(x) - \sigma'(x)\| \approx 0$ for $x \in \mathcal{F} \cup (\mathcal{F}\cdot\mathcal{F})$. Thus one may assume that σ' also satisfies the first two conditions. Note σ' has the following property: $\sigma'\alpha_s \le e^{\gamma|s|}\sigma'$, i.e., $e^{\gamma|s|}\sigma' - \sigma'\alpha_s$ is CP, which implies that σ' is α-covariant. This fact follows from Lemma 4.2 below, a version of Stinespring's theorem.

Assume that B acts on a finite-dimensional Hilbert space $\mathcal{H}_B$ such that the commutant of B is abelian. There is a covariant representation (π, U) and an isometry V from $\mathcal{H}_B$ into $\mathcal{H}_\pi$ such that $\sigma'(x) = V^*\pi(x)V$ for $x \in A$. By adding another covariant representation to (π, U) we may suppose that $\pi \times U$ is a faithful representation of $A \times_\alpha \mathbf{R}$. Since α is quasi-diagonal it follows from Theorem 1.4 of [11] that $(\pi(A), U)$ is quasi-diagonal. Hence there is a finite-rank projection F on $\mathcal{H}_\pi$ such that $F \ge VV^*$, $[F, \pi(x)] \approx 0$ for $x \in \mathcal{F}$ and $\|[F, U_t]\| \approx 0$ for $t \in [-1, 1]$. By Lemma 3.2 applied to the compact operators $\mathcal{K}(\mathcal{H}_\pi)$ and $F \in \mathcal{K}(\mathcal{H}_\pi)$ there is an Ad U-cocycle Z in $\mathcal{K}(\mathcal{H}) + \mathbf{C}1 \subset \mathcal{B}(\mathcal{H}_\pi)$ such that $Z_t \approx 1$ for $t \in [-1, 1]$ and $[F, Z_tU_t] = 0$. Define $B_1 = F\mathcal{B}(\mathcal{H}_\pi)F$ and $\beta_t = \mathrm{Ad}(Z_tU_t)$ on B_1 and let $\sigma_1 = F\pi(\,\cdot\,)F$, a CP contraction from A to B_1. Then since $(\beta_t\sigma_1 - \sigma_1\alpha_t)(x) = F\{(\mathrm{Ad}(Z_tU_t) - \mathrm{Ad}(U_t))\pi(x)\}F$ for $x \in A$, we have that $\|\beta_t\sigma_1 - \sigma_1\alpha_t\| \approx 0$ for $t \in [-1, 1]$. Note also that $\sigma_1(xy) = F\pi(xy)F \approx F\pi(x)F\pi(y)F = \sigma_1(x)\sigma_1(y)$ for $x, y \in \mathcal{F}$. Let $\tau_1(T) = \tau P_B(V^*TV)$, $T \in B_1$, where P_B is a norm-one projection from

$\mathscr{B}(\mathscr{H}_B)$ onto B. Then $\tau_1\sigma_1(x) = \tau P_B(V^*F\pi(x)FV) = \tau\sigma'(x) \approx \tau\sigma(x)$ for $x \in \mathscr{F}$. Thus one can conclude that $(B_1, \beta, \sigma_1, \tau_1)$ has the required properties.

The following is taken from Section 4 of [11] (see also the proof of Proposition 2 of [10]).

LEMMA 4.2. *Let α be a flow on a C^*-algebra A and let B be a C^*-algebra acting on $\mathscr{H}_B$ and Z a unitary flow on $\mathscr{H}_B$ such that $t \mapsto \mathrm{Ad}\, Z_t$ defines a flow on B. Let ψ be a CP contraction from A into B and $\gamma > 0$ such that $\mathrm{Ad}\, Z_{-t}\psi\alpha_t \le e^{\gamma|t|}\psi$ for $t \in \mathsf{R}$. Let (π, V) denote the Stinespring pair for ψ, i.e., π is a representation of A and V is an isometry from $\mathscr{H}_B$ into $\mathscr{H}_\pi$ such that $\psi(x) = V^*\pi(x)V$, $x \in A$ and $P\mathscr{H}_\pi$ is cyclic for $\pi(A)$ with $P = VV^*$. Then there is a unitary flow $U = e^{itH}$ on $\mathscr{H}_\pi$ such that $\mathrm{Ad}\, U_t\pi = \pi\alpha_t$ and $\|[H, P]\| \le \gamma/2$.*

PROOF. We replace A by the unitization of A and assume $\psi(1) = 1$. On the algebraic tensor product $A \otimes \mathscr{H}_B$ we define a quasi-inner product by

$$\langle x \otimes \xi, y \otimes \eta\rangle = \langle \psi(y^*x)\xi, \eta\rangle_{\mathscr{H}_B},$$

and a representation π of A by

$$\pi(a)x \otimes \xi = ax \otimes \xi.$$

We define a linear map V from $\mathscr{H}_B$ into $A \otimes \mathscr{H}_B$ by $V\xi = 1 \otimes \xi$. Then we obtain the pair (π, V) in the statement by the usual procedure.

We define a linear operator W_t on $A \otimes \mathscr{H}_B$ by

$$W_t x \otimes \xi = \alpha_t(x) \otimes Z_t\xi.$$

We compute for a finite sum $\zeta = \sum_i x_i \otimes \xi_i$

$$\begin{aligned}\|W_t\zeta\|^2 &= \sum_{i,j}\langle \psi\alpha_t(x_i^*x_j)Z_t\xi_j, Z_t\xi_i\rangle \\ &\le e^{\gamma|t|}\sum_{i,j}\langle \psi(x_i^*x_j)\xi_j, \xi_i\rangle \\ &= e^{\gamma|t|}\|\zeta\|^2.\end{aligned}$$

This implies that W_t is a well-defined bounded operator in $\mathscr{H}_\pi$ such that $(W_t)^*W_t \le e^{\gamma|t|}1$. Moreover the family W_t, $t \in \mathsf{R}$ satisfies that $W_sW_t = W_{s+t}$, $W_0 = 1$, $t \mapsto W_t$ is strongly continuous, and $W_t\pi(x) = \pi\alpha_t(x)W_t$, $x \in A$.

Let $W_t = e^{iLt}$, i.e., iL is the generator of W. Since $(W_t)^*W_t \le e^{\gamma|t|}$ it follows that for any $\xi \in D(L)$

$$\frac{\|W_t\xi\|^2 - \|\xi\|^2}{|t|} \le \frac{(e^{\gamma|t|}-1)\|\xi\|^2}{|t|}.$$

By taking the limits $t \downarrow 0$ and $t \uparrow 0$ we derive

$$-\gamma\|\xi\|^2 \le \langle iL\xi, \xi\rangle + \langle \xi, iL\xi\rangle \le \gamma\|\xi\|^2,$$

which implies that $\mathscr{D}(L^*) \supset \mathscr{D}(L)$ and $-\gamma 1 \le iL - iL^* \le \gamma 1$ as a sesquilinear form on $D(L)$. Let C be the closure of $i(L-L^*)/2$. Then $\|C\| \le \gamma/2$ and $C = C^*$, and $L+iC$ is a symmetric operator because $L+iC = L-L/2+L^*/2 = (L+L^*)/2$ on $\mathscr{D}(L)$. Since $L+iC$ generates a strongly continuous one-parameter group of bounded operators, $L+iC$ must be self-adjoint with $D(L^*) = D(L)$.

Since $W_t\pi(x)W_{-t} = \pi\alpha_t(x)$, $x \in A$, it follows that $(W_t)^*W_t \in \pi(A)'$ and hence $C \in \pi(A)'$. Let $U_t = e^{i(L+iC)t}$, which is a unitary flow implementing α. We assert that $H = L+iC$ has the required property.

By the definition of W_t we deduce $W_tV = VZ_t$, which implies that $W_tP = VZ_tV^*$ is a unitary on $P\mathscr{H}_\pi$ with $P = VV^*$. Hence $W_tPW_t^* = P$. Since $(W_t-1)PW_t^*+P(W_t^*-1) = W_tPW_t^*-P = 0$, it follows that $LP-PL^* = 0$ on $D(L)$. Using $H = L+iC = L^*-iC$ we deduce that $[H,P] = (L+iC)P-P(L^*-iC) = i(CP+PC)$ on $D(L)$. Namely $[H,P]$ is bounded by $\|CP+PC\|$. On the other hand $PW_t^*W_tP = P$, which implies $PCP = 0$. Hence $\|CP+PC\| = \|(1-P)CP\| \le \gamma/2$. This completes the proof.

We prepare three technical lemmas which can be derived by using standard techniques which may be found in [4].

LEMMA 4.3. *There exists a constant $C > 0$ satisfying: Let γ be a flow on a C^*-algebra A and let δ_γ be the generator of γ. If $x \in D(\delta_\gamma)$ is such that* $\mathrm{Sp}(x^*x) \subset \{0\} \cup [1/2, 1]$ *then the partial isometry w obtained from the polar decomposition of x belongs to $D(\delta_\gamma)$ and satisfies that $\|\delta_\gamma(w)\| \le C\|\delta_\gamma(x)\|$.*

PROOF. Let f be a C^∞-function on R with compact support such that $f(0) = 0$ and $f(t) = t^{-1/2}$, $t \in [1/2, 1]$. Then w is obtained as $xf(x^*x)$. We use the formula:

$$f(x^*x) = \int \hat{f}(t)e^{itx^*x}\,dt$$

where $\hat{f}(t) = 1/2\pi \int f(s)e^{-ist}\,ds$, to derive $f(x^*x) \in D(\delta_\gamma)$ and

$$\|\delta_\gamma(f(x^*x))\| \le \int |t\hat{f}(t)|\,dt\,\|\delta_\gamma(x^*x)\|.$$

Thus $C = \sqrt{2} + 2\int |t\hat{f}(t)|\,dt$ will do. See Section 3.2.2 of [4] for details.

LEMMA 4.4. *There exists a constant $C > 0$ satisfying: Let γ be a flow on a C^*-algebra A. Let $p \in A$ be a projection in $D(\delta_\gamma)$ such that $\delta_\gamma(p) = 0$ and let $e \in D(\delta_\gamma)$ be a projection such that $\|pe - p\| \le 1/8$. Then there is a projection $e' \in D(\delta_\gamma)$ such that $pe' = p$, $\|e - e'\| \le 12\|pe - p\|$, and $\|\delta_\gamma(e')\| \le C\|\delta_\gamma(e)\|$.*

PROOF. Since $\|pep - p\| = \|p(ep - p)\| < 1/8$, it follows that $\mathrm{Sp}(epe) = \{0\} \cup [7/8, 1]$. Let w be the partial isometry obtained from the polar decomposition of pe. Note that $\|w - p\| \le \|w - w|pe|\| + \|pe - p\| \le 2\|pe - p\|$. Note also, from Lemma 4.3, that $\|\delta_\gamma(w)\| \le C\|\delta_\gamma(e)\|$, where C is the universal constant there. Since $\|(1 - p)(1 - w^*w)(1 - p) - (1 - p)\| \le \|(1 - w^*w)(1 - p) - (1 - p)\| = \|w^*w - w^*wp\|$ and $\|w^*w - p\| \le \|w^*(w - p)\| + \|(w^* - p)p\| \le 2\|w - p\| \le 4\|pe - p\|$, it follows that the spectrum of $(1 - w^*w)(1 - p)(1 - ww^*)$ is contained in $\{0\} \cup [1/2, 1]$. Let w' be the partial isometry obtained from $(1 - p)(1 - w^*w)$ (in $A + \mathbf{C}1$ if A is not unital). Then $\|w' - (1 - p)\| \le 2\|(1 - p)(1 - w^*w) - (1 - p)\| \le 2\|w^*w - pw^*w\| = \|(1 - p)w^*w\| \le 4\|pe - p\|$. From Lemma 4.3 it follows that $\|\delta_\gamma(w')\| \le C\|\delta_\gamma(w^*w)\| \le 2C\|\delta_\gamma(w)\|$. We set $u = w + w'$, which is a unitary such that $\|u - 1\| \le \|w - p\| + \|w' - (1 - p)\| \le 6\|pe - p\|$ and $\|\delta_\gamma(u)\| \le \|\delta_\gamma(w)\| + \|\delta_\gamma(w')\| \le (2C^2 + C)\|\delta_\gamma(e)\|$. We set $e' = ueu^*$, which is a projection such that $e' \ge wew^* = ww^* = p$. Note that $\|e' - e\| = \|ueu^* - e\| \le 2\|u - 1\| \le 12\|pe - p\|$ and that $\|\delta_\gamma(e')\| \le 2\|\delta_\gamma(u)\| + \|\delta_\gamma(e)\| \le (4C^2 + 2C + 1)\|\delta_\gamma(e)\|$. This completes the proof.

LEMMA 4.5. *Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the compact operators on a Hilbert space $\mathcal{H}$ and H a self-adjoint operator on $\mathcal{H}$ which defines a flow $\gamma : t \mapsto \mathrm{Ad}\, e^{itH}$ on $\mathcal{K}$. Then the domain $D(\delta_\gamma)$ is the set of operators $x \in \mathcal{K}$ such that $xD(H) \subset D(H)$ and $[iH, x]$ on $D(H)$ extends to a compact operator, which is $\delta_\gamma(x)$. If $x \in \mathcal{K}$ is of finite rank and $xD(H) \subset D(H)$ and $[iH, x]$ is bounded on $D(H)$ then the closure of $[iH, x]$ is compact and thus $x \in D(\delta_\gamma)$ and $\delta_\gamma(x) = \overline{[iH, x]}$.*

PROOF. Let $\bar{\gamma}_t = \mathrm{Ad}\, e^{itH}$ on the bounded operators $B(\mathcal{H})$. Then $\bar{\gamma}$ is a one-parameter group of automorphisms of the type I factor $B(\mathcal{H})$ and $t \mapsto \bar{\gamma}(Q)$ is continuous in the strong operator topology for $Q \in B(\mathcal{H})$. Let L be the generator of $\bar{\gamma}$. Then $D(L)$ consists of $Q \in B(\mathcal{H})$ such that $QD(H) \subset D(H)$ and $[iH, Q]$ is bounded on $D(H)$ and if $Q \in D(L)$ then $L(Q)$ is the closure of $[iH, Q]$. (See Proposition 3.2.55 of [4].) Thus if $x \in D(\delta_\gamma)$ then it follows that $xD(H) \subset D(H)$ and $[iH, x]$ on $D(H)$ extends to a compact operator. Conversely if $x \in \mathcal{K}$ satisfies the latter conditions, then it follows that $t \mapsto$

$iHe^{itH}xe^{-itH}\xi$ is continuous for $\xi \in D(H)$. Hence if $f \in L^1(\mathbf{R})$ is such that $\hat{f}$ has compact support then the closure of $[iH, \gamma_f(x)]$ is equal to

$$\int f(t)e^{itH}\overline{[iH, x]}e^{-itH}\,dt,$$

where $\gamma_f(x) = \int f(t)\gamma_t(x)\,dt$ belongs to $D(\delta_\gamma)$ as having compact γ-spectrum. Since $\overline{[iH, \gamma_f(x)]} = \delta_\gamma(\gamma_f(x))$ and δ_γ is closed it follows that $x \in D(\delta_\gamma)$ and $\delta_\gamma(x) = \overline{[iH, x]}$.

Let $x \in \mathscr{K}$ be of finite rank. Since the range V of x is finite-dimensional and contained in $D(H)$ it follows that $H|V$ is bounded. If (ξ_n) is a bounded sequence in $D(H)$ then there is a subsequence (ξ_n') of (ξ_n) such that $x\xi_n'$ converges; so $iHx\xi_n'$ converges. Since $(xiH\xi_n')$ is a bounded sequence in V we can choose a subsequence (ξ_n'') of (ξ_n') such that $xiH\xi_n''$ converges. Thus $[iH, x]\xi_n''$ converges and $\overline{[iH, x]}$ is compact. By the way in general we have to require $[iH, x]$ to be compact (not just bounded) to ensure $x \in D(\delta_\gamma)$.

We will apply Lemma 4.4 to the situation described in Lemma 4.5 in the proof of the following lemma.

LEMMA 4.6. *Let B be a separable nuclear C^*-algebra on a Hilbert space $\mathscr{H}$ and U a unitary flow on $\mathscr{H}$ such that $B \supset \mathscr{K}(\mathscr{H})$, $t \mapsto \mathrm{Ad}\,U_t(x)$ defines a norm-continuous flow on B. Let α denote the flow on $A = B/\mathscr{K}(\mathscr{H})$ induced by $t \mapsto \mathrm{Ad}\,U_t|B$. Then if (B, U) is quasi-diagonal then (A, α) is quasi-diagonal.*

PROOF. Under the assumption we shall prove the condition (2) of Theorem 1.5 of [11]. Namely for any finite subset $\mathscr{F}$ of A and $\epsilon > 0$ we shall construct a finite-dimensional C^*-algebra D, a flow β on D, and a CP map ϕ of A into D such that $\|\phi\| \le 1$, $\|\phi(x)\| \ge (1-\epsilon)\|x\|$ and $\|\phi(x)\phi(y) - \phi(xy)\| \le \epsilon\|x\|\|y\|$ for $x, y \in \mathscr{F}$, and $\|\beta_t\phi - \phi\alpha_t\| < \epsilon$ for $t \in [-1, 1]$.

Since (B, U) is quasi-diagonal there is an increasing sequence (P_n) of finite-rank projections on $\mathscr{H}$ such that $\lim_n P_n = 1$, $\|[P_n, a]\| \to 0$ for all $a \in B$, and $\|[P_n, H]\| < 2^{-n}$ where H is the self-adjoint generator of U. Note that the last condition means that $P_n\mathscr{D}(H) \subset \mathscr{D}(H)$ and $\|[P_n, H]|\mathscr{D}(H)\| < 2^{-n}$. Let $P_0 = 0$ and let $H_0 = \sum_{n=1}^{\infty}(P_n - P_{n-1})H(P_n - P_{n-1})$, which is a well-defined self-adjoint operator. Since $H - H_0$ on $\mathscr{D}(H)$ is compact, we may take the unitary flow generated by H_0 instead of U, which still leaves B invariant and defines a flow on B dropping to the same flow α on the quotient $A = B/\mathscr{K}(\mathscr{H})$. Thus we assume now that $[P_n, H] = 0$ for all n.

The existence of the above (P_n) follows by the following arguments. Suppose that P_n was chosen. We have to define P_{n+1}. The main difficulty lies in finding one strictly bigger than P_n. First let $h = -(1-P_n)HP_n - P_nH(1-P_n)$, which is a compact operator with norm less than or equal to $\|[H, P_n]\|$. We choose a constant $C > 0$ as in Lemma 4.4. Let $\epsilon > 0$ be sufficiently small and set $\delta = \epsilon/C$. Then we find a sufficiently large finite-rank projection E such that $\|P_nE - P_n\| < \delta$, $\|hE - h\| < \delta/4$, $\|[H, E]\| < \delta/2$, and $\|[E, a]\| < \delta$ for a finite number of $a \in B$ prescribed. Note that $[H + h, P_n] = 0$ and $\|[H + h, E]\| \le \delta$. By applying Lemma 4.4 to the pair P_n, E with the derivation $i[H+h, \cdot\,]$ on the compact operators, we obtain a finite-rank projection E' such that $P_n \le E'$, $\|E - E'\| < \epsilon$, and $\|[H + h, E']\| < \epsilon$. Since $\|[h, E']\| \le 2\epsilon\|h\| + \|[h, E]\| < 2\epsilon + \delta/2$, we deduce that $\|[H, E']\| \le (3 + (2C)^{-1})\epsilon$. Thus for a sufficiently small $\epsilon > 0$ we can set $P_{n+1} = E'$.

Since A is nuclear there is a completely positive (CP) contraction ϕ of A into B such that $Q\phi = \mathrm{id}$, where Q is the quotient map of B onto $A = B/\mathscr{K}(\mathscr{H})$ [5]. Let $\phi_t = \mathrm{Ad}\, U_{-t}\phi\alpha_t$ for $t \in \mathsf{R}$, which is also a CP map. Since $Q\phi_t = \mathrm{id}$, it follows that $\phi_t(a) - \phi(a) \in \mathscr{K}(\mathscr{H})$. Since $t \mapsto \phi_t(a)$ is norm-continuous one deduces that $\|(1 - P_n)(\phi_t(a) - \phi(a))(1 - P_n)\| \to 0$ as $n \to \infty$ uniformly in t on every compact subset of R for all $a \in A$.

Let $\mathscr{F}$ be a finite subset of A and $\epsilon > 0$. Let

$$\psi = \frac{\epsilon}{2}\int e^{-\epsilon|t|}\phi_t\, dt$$

which is a CP map of A into B such that $Q\psi = \mathrm{id}$. Since $e^{-\epsilon|t|}\psi \le \mathrm{Ad}\, U_{-t}\psi\alpha_t \le e^{\epsilon|t|}\psi$ it follows that $\|\psi\alpha_t - \mathrm{Ad}\, U_t\psi\| \le e^{\epsilon|t|} - 1$.

Since $\psi(x)\psi(y) - \psi(xy) \in \mathscr{K}(\mathscr{H})$ there is an $N \in \mathsf{N}$ such that $\|(1 - P_N)(\psi(x)\psi(y) - \psi(xy))(1 - P_N)\| < \epsilon\|x\|\|y\|/2$ for $x, y \in \mathscr{F}$. There exists an $n \ge N$ such that for any $m \ge n$ $\|[P_m, \psi(x)]\| < \epsilon/4$ for $x \in \mathscr{F}$. Since $Q\psi = \mathrm{id}$ we have that $\|(1 - P_n)\psi(x)(1 - P_n)\| \ge \|x\|$ for $x \in A$. We then choose $m > n$ such that $\|(P_m - P_n)\psi(x)(P_m - P_n)\| \ge (1 - \epsilon)\|x\|$ for $x \in \mathscr{F}$. Let $E = P_m - P_n$. Since $\|[E, \psi(x)]\| \le \epsilon\|x\|/2$ for $x \in \mathscr{F}$, we obtain that $\|E\psi(x)E\psi(y)E - E\psi(xy)E\| \le \epsilon\|x\|\|y\|/2 + \|E\psi(x)\psi(y)E - E\psi(xy)E\| \le \epsilon\|x\|\|y\|$. By setting $D = E\mathscr{B}(\mathscr{H})E$, $\beta_t = \mathrm{Ad}\, U_t|D$, and $\phi(x) = E\psi(x)E$, $x \in A$, we obtain the desired triple (D, β, ϕ).

The following result is proved by mimicking the proof of Theorem 5.2.2 of [1].

THEOREM 4.7. *Let α be a flow on a separable C^*-algebra. Then the following conditions are equivalent:*

(1) *(A, α) is obtained as the inductive limit of a $*$-linear generalized inductive system of flows on finite-dimensional C^*-algebras where the coherent maps are all completely positive contractions.*

(2) *A is nuclear and α is an MF flow.*

(3) *A is nuclear and there is an essential quasi-diagonal extension B of A by the compact operators $\mathcal{K}$ and a unitary flow $U \in M(\mathcal{K})$ such that $t \mapsto \operatorname{Ad} U_t$ defines a flow on B, (B, U) is quasi-diagonal, and $Q \operatorname{Ad} U_t = \alpha_t Q$, where Q is the quotient map of B onto A.*

(4) *A is nuclear and α is quasi-diagonal.*

(5) *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a flow β on a finite-dimensional C^*-algebra B and completely positive contractions $\sigma : A \to B$ and $\tau : B \to A$ such that*

$$\|x - \tau\sigma(x)\| < \epsilon, \qquad x \in \mathcal{F},$$
$$\|\sigma(xy) - \sigma(x)\sigma(y)\| < \epsilon, \qquad x, y \in \mathcal{F},$$
$$\|\sigma\alpha_t - \beta_t\sigma\| < \epsilon, \qquad t \in [-1, 1].$$

(6) *A is nuclear and there is a continuous field of flows (A_n, α_n) over $\mathsf{N} \cup \{\infty\}$ such that A_n is finite-dimensional for $n \in \mathsf{N}$ and $(A_\infty, \alpha_\infty) \cong (A, \alpha)$.*

(7) *A is nuclear and there is a continuous field of flows (A_n, α_n) over $\mathsf{N} \cup \{\infty\}$ such that $A_n \cong M_{k_n}$ for some k_n for $n \in \mathsf{N}$ and $(A_\infty, \alpha_\infty) \cong (A, \alpha)$.*

Proof. (1) ⇒ (2): That A is nuclear follows from Proposition 5.1.3 of [1] and that α is an MF flow follows from Theorem 3.10.

(2) ⇒ (3): This follows from (2) ⇒ (3) of Theorem 3.10.

(3) ⇒ (4): This follows from Lemma 4.6.

(4) ⇒ (5): This follows from Lemma 4.1.

The equivalences between (2), (6), and (7) follow from those between (2), (4), and (5) in Theorem 3.10.

It remains to prove (5) ⇒ (1). We define a sequence (A_n, α_n) of flows on finite-dimensional C^*-algebras and sequences of CP contractions $\sigma_n : A \to A_n$ and $\tau_n : A_n \to A$ as follows. Let (x_n) be a dense sequence in A. We choose (A_1, α_1) and CP contractions $\sigma_1 : A \to A_1$ and $\tau_1 : A_1 \to A$ such that $\|x_1 - \tau_1\sigma_1(x_1)\| < 1/2$ and $\|\sigma_1\alpha_t - \alpha_{1,t}\sigma_1\| < 1/2$ for $t \in [-1, 1]$. Suppose that $(A_m, \alpha_m, \sigma_m, \tau_m)$ is defined up to $m = n$. Let $N \in \mathsf{N}$ be such that if $|t| < 1/N$ then $\|\alpha_t(x) - x\| < 2^{-n}$ for all x in the unit ball of $\tau_n(A_n)$. Let V_n be the finite-dimensional subspace generated by $\alpha_{k/N}(x)$ with $x \in \tau_n(A_n)$, $k = 0, \pm 1, \ldots, \pm N$, and xy with $x, y \in \tau_n(A_n)$ and $V_{n-1} \cup \{x_n\}$. We choose $(A_{n+1}, \alpha_{n+1}, \sigma_{n+1}, \tau_{n+1})$ such that

$$\|x - \tau_{n+1}\sigma_{n+1}(x)\| \le 2^{-n-1}\|x\|, \qquad x \in V_n,$$
$$\|\sigma_{n+1}(x)\sigma_{n+1}(y) - \sigma_{n+1}(xy)\| \le 2^{-n-1}\|x\|\|y\|, \quad x, y \in V_n,$$
$$\|\sigma_{n+1}\alpha_t - \alpha_{n+1,t}\sigma_{n+1}\| < 2^{-n-1}, \qquad t \in [-1, 1].$$

Note that (V_n) is increasing with dense union in A.

Let $\phi_n = \sigma_{n+1}\tau_n : A_n \to A_{n+1}$, a CP contraction. We can show that ϕ_n is almost multiplicative on $\phi_{n-1}(A_{n-1})$ as follows. If $x \in A_{n-1}$, then $\phi_n(\phi_{n-1}(x)\phi_{n-1}(x))$ is approximately equal to $\sigma_{n+1}\tau_n\sigma_n(\tau_{n-1}(x)\tau_{n-1}(y))$ (since σ_n is approximately multiplicative) and then to

$$\sigma_{n+1}(\tau_{n-1}(x)\tau_{n-1}(y)) \approx \sigma_{n+1}\tau_{n-1}(x)\sigma_{n+1}\tau_{n-1}(y) \approx \phi_n\phi_{n-1}(x)\phi_n\phi_{n-1}(y),$$

where the error is up to $5 \cdot 2^{-n}\|x\|\|y\|$. We can show that $\alpha_{n+1,t}\phi_n - \phi_n\alpha_{n,t}$ is almost equal to zero on $\phi_{n-1}(A_{n-1})$. If $x \in A_{n-1}$ and $t \in [-1, 1]$, then $(\alpha_{n+1,t}\phi_n - \phi_n\alpha_{n,t})\phi_{n-1}(x) = (\alpha_{n+1,t}\sigma_{n+1}\tau_n - \sigma_{n+1}\tau_n\alpha_{n,t})\sigma_n\tau_{n-1}(x)$ is approximately equal to

$$\begin{aligned}&\sigma_{n+1}\alpha_t\tau_n\sigma_n\tau_{n-1}(x) - \sigma_{n+1}\tau_n\sigma_n\alpha_t\tau_{n-1}(x)\\ &\qquad\approx \sigma_{n+1}\alpha_t\tau_{n-1}(x) - \sigma_{n+1}\alpha_t\tau_{n-1}(x) = 0,\end{aligned}$$

where the error is up to $6 \cdot 2^{-n}\|x\|$.

Now we have the following commutative diagram:

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \ldots \\ {\scriptstyle\tau_1}\downarrow & \overset{\sigma_2}{\nearrow} & {\scriptstyle\tau_2}\downarrow & \overset{\sigma_3}{\nearrow} & \downarrow & & \\ A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \ldots \end{array}$$

where the arrows represent CP contractions. Hence the upper sequence and the lower sequence define the same object as Banach spaces (at least). Let

$$\psi_{m,n} = \tau_{n-1}\sigma_{n-1}\tau_{n-2}\sigma_{n-2}\cdots\tau_{m+1}\sigma_{m+1}$$

for $n > m$, a CP contraction from the m'th A into n'th A. Since $(\psi_{n,m}(x))_{n\geq m}$ is a Cauchy sequence for each $x \in A$ we denote the limit by $\Psi_m(x)$. Then (Ψ_n) defines a sequence of CP contractions from A into A and satisfies $\Psi_n\psi_{m,n} = \Psi_m$ for $n \geq m$. Since $\bigcup_n \Psi_n(A)$ is dense in A it follows that the lower sequence defines A as a Banach space. From the way to define product in the inductive limit, one concludes that the lower sequence defines A as a C^*-algebra. Since $\Psi_n\alpha_t\psi_{m,n}(x)$ converges to $\alpha_t\Psi_m(x)$ as $n \to \infty$, the lower sequence defines (A, α) as a flow. Then one argues the upper sequence defines (A, α) as well.

We will call α an *NF flow* if it satisfies the conditions described in the above theorem. Since quasi-diagonality is preserved under cocycle perturbations (2.2 of [4]), a cocycle perturbation of an NF flow is also an NF flow.

5. Strongly quasi-diagonal flows

DEFINITION 5.1. Let A be a C^*-algebra and let α be a flow on A. We call α *strongly quasi-diagonal* if $(\pi(A), U)$ is quasi-diagonal for any covariant representation (π, U).

Note that the C^*-algebra A is called strongly quasi-diagonal if $\pi(A)$ is quasi-diagonal for any representation π of A.

A quasi-diagonal flow need not be strongly quasi-diagonal. If α is an arbitrary flow on a quasi-diagonal C^*-algebra A, the flow β on $B = A \otimes C[0, 1]$ defined by $\beta_t(x)(s) = \alpha_{st}(x(s))$ is quasi-diagonal and has (A, α) as a quotient (see Proposition 2.15 of [11]). Hence if (A, α) is not quasi-diagonal then (B, β) is not strongly quasi-diagonal.

In a similar fashion we can define a notion of strong pseudo-diagonality. Then it follows that an approximately inner flow on a quasi-diagonal C^*-algebra is strongly pseudo-diagonal (see the proof of Proposition 2.17 of [11]). But we do not know if they are strongly quasi-diagonal or not.

The following shows the above definition is not empty.

LEMMA 5.2. *Let A be a strongly quasi-diagonal C^*-algebra. Then the trivial flow $\alpha = \mathrm{id}$ is strongly quasi-diagonal.*

PROOF. Let (π, U) be a covariant representation of (A, α), i.e., U is a unitary flow on $\mathscr{H}_\pi$ such that $U_t \in \pi(A)'$. Let H be the self-adjoint generator of U and E the spectral measure of H.

Let $\mathscr{F}$ be a finite subset of A, let $\mathscr{G}$ be a finite subset of $\mathscr{H}_\pi$, and let $\epsilon > 0$. We may suppose that all $\xi \in \mathscr{G}$ belong to $E(a, b]\mathscr{H}_\pi$ for some $a < b$. Let $(a_i)_{i=0}^N$ be an increasing sequence in R such that $a_0 = a$, $a_N = b$, and $a_i - a_{i-1} < \epsilon$ for $i = 1, 2, \ldots, N$. Let $\mathscr{G}_i = \{E(a_{i-1}, a_i]\xi \mid \xi \in \mathscr{G}\}$. Since $\pi(A)E(a_{i-1}, a_i]$ is quasi-diagonal on the subspace $\mathscr{H}_i = E(a_{i-1}, a_i]\mathscr{H}_\pi$, there is a finite-rank operator E_i on $\mathscr{H}_i$ such that $\|[E_i, \pi(x)E(a_{i-1}, a_i]]\| \le \epsilon\|x\|$ for $x \in \mathscr{F}$ and $\|(E(a_{i-1}, a_i] - E_i)\xi\| \le \epsilon\|\xi\|$ for $\xi \in \mathscr{G}_i$. Let $E = \sum_{i=1}^N E_i$, which is a finite-rank projection on $\mathscr{H}_\pi$. Since $[E, \pi(x)] = \sum_i E(a_{i-1}, a_i][E_i, \pi(x)]$, we deduce that

$$\|[E, \pi(x)]\| = \max_i \|E(a_{i-1}, a_i][E_i, \pi(x)]\| \le \epsilon\|x\|$$

for $x \in \mathscr{F}$. Since $(1 - E)\xi = \sum_i (E(a_{i-1}, a_i] - E_i)E(a_{i-1}, a_i]\xi$, we deduce that

$$\|(1 - E)\xi\|^2 = \sum_i \|(E(a_{i-1}, a_i] - E_i)E(a_{i-1}, a_i]\xi\|^2 \le \epsilon^2\|\xi\|^2$$

for $\xi \in \mathcal{G}$. Since $U_tEU_t^* - E = \sum_i E(a_{i-1}, a_i](U_tE_iU_t^* - E_i)$ we deduce that

$$\|U_tEU_t^* - E\| = \max_i \|U_tE_iU_t^* - E_i\| \le \epsilon|t|.$$

This shows that $(\pi(A), U)$ is quasi-diagonal.

PROPOSITION 5.3. *Let α be a strongly quasi-diagonal flow on A and let u be an α-cocycle. Then* $\mathrm{Ad}\, u\alpha$ *is also strongly quasi-diagonal.*

PROOF. Let (π, U) be a covariant representation of $(A, \mathrm{Ad}\, u\alpha)$. Then $t \mapsto V_t = \pi(u_t^*)U_t$ is a unitary flow implementing α. Hence by assumption $(\pi(A), V)$ is quasi-diagonal. Then it follows from the proof of Proposition 2.2 of [11] that $(\pi(A), U)$ is quasi-diagonal.

COROLLARY 5.4. *Let α be a flow on A. Let B be an α-invariant hereditary C^*-subalgebra of A such that B generates A as a closed ideal. Then α is strongly quasi-diagonal if and only if $\alpha|B$ is strongly quasi-diagonal.*

PROOF. Any covariant representation of $(B, \alpha|B)$ extends to a covarint representation of (A, α). Hence if (A, α) is strongly quasi-diagonal then so is $(B, \alpha|B)$.

Suppose that $(B, \alpha|B)$ is strongly quasi-diagonal. Then $(B\otimes\mathcal{K}, \alpha|B\otimes\mathrm{id})$ is also strongly quasi-diagonal, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space. If A is separable then $(A \otimes \mathcal{K}, \alpha \otimes \mathrm{id})$ is isomorphic to a cocycle perturbation of $(B \otimes \mathcal{K}, \alpha|B \otimes \mathrm{id})$. Thus one concludes that (A, α) is strongly quasi-diagonal in this case. One can reduce the general case to this case (see the proof of 2.7 of [11]).

PROPOSITION 5.5. *Let A be a C^*-algebra and let α be a flow on A. Suppose that there is an increasing sequence (A_n) of α-invariant C^*-subalgebras of A with dense union such that A_n is strongly quasi-diagonal and the restriction of α to A_n is inner, i.e., $\alpha|A_n =$* $\mathrm{Ad}\, u_t$ *for some unitary flow u in $M(A_n)$. Then α is strongly quasi-diagonal.*

PROOF. Let (π, U) be a covariant representation of (A, α). Then by assumption $(\pi(A_n), U)$ is quasi-diagonal for any n. Hence $(\pi(A), U)$ is also quasi-diagonal.

COROLLARY 5.6. *Any AF flow is strongly quasi-diagonal.*

PROOF. Let α be an AF flow on A. Then A is an AF C^*-algebra and there is an increasing sequence (A_n) of finite-dimensional α-invariant C^*-algebras of A with dense union. Since $\alpha|A_n$ is inner and A_n is strongly quasi-diagonal this follows from the above proposition.

LEMMA 5.7. *Let α be a flow on a separable C^*-algebra A. Suppose that there is a sequence (π_i, U^i) of covariant irreducible representations of (A, α) such that $\bigoplus_i \pi_i$ is faithful, (π_i) are mutually disjoint, and $(\pi_i(A), U^i)$ is quasi-diagonal for all i. Then there is an α-cocyle u and an increasing sequence (A_n) of* $\mathrm{Ad}\, u\alpha$*-invariant residually finite-dimensional* (*RFD*) *C^*-subalgebras of A with dense union such that $\pi_i|A_n$ is equivalent to a direct sum of* $\mathrm{Ad}\, u\alpha$*-covariant finite-dimensional irreducible representations for all i and n.*

PROOF. Let (x_i) be a dense sequence of the unit ball of $A_{sa} = \{x \mid x = x^* \in A\}$. Let $\mathscr{H}_i$ denote the representation Hilbert space for π_i and $(\xi_k^{(i)})$ be an orthonormal basis of $\mathscr{H}_i$. Let H_i denote the self-adjoint generator of U^i and $\epsilon > 0$.

Let E_{11} be a finite-rank projection on $\mathscr{H}_1$ such that $\|(1 - E_{11})\xi_1^{(1)}\| < \epsilon/2$, $\|[E_{11}, \pi_1(x_1)]\| < \epsilon/2$, and $\|[E_{11}, H_1]\| < \epsilon/2$.

Let E'_{11} be the range projection of $(1 - E_{11})x_1 E_{11}$, which is a finite-rank projection orthogonal to E_{11}. We apply Kadison's transitivity theorem to an operator on the finite-dimensional space $(E_{11} + E'_{11})\mathscr{H}_1$ to find a $y_{11} \in A_{sa}$ such that $\|y_{11}\| = \|E'_{11}\pi_1(x_1)E_{11}\| < \epsilon/2$ and

$$\pi_1(y_{11})(E_{11} + E'_{11}) = E'_{11}\pi_1(x_1)E_{11} + E_{11}\pi_1(x_1)E'_{11}.$$

Note that $[E_{11}, \pi_1(x_1 - y_{11})] = 0$. Similarly there is an $h_1 \in A_{sa}$ such that $\|h_1\| < \epsilon/2$ and $[E_{11}, H_1 - \pi_1(h_1)] = 0$. We set $y_{i1} = 0$ for $i > 1$.

Next we find finite-rank projections E_{12} in $\mathscr{H}_1$ and E_{22} in $\mathscr{H}_2$ such that $E_{11} \leq E_{12}$, $\|(1 - E_{12})\xi_i^{(1)}\| < \epsilon/4$ and $\|(1 - E_{22})\xi_i^{(2)}\| < \epsilon/4$ for $i = 1, 2$, $\|[E_{12}, \pi_1(x_i - y_{i1})]\| < \epsilon/4$ and $\|[E_{22}, \pi_2(x_i - y_{i1})]\| < \epsilon/4$ for $i = 1, 2$, and $\|[E_{12}, H_1 - \pi_1(h_1)]\| < \epsilon/4$, and $\|[E_{22}, H_2 - \pi_2(h_1)]\| < \epsilon/4$. (Since $[E_{11}, H_1 - \pi_1(h_1)] = 0$, we can impose the strict inequality $E_{11} \leq E_{12}$ from an approximate one as follows. If $E_{11} \lessapprox E_{12}$ let F be the projection obtained from $E_{12}E_{11}E_{12} \approx E_{11}$ by continuous functional calculus and define $X = E_{11}F + (1 - E_{11})(E_{12} - F) \approx E_{12}$ and let $X = VE_{12}$ be the polar decomposition of X. We take $VE_{12}V^*$ (which dominates E_{11}) instead of E_{12}. Since $\|[F, H_1 - \pi_1(h_1)]\| \approx 0$ and $\|[X, H_1 - \pi_1(h_1)]\| \approx 0$ depending only on $\|[E_{12}, H_1 - \pi_1(h_1)]\| \approx 0$, we conclude that $\|[VE_{12}V^*, H_1 - \pi_1(h_1)]\| \approx 0$.) Let $E_{k2}^{(i)}$ be the range projection of $(1 - E_{k2})\pi_k(x_i - y_{i1})E_{k2}$ for $k = 1, 2$ and $i = 1, 2$. There is an $y_{i2} \in A_{sa}$ for $i = 1, 2$ such that $\|y_{i2}\| < \epsilon/4$ and

$$\pi_k(y_{i2})(E_{k2} + E_{k2}^{(i)}) = E_{k2}^{(i)}\pi_k(x_i - y_{i1})E_{k2} + E_{k2}\pi_k(x_i - y_{i1})E_{k2}^{(i)}$$

for $k = 1, 2$, where we have used the fact that π_1 and π_2 are mutually disjoint. Note that $[E_{12}, \pi(x_1 - y_{11} - y_{12})] = 0$ and $[E_{12}, \pi(x_2 - y_{21} - y_{22})] = 0$. Since $\pi(y_{12})E_{11} = 0$ it also follows that $[E_{11}, \pi(x_1 - y_{11} - y_{12})] = 0$. Similarly there is an $h_2 \in A_{sa}$ such that $\|h_2\| < \epsilon/4$ and $[E_{k2}, H_k - \pi_k(h_1 + h_2)] = 0$ for $k = 1, 2$. Note also that $[E_{11}, H_1 - \pi_1(h_1 + h_2)] = 0$. We set $y_{i2} = 0$ for $i > 2$. Note that we have defined $E_{11} \le E_{12}$ on $\mathscr{H}_1$ and E_{22} on $\mathscr{H}_2$. We will set $E_{kj} = 0$ for $k > j$.

We repeat this process. After n steps we find $y_{ij} \in A_{sa}$ for $1 \le j \le n$ and $h_i \in A_{sa}$ for $1 \le i \le n$ and finite rank projections E_{kj} in $\mathscr{H}_k$ for $1 \le j \le n$ satisfying the following conditions: $y_{ij} = 0$ for $i > j$, $\|y_{ij}\| < 2^{-j}\epsilon$, $E_{kj} = 0$ for $k > j$, $(E_{kj})_j$ is an increasing sequence of finite-rank projections on $\mathscr{H}_k$ strongly converging to 1, and

$$\left[E_{kj}, \pi_k\left(x_i - \sum_{m=1}^{n} y_{im}\right)\right] = 0, \quad 1 \le i \le j, \qquad \left[E_{kj}, H_k - \pi_k\left(\sum_{m=1}^{n} h_m\right)\right] = 0$$

for $k \le j \le n$. Thus by setting $y_i = x_i - \sum_{m=1}^{\infty} y_{im}$ and $h = \sum_{m=1}^{\infty} h_m$ we obtain the following equalities: $[E_{kj}, \pi_k(y_i)] = 0$ for $i \le j$, $[E_{kj}, H_k - \pi_k(h)] = 0$, where $\|x_i - y_i\| < 2^{-i+1}\epsilon$ and $\|h\| < \epsilon$.

Let β be the flow generated by $\delta_\alpha - \mathrm{ad}\, ih$, where δ_α is the generator of α.

Let A_i be the β-invariant C^*-subalgebra of A generated by $y_1, \dots, y_i$. Then $A_i \subset A_{i+1}$ and the union of A_i is dense in A. Note that $E_{kj} \in \pi_k(A_i)'$ for $j \ge \max\{k, i\}$. Since all E_{kj} are of finite rank and a finite-dimensional covariant representation is a direct sum of finite-dimensional covariant irreducible representations, one can conclude that $\pi_k|A_i$ is a direct sum of finite-dimensional covariant irreducible representations for all k, which in particular implies that A_i is residually finite-dimensional.

When α is a flow on a C^*-algebra A we denote by $FR(\alpha)$ the set of equivalence classes of finite-dimensional α-covariant irreducible representations of A. Thus α is an RF flow if the intersection of all $\mathrm{Ker}(\pi)$, $\pi \in FR(\alpha)$ is zero. If ϕ is an injection of (A, α) into (B, β) we denote by $\phi'(FR(\beta))$ the set of $\pi \in FR(\alpha)$ which is obtained as a sub-representation of $\rho\phi|A$ for some $\rho \in FR(\beta)$. Suppose that we are given an increasing sequence (A_n, α_n) of RF flows; we denote by ϕ_{mn} the embedding of A_m into A_n for $m < n$ intertwining α_m and α_n. For each $m \in \mathsf{N}$ let FR'_m denote the intersection of all $\phi'_{mn}(FR(\alpha_n))$ with $n > m$. When the intersection of all $\mathrm{Ker}(\pi)$, $\pi \in FR'_m$ is zero for all m we say that the increasing sequence (A_n, α_n) of RF flows is *canonical*.

LEMMA 5.8. *Let (A_n, α_n) be a canonical increasing sequence of RF flows and let (A, α) be the inductive limit of (A_n, α_n). There exists a family S of α-invariant pure states of A such that if $\phi \in S$ then $\pi_\phi|A_n$ is equivalent to a direct sum of finite-dimensional covariant irreducible representations of A_n for all $n \in \mathsf{N}$ and such that $\bigoplus_{\phi \in S} \pi_\phi$ is faithful.*

PROOF. By using the notation before this lemma one finds, for any m and $\pi \in FR'_m$, a sequence $(\rho_n)_{n\geq m}$ such that $\rho_n \in FR(\alpha_n)$, $\rho_m = \pi$, and $\rho_{n+1}|A_n$ contains ρ_n as a subrepresentation. Fix a α_m-invariant pure state ϕ_m of A_m which induces ρ_m as a GNS representation. One then finds a α_{m+1}-invariant pure state ϕ_{m+1} of A_{m+1} which induces ρ_{m+1} and $\phi_{m+1}|A_m = \phi_m$. (Consider the embedding of $C = A_m/\mathrm{Ker}\,\rho_{m+1} \cap A_m$ into $D = A_{m+1}/\mathrm{Ker}\,\rho_{m+1}$; ϕ_m is an α_m-invariant pure state on a factor of the finite-dimensional C^*-algebra C. We pick up a factor E of D to which the factor of C is mapped and then find an α_{m+1}-invariant pure state ϕ_{m+1} of E, which we regard as a pure state on A_{m+1}.) By repeating this process we find a sequence $(\phi_n)_{n\geq m}$ such that ϕ_n is a α_n-invariant pure state of A_n which induces ρ_n and $\phi_n|A_{n-1} = \phi_{n-1}$. Thus we can define a state ϕ of A by $\phi|A_n = \phi_n$. One concludes that ϕ is an α-invariant pure state. We denote by U the unitary flow on $\mathscr{H}_\pi$ defined by $U_t\pi_\phi(x)\Omega_\phi = \pi_\phi(\alpha_t(x))\Omega_\phi$. Note that $\mathscr{H}_n = \pi_\phi(A_n)\Omega_\phi$ is finite-dimensional and U-invariant. Since $(\mathscr{H}_n)$ is increasing and the union of all $\mathscr{H}_n$ is dense in $\mathscr{H}_\phi$ one concludes that $\pi_\phi|A_n$ is equivalent to a direct sum of covariant finite-dimensional irreducible representations. Let S denote the set of all ϕ for all the choices of $m, \pi \in FR'_m$. Then the direct sum of π_ϕ is faithful on A_m for any m and thus it is faithful on A.

PROPOSITION 5.9. *Let α be a flow on a separable C^*-algebra. Then the following conditions are equivalent:*

(1) *There exists a faithful family of covariant irreducible representations of (A, α) which are quasi-diagonal.*
(2) *There exists an α-cocycle u and a canonical increasing sequence (A_n, α_n) of RF flows whose inductive limit is isomorphic to $(A, \mathrm{Ad}\, u\alpha)$.*

PROOF. Since A is separable it follows from (1) that there is a countable family of covariant irreducible representations; (1) $\Rightarrow$ (2) follows from Lemma 5.7. The converse follows from Lemma 5.8.

Let A be a unital separable simple quasi-diagonal C^*-algebra (e.g., a UHF algebra) and let α be an approximately inner flow on A whose Connes spectrum is the whole $\mathbf{R}$. Then one can apply the above proposition to conclude that there is a α-cocycle u and a canonical increasing sequence (A_n, α_n) of RF flows whose inductive limit is isomorphic to $(A, \mathrm{Ad}\, u\alpha)$. This is because such a system has a covariant irreducible representation which induces a faithful representation of the crossed product (see [6]) and hence must be quasi-diagonal.

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DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY
SAPPORO
JAPAN
E-mail: kishi@math.sci.hokudai.ac.jp