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# ON THE VANISHING OF HOMOLOGY WITH MODULES OF FINITE LENGTH

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## Abstract

We study the vanishing of homology and cohomology of a module of finite complete intersection dimension over a local ring. Given such a module of complexity $c$, we show that if $c$ (co)homology groups with a module of finite length vanish, then all higher (co)homology groups vanish.


## 1. Introduction

With his 1961 paper [1], Auslander initiated the study of vanishing of homology for modules over commutative Noetherian local rings. In that paper, he proved his famous rigidity theorem: if $M$ and $N$ are finitely generated modules over an unramified regular local ring $A$, then the implication

$$\operatorname{Tor}_n^A(M,N)=0 \implies \operatorname{Tor}_i^A(M,N)=0 \qquad \text{for} \quad i \geq n$$

holds for any $n \geq 0$. Lichtenbaum settled the ramified case in [10], and Murthy generalized this to arbitrary complete intersection rings in [11]. Namely, he proved that if $M$ and $N$ are finitely generated modules over a local complete intersection $A$ of codimension $c$, and

$$\operatorname{Tor}_n^A(M,N)=\operatorname{Tor}_{n+1}^A(M,N)=\cdots=\operatorname{Tor}_{n+c}^A(M,N)=0$$

for some $n \geq 0$, then $\operatorname{Tor}_i^A(M,N)=0$ for $i \geq n$. This was vastly generalized in [8], [9], where Jorgensen focused on the *complexities* of the modules involved, rather than the codimension of the ring. He showed that if $M$ is a module of finite complete intersection dimension and complexity $c$ over a local ring, then the vanishing of $c+1$ homology (respectively, cohomology) groups forces the vanishing of all the higher homology (respectively, cohomology) groups.

In all of the above mentioned vanishing results, one assumes the vanishing of *consecutive* (co)homology groups. However, the author showed in [4] that this is not necessary. In fact, the (co)homology groups assumed to vanish may be arbitrarily far apart from each other. Namely, let $M$ and $N$ be modules over a

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local ring $A$, with $M$ of finite complete intersection dimension and complexity $c$. It was shown that if there exists an odd number $q$ such that

$$\operatorname{Tor}_n^A(M, N) = \operatorname{Tor}_{n+q}^A(M, N) = \cdots = \operatorname{Tor}_{n+cq}^A(M, N) = 0$$

for some $n > \operatorname{depth} A - \operatorname{depth} M$, then $\operatorname{Tor}_i^A(M, N) = 0$ for $i > \operatorname{depth} A - \operatorname{depth} M$ (and similarly for cohomology). The above vanishing result of Jorgensen is the special case $q = 1$.

In this paper, we show that when the module $N$ has finite length, then we may reduce the number of (co)homology groups assumed to vanish by one. Namely, let $M$ and $N$ be modules over a local ring $A$, with $N$ of finite length and $M$ of finite complete intersection dimension and complexity $c$. In this situation, we show that if there exists an odd number $q$ such that

$$\operatorname{Tor}_n^A(M, N) = \operatorname{Tor}_{n+q}^A(M, N) = \cdots = \operatorname{Tor}_{n+(c-1)q}^A(M, N) = 0$$

for some $n > \operatorname{depth} A - \operatorname{depth} M$, then $\operatorname{Tor}_i^A(M, N) = 0$ for $i > \operatorname{depth} A - \operatorname{depth} M$ (and similarly for cohomology). The special case when $q = 1$ and the ring is a complete intersection was proved by Jorgensen in the above mentioned papers.

## 2. Complete intersection dimension

Throughout this paper, we assume that all modules encountered are finitely generated. In this section, we fix a local (meaning commutative Noetherian local) ring $(A, \mathfrak{m}, k)$.

Every $A$-module $M$ admits a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

which is unique up to isomorphism. The rank of the free $A$-module $F_n$ is the $n$th *Betti number* of $M$; we denote it by $\beta_n^A(M)$. The *complexity* of $M$, denoted $\operatorname{cx} M$, is defined as

$$\operatorname{cx} M \stackrel{\text{def}}{=} \inf\{t \in \mathsf{N} \cup \{0\} \mid \exists a \in \mathsf{R} \text{ such that } \beta_n^A(M) \leq an^{t-1} \text{ for all } n \gg 0\}.$$

Thus, the complexity of a module is the polynomial rate of growth of its Betti sequence.

An arbitrary local ring may have many modules with infinite complexity. In fact, by a theorem of Gulliksen (cf. [7, Theorem 2.3]), a local ring is a complete intersection if and only if all its modules have finite complexity. In order to study modules behaving homologically as modules over such rings, Avramov, Gasharov and Peeva defined in [2] the notion of modules with finite *complete* *intersection dimension*. Recall that a *quasi-deformation* of $A$ is a diagram $A \to R \leftarrow Q$ of local homomorphisms, in which $A \to R$ is faithfully flat, and $R \leftarrow Q$ is surjective with kernel generated by a regular sequence. The length of this regular sequence is the *codimension* of the quasi-deformation. An $A$-module $M$ has finite complete intersection dimension if there exists such a quasi-deformation in which the $Q$-module $R \otimes_A M$ has finite projective dimension. We shall write "CI-dimension" instead of "complete intersection dimension".

As mentioned, the homological behavior of modules of finite CI-dimension reflects the behavior of modules over local complete intersections. For example, such a module has finite complexity. Moreover, the cohomology groups are finitely generated over a ring of cohomology operators of degree two, a notion we now explain (cf. [2, 4.1]). Let $Q$ be a local ring, and let $x_1, \dots, x_c$ be a regular sequence of length $c$. Denote the ring $Q/(x_1, \dots, x_c)$ by $R$, and let $M$ be an $R$-module with a free resolution $\mathbf{F}$. The regular sequence gives rise to chain maps $\{t_i \in \operatorname{Hom}_R(\mathbf{F}, \mathbf{F})\}_{i=1}^c$ of degree $-2$, namely the *Eisenbud operators*. These are uniquely defined up to homotopy, and are therefore elements of $\operatorname{Ext}_R^2(M, M)$. Moreover, these chain maps commute up to homotopy. Thus, the Eisenbud operators give rise to a polynomial ring $R[\chi_1, \dots, \chi_c]$ of operators with the following properties:

(1) The degree of $\chi_i$ is two for all $i$.

(2) There is a graded ring homomorphism $R[\chi_1, \dots, \chi_c] \xrightarrow{\varphi_M} \operatorname{Ext}_R^*(M, M)$ for every $R$-module $M$, where $\operatorname{Ext}_R^*(M, M) = \oplus_{n=0}^{\infty} \operatorname{Ext}_R^n(M, M)$.

(3) For every pair $(X, Y)$ of $R$-modules, the $R[\chi_1, \dots, \chi_c]$-module structure on $\operatorname{Ext}_R^*(X, Y)$ through $\varphi_X$ and $\varphi_Y$ coincide (cf. [3, Theorem 3.3]).

(4) For every pair $(X, Y)$ of $R$-modules, the $R[\chi_1, \dots, \chi_c]$-module $\operatorname{Ext}_R^*(X, Y)$ is finitely generated whenever $\operatorname{Ext}_Q^n(X, Y) = 0$ for $n \gg 0$ (cf. [6, Theorem 3.1]).

As mentioned above, modules of finite CI-dimension have finitely generated cohomology, as we see from property (4) above. Namely, let $M$ and $N$ be $A$-modules, with $M$ of finite CI-dimension over a quasi-deformation $A \to R \leftarrow Q$, say. Furthermore, let $S$ be the polynomial ring induced by the Eisenbud operators coming from the surjection $Q \to R$. Then since $\operatorname{pd}_Q(R \otimes_A M)$ is finite, the $S$-module $\operatorname{Ext}_R^*(R \otimes_A M, R \otimes_A N)$ is finitely generated.

Having defined Eisenbud operators, we end this section with the following result. It shows that a module of finite CI-dimension and complexity one can be "realized" by a codimension one quasi-deformation.

LEMMA 2.1. *Let $A$ be a local ring, and let $M$ be an $A$-module of finite*

*CI-dimension and complexity one. Then there exists a codimension one quasi-deformation $A \to R \leftarrow Q$ with* $\mathrm{pd}_Q(R \otimes_A M) < \infty$.

PROOF. By [2, Proposition 7.2(2)], there exists a quasi-deformation $A \to R \leftarrow Q$ of codimension one, such that the Eisenbud operator $\chi$ on the minimal free resolution of $R \otimes_A M$ is eventually surjective. Thus, if

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

is the minimal free resolution of $M$, and

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & R \otimes_A F_{n+2} & \longrightarrow & R \otimes_A F_{n+1} & \longrightarrow & R \otimes_A F_n & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle f_{n+2}} & & \big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} & & \\
\cdots & \longrightarrow & R \otimes_A F_n & \longrightarrow & R \otimes_A F_{n-1} & \longrightarrow & R \otimes_A F_{n-2} & \longrightarrow & \cdots
\end{array}$$

is the chain map corresponding to $\chi$, then $f_n$ is surjective for large $n$. By [2, Theorem 7.3], the minimal free resolution of $M$ is eventually periodic of period 2, so that $F_n$ is isomorphic to $F_{n+2}$ for $n \gg 0$. Consequently, the map $f_n$ is an isomorphism for large $n$.

For any $R$-module $N$ and integer $n$, the action of $\chi$ on $\mathrm{Ext}_R^n(R \otimes_A M, N)$ is induced by $f_n$. Therefore, there exists an integer $n_0$ such that

$$\chi \,\mathrm{Ext}_R^n(R \otimes_A M, N) = \mathrm{Ext}_R^{n+2}(R \otimes_A M, N)$$

for all $n \geq n_0$. Consequently, the $R[\chi]$-module $\mathrm{Ext}_R^n(R \otimes_A M, N)$ is Noetherian, and so it follows from [2, Theorem 4.2] that $\mathrm{Ext}_Q^n(R \otimes_A M, N)$ vanishes for large $n$. In particular, this holds if we take $N$ to be the residue field of $Q$, hence $\mathrm{pd}_Q(R \otimes_A M)$ is finite.

## 3. Vanishing of homology

We start by showing the following special case of the main result. It shows that when a module has finite CI-dimension and complexity one, then the vanishing of a single homology group with a module of finite length implies the vanishing of all higher homology groups.

PROPOSITION 3.1. *Let $A$ be a local ring, and let $M$ be an $A$-module of finite CI-dimension and complexity one. Furthermore, let $N$ be an $A$-module of finite length. If* $\mathrm{Tor}_n^A(M, N) = 0$ *for some* $n >$ depth $A -$ depth $M$*, then* $\mathrm{Tor}_n^A(M, N) = 0$ *for all* $n >$ depth $A -$ depth $M$.

PROOF. Choose, by Lemma 2.1, a codimension one quasi-deformation $A \to R \leftarrow Q$ for which the projective dimension of the $Q$-module $R \otimes_A M$ is finite. Thus $R = Q/(x)$ for a regular element $x$ in $Q$. Denote the $R$-modules

$R \otimes_A M$ and $R \otimes_A N$ by $M'$ and $N'$, respectively, and note that $N'$ has finite length.

The change of rings spectral sequence

$$\operatorname{Tor}_p^R\left(M', \operatorname{Tor}_q^Q(N', R)\right) \underset{p}{\Rightarrow} \operatorname{Tor}_{p+q}^Q(M', N')$$

degenerates into a long exact sequence

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \operatorname{Tor}_1^R(M', N') & \longrightarrow & \operatorname{Tor}_2^Q(M', N') & \longrightarrow & \operatorname{Tor}_2^R(M', N') & \longrightarrow & \\ \operatorname{Tor}_0^R(M', N') & \longrightarrow & \operatorname{Tor}_1^Q(M', N') & \longrightarrow & \operatorname{Tor}_1^R(M', N') & \longrightarrow & \end{array}$$

in homology. Let

$$0 \to Q^{\beta_d^Q(M')} \to \cdots \to Q^{\beta_0^Q(M')} \to M' \to 0$$

be a minimal $Q$-free resolution of $M'$. Localizing this resolution at $x$, keeping in mind that $xM' = 0$, we see that $\sum_{i=0}^d (-1)^i \beta_i^Q(M') = 0$. Now $\ell(\operatorname{Tor}_i^Q(M', l)) = \beta_i^Q(M')$, where $l$ is the residue field of $Q$. Moreover, the Euler characteristic $\sum_{i=0}^\infty (-1)^i \ell(\operatorname{Tor}_i^Q(M', -))$ is well defined and additive on the category of finite length $Q$-modules. Therefore, induction on length shows that $\sum_{i=0}^\infty (-1)^i \ell(\operatorname{Tor}_i^Q(M', X)) = 0$ for every $Q$-module of finite length.

By the Auslander-Buchsbaum formula, the projective dimension of $M'$ over $Q$ is given by

$$\begin{aligned} \operatorname{depth}_Q Q - \operatorname{depth}_Q M' &= \operatorname{depth}_R R - \operatorname{depth}_R M' + 1 \\ &= \operatorname{depth}_A A - \operatorname{depth}_A M + 1. \end{aligned}$$

Let $m > \operatorname{depth}_A A - \operatorname{depth}_A M$ be an integer, and consider the long exact sequence in homology starting in $\operatorname{Tor}_{m+1}^Q(M', N') = 0$, i.e. the long exact sequence obtained from the above spectral sequence. By taking the alternating sum of the lengths of the terms, and identifying $\operatorname{Tor}_0^R(M', N')$ with $\operatorname{Tor}_0^Q(M', N')$, we see that $\ell(\operatorname{Tor}_m^R(M', N')) = \ell(\operatorname{Tor}_{m+1}^R(M', N'))$.

Suppose that $\operatorname{Tor}_n^A(M, N) = 0$ for some $n > \operatorname{depth} A - \operatorname{depth} M$. Since $\operatorname{Tor}_n^R(M', N')$ is isomorphic to $R \otimes_A \operatorname{Tor}_n^A(M, N)$, the above shows that $\operatorname{Tor}_n^R(M', N') = 0$ for all $n > \operatorname{depth} A - \operatorname{depth} M$. Then by faithful flatness, the homology group $\operatorname{Tor}_n^A(M, N)$ vanishes for all $n > \operatorname{depth} A - \operatorname{depth} M$.

Having proved the complexity one case, we now prove the main result, the case when one of the modules has finite CI-dimension and complexity $c$, and the other has finite length. In this situation, the vanishing of $c$ homology groups implies the vanishing of all higher homology groups. Moreover, the "vanishing gap" can be any odd number. This result therefore generalizes [4, Theorem 3.5] to rings which are not necessarily complete intersections.

THEOREM 3.2. *Let $A$ be a local ring, and let $M$ be an $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be an $A$-module of finite length. If there exist an odd number $q \geq 1$ and a number $n >$ depth $A -$ depth $M$ such that $\operatorname{Tor}_i^A(M, N) = 0$ for $i \in \{n, n+q, \dots, n+(c-1)q\}$, then $\operatorname{Tor}_i^A(M, N) = 0$ for all $i >$ depth $A -$ depth $M$.*

PROOF. We argue by induction on the complexity $c$ of $M$. If $c = 0$, then $\operatorname{pd} M = \operatorname{depth} A - \operatorname{depth} M$ by the Auslander-Buchsbaum formula, and the result follows. The case $c = 1$ is Proposition 3.1, so we assume that $c \geq 2$.

By [5, Lemma 3.1], there exists a faithfully flat extension $A \to R$ of local rings with the following properties:

(1) There is an exact sequence

$$0 \to R \otimes_A M \to K \to \Omega_R^q(R \otimes_A M) \to 0$$

of $R$-modules.

(2) The $R$-modules $R \otimes_A M$ and $K$ have finite CI-dimension.

(3) The complexity of $K$ is $c - 1$.

(4) $\operatorname{depth}_R R - \operatorname{depth}_R K = \operatorname{depth}_A A - \operatorname{depth}_A M$.

As in the previous proof, denote the $R$-modules $R \otimes_A M$ and $R \otimes_A N$ by $M'$ and $N'$, respectively, and note that $N'$ has finite length. Since $\operatorname{Tor}_i^R(M', N')$ is isomorphic to $R \otimes_A \operatorname{Tor}_i^A(M, N)$ for all $i$, the vanishing assumption implies that $\operatorname{Tor}_i^R(M', N') = 0$ for $i \in \{n, n+q, \dots, n+(c-1)q\}$. From (1), we obtain a long exact sequence

$$\cdots \to \operatorname{Tor}_{i+q+1}^R(M', N') \to \operatorname{Tor}_i^R(M', N') \\ \to \operatorname{Tor}_i^R(K, N') \to \operatorname{Tor}_{i+q}^R(M', N') \to \cdots$$

of homology groups, from which we obtain that $\operatorname{Tor}_i^R(K, N') = 0$ for $i \in \{n, n+q, \dots, n+(c-2)q\}$. Therefore by induction and properties (2), (3) and (4), the homology group $\operatorname{Tor}_i^R(K, N')$ vanishes for all $i > \operatorname{depth}_A A - \operatorname{depth}_A M$. The long exact sequence then shows that $\operatorname{Tor}_i^R(M', N') \simeq \operatorname{Tor}_{i+j(q+1)}^R(M', N')$ for all $i > \operatorname{depth}_A A - \operatorname{depth}_A M$ and $j \geq 0$.

By considering the pairs $(i, j) \in \{(n, c), (n+q, c-1), \dots, (n+(c-1)q, 1)\}$, we see that $\operatorname{Tor}_i^R(M', N') = 0$ for $n+cq+1 \leq i \leq n+cq+c$. Faithful flatness then implies that $\operatorname{Tor}_i^A(M, N) = 0$ for $n+cq+1 \leq i \leq n+cq+c$. Thus we have reduced to the case when $c$ *consecutive* homology groups vanish.

Now we use [5, Lemma 3.1] once more: there exists a faithfully flat local homomorphism $A \to S$ and an exact sequence

$$0 \to S \otimes_A M \to K' \to \Omega^1_S(S \otimes_A M) \to 0$$

of $S$ modules, in which $S \otimes_A M$ and $K'$ satisfy properties (2), (3) and (4). Arguing as above, we obtain an isomorphism

$$\operatorname{Tor}_i^S(S \otimes_A M, S \otimes_A N) \simeq \operatorname{Tor}_{i+2}^S(S \otimes_A M, S \otimes_A N)$$

for all $i > \operatorname{depth}_A A - \operatorname{depth}_A M$. Since $\operatorname{Tor}_i^S(S \otimes_A M, S \otimes_A N) = 0$ for $n + cq + 1 \leq i \leq n + cq + c$ and $c$ is at least two, we conclude that $\operatorname{Tor}_i^S(S \otimes_A M, S \otimes_A N) = 0$ for all $i > \operatorname{depth}_A A - \operatorname{depth}_A M$. Finally, faithful flatness implies that $\operatorname{Tor}_i^A(M, N) = 0$ for all $i > \operatorname{depth}_A A - \operatorname{depth}_A M$.

We record the special case $q = 1$ in the following corollary, i.e. the case when $c$ consecutive homology groups vanish. Note that the case when the ring is a complete intersection follows from [8, Theorem 2.6].

Corollary 3.3. *Let $A$ be a local ring, and let $M$ be an $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be an $A$-module of finite length. If there exists a number $n >$ depth $A -$ depth $M$ such that $\operatorname{Tor}_i^A(M, N) = 0$ for $n \leq i \leq n + c - 1$, then $\operatorname{Tor}_i^A(M, N) = 0$ for all $i >$ depth $A -$ depth $M$.*

We also include the cohomology versions of Theorem 3.2 and Corollary 3.3. We do not include a proof; the proofs of Proposition 3.1 and Theorem 3.2 carry over verbatim to the cohomology case.

Theorem 3.4. *Let $A$ be a local ring, and let $M$ be an $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be an $A$-module of finite length. If there exist an odd number $q \geq 1$ and a number $n >$ depth $A -$ depth $M$ such that $\operatorname{Ext}_A^i(M, N) = 0$ for $i \in \{n, n+q, \dots, n+(c-1)q\}$, then $\operatorname{Ext}_A^i(M, N) = 0$ for all $i >$ depth $A -$ depth $M$.*

Corollary 3.5. *Let $A$ be a local ring, and let $M$ be an $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be an $A$-module of finite length. If there exists a number $n >$ depth $A -$ depth $M$ such that $\operatorname{Ext}_A^i(M, N) = 0$ for $n \leq i \leq n + c - 1$, then $\operatorname{Ext}_A^i(M, N) = 0$ for all $i >$ depth $A -$ depth $M$.*

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