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ON BOUNDED WEAK AND STRONG SOLUTIONS OF NON LINEAR DIFFERENTIAL EQUATIONS WITH AND WITHOUT DELAY IN BANACH SPACES

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Abstract

Assume that E is a Banach space, $B_r = \{x \in E : \|x\| \leq r\}$ and $C([-d, 0], B_r)$ is the Banach space of continuous functions from $[-d, 0]$ into B_r. Consider $f : \mathsf{R}^+ \times E \to E$; $f^d : [0, T] \times C([-d, 0], B_r) \to E$; for each $t \in [0, T]$ the mapping $\theta_t \in C([-d, 0], B_r)$ is defined by $\theta_t x(s) = x(t + s)$, $s \in [-d, 0]$ and let $A(t)$ be a linear operator from E into itself. In this paper we give existence theorems for bounded weak and strong solutions of the nonlinear differential equation

$$\text{(P)} \qquad \dot{x}(t) = A(t)x + f(t, x), \qquad t \in \mathsf{R}^+,$$

and we prove that, with certain conditions, the differential equation with delay

$$\text{(Q)} \qquad \dot{x}(t) = L(t)x(t) + f^d(t, \theta_t x), \qquad \text{if} \quad t \in [0, T]$$

has at least one weak solution where $L(t)$ is a linear operator from E into E. Moreover, under suitable assumptions, the problem (Q) has a solution. Furthermore under a generalization of the compactness assumptions, we show that (Q) has a solution too.

1. Introduction and preliminaries

In this paper the dual space of an infinite dimensional Banach space E will be denoted by E^* and the pairing between E and E^* is denoted by $\langle\rangle$. Denote by E_w the Banach space E endowed with the weak topology. We denote the closed unit sphere in E by $B_1 = \{x \in E : \|x\| \leq 1\}$. Further, let $\mathscr{L}(\mathsf{R}^+, E)$ be the space of measurable functions $u : \mathsf{R}^+ \to E$, $\mathscr{L}(E)$ be the space of linear operators from E into itself and λ be the Lebesgue measure on $I = [0, T]$. Furthermore, let $C(I, E)$ be the space of all continuous functions from I to E with the usual supremum norm and $C_w(I, E)$ be the space of all weakly continuous functions from I to E endowed with the topology of weak uniform convergence. Let $C([-d, 0], E)$ be the Banach space of continuous functions from the closed interval $[-d, 0]$ ($d \geq 0$) into E and $\mathscr{B}$ be the family of all bounded subsets of E.

Let $\mathscr{M} = \mathscr{M}(\mathsf{R}^+, E)$ be a Banach space of measurable functions $x : \mathsf{R}^+ \to E$ with $\|x\| \in \mathscr{M}(\mathsf{R}^+, \mathsf{R})$, $\|x\|_{\mathscr{M}} = \| \|x\| \|_{\mathscr{M}(\mathsf{R}^+, \mathsf{R})}$, where

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(1) $\mathscr{M}(\mathsf{R}^+, \mathsf{R}) \subset \mathscr{L}(\mathsf{R}^+, \mathsf{R})$,

(2) $\mathscr{M}(\mathsf{R}^+, \mathsf{R})$ contains all essentially bounded functions with compact support,

(3) if $x \in \mathscr{M}(\mathsf{R}^+, \mathsf{R})$, $y : \mathsf{R}^+ \to \mathsf{R}$ is measurable with $|y| \leq |x|$, then $y \in \mathscr{M}(\mathsf{R}^+, \mathsf{R})$ and $\|y\|_{\mathscr{M}(\mathsf{R}^+,\mathsf{R})} \leq \|x\|_{\mathscr{M}(\mathsf{R}^+,\mathsf{R})}$,

(4) if $x \in \mathscr{M}(\mathsf{R}^+, \mathsf{R})$, $x_n \in \mathscr{M}(\mathsf{R}^+, \mathsf{R})$, $|x_n| \leq |x|$ and $\lim_{n\to\infty} x_n(t) = 0$ a.e. on R^+, then $\lim_{n\to\infty} \|x_n\|_{\mathscr{M}(\mathsf{R}^+,\mathsf{R})} = 0$.

Let $\mathscr{M}'$ denote the associate space to $\mathscr{M}$ [20].

Definition 1.1. The map $\gamma : \mathscr{B} \to \mathsf{R}^+$ is called a measure of strong (weak) noncompactness on $\mathscr{B}$ if, for $U, V \in \mathscr{B}$,

(M_1) $U \subset V \longrightarrow \gamma(U) \leq \gamma(V)$,

(M_2) $\gamma(U \cup V) \leq \max(\gamma(U), \gamma(V))$,

(M_3) $\gamma(\overline{\operatorname{conv}} U) = \gamma(U)$,

(M_4) $\gamma(U + V) \leq \gamma(U) + \gamma(V)$,

(M_5) $\gamma(cU) = |c|\gamma(U)$, $c \in \mathsf{R}$,

(M_6) $\gamma(U) = 0 \Longleftrightarrow U$ is relatively strongly (weakly) compact in E,

(M_7) $\gamma(U \cup \{x\}) = \gamma(U)$, $x \in E$.

Definition 1.2. A function $u : [a, b] \to E$, $(a, b) \in \mathsf{R}^2$, is called:

(a) Pettis integrable if for any measurable subset D of $[a, b]$ there is an element v_D in E such that $\langle v_D, f\rangle = \int_D \langle u(s), f\rangle\, ds$, for all $f \in E^*$, we write $v_D = \int_D u(s)\, ds$,

(b) Bochner integrable if there exists a sequence of countable-valued functions $\{u_n\}$ converging almost everywhere on $[a, b]$ such that $\lim_{n\to\infty} \int_a^b \|u_n(s) - u(s)\|\, ds = 0$.

We note that every Bochner integrable function is Pettis integrable (see [14]).

Definition 1.3. The Hausdorff measure of weak noncompactness $\beta : \mathscr{B} \to \mathsf{R}^+$ and the Kuratowski measure of noncompactness $\alpha : \mathscr{B} \to \mathsf{R}^+$ are defined as follows: for each $U \in \mathscr{B}$,

(i) $\beta(U) = \inf\{\varepsilon > 0 : \exists\ K = \text{weakly compact subset of } E, U \subseteq K + \varepsilon B_1\}$,

(ii) $\alpha(U) = \inf\{\varepsilon > 0 : U \text{ admits a finite cover of sets with diameter} < \varepsilon\}$.

For more details of β and α we refer the reader to [1], [8].

DEFINITION 1.4. By a Kamke function we mean a function $w : I \times \mathsf{R}^+ \to \mathsf{R}^+$ such that:

(i) w is a Carathéodory function,

(ii) for all $t \in I$; $w(t, 0) = 0$,

(iii) for any $c \in (0, b]$, $u \equiv 0$ is the only absolutely continuous function on $[0, c]$ which satisfies $\dot{u}(t) \leq w(t, u(t))$ a.e. on $[0, c]$ and such that $u(0) = 0$.

DEFINITION 1.5. A continuous function $x : [-d, T] \to E_w$ is called a weak solution of problem (P) if, for some $\xi \in C([-d, 0], E)$,

$$x = \xi \qquad \text{on} \quad [-d, 0]$$

and

$$x(t) = G(t, 0)\xi(0) + \int_0^t G(t, s) f(s, x(s))\, ds \qquad \text{for all} \quad t \in I.$$

LEMMA 1.6. *Let $\mathscr{F}$ be a continuous mapping from a compact interval I to $\mathscr{L}(E)$ and $\mathscr{U}$ be a bounded subset of E, then*

$$\gamma\left(\bigcup_{t \in I} \mathscr{F}(t)\mathscr{U}\right) \leq \sup_{t \in I} \|\mathscr{F}(t)\| \gamma(\mathscr{U}).$$

PROOF. $\mathscr{U}$ is bounded, so $\exists\ c > 0$; $\|\mathscr{U}\| = \sup\{\|u\| : u \in \mathscr{U}\} \leq c$. From the continuity of $\mathscr{F}$, for $\varepsilon > 0$ there exists $\delta > 0$ such that if $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of I, that is, $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ with $\|P\| = \sup\{|x_{i+1} - x_i| : i = 0, 1, 2, \ldots, n-1\} < \delta$, then $\|\mathscr{F}(x_{i+1}) - \mathscr{F}(x_i)\| < \frac{\varepsilon}{c}$. Since B_1 is the closed unit ball in E, there exists a weakly compact subset $\mathscr{K}$ of E such that $\mathscr{U} \subset \mathscr{K} + \frac{(\gamma(\mathscr{U})+\varepsilon)}{\gamma(B_1)} B_1$. But for each $t \in I_i = [x_i, x_{i+1}]$, $\mathscr{F}(t)\mathscr{U} \subset \{\mathscr{F}(t)u - \mathscr{F}(t_{i+1})u : u \in \mathscr{U}\} + \mathscr{F}(t_{i+1})\mathscr{U}$ and $\|\mathscr{F}(t) - \mathscr{F}(t_{i+1})\| \|\mathscr{U}\| < \frac{\varepsilon}{c} \cdot c = \varepsilon$. Hence $\{\mathscr{F}(t)u - \mathscr{F}(t_{i+1})u : u \in \mathscr{U}\} \subset \varepsilon B_1$ and $\mathscr{F}(t)\mathscr{U} \subseteq \varepsilon B_1 + \mathscr{F}(t_{i+1})\mathscr{U}$. Therefore

$$\begin{aligned}\bigcup_{t \in I} \mathscr{F}(t)\mathscr{U} &= \bigcup_{t=0}^{n-1} \bigcup_{t \in I} \mathscr{F}(t)\mathscr{U} \\ &\subseteq \varepsilon B_1 + \bigcup_{t=0}^{n-1} \bigcup_{t \in I} \mathscr{F}(t)\mathscr{U} \subseteq \varepsilon B_1 + \bigcup_{t=0}^{n-1} (\varepsilon B_1 + \mathscr{F}(t_{i+1})\mathscr{U})\end{aligned}$$

$$\subseteq 2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathscr{F}(t_{i+1})(\mathscr{K} + \frac{(\gamma(\mathscr{U}) + \varepsilon)}{\gamma(B_1)} B_1)$$

$$\subseteq 2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathscr{F}(t_{i+1})\mathscr{K} + \bigcup_{t=0}^{n-1} \mathscr{F}(t_{i+1}) \frac{(\gamma(\mathscr{U}) + \varepsilon)}{\gamma(B_1)} B_1$$

$$\subseteq 2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathscr{F}(t_{i+1})\mathscr{K} + \sup_{t\in I} \|\mathscr{F}(t)\| \frac{(\gamma(\mathscr{U}) + \varepsilon)}{\gamma(B_1)} B_1.$$

Moreover

$$\gamma\left(\bigcup_{t\in I} \mathscr{F}(t)\mathscr{U}\right) \leq \gamma(2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathscr{F}(t_{i+1})\mathscr{K} + \sup_{t\in I} \|\mathscr{F}(t)\| \frac{(\gamma(\mathscr{U}) + \varepsilon)}{\gamma(B_1)} B_1)$$

$$\leq 2\varepsilon\gamma(B_1) + \gamma(\sup_{t\in I} \|\mathscr{F}(t)\| \frac{(\gamma(\mathscr{U}) + \varepsilon)}{\gamma(B_1)} B_1)$$

$$\leq 2\varepsilon\gamma(B_1) + \sup_{t\in I} \|\mathscr{F}(t)\| \frac{(\gamma(\mathscr{U}) + \varepsilon)}{\gamma(B_1)} \gamma(B_1)$$

$$\leq 2\varepsilon\gamma(B_1) + \sup_{t\in I} \|\mathscr{F}(t)\| (\gamma(\mathscr{U}) + \varepsilon)$$

where $\bigcup_{t=0}^{n-1} \mathscr{F}(t_{i+1})\mathscr{K}$ is weakly compact. Since ε is arbitrary the result follows.

LEMMA 1.7 ([3]). *Let Y and E be two Banach spaces, $P_{fc}(Y)$ be the set of all closed and convex subsets of Y and $F : E \to P_{fc}(Y)$ be weakly sequentially upper hemicontinuous. Further let $(x_n)_{n\in\mathsf{N}} \subset C(I, E)$, $x_n(t) \to x_0(t)$ weakly a.e. on I and $(y_n)_{n\in\mathsf{N}\cup\{0\}} \subset L^1(I, E)$, $y_n \to y_0$ weakly. Suppose that there exists $a \in L^1(I, \mathsf{R})$ such that $\|F(x)\| \leq a(t)$ for all $x \in C(I, E)$ and $y_n(t) \in F(x_n(t))$ a.e. on I. Then $y_0(t) \in F(x_0(t))$ a.e. on I.*

LEMMA 1.8 ([18], [1]). *If $\gamma : \mathscr{B} \to \mathsf{R}^+$ satisfies conditions* (M$_2$), (M$_4$) *and* (M$_6$) *then, for any nonempty $U \in \mathscr{B}$,*

$$\gamma(U) \leq \gamma(B_1)\alpha(U) \leq 2\gamma(B_1)\beta(U).$$

LEMMA 1.9 ([21], [17]). *If γ is a measure of weak* (*strong*) *noncompactness and $A \subset C_w(I, E)$ is a family of strongly equicontinuous functions, then*

$$\gamma(A(I)) = \sup\{\gamma(A(t)) : t \in I\}.$$

If for each $t \in \mathsf{R}^+$, $A(t) \in \mathscr{L}(E)$ and $\dot{x}(t)$ denotes the weak derivative of x at t, then we consider the differential equation

$$\dot{x}(t) = A(t)x(t). \tag{1}$$

Let E be the direct sum of $\mathscr{E}_0$ and $\mathscr{E}_1$, where

$$\mathscr{E}_0 = \{x_0 \in E : \exists \text{ a bounded weak solution } x \text{ of (1) and } x(0) = x_0\}$$

is closed and has a closed complement $\mathscr{E}_1$.

Let $G \in C(\mathsf{R}^+ \times \mathsf{R}^+, E)$ be the Green function corresponding to (1):

$$G(t,s) = \begin{cases} S(t)PS^{-1}(s) & \text{if } 0 \le s \le t \\ -S(t)(id - P)S^{-1}(s) & \text{if } 0 \le t \le s, \end{cases} \tag{2}$$

where $S : \mathsf{R}^+ \to \mathscr{L}(E)$ is a solution of the differential equation

$$\dot{S}(t) = A(t)S(t), \qquad S(0) = id,$$

and P is the projection of E onto $\mathscr{E}_0$; hence $P(\mathscr{E}_1) = \{0\}$.

2. Existence results for problem (P)

In this section we shall consider the nonlinear differential equation

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \qquad t \in \mathsf{R}^+. \tag{P}$$

This problem was studied by many authors (see, for instance, [5], [19], [6], [16], [10]). The next theorem is a generalization of Theorem 8 in [13]. Moreover we use here a general weak noncompactness measure, in contrast with the Hausdorff noncompactness measure used in [10]; hence, the result below is at the same time a generalization of Theorem 5 in [10].

THEOREM 2.1. *Let $A : \mathsf{R}^+ \to \mathscr{L}(E)$ be strongly measurable and Bochner integrable on every subinterval I of R^+. Let γ be a weak measure of noncompactness, for each $t \in \mathsf{R}^+$, let $G(t, .) \in \mathscr{M}'$ with $\|G(t, .)\|_{\mathscr{M}} \le c$ where $c > 0$. Let f be a continuous function from $\mathsf{R}^+ \times E_w$ to E_w and $m : \mathsf{R}^+ \to \mathsf{R}^+$ belongs to $\mathscr{M}'$ such that $\|f(t, x)\| \le m(t)$ for every $(t, x) \in \mathsf{R}^+ \times B_r$. Assume that $c\|m\|_{\mathscr{M}} < r$ and for each T, $\varepsilon > 0$ there exists a closed subset I_ε of I with $\lambda(I - I_\varepsilon) < \varepsilon$ such that for any nonempty bounded subset U of E one has $\beta(f(J \times U)) \le \sup_{t \in J} w(t, \beta(U))$, for any compact subset J of I_ε. Then, for each $x_0 \in \mathscr{E}_0$ such that $\|x_0\| \le \frac{r - c\|m\|_{\mathscr{M}}}{\|G(t,0)\|}$, there exists a bounded weak solution of* (P).

PROOF. Let

$$S = \left\{ x \in C_w(\mathsf{R}^+, E) : \|x(t) - x(\tau)\| \le r \int_t^\tau |A(s)|\, ds + \int_t^\tau m(s)\, ds, 0 \le t \le \tau \right\}.$$

From (2) and by results from [20] there exists a positive number d such that $\|G(t, 0)\| \le d$. Let $x_0 \in \mathscr{E}_0$ with $\|x_0\| \le \frac{r - c\|m\|_{\mathscr{M}}}{d}$. Then $G(t, 0)x_0$ is a solution of (1) and $\|G(t, 0)x_0\| \le d\|x_0\| \le r - c\|m\|_{\mathscr{M}}$. If ϕ is defined by

$$\phi(x)(t) = G(t, 0)x_0 + \int_0^\infty G(t, s) f(s, x(s))\, ds \qquad \text{for} \quad t \in \mathsf{R}^+ \text{ and } x \in S,$$

then

$$\|\phi(x)(t)\| \le d\|x_0\| + c\|m\|_{\mathscr{M}} \le r.$$

Since $y = \phi(x)$ is a weak solution of the equation $\dot{y}(t) = A(t)y(t) + f(t, x(t))$, we have

$$\|\phi(x)(t) - \phi(x)(\tau)\| \le \int_t^\tau \|A(s)\phi(x)(s) + f(s, x(s))\|\, ds \le r \int_t^\tau |A(s)|\, ds + \int_t^\tau m(s)\, ds.$$

Therefore ϕ is a continuous mapping from S into S [4]. Let $(x_n)_{n \in \mathsf{N} \cup \{0\}}$ be a sequence such that $\phi(x_n) = x_{n+1}$ with x_0 is an arbitrary element in S. Thus $D \subset S$ and, from (M_4), $\gamma(D) = \gamma(\phi(D))$. If G is the set of all limit points of the sequence (x_n), then $\phi(G) = G$. Put $R(X) = \operatorname{conv} \phi(X)$ for $X \subset S$ and consider the family Ω of all subsets X of S such that $G \subset X$ and $R(X) \subset X$. Now $S \in \Omega$ and so $\Omega \neq \emptyset$. Let V be the intersection of all sets of the family Ω. Then $V \in \Omega$. Moreover the mapping $t \to \gamma(\phi(V)(t))$ is absolutely continuous. Assume that $t \ge 0$ and $\varepsilon > 0$ thus from the assumptions on the function m we can find $T_0 \ge t$ such that $\|m\chi_{[T_0, \infty[}\|_{\mathscr{M}} < \frac{\varepsilon}{2c}$. If we put $I_0 := [0, T_0]$, then by the Scorza-Dragoni theorem there exists a closed subset I_ε of I_0 such that $\lambda(I_0 - I_\varepsilon) < \delta$ and the function w is uniformly continuous on $I_\varepsilon \times [0, 2T_0]$. From our last assumption, we can find a closed subset J_ε of I_0 such that $\lambda(I_0 - J_\varepsilon) < \delta$ and such that for any compact subset $\mathscr{C}$ of J_ε and any bounded subset Z of E,

$$\gamma(f(\mathscr{C} \times Z)) \le \sup_{s \in \mathscr{C}} w(s, \gamma(Z)).$$

Since ϕ is continuous and w is Carathéodory we can find a closed subset I_ε of I, $\delta > 0$, $\eta > 0$ ($\eta < \delta$) such that if $s_1, s_2 \in I_\varepsilon$ and $r_1, r_2 \in [0, 2T_0]$

satisfy $|s_1 - s_2| < \delta$, $|r_1 - r_2| < \delta$, then $|w(s_1, r_1) - w(s_2, r_2)| < \varepsilon$ and if $|s_1 - s_2| < \eta$, then $|\gamma(V(s_1)) - \gamma(V(s_2))| < \delta$. Let us fix τ such that $0 \le t \le \tau \le T$ and consider the partition, to $[t, \tau]$, $t = t_0 < t_1 < \cdots < t_m = \tau$ such that $t_i - t_{i-1} < \eta$ for $i = 1, \dots, n$. Let $T_i = J_\varepsilon \cap [t_{i-1}, t_i] \cap I_\varepsilon$, $P = \sum_{i=1}^m T_i = [t, \tau] \cap J_\varepsilon \cap I_\varepsilon$ and $Q = [t, \tau] - P$. Since $G(t, .)$ is uniformly continuous on P, we can find $\eta' > 0$ ($\eta' < \delta$) such that if $r_1, r_2 \in P$ and $|r_1 - r_2| < \eta'$, then

$$\|G(t, r_1) - G(t, r_2)\| < \varepsilon$$

and we can find s_i in T_i with

$$\sup_{s \in T_i} \|G(t, s)\| = \|G(t, s_i)\|.$$

Let $S_i = \{x(t) : x \in S, t \in T_i\}$. In virtue of Lemma 1.6, Lemma 1.9, the mean value theorem and Lemma 1.8 if $\rho(t) := \gamma(V(t))$ we get

$$\begin{aligned}\rho(\tau) - \rho(t) &\le \gamma \int_t^\tau G(t, s) f(s, V(s))\, ds \\ &\le 2\gamma(B_1) \int_t^\tau \|G(t, s)\| w(s, \rho(s))\, ds.\end{aligned}$$

Therefore $\dot\rho(t) \le c w(t, \rho(t))$ a.e. [12] and since $\rho(0) = 0$, then $\rho \equiv 0$ and so $\overline{V}^w$ is weakly compact in $C_w(\mathsf{R}^+, E)$. But V is closed, hence it is a convex and compact subset in $C_w(\mathsf{R}^+, E)$. From the Schauder-Tichonov theorem, since ϕ is a continuous mapping from V to V, there is a fixed point y of ϕ such that y is the desired weak solution of (P) and satisfies $\sup_{t \in \mathsf{R}^+} \|y(t)\| \le r$.

In the following theorem we will deal with the differential equation

$$\text{(P}'\text{)} \qquad \dot x(t) = L(t)x(t) + f'(t, x(t)), \qquad t \in I$$

where $f' : I \times B_r \to E$ is a Carathéodory function, $L : I \to \mathscr{L}(E)$ is strongly measurable and Bochner integrable operator on I and γ is a measure of strong noncompactness. We get a generalization of Theorem 2 in [26] and Theorem 9 in [13].

THEOREM 2.2. *In the setting of Theorem 2.1 we replace the function f by f' such that for each $x \in B_r$, $f'(I \times \{x\})$ is separable; the function m by $m' \in L^1(I, \mathsf{R}^+)$ and the operator A by L. Then problem* (P′) *has a solution.*

PROOF. Let

$$S = \left\{ x \in C(I, E) : \|x(t) - x(\tau)\| \le r \int_t^\tau |A(s)|\, ds + \int_t^\tau m'(s)\, ds,\ 0 \le t \le \tau \right\}.$$

Suppose that the mapping $\phi : S \to S$ is defined by

$$\phi(x)(t) = G(t,0)x_0 + \int_0^t G(t,s)f(s,x(s))\,ds \quad \text{for } t \in I \text{ and } x \in S.$$

As in Theorem 2.1 we let $(x_n)_{n\in \mathsf{N}\cup\{0\}}$ be a sequence such that $\phi(x_n) = x_{n+1}$ where x_0 is an arbitrary element in S, $V = \{x_n : n = 0, 1, 2, \ldots\}$, $V \subset S$, $\gamma(V) = \gamma(\phi(V))$ and $\rho(t) = \gamma(V(t))$. Then by the same argument we get

$$\begin{aligned}\rho(\tau) - \rho(t) &\le \gamma\left(\int_t^\tau G(t,s)f(s,V(s))\,ds\right)\\ &\le \gamma(B_1)\int_t^\tau \|G(t,s)\|w(s,\rho(s))\,ds,\end{aligned}$$

ρ is differentiable a.e. on I and $\rho \equiv 0$. Thus the closure of V is compact in $C(I,E)$ and so we can find a subsequence (x_{n_k}) of (x_n) which converges to a limit x in $C(I,E)$. Since $\|x_n - \phi(x_n)\| \to 0$ as $n \to \infty$ and ϕ is continuous, then $x = \phi(x)$ so as x is the desired solution of (P′) and $\|x\| \le r$.

In the following theorem we let $h : I \times \mathsf{R}^+ \to \mathsf{R}^+$ be a Carathéodory function, such that for each bounded subset Z of $I \times \mathsf{R}^+$ there exists a function $\varphi : I \to \mathsf{R}^+$ such that $h(t,s) \le \varphi(t)$, $(t,s) \in Z$ and φ is integrable function on $[c,T]$ for each c, $0 < c \le T$. Moreover we assume that the identically zero function is the only absolutely continuous function on $[0,c]$ which satisfies $\dot{u}(t) = h(t,u(t))$ a.e. on $[0,c]$ such that the right derivative $D_+u(0)$ of $u(t)$ at $t = 0$ exists and $D_+u(0) = u(0) = 0$.

We note that the assumptions on h are weaker than that on a Kamke function w.

THEOREM 2.3. *If we replace in the setting of Theorem 2.2 a Kamke function w by a function h and we suppose that f' is bounded and continuous, then problem* (P′) *has a solution.*

PROOF. By the same argument as in Theorem 2.2 we get

$$\begin{aligned}\rho(\tau) - \rho(t) &\le \gamma\int_t^\tau G(t,s)f(s,V(s))\,ds\\ &\le \gamma(B_1)\int_t^\tau \|G(t,s)\|h(s,\rho(s))\,ds\end{aligned} \tag{3}$$

where $\rho(t) = \gamma(V(t))$. Since f' is a bounded function, we can find a constant $M > 0$ such that $\|f'(t,x)\| \le M$ for each $(t,x) \in I \times B_r$. Let $\mathcal{N} : I \to \mathsf{R}$ be defined by $\mathcal{N}(t) = \sup_{\|x\|,\|y\| \le Mt} \|f'(t,x) - f'(t,y)\|$. We see that $\mathcal{N}$ is lower semicontinuous on $]0, T]$ and continuous at 0 [22]. Let $\varepsilon > 0$ and t_0 be fixed in I. Then, there exist $x_1, y_1 \in B_r$; $\|x_1\|, \|y_1\| \leq Mt$ such that

$$\mathcal{N}(t_0) - \frac{\varepsilon}{2} \leq \|f'(t_0, x_1) - f'(t_0, y_1)\|. \tag{4}$$

Moreover, f' is continuous. Thus $\exists\ \delta > 0$ such that if $|t - t_0| < \delta$, $\|x_1 - x\| < \delta$, $\|y_1 - y\| < \delta$, we have

$$\|f'(t_0, x_1) - f'(t, x)\| < \frac{\varepsilon}{4} \quad \text{and} \quad \|f'(t_0, y_1) - f'(t, y)\| < \frac{\varepsilon}{4}. \tag{5}$$

From relations (4) and (5), we get

$$\begin{aligned}\mathcal{N}(t_0) - \frac{\varepsilon}{2} &\leq \|f'(t_0, x_1) - f'(t_0, y_1)\| \\ &\leq \|f'(t_0, x_1) - f'(t, x)\| \\ &\qquad + \|f'(t, x) - f'(t, y)\| + \|f'(t, y) - f'(t_0, y_1)\| \\ &\leq \|f'(t, x) - f'(t, y)\| + \frac{\varepsilon}{2},\end{aligned}$$

and so,

$$\mathcal{N}(t_0) - \varepsilon \leq \|f'(t, x) - f'(t, y)\|.$$

Thus, for each t with $|t - t_0| < \delta$, there exist x_1, y_1 with $\|x_1\|, \|y_1\| \leq Mt$ such that $\mathcal{N}(t_0) - \varepsilon \leq \|f'(t, x_1) - f'(t, y_1)\| \leq \mathcal{N}(t)$. We conclude that $\mathcal{N}$ is lower semicontinuous. Moreover from the continuity of f', $\mathcal{N}$ is continuous at 0. Consequently we can say that $\| \int_t^\tau f'(s, x(s)) - \int_t^\tau f'(s, y(s))\, ds\| \leq \int_t^\tau \mathcal{N}(s)\, ds$ for each $x, y \in V$. Then from relation (3) we have

$$\begin{aligned}&\rho(\tau) - \rho(t) \\ &\qquad \leq \min\left(\int_t^\tau \|G(t, s)\| \mathcal{N}(s)\, ds, \gamma(B_1) \int_t^\tau \|G(t, s)\| h(s, \rho(s))\, ds\right),\end{aligned}$$

where $0 < t \leq \tau \leq T$. Therefore ρ is an absolutely continuous function on I and so

$$\dot{\rho}(t) \leq \min\big(\|G(t, s)\| \mathcal{N}(t), \|G(t, s)\| h(t, \rho(t))\big), \qquad \text{a.e. on} \quad I.$$

Thus $\rho \equiv 0$ on I, see Lemma 1 in [22]. We can complete the proof as in the proof of Theorem 2.2.

3. Existence results for problem (Q)

We consider the problem

$$\text{(Q)} \qquad \dot{x}(t) = L(t)x(t) + f^d(t, \theta_t x), \qquad t \in I.$$

Let $B_r = \{x \in E : \|x\| \le r\}$, $L(t) \in \mathscr{L}(E)$ and for $t \in I$ we define $\theta_t x(s) = x(t+s)$ for all $s \in [-d, 0]$. We assume that $C([-d,0], B_r)$ is the Banach space of continuous functions from $[-d, 0]$ into B_r and $f^d : I \times C([-d,0], B_r) \to E$.

In the following theorem we deal with problem (Q) and we have a generalization of Theorem 2.1.

THEOREM 3.1. *If we replace in the setting of Theorem 2.1 the function f by f^d; the function m by $m' \in L^1(I, \mathsf{R}^+)$ and the operator A by L, then problem* (Q) *has a weak solution.*

PROOF. We apply some methods for functional equations similar to those of [10]. For any arbitrary $n \in \mathsf{N}$, we define $\gamma_1 : \left[-d, \frac{T}{n}\right] \times E \to E$ by

$$\gamma_1(t,x) = \begin{cases} \xi(t) & \text{if } t \in [-d, 0] \\ \xi(0) + nt(x - \xi(0)) & \text{if } t \in \left[0, \frac{T}{n}\right] \end{cases}$$

and also we define $f_1 : \left[0, \frac{T}{n}\right] \times E \to E$ by $f_1(t,x) = f^d(t, \theta_{\frac{T}{n}}(\gamma_1(.,x)))$. Arguing as in the proof of Theorem 2.1, there is a continuous function y_1 such that $y_1 = \xi$ on $[-d, 0]$ and for each $t \in \left[0, \frac{T}{n}\right]$

$$y_1(t) = G(t,0)\xi(0) + \int_0^t G(t,s) f_1(s, y_1(s))\, ds.$$

Moreover $\sup_{t \in [0, \frac{T}{n}]} \|y_1(t)\| \le r$. Set $k' = k - 1$. By induction, for each $k \in \{2, 3, \ldots, n\}$, there exists a bounded function $y_{k'}$ such that $y_{k'} = \xi$ on $[-d, 0]$ and for each $t \in \left[0, \frac{k'T}{n}\right]$

$$y_{k'}(t) = G(t,0)\xi(0) + \int_0^t G(t,s) f_{k'}(s, y_{k'}(s))\, ds,$$

where $f_{k'}(t,x) = f^d\big(t, \theta_{\frac{k'T}{n}} \gamma_{k'}(.,x)\big)$. Assume that $\gamma_k : \left[-d, \frac{kT}{n}\right] \times E \to E$ is such that

$$\gamma_k(t,x) = \begin{cases} y_{k'}(t) & \text{if } t \in \left[-d, \frac{k'T}{n}\right] \\ y_{k'}\big(\frac{k'T}{n}\big) + n\big(t - \frac{k'T}{n}\big)\big(x - y_{k'}\big(\frac{k'T}{n}\big)\big) & \text{if } t \in \left[\frac{k'T}{n}, \frac{kT}{n}\right]. \end{cases}$$

Thus if $f_k : \left[\frac{k'T}{n}, \frac{kT}{n}\right] \times E \to E$ is defined by $f_k(t, x) = f^d\big(t, \theta_{\frac{kT}{n}}(\gamma_k(., x))\big)$, then we have a continuous function y_k defined on $\left[\frac{k'T}{n}, \frac{kT}{n}\right]$ by

$$y_k(t) = G\big(t, \tfrac{k'T}{n}\big) y_{k'}\big(\tfrac{k'T}{n}\big) + \int_{\frac{k'T}{n}}^{t} G(t, s) f_k(s, y_k(s))\, ds.$$

Further, for $0 \le s \le r \le t$, $G(t, s)G(s, r) = G(t, r)$ and for each $t \in \left[\frac{k'T}{n}, \frac{kT}{n}\right]$ we have

$$y_{k'}\big(\tfrac{k'T}{n}\big) = G\big(\tfrac{k'T}{n}, 0\big)\xi(0) + \int_0^{\frac{k'T}{n}} G\big(\tfrac{k'T}{n}, s\big) f_{k'}(s, y_{k'}(s))\, ds.$$

Hence

$$\begin{aligned} y_k(t) &= G\big(t, \tfrac{k'T}{n}\big) G\big(\tfrac{k'T}{n}, 0\big)\xi(0) + \int_0^{\frac{k'T}{n}} G\big(t, \tfrac{k'T}{n}\big) G\big(\tfrac{k'T}{n}, s\big) f_{k'}(s, y_{k'}(s))\, ds \\ &\quad + \int_{\frac{k'T}{n}}^{t} G(t, s) f_k(s, x(s))\, ds \\ &= G(t, 0)\xi(0) + \int_0^{\frac{k'T}{n}} G(t, s) f_{k'}(s, y_{k'}(s))\, ds \\ &\quad + \int_{\frac{k'T}{n}}^{t} G(t, s) f_k(s, y_k(s))\, ds \\ &= G(t, 0)\xi(0) + \int_0^{t} G(t, s) g_k(s, y_k(s))\, ds, \end{aligned}$$

where

$$g_k(t, y_k(t)) = \begin{cases} f_{k'}(t, y_{k'}(t)) & \text{if } t \in \left[0, \frac{k'T}{n}\right] \\ f_k(t, y_k(t)) & \text{if } t \in \left[\frac{k'T}{n}, \frac{kT}{n}\right]. \end{cases}$$

Consequently, for all $n \in \mathsf{N}$, we have a continuous bounded function v_n such that $v_n = \xi$ on $[-d, 0]$ and for each $t \in I$, $\frac{k'T}{n} \le t \le \frac{kT}{n}$ for some $k \in \{1, 2, 3, \ldots, n\}$, we have

$$v_n(t) = G(t, 0)\xi(0) + \int_0^{t} G(t, s) h_n(s)\, ds$$

where $h_n(t) = f^d\big(t, \theta_{\frac{kT}{n}}\gamma_k(., v_n(t))\big)$. Let $t_1, t_2 \in I$ and $t_1 < t_2$. Then

$$\begin{aligned}
&\|v_n(t_1) - v_n(t_2)\| \\
&\leq \|G(t_1,0) - G(t_2,0)\|\|\xi(0)\| + \int_0^{t_1} \|G(t_1,s) - G(t_2,s)\|\|h_n(s, v_n(s))\|\,ds \\
&\qquad + \int_{t_1}^{t_2} \|G(t_2,s)\|\|h_n(s, v_n(s))\|\,ds \\
&\leq \|G(t_1,0) - G(t_2,0)\|\|\xi(0)\| + \int_0^{t_1} \|G(t_1,s) - G(t_2,s)\|\|m'(s)\|\,ds \\
&\qquad + c\int_{t_1}^{t_2} \|m'(s)\|\,ds,
\end{aligned}$$

since $v_n = \xi$ on $[-d, 0]$ and for all $s \in I$ $G(., s)$ is uniformly continuous, then A is equicontinuous in $C([-d, T], E)$. $\gamma(A(t)) = \gamma(\{v_n(t) : n \in \mathsf{N}\})$ is such that $\gamma(A(0)) = 0$ and, as in the proof of Theorem 2.1, $\gamma(A(t)) = 0$ for all $t \in I$. Thus by Ascoli's theorem, the sequence $\{v_n : n \in \mathsf{N}\}$ converges uniformly to a function v which belongs to $C([-d, T], E)$ such that $y = \xi$ on $[-d, 0]$. But $\gamma(\{h_n(t) : n \in \mathsf{N}\}) = 0$ and so $\{h_n(t) : n \in \mathsf{N}\}$ is relatively compact. Let $\mathscr{F}(t) = \overline{\text{conv}}\{h_n(t) : n \in \mathsf{N}\}$. Thus $\mathscr{F}(t)$ is nonempty convex and compact. Moreover $\delta^1_{\mathscr{F}} = \{l \in L^1(I, E) : l(t) \in \mathscr{F}(t)\}$ is nonempty convex and weakly compact. Therefore, there exists a subsequence (h_{n_k}) of (h_n) such that $h_{n_k} \to l$ weakly, $l \in \delta^1_{\mathscr{F}}$. Thus $\{v_n : n \in \mathsf{N}\}$ tends weakly to $v(t) := G(t,0)\xi(0) + \int_0^t G(t,s)l(s)\,ds$. Now v is uniformly continuous on $[-d, 0]$ and for each $t \in I$, there exists $n > \frac{T}{d}$ with $t \in \big[\frac{k'T}{n}, \frac{kT}{n}\big]$ for $k \in \{1, 2, \ldots, n-1\}$. Hence

$$\begin{aligned}
&\big\|\theta_{\frac{kT}{n}}\gamma_k(., v_n(t)) - \theta_t v\big\| \\
&\leq \sup_{s\in[-d,-\frac{T}{n}]}\Big[\big\|\gamma_k\big(\tfrac{kT}{n} + s, v_n(t)\big) - v\big(\tfrac{kT}{n} + s\big)\big\| + \big\|v\big(\tfrac{kT}{n} + s\big) - v(t+s)\big\|\Big] \\
&\quad + \sup_{s\in[-\frac{T}{n},0]}\Big[\big\|v_n\big(\tfrac{k'T}{n}\big) + n\big(\tfrac{kT}{n} + s - \tfrac{k'T}{n}\big)\big(v_n(t) - v_n\big(\tfrac{k'T}{n}\big)\big) - v\big(\tfrac{kT}{n} + s\big)\big\| \\
&\qquad\qquad + \big\|v\big(\tfrac{kT}{n} + s\big) - v(t+s)\big\|\Big] \\
&\leq \sup_{s\in[-d,-\frac{T}{n}]}\Big[\big\|v_n\big(\tfrac{kT}{n} + s\big) - v\big(\tfrac{kT}{n} + s\big)\big\| + \big\|v\big(\tfrac{kT}{n} + s\big) - v(t+s)\big\|\Big] \\
&\quad + \sup_{s\in[-\frac{T}{n},0]}\Big[T\big\|\big(v_n(t) - v_n\big(\tfrac{k'T}{n}\big)\big)\big\| + \big\|v_n\big(\tfrac{k'T}{n}\big) - v\big(\tfrac{kT}{n} + s\big)\big\| \\
&\qquad\qquad + \big\|v\big(\tfrac{kT}{n} + s\big) - v(t+s)\big\|\Big]
\end{aligned}$$

as $n \to \infty$. So from Lemma 1.7, problem (Q) has a weak solution v.

In the following theorem we use a measure of strong noncompactness γ so we have a generalization of Theorem 3.1 and an improvement to Theorem 2 in [26] and Theorem 9 in [13].

THEOREM 3.2. *In the setting of Theorem 2.2 if we replace the function f' by f^d such that for all $\varphi \in C([-d,0], B_r)$ $f^d(I \times \{\varphi\})$ is separable, then problem* (Q) *has a solution.*

PROOF. For $n \in \mathsf{N}$ we define $\gamma_1 : \left[-d, \frac{T}{n}\right] \times E \to E$, as in the proof of Theorem 3.1, by

$$\gamma_1(t,x) = \begin{cases} \xi(t) & \text{if } t \in [-d,0] \\ \xi(0) + nt(x - \xi(0)) & \text{if } t \in \left[0, \frac{T}{n}\right] \end{cases}$$

and $f_1 : \left[0, \frac{T}{n}\right] \times E \to E$ by $f_1(t,x) = f^d(t, \theta_{\frac{T}{n}}(\gamma_1(.,x)))$. By Theorem 2.2 there exists a continuous function y_1 such that $y_1 = \xi$ on $[-d,0]$ and for each $t \in \left[0, \frac{T}{n}\right]$

$$y_1(t) = G(t,0)\xi(0) + \int_0^t G(t,s) f_1(s, y_1(s))\, ds.$$

Then we can construct, for each $n \in \mathsf{N}$, a continuous bounded function v_n such that $v_n = \xi$ on $[-d,0]$ and for each $t \in I$ v_n is defined by

$$v_n(t) = G(t,0)\xi(0) + \int_0^t G(t,s) h_n(s)\, ds,$$

where $h_n(t) = f^d(t, \theta_{\frac{kT}{n}} \gamma_k(., v_n(t)))$ with $k \in \{1,2,3,\ldots,n\}$ and $\frac{(k-1)T}{n} \leq t \leq \frac{kT}{n}$. We can complete the proof as in the proof of Theorem 3.1.

In the next theorem we let $h : I \times \mathsf{R}^+ \to \mathsf{R}^+$ be a Carathéodory function. Also for each bounded subset Z of $I \times \mathsf{R}^+$ we suppose that there exists a function $m : I \to \mathsf{R}^+$ such that $h(t,s) \leq m(t)$, $(t,s) \in Z$ and m is integrable on $[c,T]$ for each c, $0 < c \leq T$. Moreover, assume that the identically zero function is the only absolutely continuous function on $[0,c]$ which satisfies $\dot{u}(t) = h(t, u(t))$ a.e. on $[0,c]$ and for which the right derivative $D_+u(0)$ of $u(t)$ at $t = 0$ exists and is 0.

THEOREM 3.3. *If we replace in the setting of Theorem 3.2 a Kamke function w by a function h and we suppose that f^d is bounded and continuous, then problem* (Q) *has a solution.*

We omit the proof since it runs as in the proof of Theorem 3.2 except that we replace the use of Theorem 2.2 by that of Theorem 2.3 to find a continuous function y_1 such that $y_1 = \xi$ on $[-d, 0]$ and for each $t \in \left[0, \frac{T}{n}\right]$

$$y_1(t) = G(t, 0)\xi(0) + \int_0^t G(t, s) f_1(s, y_1(s))\, ds.$$

In fact, if $L(t) \neq 0$ our results generalize that of Gomaa [10] and Cichon [4], since we have a generalization of the compactness assumptions and in [4] the results are stated without delay. For the important case $L(t) = 0$ we have, as a special case, a generalization of the existence theorems of Gomaa [13], Ibrahim-Gomaa [15], Papageorgiou [23], Cramer-Lakshmikantham-Mitchell [7], Szep [25] and Boundourides [2] in all of which the results are stated without delay. Szep in [25] studied the special case of problem (P) in a reflexive Banach space, Boundourides [2] and Cramer-Lakshmikantham-Mitchell [7] studied the special case of problem (P) in a nonreflexive Banach space, Papageorgiou [23] found weak solutions for the special case of problem (P) on a finite interval I with $0 < T < \infty$, Ibrahim-Gomaa [15] found weak solutions for the special case of problem (P) on a finite interval I and in [13] we give a generalization to recent results on the Cauchy problem by using weak and strong measures of noncompactness. Moreover in [11], [12] we study the nonlinear differential equations with and without delay while in [9] we study the differential inclusions with moving constraints.

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REFERENCES

1. Banas, J., and Goebel, K., *Measure of noncompactness in Banach spaces*, Lecture notes pure appl. math. 60, Marcel Dekker, New York 1980.
2. Boudourides, M., *An existence theorem for ordinary differential equations in Banach spaces*, Bull. Austral. Math. Soc. 22 (1980), 457–463.
3. Cichon, M., *Differential inclusions and abstract control problems*, Bull. Austral Math. Soc. 53 (1996), 109–122.
4. Cichon, M., *On bounded weak solutions of a nonlinear differential equation in Banach space*, Funct. Approx. Comment. Math. 21 (1992), 27–35.
5. Cichon, M., *On measures of weak noncompactness*, Publ. Math. Debrecen 45 (1994), 93–102.
6. Cichon, M., *Trichotomy and bounded solutions of nonlinear differential equations*, Math. Bohem. 119 (1994), 275–284.
7. Cramer, E., Lakshmikantham, V., and Mitchell, A. R., *On the existence of weak solutions of differential equations in nonreflexive Banach spaces*, Nonlinear Anal. 2 (1978), 169–177.
8. Deimling, K., *Ordinary Differential Equations in Banach Spaces* , Lecture Notes in Math. 596, Springer, Berlin 1977.

9. Gomaa, A. M., *Existence of solutions for differential inclusions on closed moving constraints in Banach spaces*, Electron. J. Differential Equations 22 (2009), 10 pp.
10. Gomaa, A. M., *Existence solutions for differential equations with delay in Banach spaces*, Proc. Math. Phys. Soc. Egypt 84 (2006), 1–12.
11. Gomaa, A. M., *On bounded weak and pseudo-solutions of nonlinear differential equations having trichotomy with and without delay in Banach spaces*, Int. J. Geom. Methods Mod. Phys. 7 (2010), 357–366.
12. Gomaa, A. M., *On theorem for weak solutions of nonlinear differential equations with and without delay in Banach Spaces*, Comment. Math. Prace Mat. 47 (2007), 179–191.
13. Gomaa, A. M., *Weak and strong solutions for differential equations in Banach spaces*, Chaos Solitons Fractals 18 (2003), 687–692.
14. Hille, E., and Phillips, R. G., *Functional analysis and semi-groups*, rev. ed. Colloq. Publ. 31, Amer. Math. Soc., Providence, RI 1957.
15. Ibrahim, A. G., and Gomaa, A. M., *Strong and weak solutions for differential inclusions with moving constraints in Banach spaces*, Pure Math. Appl. 8 (1997), 53–65.
16. Krzyska, S., and Kubiaczyk, I., *On bounded pseudo and weak solutions of a nonlinear differential equation in Banach spaces*, Demonstratio Math. 32 (1999), 323–330.
17. Kubiaczyk, I., *On the existence of solutions of differential equations in Banach spaces*, Bull. Polish Acad. Sci. Math. 33 (1985), 607–614.
18. Kubiaczyka, R., *Zastosowanic miar niezwartośći w teorii róńrózniczkowych i mathcalkowych, Oh D*, thesis, Lublin 1978.
19. Makowiak, M., *On some bounded solutions of a nonlinear differential equations*, Demonstratio Math. 30 (1997), 801–808.
20. Massera, J. L., and Shaffer, J. J., *Linear differential equations and function spaces*, Pure and Appl. Math. 21, Academic Press, New York 1966.
21. Mitchell, A. R., and Smith, C., *An existence theorem for weak solutions of differential equations in Banach spaces*, pp. 387–403 in: *Nonlinear equations in abstract spaces*, Proc. Int. Symp., Arlington 1977, Academic Press, New York 1978.
22. Olech, O., *On the existence and uniqueness of solutions of an ordinary differential equation in the case of Banach space*, Bull. Acad. Polon. Sci. 8 (1960), 667–673.
23. Papageorgiou, N. S., *Weak solutions of differential equations in Banach spaces*, Bull. Austral. Math. Soc. 33 (1986), 407–418.
24. Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. 44, Springer, Berlin 1983.
25. Szep, A., *Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces*, Studia Sci. Math. Hungar. 6 (1971), 197–203.
26. Szufla, S., *On the existence of solutions of differential equations in Banach spaces*, Bull. Acad. Polon. Sci. 30 (1982), 507–515.

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