MATH. SCAND. 112 (2013), 275–301

DIMENSION OF THE BOUNDARY IN DIFFERENT METRICS

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Abstract

We consider metrics on Euclidean domains $\Omega \subset \mathsf{R}^n$ that are induced by continuous densities $\rho\colon \Omega \to (0,\infty)$ and study the Hausdorff and packing dimensions of the boundary of Ω with respect to these metrics.

1. Introduction

Let $\Omega \subset \mathsf{R}^n$ be a domain. For $x, y \in \Omega$, we denote by $d(x,y)$ the internal Euclidean distance between x and y defined as

$$d(x,y) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken over all rectifiable curves in Ω with endpoints x and y and ℓ refers to the standard Euclidean length. It is well known and easy to see that d defines a metric on Ω called the internal metric. Furthermore, we may extend this metric to the internal boundary $\partial\Omega_d = \overline{\Omega}_d \setminus \Omega$, where $\overline{\Omega}_d$ is the standard metric completion of Ω with respect to d.

Let $\rho\colon \Omega \to (0,\infty)$ be a continuous function. We define the ρ-length of a rectifiable curve $\gamma \subset \Omega$ as

$$\ell_\rho(\gamma) = \int_\gamma \rho(z)\,|dz|$$

where $|dz|$ denotes integration with respect to arclength. The ρ-distance between $x, y \in \Omega$ is then given by

$$d_\rho(x,y) = \inf_{\gamma} \ell_\rho(\gamma),$$

where the infimum is again over all curves joining x to y in Ω. This defines a metric on Ω and as with the internal metric, we may extend it to the ρ-boundary of Ω defined as $\partial_\rho\Omega = \overline{\Omega}_\rho \setminus \Omega$, where $\overline{\Omega}_\rho$ is the standard metric completion

Received 20 April 2011, in final form 2 April 2012.

of Ω with respect to d_ρ. Observe that the internal metric d corresponds to d_ρ for the constant function $\rho \equiv 1$.

Thus, given ρ as above (*a density* in what follows), we have two complete metric spaces $(\overline{\Omega}_d, d)$ and $(\overline{\Omega}_\rho, d_\rho)$ which need not be topologically equivalent. For simplicity, however, we only deal with cases in which $\partial_\rho\Omega$ may be naturally identified with a metric subspace of $\partial\Omega_d$.

In this paper, we will consider $\dim_\rho(\partial_\rho\Omega)$ and $\mathrm{Dim}_\rho(\partial_\rho\Omega)$, the Hausdorff and packing dimensions of $\partial_\rho\Omega$ with respect to d_ρ (For more comprehensive notation and definitions, we refer to Section 2 below). Classically, this sort of problems arise in connection to harmonic measures and the boundary behaviour of conformal maps [8], [9], [13], [5], [7]. In that setting, $\rho = |f'|$ for a conformal map f and d_ρ corresponds to the internal metric on the image domain. The Hausdorff dimension, $\dim_\rho(\partial_\rho\Omega)$, has been analysed also for a much larger collection of so called *conformal densities* on the unit ball $\mathsf{B}^n \subset \mathsf{R}^n$. See [2], [1], [11]. Although we provide some estimates in the setting of conformal densities, our main goal is to study general densities defined on John domains in R^n, and to provide tools to estimate the values of the dimensions $\dim_\rho(\partial_\rho\Omega)$ and $\mathrm{Dim}_\rho(\partial_\rho\Omega)$. Because of this, our methods are perhaps more geometric than analytic.

Given $A \subset \overline{\Omega}_d$, we denote by $d(x, A) = \inf_{a\in A} d(x, a)$ the internal distance from x to A and, moreover, abbreviate $d(x) = d(x, \partial\Omega_d)$. Of course, $d(x)$ is just the Euclidean distance to the boundary of Ω.

Let us consider the following simple example: Suppose that $\Omega \subsetneq \mathsf{R}^n$ has smooth boundary, $-1 < \beta < 0$, and define a density $\rho(x) = d(x)^\beta$. Then it is well known, and easy to see that $\partial_\rho\Omega$ is a "snowflake". More precisely, $d_\rho(x, y) \approx d(x, y)^{1+\beta}$ for all $x, y \in \partial_\rho\Omega$. Thus, the effect of ρ on the dimensions of the boundary is described by a power law

$$\begin{aligned}\mathrm{Dim}_d(\partial\Omega_d)/\mathrm{Dim}_\rho(\partial_\rho\Omega) &= \dim_d(\partial\Omega_d)/\dim_\rho(\partial_\rho\Omega)\\ &= 1 + \log\rho(x)/\log d(x).\end{aligned}$$

Keeping this example in mind, it is now natural to consider (the upper and lower) limits of the quantity $\log\rho(y)/\log d(y)$ as y approaches the boundary of Ω. Under sufficient assumptions, this leads to multifractal type formulas for the dimension of $\partial_\rho\Omega$. For instance, we obtain the following result.

THEOREM 1.1. *Let $\Omega \subset \mathsf{R}^n$ be a John domain and $\rho > c > 0$ a density. Suppose that*

$$i(x) = \lim_{y\in\Omega, y\to x} \frac{\log\rho(y)}{\log d(y)}$$

exists at all points $x \in \partial\Omega_d$ and satisfies $i(x) > -1$. Then

$$\dim_\rho(\partial_\rho\Omega) = \sup_{\beta > -1} (1+\beta)^{-1} \dim_d(\{x \in \partial\Omega_d : i(x) \le \beta\}).$$

An analogous formula holds for the packing dimension.

This theorem is a simple special case of a more general result, Theorem 4.2, and it can be used to obtain a formula for the dimensions $\dim_\rho(\partial_\rho\Omega)$ and $\mathrm{Dim}_\rho(\partial_\rho\Omega)$ in many situations. A generic case is the following: $\Omega = \mathsf{B}^n$, $C \subset \partial\mathsf{B}^n$ is a Cantor set with $0 < \dim_d C < \mathrm{Dim}_d C < n$ and $\rho(x) = d(x, C)^\beta$ for some $\beta > -1$ (Example 4.4).

In Theorem 1.1, there is an annoying lack of generality since we have to consider inner limits in the definition of $i(x)$. The situation is different if we know that the distance $d_\rho(x, y)$ between points $x, y \in \partial_\rho\Omega$ is realised along curves that are "non-tangential". If the density satisfies a suitable Harnack inequality together with a Gehring-Hayman type estimate, then it is enough to consider limits along some fixed cones. For conformal densities, for instance, we may replace the quantity $i(x)$ by a radial version $k(x) = \lim_{t\uparrow 1} \log\rho(tx)/\log(1-t)$; see Section 5 where we actually consider upper and lower limits as $t \uparrow 1$.

Section 6 contains several examples and some open questions. Most notably, in Example 6.3 we construct a new nontrivial example of a conformal density with multifractal type boundary behavior.

As our results indicate, a careful inspection of the power exponents and the size of certain sub and super level sets of these quantities can be used to study the dimensions $\dim_\rho(\partial_\rho\Omega)$ and $\mathrm{Dim}_\rho(\partial_\rho\Omega)$. Although the main idea in most of our results is the same, it is perhaps not possible to find a general statement which would fit into all, or even most, of the interesting situations. Often, a suitable case study and a combination of different ideas is needed in order to deduce the relevant information (for instance, see Examples 4.7, 6.2, and 6.3). We strongly believe that the ideas we have used can be applied also elsewhere, beyond the results of this paper.

2. Notation

Let $\Omega \subset \mathsf{R}^n$ be a domain. For technical reasons, we want to be able to naturally identify $\partial_\rho\Omega$ with a subset of $\partial\Omega_d$. To ensure this, we assume throughout this paper that for all sequences (x_i), $x_i \in \Omega$, the following two conditions are satisfied:

(A1) If (x_i) converges in $\overline{\Omega}_\rho$, then it converges in $\overline{\Omega}_d$.

(A2) If (x_i) converges in $\overline{\Omega}_d$, it has at most one accumulation point in $\partial_\rho\Omega$.

In other words, (A1) means that the identity mapping $(\Omega, d_\rho) \to (\Omega, d)$ has a continuous extension $f\colon \overline{\Omega}_\rho \to \overline{\Omega}_d$ and, furthermore, (A2) means that this f is injective.

DEFINITION 2.1. *A density* is a continuous function $\rho\colon \Omega \to (0, \infty)$ satisfying (A1) and (A2). For simplicity, we also require that $\partial_\rho \Omega \neq \emptyset$.

Whenever we talk about a curve γ, we assume that it is rectifiable, is arc-length parametrized, and that $\gamma(t) \in \Omega$ for all $0 < t < \ell(\gamma)$ (the endpoints may or may not belong to $\partial\Omega_d$). Note that the internal length of a curve equals the Euclidean length of the curve. We say that $\Omega \subset \mathsf{R}^n$ is an α-John domain for $0 < \alpha \leq 1$, if there is $x_0 \in \Omega$ such that all points $x \in \Omega$ may be joined to x_0 by an α-cone, i.e. by a curve γ joining x to x_0 such that $d(\gamma(t)) \geq \alpha\, t$ for all $0 \leq t \leq \ell(\gamma)$. If α is not important, we simply talk about John domains. Let $\gamma \subset \Omega$ be a curve. We say that γ is an α-cigar if

$$d(\gamma(t)) \geq \alpha \min\{t, \ell(\gamma) - t\} \qquad \text{for all} \quad 0 \leq t \leq \ell(\gamma). \tag{2.1}$$

For technical purposes, we define an α-distance between points $x, y \in \Omega$ as

$$d_\alpha(x, y) = \inf_\gamma \ell(\gamma)$$

and this time the infimum is taken over all α-cigars γ joining x and y. It is easy to see that if Ω is an α-John domain, then any two points $x, y \in \overline{\Omega}_d$ may be joined by an α-cigar. Thus $d_\alpha(x, y) < \infty$ for all $x, y \in \overline{\Omega}_d$. Note however that d_α is not necessarily a metric since it may be infinite and even if it happens to be finite, it may fail to satisfy the triangle inequality.

Let $X = (X, d_X)$ be a separable metric space. We denote balls $B_X(x, r) = \{y \in X : d_X(y, x) < r\}$ and spheres $S_X(x, r) = \{y \in X : d_X(x, y) = r\}$. Given $A \subset X$, we define its s-dimensional Hausdorff and packing measures, $\mathscr{H}^s_X(A)$ and $\mathscr{P}^s_X(A)$, respectively, by the following procedure:

$$\mathscr{H}^{s,\varepsilon}_X(A) = \inf\left\{\sum_{i=1}^\infty \operatorname{diam}_X(A_i)^s \colon A \subset \bigcup_{i\in\mathsf{N}} A_i \text{ and } \operatorname{diam}_X(A_i) < \varepsilon \text{ for all } i\right\},$$

$$\mathscr{H}^s_X(A) = \lim_{\varepsilon\downarrow 0} \mathscr{H}^{s,\varepsilon}_X(A),$$

$$P^{s,\varepsilon}_X(A) = \sup\left\{\sum_{i=1}^\infty r_i^s \colon \{B_X(x_i, r_i)\} \text{ is a packing of } A \text{ with } r_i \leq \varepsilon\right\},$$

$$P^s_X(A) = \lim_{\varepsilon\downarrow 0} P^{s,\varepsilon}_X(A),$$

$$\mathscr{P}^s_X(A) = \inf\left\{\sum_{i=1}^\infty P^s_X(A_i) \colon A \subset \bigcup_{i=0}^\infty A_i\right\},$$

where $0 < \varepsilon, s < \infty$ and a packing of A is a disjoint collection of balls with centres in A. We define the Hausdorff and packing dimensions of $A \subset X$, respectively, as

$$\dim_X(A) = \sup\{s \geq 0 \colon \mathscr{H}^s_X(A) = \infty\} = \inf\{s \geq 0 : \mathscr{H}^s_X(A) = 0\},$$
$$\mathrm{Dim}_X(A) = \sup\{s \geq 0 \colon \mathscr{P}^s_X(A) = \infty\} = \inf\{s \geq 0 : \mathscr{P}^s_X(A) = 0\},$$

with the conventions $\sup\emptyset = 0$, $\inf\emptyset = \infty$.

When the domain $\Omega \subset \mathsf{R}^n$ has been fixed, we use all the notation introduced above with the subscript d when referring to the internal metric. Moreover, given a density $\rho\colon \Omega \to (0, \infty)$, we use the subscript ρ to refer to the corresponding notions in terms of the metric d_ρ. For example, given $x \in \overline{\Omega}_d$, $y \in \overline{\Omega}_\rho$, and $r > 0$ we have $B_d(x, r) = \{z \in \overline{\Omega}_d : d(z, x) < r\}$ and $S_\rho(y, r) = \{z \in \overline{\Omega}_\rho : d_\rho(z, y) = r\}$. We also use the notation $B_\alpha(x, r)$ for balls in terms of the "distance" d_α. When referring to "round" Euclidean balls we use a subindex e, so $B_e(x, r) = \{y \in \mathsf{R}^n : |y - x| < r\}$ where $|\cdot|$ is the usual Euclidean distance. We also denote $\mathsf{B}^n = B_e(0, 1) \subset \mathsf{R}^n$ and $S^{n-1} = S_e(0, 1) \subset \mathsf{R}^n$. Observe that if $A \subset \overline{\Omega}_\rho$, both notations $\mathrm{diam}_d(A)$ and $\mathrm{diam}_\rho(A)$ make sense, since by (A1) and (A2), if $x, y \in A$, then $d(x, y)$, $d_\rho(x, y) < \infty$ are well defined.

To finish this section, we introduce various limits that are used later to obtain dimension bounds for $\partial_\rho\Omega$. For a domain $\Omega \subset \mathsf{R}^n$, a density ρ and $x \in \partial\Omega_d$, we define

$$i^-(x) = \liminf_{\substack{y\in\Omega \\ y\to x}} \frac{\log\rho(y)}{\log d(y)}, \qquad i^+(x) = \limsup_{\substack{y\in\Omega \\ y\to x}} \frac{\log\rho(y)}{\log d(y)}, \tag{2.2}$$

where the limits are considered with respect to the internal metric. Observe that $i^+(x) \geq -1$ for all $x \in \partial_\rho\Omega$, but $i^-(x)$ does not have to be bounded from below.

For a domain $\Omega \subset \mathsf{R}^n$, a density ρ and $\beta > -1$, we define

$$d^+(\beta) = \dim_d\{x \in \partial_\rho\Omega : i^+(x) \leq \beta\}, \tag{2.3}$$

$$D^+(\beta) = \mathrm{Dim}_d\{x \in \partial_\rho\Omega : i^+(x) \leq \beta\}, \tag{2.4}$$

$$d^-(\beta) = \dim_d\{x \in \partial_\rho\Omega : i^-(x) \leq \beta\}, \tag{2.5}$$

$$D^-(\beta) = \mathrm{Dim}_d\{x \in \partial_\rho\Omega : i^-(x) \leq \beta\}. \tag{2.6}$$

For a density ρ on B^n and $x \in S^{n-1}$, we set

$$k^-(x) = \liminf_{r\uparrow 1} \frac{\log\rho(rx)}{\log(1-r)}, \qquad k^+(x) = \limsup_{r\uparrow 1} \frac{\log\rho(rx)}{\log(1-r)}. \tag{2.7}$$

TABLE 1.

ρ	density: A continuous $\rho\colon \Omega \to (0,\infty)$ satisfying (A1) and (A2)
$\ell_\rho(\gamma)$	the ρ-length of a rectifiable curve γ
d	internal metric
d_ρ	ρ-metric
d_α	α-distance
$d(x)$	Euclidean distance from x to the boundary
$B_d(x,r)$, $S_d(x,r)$	ball and sphere with respect to the internal metric d
$B_\rho(x,r)$, $S_\rho(x,r)$	ball and sphere with respect to d_ρ
$B_e(x,r)$, $S_e(x,r)$	ball and sphere with respect to the Euclidean distance
$B_\alpha(x,r)$, $S_\alpha(x,r)$	ball and sphere with respect to d_α
$\dim_d$	Hausdorff dimension with respect to d
Dim_d	packing dimension with respect to d
$\dim_\rho$	Hausdorff dimension with respect to d_ρ
Dim_ρ	packing dimension with respect to d_ρ
$\overline{\Omega}_d$	metric completion of Ω with respect to d
$\partial\Omega_d$	internal boundary $\overline{\Omega}_d \setminus \Omega$
$\overline{\Omega}_\rho$	metric completion of Ω with respect to d_ρ
$\partial_\rho\Omega$	ρ-boundary $\overline{\Omega}_\rho \setminus \partial_\rho\Omega$.
$i^\pm$, $k^\pm$	limits used for dimension bounds

Note that for $\Omega = \mathsf{B}^n$ we have $i^-(x) \le k^-(x) \le k^+(x) \le i^+(x)$ for $x \in S^{n-1}$.

Occasionally, we need to make the following technical assumption for the metric d_ρ:

ASSUMPTION 2.2. For each $x \in \partial_\rho\Omega$ and each $\varepsilon > 0$, there is $r > 0$ such that for all $y \in B_\rho(x,r)$ there is a curve γ joining x to y in Ω such that $h(\gamma) \ge d(x,y)^{1+\varepsilon}$ and $\ell_\rho(\gamma) \le d_\rho(x,y)^{1-\varepsilon}$.

Here $h(\gamma) = \sup_{y\in\gamma} d(y)$ is the maximal distance of γ from the boundary (the "height" of γ). This assumption should be understood as a very mild monotonicity condition with respect to $d(x)$. It is used to obtain dimension lower bounds for the part of $\partial_\rho\Omega$ where $i^+ \ge 0$. Close to such points, it is hard to obtain lower estimates for the ρ-length of curves that stay very close to $\partial\Omega$. In fact, if Assumption 2.2 fails, it may happen that $\dim_\rho \partial_\rho\Omega = \mathrm{Dim}_\rho \partial_\rho\Omega = 0$ even if Ω is a half-space, $\dim_d \partial_\rho\Omega > 0$, and i^+ is uniformly bounded. See Example 4.6.

Assumption 2.2 is a natural generalisation of the Gehring-Hayman condition valid for conformal densities, see (5.3).

We summarise our main notation in Table 1.

3. Preliminary lemmas

We start by recalling the following simple lemma giving estimates on expansion and compression behaviour of Hölder type maps.

LEMMA 3.1. *Suppose that* Z *and* Y *are separable metric spaces and let* $f\colon Z \to Y$, $0 < \delta < \infty$ *and* $X \subset Z$.

(1) *If for each* $x \in X$ *there are* $0 < r_x, C_x < \infty$ *so that* $f(B_Z(x,r)) \subset B_Y(f(x), C_x r^\delta)$ *for all* $0 < r < r_x$, *then*

$$\delta \dim_Y(f(X)) \le \dim_Z(X), \tag{3.1}$$

$$\delta \operatorname{Dim}_Y(f(X)) \le \operatorname{Dim}_Z(X). \tag{3.2}$$

(2) *If for each* $x \in X$ *there are* $0 < C_x < \infty$ *and a sequence* $r_{x,i} > 0$ *such that* $\lim_{i\to\infty} r_{x,i} = 0$ *and* $f(B_Z(x, r_{x,i})) \subset B_Y(f(x), C_x r_{x,i}^\delta)$ *for all* i, *then*

$$\delta \dim_Y(f(X)) \le \operatorname{Dim}_Z(X). \tag{3.3}$$

PROOF. The proof of (3.1) is standard. We give some details for (3.2) and (3.3).

To prove (3.2), we first observe that $X = \bigcup_{n\in\mathbb{N}} X_n$ where

$$X_n = \{x \in X : f(B_Z(x,r)) \subset B_Y(f(x), nr^\delta) \text{ for all } 0 < r < 1/n\}.$$

Let $0 < \varepsilon, s < \infty$, $A \subset X_n$ and suppose that $B_Y(x_i, r_i)$, $i \in \mathbb{N}$ is a packing of $f(A)$ so that $r_i < \min\{\varepsilon, n^{1-\delta}\}$ for each i. If $y_i \in A \cap f^{-1}\{x_i\}$ it follows that $f(B_Z(y_i, n^{-1/\delta} r_i^{1/\delta})) \subset B_Y(x_i, r_i)$ (note that there can be more than one y_i with $f(y_i) = x_i$, choosing any of them will do). Thus, $B_Z(y_i, n^{-1/\delta} r_i^{1/\delta})$ is a packing of A. Letting $\varepsilon \downarrow 0$, this implies $P_Y^s(f(A)) \le n^s P_Z^{s\delta}(A)$ for all $A \subset X_n$. As $A \subset X_n$ is arbitrary, we also get $\mathscr{P}_Y^s(f(X_n) \le n^s \mathscr{P}_Z^{s\delta}(X_n)$, in particular $\operatorname{Dim}_Y(f(X_n)) \le \operatorname{Dim}_Z(X_n)/\delta$. The claim (3.2) now follows as $X = \cup_{n\in\mathbb{N}} X_n$.

In order to prove (3.3), let

$$X_n = \{x \in X : f(B_Z(x, r_{x,i})) \subset B_Y(f(x), nr_{x,i}^\delta) \text{ for some sequence } r_{x,i} \downarrow 0\}.$$

Then $X = \cup_{n\in\mathbb{N}} X_n$. Choose $A \subset X_n$ and fix $s, \varepsilon > 0$. Applying the standard $5R$-covering theorem (see e.g. [10, Theorem 2.1]) to the collection

$$\mathscr{B} = \{B_Y(f(x), nr^\delta) : x \in A, 0 < r < \varepsilon, f(B_Z(x,r)) \subset B_Y(f(x), nr^\delta)\}$$

we find a pairwise disjoint subcollection $\{B_Y(f(x_i), nr_i^\delta)\}_i$ of $\mathscr{B}$ so that $f(A) \subset \cup_i B_Y(f(x_i), 5nr_i^\delta)$. As $\{B_Z(x_i, r_i)\}_i$ is a packing of A, we get $\mathscr{H}_Y^{s/\delta, 5n\varepsilon^\delta}(f(A))$

$\leq (5n)^{s/\delta} P_Z^{s,\varepsilon}(A)$ and letting $\varepsilon \downarrow 0$, $\mathscr{H}_Y^{s/\delta}(f(A)) \leq (5n)^{s/\delta} P_Z^s(A)$. As $A \subset X_n$ is arbitrary, we also get $\mathscr{H}_Y^{s/\delta}(f(X_n)) \leq (5n)^{s/\delta} \mathscr{P}_Z^s(X_n)$ and finally $\dim_Y(f(X)) \leq \mathrm{Dim}_Z(X)/\delta$ since $X = \cup_{n \in \mathsf{N}} X_n$.

Below, we give a variant of Lemma 3.1 in terms of the metrics d and d_ρ.

LEMMA 3.2. *Suppose that $\Omega \subset \mathsf{R}^n$ is a domain and $\rho\colon \Omega \to (0, \infty)$ is a density. Let $A \subset \partial_\rho \Omega$ and $0 \leq \delta \leq \infty$.*

(1) *If*
$$\liminf_{r \downarrow 0} \frac{\log(\operatorname{diam}_\rho(B_d(x, r)))}{\log r} \geq \delta$$
for all $x \in A$, then $\delta \dim_\rho(A) \leq \dim_d(A)$ and $\delta \,\mathrm{Dim}_\rho(A) \leq \mathrm{Dim}_d(A)$.

(2) *If*
$$\liminf_{r \downarrow 0} \frac{\log(\operatorname{diam}_d(B_\rho(x, r)))}{\log r} \geq \delta$$
for all $x \in A$, then $\dim_\rho(A) \geq \delta \dim_d(A)$ and $\mathrm{Dim}_\rho(A) \geq \delta\, \mathrm{Dim}_d(A)$.

(3) *If*
$$\limsup_{r \downarrow 0} \frac{\log(\operatorname{diam}_\rho(B_d(x, r)))}{\log r} \geq \delta$$
for all $x \in A$, then $\delta \dim_\rho(A) \leq \mathrm{Dim}_d(A)$.

(4) *If*
$$\limsup_{r \downarrow 0} \frac{\log(\operatorname{diam}_d(B_\rho(x, r)))}{\log r} \geq \delta$$
for all $x \in A$, then $\mathrm{Dim}_\rho(A) \geq \delta \dim_d(A)$.

PROOF. All the claims (1)–(4) follow easily from Lemma 3.1 applied to the mapping $f\colon (\overline{\Omega}_\rho, d) \to (\overline{\Omega}_\rho, d_\rho)$, $x \mapsto x$ and its inverse. To prove (1), for instance, fix $\lambda < \delta$. Then for all $x \in A$, there is $r_x > 0$ so that $B_d(x, r) \subset B_\rho(x, r^\lambda)$ when $0 < r < r_x$. Thus, Lemma 3.1 (1) implies $\lambda \dim_\rho(A) \leq \dim_d(A)$ and $\lambda\, \mathrm{Dim}_\rho(A) \leq \mathrm{Dim}_d(A)$. Letting $\lambda \uparrow \delta$, yields (1).

We end the preliminaries with the following lemma.

LEMMA 3.3. *For all $0 < \alpha \leq 1$ and $n \in \mathsf{N}$, there exists constants $N = N(\alpha, n) \in \mathsf{N}$ and $c = c(\alpha) < \infty$ so that for all α-John domains $\Omega \subset \mathsf{R}^n$ the following holds: For all $x \in \overline{\Omega}_d$ and $r > 0$, there are points $x_1, \ldots, x_N \in \overline{\Omega}_d$ so that $B_d(x, r) \subset \cup_{i=1}^N B_{\alpha/2}(x_i, cr)$.*

PROOF. For all $y \in B_d(x, r)$, let γ_y be an α-cone that joins y to x_0, where $x_0 \in \Omega$ is a fixed John centre of Ω. Moreover, we let
$$A_y = \{z \in \Omega : d(z, \gamma_y(t)) < \tfrac{\alpha}{3} t \text{ for some } 0 < t < \ell(\gamma_y)\}.$$

We may assume that $d(x, x_0) \geq 2r$ since otherwise $B_d(x, r) \subset B_\alpha(x_0, 2r)$.

We first claim that if $y, z \in B_d(x, r)$ such that $B_d(x, 2r) \cap A_y \cap A_z \neq \emptyset$, then y and z may be joined by an $(\alpha/2)$-cigar γ with $\ell(\gamma) \leq c(\alpha)r$. For this, we may assume that $d(x) < 2r$ as otherwise the Euclidean line segment joining y to z suites as γ. Assume that $w \in B_d(x, 2r) \cap A_y \cap A_z$ and choose $t_y, t_z > 0$ so that $d(w, \gamma_y(t_y)) < \frac{\alpha}{3}t_y$ and $d(w, \gamma_z(t_z)) < \frac{\alpha}{3}t_z$. Let γ denote the curve which consists of $\gamma_y|_{0<t\leq t_y}$, $\gamma_z|_{0<t<t_z}$ and the two (Euclidean) line segments joining w to $\gamma_y(t_y)$ and $\gamma_z(t_z)$. As $\big(t_y + \frac{\alpha}{3}t_y\big)\frac{\alpha}{2} \leq \alpha t_y - \frac{\alpha}{3}t_y$ (and similarly for t_z), it follows that γ is an $\frac{\alpha}{2}$-cigar. Now $B_e\big(w, \frac{2}{3}\alpha t_y\big) \subset \Omega$, $B_e\big(w, \frac{2}{3}\alpha t_z\big) \subset \Omega$ by the α-cone condition. Combining this with the fact $d(w) \leq |w - x| + d(x) \leq 4r$ implies $t_y, t_z \leq \frac{6}{\alpha}r$ and consequently

$$\ell(\gamma) \leq \left(1 + \frac{\alpha}{3}\right)(t_y + t_z) \leq \left(\frac{4}{\alpha} + 1\right)r = c(\alpha)r.$$

Let $x_1, \ldots, x_N \in B_d(x, r)$ be such that $B_d(x, 2r) \cap A_{x_j} \cap A_{x_i} = \emptyset$ whenever $i \neq j$. It suffices to show that $N \leq N(n, \alpha)$. For each i, let $y_i = \gamma_{x_i}(r)$. Then $B_d(y_i, \alpha r/3) = B_e(y_i, \alpha r/3) \subset A_{x_i} \cap B_d(x, 2r)$ and a volume comparison yields $N(r\alpha/3)^n \leq 2^n r^n$ implying the claim for $N(n, \alpha) = (6/\alpha)^n$.

REMARK 3.4. A subset of the boundary of a John domain has the same Hausdorff dimension both in the internal and the Euclidean metric. Indeed, it follows as in the above proof that for any x which is an Euclidean boundary point of Ω, the set $B_e(x, r) \cap \Omega$ may be covered by $N = N(\alpha, r)$ balls of radius $c(\alpha)r$ in the internal metric. A slightly more detailed argument implies a similar statement for the packing dimension.

4. Dimension estimates on general domains

We first derive some straightforward dimension bounds arising from the local power law behaviour of the density ρ near $\partial\Omega$. For the definition of $i^-(x)$ and $i^+(x)$ recall (2.2). The relevant assumptions are slightly different for the upper and lower bounds, and also depend on the sign of $i^\pm$. Roughly speaking, the positive values of $i^\pm$ correspond to expansion behaviour (of d_ρ compared to d), whereas the negative values are related to compression of dimensions. If we aim to find the exact values of $\dim_\rho(\partial_\rho\Omega)$ and $\mathrm{Dim}_\rho(\partial_\rho\Omega)$, then we are usually more interested in the set where $i^\pm$ are negative.

LEMMA 4.1. *Suppose that* $\Omega \subset \mathbb{R}^n$, ρ *is a density on* Ω, $\beta > -1$, $A \subset \{x \in \partial_\rho\Omega : i^+(x) \leq \beta\}$ *and* $B \subset \{x \in \partial_\rho\Omega : i^-(x) \geq \beta\}$.

If $\beta < 0$ *or if Assumption 2.2 holds, then*

(1) $(1 + \beta)\dim_\rho(A) \geq \dim_d(A)$,

(2) $(1+\beta)\,\mathrm{Dim}_\rho(A) \geq \mathrm{Dim}_d(A)$.

If Ω is a John domain, or if $\beta > 0$, we have

(3) $(1+\beta)\dim_\rho(B) \leq \dim_d(B)$,

(4) $(1+\beta)\,\mathrm{Dim}_\rho(B) \leq \mathrm{Dim}_d(B)$.

PROOF. Assume first that $\beta < 0$ and choose $\beta < s < 0$. Now, for all $x \in A$, there is $q > 0$ so that $\rho(y) > d(y)^s$ for all $y \in B_d(x,q)$. Let $r < (q/2)^{1+s}$ and choose $y \in B_\rho(x,r)$ such that $d(x,y) > \mathrm{diam}_d(B_\rho(x,r))/3$. Also, let γ be a curve joining x to y such that $\ell_\rho(\gamma) < r$. Then $\gamma \subset B_d(x,q)$ as otherwise there is a curve $\gamma' \subset \gamma \cap \overline{B}_d(x,q)$ connecting x to $\partial B_d(x,q)$, and then

$$\ell_\rho(\gamma) \geq \ell_\rho(\gamma') = \int_{\gamma'} \rho(z)\,|dz| \geq \int_{\gamma'} d(\gamma(t))^s\,dt \geq \ell(\gamma')^{1+s} \geq q^{1+s},$$

which is impossible. Now $\rho(z) > d(z)^s$ for all $z \in \gamma$ and combining this with the fact $\ell(\gamma) \geq d(x,y)$, we obtain

$$r > \ell_\rho(\gamma) = \int_\gamma \rho(z)\,|dz| > \left(\frac{d(x,y)}{2}\right)^{1+s}.$$

This yields $\mathrm{diam}_d(B_\rho(x,r)) < 3d(x,y) < 6r^{1/(1+s)}$. As this holds for all $0 < r < (q/2)^{1/(1+s)}$, we get

$$\liminf_{r\downarrow 0} \frac{\log \mathrm{diam}_d(B_\rho(x,r))}{\log r} \geq \frac{1}{1+s} \tag{4.1}$$

for all $x \in A$.

Assume now that $s > \beta \geq 0$ and that Assumption 2.2 holds. Let $x \in A$, $\varepsilon > 0$, $y \in \overline{\Omega}_\rho$. If $d_\rho(x,y)$ is small, then Assumption 2.2 gives a curve γ joining x to y with $h(\gamma) \geq d(x,y)^{1+\varepsilon}$ and $\ell_\rho(\gamma) \leq d_\rho(x,y)^{1-\varepsilon}$. Thus, for $r > 0$ small enough, and all $y \in B_\rho(x,r)$, we have

$$d_\rho(x,y)^{1-\varepsilon} \geq \ell_\rho(\gamma) = \int_\gamma \rho(z)\,|dz| \geq h(\gamma)(h(\gamma)/2)^s \geq 2^{-s} d(x,y)^{(1+s)(1+\varepsilon)}$$

for some curve joining x and y. This shows that under Assumption 2.2, (4.1) holds true also if $\beta \geq 0$. The claims (1) and (2) now follow using Lemma 3.2 (2) and letting $s \downarrow \beta$.

In order to prove the claims (3) and (4), in view of Lemma 3.2 (1), it suffices to show that

$$\liminf_{r\downarrow 0} \frac{\log \mathrm{diam}_\rho(B_d(x,r))}{\log r} \geq 1+\beta \tag{4.2}$$

for all $x \in B$. Let $x \in B$ and $s < \beta$. Then there is $q > 0$ so that $\rho(y) < d(y)^s$ for all $y \in B_d(x, q)$. Let $r < q/(2c + 1)$, where $c = c(\alpha) < \infty$ is the constant of Lemma 3.3 and where $\alpha > 0$ is such that Ω is an α-John domain. By Lemma 3.3, we find $x_1, \ldots, x_N \in B_d(x, (c + 1)r)$, $N = N(n, \alpha)$, such that $B_d(x, r) \subset \bigcup_{i=1}^{N} B_{\alpha/2}(x_i, cr)$.

Let $x_i \in \{x_1, \ldots, x_N\}$ and $y \in B_{\alpha/2}(x_i, cr)$. Then there is an $(\alpha/2)$-cigar γ joining x_i to y with $\ell(\gamma) < cr$. Assume that $s \le 0$. Since $r < q/(2c + 1)$, we have $\rho(\gamma(t)) < d(\gamma(t))^s \le \alpha^s \min\{t, \ell(\gamma) - t\}^s$ for all $0 < t < \ell(\gamma)$ and thus

$$d_\rho(x_i, y) \le \int_\gamma \rho(z)\,|dz| \le 2\alpha^s \int_{t=0}^{\ell(\gamma)/2} t^s\,dt = c_1 \ell(\gamma)^{1+s} < c^{1+s} c_1 r^{1+s}$$

giving $\operatorname{diam}_\rho(B_{\alpha/2}(x_i, r)) \le 2 \cdot c^{1+s} c_1 r^{1+s} = c_2 r^{1+s}$, where $c_2 < \infty$ depends only on α, n, and s. As $B_d(x, r)$ is connected, we arrive at

$$\operatorname{diam}_\rho(B_d(x, r)) \le \sum_{i=1}^{N} \operatorname{diam}_\rho(B_{\alpha/2}(x_i, cr)) \le N c_2 r^{1+s}. \tag{4.3}$$

If $s \ge 0$, we arrive at the same estimate by using the trivial estimate $\rho(z) \le \ell(\gamma)^s$ for all $z \in \gamma$. Since (4.3) holds for all sufficiently small $r > 0$ and $s < \beta$ is arbitrary, we get (4.2).

Next we will use the Lemma 4.1 to obtain multifractal type formulas for estimating the dimension of $\partial_\rho \Omega$. To recall the definitions of $d^\pm(\beta)$ and $D^\pm(\beta)$, see (2.3)–(2.6).

THEOREM 4.2. *Let $\Omega \subset \mathsf{R}^n$ be a John domain, and ρ a density on Ω so that Assumption 2.2 is satisfied. Then*

$$\dim_\rho(\partial_\rho \Omega) \ge \sup_{\beta > -1} \frac{d^+(\beta)}{1 + \beta}, \tag{4.4}$$

$$\operatorname{Dim}_\rho(\partial_\rho \Omega) \ge \sup_{\beta > -1} \frac{D^+(\beta)}{1 + \beta}, \tag{4.5}$$

$$\dim_\rho\left(\partial_\rho \Omega \cap \{x : i^-(x) > -1\}\right) \le \sup_{\beta > -1} \frac{d^-(\beta)}{1 + \beta}, \tag{4.6}$$

$$\operatorname{Dim}_\rho\left(\partial_\rho \Omega \cap \{x : i^-(x) > -1\}\right) \le \sup_{\beta > -1} \frac{D^-(\beta)}{1 + \beta}. \tag{4.7}$$

PROOF. Let us prove (4.4) and (4.6). The other estimates are obtained similarly with the help of the corresponding statements of Lemma 4.1. Let

$$s < \sup_{\beta > -1} \frac{d^+(\beta)}{1+\beta}$$

and pick $\beta > -1$ such that $\dim_d\{x \in \partial_\rho \Omega : i^+(x) \le \beta\} > s(1+\beta)$. Combining this with Lemma 4.1 (1) gives

$$\begin{aligned} \dim_\rho\left(\partial_\rho \Omega\right) &\ge \dim_\rho\{x \in \partial_\rho \Omega : i^+(x) \le \beta\} \\ &\ge \frac{\dim_d\{x \in \partial_\rho \Omega : i^+(x) \le \beta\}}{1+\beta} > s \end{aligned}$$

proving (4.4).

To prove (4.6), we observe that given an interval $[a,b] \subset (-1,\infty)$, Lemma 4.1 (3) gives

$$\begin{aligned} \dim_\rho\{x \in \partial_\rho \Omega : i^-(x) \in [a,b]\} &\le \dim_d\{x \in \partial \Omega_d : i^-(x) \in [a,b]\}/(1+a) \\ &\le \frac{1+b}{1+a} \sup_{\beta > -1} \frac{d^-(\beta)}{1+\beta}. \end{aligned}$$

For any $\varepsilon > 0$ we may cover the interval $(-1,\infty)$ with intervals $[a_i,b_i]_{i \in \mathsf{N}}$ so that $1+b_i < (1+\varepsilon)(1+a_i)$ for all i. Consequently,

$$\begin{aligned} \dim_\rho(\partial_\rho \Omega \cap \{x : i^-(x) > -1\}) &\le \sup_{i \in \mathsf{N}} \dim_\rho(\partial_\rho \Omega \cap \{x : i^-(x) \in [a_i,b_i]\}) \\ &< (1+\varepsilon) \sup_{\beta > -1} \frac{d^-(\beta)}{1+\beta}. \end{aligned}$$

Now (4.6) follows as $\varepsilon \downarrow 0$.

REMARKS 4.3. a) Suppose that Ω is a John domain, ρ satisfies Assumption 2.2 and $i^-(x) > -1$ for all $x \in \partial_\rho \Omega$. Then Theorem 4.2 gives a formula for calculating $\dim_\rho(\partial_\rho \Omega)$ provided that $\sup_{\beta > -1} d^+(\beta)/(1+\beta)$ and $\sup_{\beta > -1} d^-(\beta)/(1+\beta)$ coincide. In particular, this is the case if $-1 < i^-(x) = i^+(x)$ for all $x \in \partial_\rho \Omega$. A similar statement is, of course, true for the packing dimension. See also the examples below.

b) In general it is not possible to control $\dim_\rho\{x \in \partial_\rho \Omega : i^-(x) \le -1\}$ in terms of $\dim_d\{x \in \partial_\rho \Omega : i^-(x) \le -1\}$. Let $\Omega = \mathsf{B}^n$ and choose a continuous $f\colon (0,\infty) \to (0,\infty)$ such that $\int_{t=0}^1 f(t)\,dt < \infty$ and $\log f(t)/\log t \to -1$ as $t \downarrow 0$. Then it is possible to construct a Cantor set $C \subset S^{n-1}$ such that $\dim_d(C) = 0$ and $\dim_\rho(C) = \infty$ for $\rho(x) = f(\operatorname{dist}(x,C))$. See also [2, Proposition 7.1], where a similar type of example is considered.

In the following example, all four of the inequalities (4.4)–(4.7) hold with equalities.

EXAMPLE 4.4. Let $\Omega = B_e(0,1) \subset \mathsf{R}^n$ and let $C \subset S^{n-1}$ be a Cantor set with $\dim_d C = s$ and $\mathrm{Dim}_d C = t$. Let $\beta > -1$ and $\rho(x) = d(x,C)^\beta$. Then $\dim_\rho(C) = s/(1+\beta)$, $\mathrm{Dim}_\rho(C) = t/(1+\beta)$, and $\dim_\rho(\partial\Omega_d \setminus C) = \mathrm{Dim}_\rho(\partial\Omega_d \setminus C) = n-1$. Thus $\dim_\rho(\partial_\rho\Omega) = \max\{n-1, s/(1+\beta)\}$ and $\mathrm{Dim}_\rho(\partial_\rho\Omega) = \max\{n-1, t/(1+\beta)\}$.

Below, we construct an example to show that all inequalities in Theorem 4.2 can be strict.

EXAMPLE 4.5. There exist domains Ω and densities ρ such that all four of the inequalities (4.4)–(4.7) are strict.

Let $\Omega = \{(x,y) \in \mathsf{R}^2 : y > 0\}$ be the upper half-plane and fix $-1 < q < s < p < 0$. Define $A_k = \{(n2^{-2k}, 2^{-2k}) : n \in \mathsf{Z}\}$, $B_k = \{(n2^{-2k+1}, 2^{-2k+1}) : n \in \mathsf{Z}\}$, and $r_k = 2^{-100k^2}$ for all $k \in \mathsf{N}$. Then choose a continuous density $\rho\colon \Omega \to (0,\infty)$ so that $\rho(z) = 2^{-2kq}$ if $z \in A_k$, $\rho(z) = 2^{-(2k+1)p}$ if $z \in B_k$ and $\rho(z) = d(z)^s$ if $z \in \Omega \setminus \big(\cup_{k\in\mathsf{N}} \cup_{x\in A_k\cup B_k} B_d(x,r_k)\big)$. Then $i^+(x) \geq p$ and $i^-(x) \leq q$ for all $x \in \partial\Omega_d$. Thus

$$\sup_{\beta>-1} d^+(\beta)/(1+\beta) = \sup_{\beta>-1} D^+(\beta)/(1+\beta) \leq 1/(1+p),$$
$$\sup_{\beta>-1} d^-(\beta)/(1+\beta) = \sup_{\beta>-1} D^-(\beta)/(1+\beta) \geq 1/(1+q).$$

On the other hand, it is easy to see that $\dim_\rho(\partial_\rho\Omega) = \mathrm{Dim}_\rho(\partial_\rho\Omega) = 1/(1+s)$.

Our next example shows that the claims (1) and (2) of Lemma 4.1 do not necessarily hold without Assumption 2.2.

EXAMPLE 4.6. Let $0 < \alpha_n < 1$ be a sequence satisfying $\sum_{n=1}^\infty \alpha_n < \infty$. We construct a Cantor set $C \subset [0,1]$ with the following procedure: Let $I_\varnothing = [0,1]$, $\ell_0 = 1$, $I_0 = [0, (1-\alpha_1)/2]$, $I_1 = [(1+\alpha_1)/2, 1]$ and $\ell_1 = (1-\alpha_1)/2$. Suppose $n \in \mathsf{N}$, $\mathtt{i} \in \{0,1\}^n$, and that $I_\mathtt{i}$ with $\mathrm{diam}(I_\mathtt{i}) = \ell_n$ has been defined. We then define inductively $I_{\mathtt{i}0}$ and $I_{\mathtt{i}1}$ to be the subintervals of $I_\mathtt{i}$ with length $\ell_{n+1} = \ell_n(1-\alpha_n)/2$ such that $I_{\mathtt{i}0}$ has the same left endpoint as $I_\mathtt{i}$ and $I_{\mathtt{i}1}$ has the same right endpoint as $I_\mathtt{i}$. We also denote by $J_\mathtt{i}$ the interval between $I_{\mathtt{i}0}$ and $I_{\mathtt{i}1}$. The (α_n)-Cantor set $C = C(\alpha_n)$ is then defined as

$$C = \bigcap_{n\in\mathsf{N}} \bigcup_{\mathtt{i}\in\{0,1\}^n} I_\mathtt{i}.$$

For each $n \in \mathsf{N}$, we may choose $0 < h_n < \ell_n$ such that

$$2^n \ell_n^{1/n} h_n^{1/n} \leq 1. \tag{4.8}$$

We also require that $h_{n+1} \leq h_n$.

Next we define a density ρ on the upper half-plane H. For each n, and $\mathtt{i} \in \{0,1\}^n$, let $T_{\mathtt{i}}$ and $U_{\mathtt{i}}$ be the isosceles triangles with base $J_{\mathtt{i}}$ and heights h_n and $h_n/2$ respectively. For $\mathtt{i} = \varnothing$, we define $T_\varnothing = \{(x,y) \in H : x < 0 \text{ and } y < -2x\} \cup \{(x, y \in H : x > 1 \text{ and } y < 2x-2\}$ and $U_\varnothing = \{(x,y) \in H : (x, 2y) \in T\}$. We define

$$\rho(z) = \begin{cases} d(z)^{-1}, & \text{if } z \in \cup_{\mathtt{i}} U_{\mathtt{i}} \\ d(z), & \text{if } z \notin \cup_{\mathtt{i}} T_{\mathtt{i}}, \end{cases}$$

where the union is over all $\mathtt{i} \in \{\varnothing\} \cup_{n\in\mathbb{N}} \{0,1\}^n$ Moreover, we extend ρ continuously into the strips $T_{\mathtt{i}} \setminus U_{\mathtt{i}}$ such that it is monotone in the y-coordinate.

It is now easy to see that $\partial_\rho H = C$ and that $i^+ = 1$ on $\partial_\rho H$. Since $\sum_n \alpha_n < \infty$, it follows that $\mathscr{L}(C) > 0$ and thus in particular $\dim_d(C) = \mathrm{Dim}_d(C) = 1$. If $n \in \mathbb{N}$ and $\mathtt{i} \in \{0,1\}^n$, we can connect any two points of $C \cap I_{\mathtt{i}}$ by two vertical segments of length h_n and the horisontal segment between their tops such that apart from endpoints, these segments lie completely outside $\cup_{\mathtt{i}} T_{\mathtt{i}}$. This implies $\mathrm{diam}_\rho(C \cap I_{\mathtt{i}}) \le h_n^2 + \ell_n h_n \le 2\ell_n h_n$ and thus for each n, there is a covering of $\partial_\rho H$ by 2^n sets of ρ-diameter $2\ell_n h_n$. Combining with (4.8) and letting $n \to \infty$ yields $\dim_\rho(\partial_\rho H) = \mathrm{Dim}_\rho(\partial_\rho H) = 0$. This shows that the claims (1) and (2) of Lemma 4.1 are not valid.

The final example of this section shows that neither the estimates (3)–(4) of Lemma 4.1 nor (4.6)–(4.7) of Theorem 4.2 need hold if Ω is not a John domain.

EXAMPLE 4.7. We construct a snowflake type domain $\Omega \subset \mathbb{R}^2$ that does not satisfy (3) nor (4) of Lemma 4.1.

To begin with, we fix $0 < s < 1/2$ and let $0 < \alpha_1 < 1/2$. We start with an equilateral triangle with sides of length $l_0 = 1$ and replace the middle α_1-th portion of each of the sides by two segments of length $l_1 = (1-\alpha_1)/2$. We continue inductively. At the step k, we have $3 \cdot 4^k$ segments of length l_k and we replace the middle α_k-th portion of each of these segments by two line segments of length $l_{k+1} = l_k(1-\alpha_{k+1})/2$, see Figure 1. The numbers α_k are defined so that

$$\alpha_{k+1} = l_k^{1-2s}(1-\alpha_{k+1})/2. \tag{4.9}$$

Observe that $\alpha_k \downarrow 0$ as $k \to \infty$. We denote by Ω_k the domain bounded by the line segments at step k and define $\Omega = \cup_{k\in\mathbb{N}} \Omega_k$. We denote by $\Sigma \subset \partial\Omega_d$ the part of the boundary that joins two vertexes of the original equilateral triangle and does not contain the third vertex. For notational convenience, we consider only points of Σ. This does not affect the generality as $\partial\Omega_d \setminus \Sigma$ consist of two

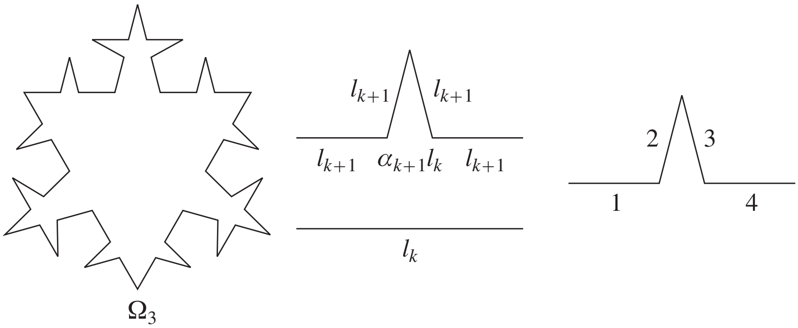

FIGURE 1. Three figures concerning Example 4.7: domain Ω_3 (left), construction at level $k+1$ (middle) and enumeration of the segments (right).

translates of Σ. For $x \in \Sigma$, we let $a(x) \in \{1, 2, 3, 4\}^{\mathsf{N}}$ denote its coding or "address" arising from the enumeration of the segments in each level as in the Figure 1. Note that this address is unique outside a countable set of points.

Next we define $\rho(z) = d(z)^{-1/2}$ for all $z \in \Omega$ and consider the set $A = \{x \in \Sigma : a(x) \in \{2, 3\}^{\mathsf{N}}\}$. It is easy to see that there are numbers $0 < D_1 < D_2 < \infty$, so that $\dim_d(A) = D_1 = \mathrm{Dim}_d(A)$ and $\dim_d(\partial\Omega_d) = D_2 = \mathrm{Dim}_d(\partial\Omega_d)$ (actually $D_1 = 1$ and $D_2 = 2$ but this is not essential). If we show that

$$\dim_\rho(A) = \mathrm{Dim}_\rho(A) = D_1/s, \tag{4.10}$$

then it follows that the claims (3) and (4) of Lemma 4.1 do not hold. Observe that $i^-(x) = i^+(x) = -1/2$ for all $x \in \partial\Omega_d$.

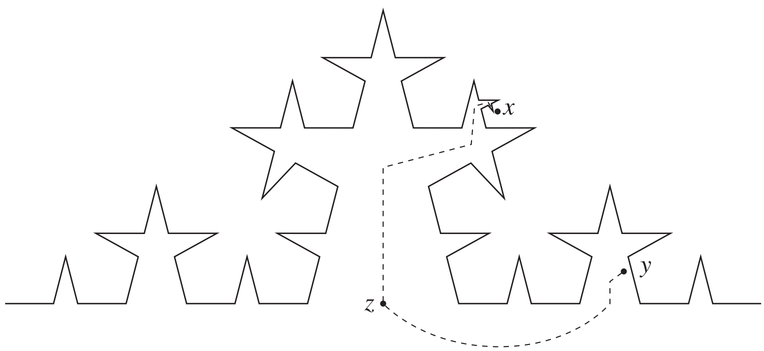

FIGURE 2. Selection of the "base point" z in Example 4.7.

Let $x \in A$ and $y \in \overline{\Omega}_d$ and choose the smallest $k \in \mathsf{N}$ so that $l_k < 2d(x, y)$. Let z be as in Figure 2, *i.e.* z is the "base point" of a cone of Ω_k with "side-length"

l_k which is closest to x. Then

$$d_\rho(x, z) \leq c_0 \sum_{n=k}^{\infty} \alpha_n^{-1/2} l_n^{1/2} = c_0 \sum_{n=k}^{\infty} l_{n-1}^s \leq c_1 l_k^s \leq 2^s c_1 d(x, y)^s.$$

Here the first equality follows from (4.9) and the former estimate holds because $l_k/4 < l_{k+1} < l_k/2$ for all $k \in \mathsf{N}$. By a similar argument, it follows that $d_\rho(z, y) \leq c_2 l_k^s \leq c_3 d(x, y)^s$. Thus $d_\rho(x, y) \leq c d(x, y)^s$. On the other hand, it is clear that $d_\rho(x, y) \geq c_4 l_k^s \geq c' d(x, y)^s$, since $a(x) \in \{2, 3\}^{\mathsf{N}}$. Thus, we have $c' d_\rho(x, y) \leq d(x, y)^s \leq c d_\rho(x, y)$, in other words $B_\rho(x, c'r) \subset B_d(x, r^s) \subset B_\rho(x, cr)$, for all $x \in A$ and $y \in \overline{\Omega}_d$ where the constants $0 < c', c < \infty$ are independent of the points x and y. The claim (4.10) now follows from Lemma 3.2.

REMARK 4.8. Suppose that $A \subset \{x \in \partial_\rho \Omega : i^-(x) \geq \beta\}$ has the following accessibility property for some $1 \leq \lambda < -1/\beta$: For each $x \in \partial \Omega_d$ there are $0 < r, c < \infty$ such that for all $y \in B_d(x, r) \cap A$ there exists a curve γ joining x and y so that $d(\gamma(t), \partial \Omega_d) \geq c \min\{t^\lambda, (\ell(\gamma) - t)^\lambda\}$ for all $0 < t < \ell(\gamma)$. Then the proof of Lemma 4.1 with trivial modifications implies $(1 + \lambda\beta) \dim_\rho(A) \leq \dim_d(A)$ and $(1 + \lambda\beta) \operatorname{Dim}_\rho(A) \leq \operatorname{Dim}_d(A)$. On the other hand, if for each $x \in B \subset \{x \in \partial_\rho \Omega : i^+(x) \leq \beta\}$ there are $0 < r, c < \infty$ so that for all curves γ with $\gamma(0) = x$ we have $d(\gamma(t), \partial \Omega_d) < ct^\lambda$ for $0 < t < r$, then we get $(1 + \lambda\beta) \dim_\rho(B) \geq \dim_d(B)$, $(1 + \lambda\beta) \operatorname{Dim}_\rho(B) \geq \operatorname{Dim}_d(B)$. The previous example shows that these estimates are sharp.

5. Conformal densities

The results in the last section, are based on estimates of the quantities $i^+(x)$ and $i^-(x)$ which are defined as internal limits when $\Omega \ni y \to x \in \partial \Omega_d$. This causes a lack of the generality; it is quite possible that $i^+(x) = 0$ and $i^-(x) = -1$ for all $x \in \partial \Omega_d$. (For instance, choose $\beta = -1$, $\lambda = 0$ in the forthcoming Example 6.3.) However, if we have additional information on the geometry of $(\overline{\Omega}_\rho, d_\rho)$, then it might be enough to consider the ratios $\log \rho(y)/\log d(y)$ along some fixed curves or cones. The purpose of this section is to show that this is the case for so called conformal densities which arise naturally in connection with conformal and quasiconformal mappings and their generalisations, see [2].

A density ρ on B^n is called a conformal density if there are constants $1 \leq c_0, c_1 \leq \infty$ such that for each $x \in \mathsf{B}^n$ and for all $y \in B_e(x, d(x)/2)$ we have

$$c_0^{-1} \leq \rho(y)/\rho(x) \leq c_0, \tag{5.1}$$

and moreover,

$$\mu_\rho(B_\rho(x,r)) \le c_1 r^n \tag{5.2}$$

for all $r > 0$. Here μ_ρ is the measure given by $\mu_\rho(E) = \int_E \rho^n \, d\mathscr{L}^n$ for $E \subset \mathsf{B}^n$. In the literature, (5.1) is often called the Harnack inequality, and one refers to (5.2) as a volume growth condition. An important corollary of the conditions (5.1)–(5.2) is the following Gehring-Hayman type estimate: There exists $1 \le c < \infty$ such that

$$\begin{aligned} c^{-1} d_\rho(x,y) \le \int_{t=0}^{d(x,y)} & \rho\,((1-t)x)\; dt \\ & + \int_{t=0}^{d(x,y)} \rho\,((1-t)y)\; dt \le c d_\rho(x,y) \end{aligned} \tag{5.3}$$

for all $x, y \in \partial_\rho \mathsf{B}^n$. See [2, Theorem 3.1] and also [6].

Motivated by this estimate, we consider variants k^- and k^+ of the quantities i^- and i^+ for a density ρ on B^n at $x \in S^{n-1}$. Recall that $k^-(x) = \liminf_{r\uparrow 1} \log \rho(rx)/\log(1-r)$, and $k^+(x) = \limsup_{r\uparrow 1} \log \rho(rx)/\log(1-r)$. Occasionally we also use k^- and k^+ when $\Omega = \mathsf{H}$ is an open half space and then the limits are considered along straight lines orthogonal to the boundary of H. The reduction to $k^\pm$ is possible since (5.3) is a much stronger condition than Assumption 2.2 that was used earlier for the same purpose.

In the following result we only assume that (5.1) and (5.3) hold. Thus, the result applies to a slightly larger collection of densities than the conformal densities. See [12], and also Example 6.3 to follow.

THEOREM 5.1. *Suppose that ρ is a density on B^n that satisfies the conditions* (5.1) *and* (5.3). *Let $\beta > -1$,*

$$\begin{aligned} A &\subset \{x \in \partial_\rho \mathsf{B}^n : k^+(x) \le \beta\}, \\ B &\subset \{x \in \partial_\rho \mathsf{B}^n : k^-(x) \ge \beta\}, \\ C &\subset \{x \in \partial_\rho \mathsf{B}^n : k^-(x) \le \beta\}. \end{aligned}$$

Then

(1) $(1+\beta)\dim_\rho(A) \ge \dim_d(A)$,

(2) $(1+\beta)\dim_\rho(B) \le \dim_d(B)$,

(3) $(1+\beta)\operatorname{Dim}_\rho(A) \ge \operatorname{Dim}_d(A)$,

(4) $(1+\beta)\operatorname{Dim}_\rho(B) \le \operatorname{Dim}_d(B)$,

(5) $(1+\beta)\operatorname{Dim}_\rho(C) \ge \dim_d(C)$.

PROOF. The claims (1)–(4) have proofs very similar to the proofs of the corresponding statements of Lemma 4.1. We first apply (5.3) to conclude that for each $x \in A$ and $y \in \mathsf{B}^n \setminus B_d(x, r)$, we have

$$d_\rho(x, y) \geq c^{-1} \int_{t=0}^{r} \rho((1-t)x)\, dt \geq c^{-1} \int_{t=0}^{r} t^s \geq c_0 r^{1+s}$$

if $s > \beta$ and $r > 0$ is small. This implies $\operatorname{diam}_d(B_\rho(x, r)) \leq c_1 r^{1/(1+s)}$ and the claims (1) and (3) now follow by Lemma 3.2 (2).

To prove (2) and (4), let $s < \beta$ and for $n \in \mathsf{N}$, denote

$$B_n = \{x \in B : \rho((1-t)x) < t^s \text{ for all } 0 < t < 1/n\}.$$

Using (5.3), we find $r_0 > 0$ so that

$$d_\rho(x, y) \leq c_2 \int_{t=0}^{d(x,y)} t^s \leq c_3 d(x, y)^{1+s}$$

whenever $x, y \in B_n$ and $d(x, y) < r_0$. In other words, $\operatorname{diam}_\rho(B_d(x, r) \cap B_n) \leq c_4 r^{1+s}$ when $0 < r < r_0^{1/(1+s)}$. Now the Lemma 3.2 (1) implies $(1+s)\dim_\rho(B_n) \leq \dim_d(B_n)$ and $(1+s)\operatorname{Dim}_\rho(B_n) \leq \operatorname{Dim}_d(B_n)$. Note that it is enough to assume $\liminf_{r\downarrow 0}(\log\big(\operatorname{diam}_\rho(B_d(x, r) \cap A)\big))/(\log r) \geq \delta$ in Lemma 3.2 (1) (since $\dim_{\partial_\rho \mathsf{B}^n}(A) = \dim_{(A,d_\rho)}(A)$ and $\operatorname{Dim}_{\partial_\rho \mathsf{B}^n}(A) = \operatorname{Dim}_{(A,d_\rho)}(A)$). The claims (2) and (4) now follow since $B = \cup_{n\in\mathsf{N}}$ and $s < \beta$ is arbitrary.

It remains to prove (5). Let $x \in C$ and $s > \beta$. Then there is a sequence $0 < r_i \downarrow 0$ such that $\rho((1-r_i)x) > r_i^s$ for all i. Combined with (5.1), this gives

$$\int_{t=0}^{r_i} \varrho((1-t)x)\, dt \geq c_5 r_i^{1+s}$$

and using also (5.3), $\operatorname{diam}_d(B_\rho(x, c_6 r_i^{1+s})) \leq r_i$. Thus

$$\limsup_{r\downarrow 0} \frac{\log \operatorname{diam}_d(B_\rho(x, r))}{\log r} \geq \frac{1}{1+s}$$

and (5) follows from Lemma 3.2 (4).

REMARKS 5.2. a) Using the claims (1)–(4) of Theorem 5.1 one may derive multifractal type formulas completely analogous to (4.4)–(4.7). Using (5), we have moreover, that $\operatorname{Dim}_\rho(\partial_\rho B_d(0, 1)) \geq \sup_{\beta > -1} \frac{e^-(\beta)}{1+\beta}$ where

$$e^-(\beta) = \dim_d(\{x \in \partial_\rho B_d(0, 1) : k^-(x) \leq \beta\}). \tag{5.4}$$

Example 6.2 shows that this is sharp in the sense that one can not replace $\dim_d$ by Dim_d in defining $e^-(\beta)$ even if ρ is a conformal density.

b) We formulated the above result for densities defined on B^n. The same proof goes through for any John domain $\Omega \subset \mathsf{R}^n$ if the condition (5.3) is replaced by

$$c^{-1} d_\rho(x, y) \le \int_{t=0}^{d(x,y)} \rho(\gamma_x(t))\, dt + \int_{t=0}^{d(x,y)} \rho(\gamma_y(t))\, dt \le c d_\rho(x, y),$$

where γ_x is a fixed α-cone with $\gamma_x(0) = x$ for each $x \in \partial_\rho \Omega$. Actually, we could even weaken this in the spirit of Assumption 2.2 and assume only that for all $\varepsilon > 0$, we have

$$d_\rho(x, y)^{1+\varepsilon} \le \int_{t=0}^{d(x,y)} \rho(\gamma_x(t))\, dt + \int_{t=0}^{d(x,y)} \rho(\gamma_y(t))\, dt \le d_\rho(x, y)^{1-\varepsilon}$$

when $d(x, y)$ is small enough.

c) Makarov [9, Theorems 0.5, 0.6] proved results essentially similar to Theorem 5.1 (1)–(2) in case $\beta > 0$ and $\rho = |f'|$ for f conformal. He also showed [9, Theorem 0.8] that k^- cannot be replaced by k^+ in (2).

d) In [2], Bonk, Koskela, and Rohde proved the following deep fact. If ρ is a conformal density on B^n, then:

(5.5) There is $E \subset S^{n-1}$ with $\dim_d E = 0$ such that $\dim_\rho(\partial_\rho \mathsf{B}^n \setminus E) \le n$.

See [2, Theorem 7.2]. As a central tool, they used an estimate analogous to Theorem 5.1 (2). In fact, combining Theorem 5.1 (2) and [2, Theorem 5.2] gives a simpler proof for (5.5) than the one given in [2]. However, their result is quantitatively stronger than (5.5).

e) A generic situation in which Theorem 5.1 is stronger than Theorem 4.2 will be discussed in Example 6.3.

6. Further examples, remarks, and questions

We first give the example mentioned in Remark 5.2 e) showing that one can not replace $\dim_d$ by Dim_d in defining $e^-(\beta)$. We will make use of the following lemma. We formulate it in a more general setting, for future reference.

LEMMA 6.1. *Let $\Omega \subset \mathsf{R}^n$ be a (2α)-John domain and $C \subset \partial\Omega_d$. Suppose that $\widetilde{\rho}\colon (0, \infty) \to (0, \infty)$ is nonincreasing and and satisfies $\int_0^1 \widetilde{\rho}(t)\, dt < \infty$. Define $\rho(x) = \widetilde{\rho}(d(x, C))$ for $x \in \Omega$. Then for all $x \in C$ and $0 < r < \mathrm{diam}_d(\Omega)/2$, it holds*

$$\mathrm{diam}_d \left(B_\rho \left(x, \frac{1}{2} \int_{t=0}^{r} \widetilde{\rho}(t)\, dt \right) \right) \le 2r, \tag{6.1}$$

$$\operatorname{diam}_\rho \left(B_d(x,r)\right) \le c_1 \int_{t=0}^{c_2 r} \widetilde{\rho}(t)\,dt \tag{6.2}$$

for some constants $0 < c_1, c_2 < \infty$ *that depend only on* α *and* n.

PROOF. Let $x \in C$ and $y \in \overline{\Omega}_d$. Denote $d = d(x, y)$ and suppose that γ is a curve joining x and y. To prove (6.1), it suffices to show that

$$\ell_\rho(\gamma) \ge \frac{1}{2}\int_{t=0}^{d} \widetilde{\rho}(t)\,dt. \tag{6.3}$$

Let $h = h(\gamma) = \max_{0 \le t \le L} d(\gamma(t))$ where $L = \ell(\gamma)$. Then $\ell_\rho(\gamma) \ge \frac{1}{2}\int_{t=0}^{h} \widetilde{\rho}(t)\,dt + \frac{1}{2}\,d\rho(h)$. If $h \ge d$ the estimate (6.3) clearly holds. If $h < d$, then $d\widetilde{\rho}(h) \ge \int_{t=h}^{d} \widetilde{\rho}(t)\,dt$ since $\widetilde{\rho}$ is nonincreasing and consequently

$$\ell_\rho(\gamma) \ge \frac{1}{2}\left(\int_{r=0}^{h} \widetilde{\rho}(r)\,dr + d\widetilde{\rho}(h)\right) \ge \frac{1}{2}\int_{t=0}^{d} \widetilde{\rho}(t)\,dt.$$

This settles the proof of (6.1).

To prove (6.2), let $x \in C$ and $r > 0$. We use Lemma 3.3 to cover $B_d(x, r)$ with sets $B_\alpha(x_i, cr)$, $i = 1, \dots, N = N(n, \alpha)$. Let $y \in B_\alpha(x_i, cr)$ and pick an α-cigar γ with $\ell(\gamma) \le cr$ joining y to x_i. Now

$$d_\rho(y, x_i) \le \int_\gamma \widetilde{\rho}(d(z, C))\,|dz| \le 2\int_{t=0}^{cr/2} \widetilde{\rho}(\alpha t)\,dt = \frac{2}{\alpha}\int_{t=0}^{\alpha cr/2} \widetilde{\rho}(t)\,dt. \tag{6.4}$$

As $B_d(x, r)$ is (path-)connected and is covered by N sets of the type $B_\alpha(x_i, cr_i)$, we arrive at $\operatorname{diam}_\rho(B_d(x, r)) \le (4N/\alpha)\int_{t=0}^{\alpha cr/2} \widetilde{\rho}(t)\,dt$ proving the claim.

EXAMPLE 6.2. We show that $\dim_d$ cannot be replaced by Dim_d in (5.4) even if ρ is a conformal density.

We first fix numbers $0 < a < b < 1/2$, $-1 < \lambda < \eta < 0$, and ξ such that

$$a^{1+\lambda} = b^{1+\eta} = \xi, \tag{6.5}$$

and

$$-\log 2 < \log \xi < -\tfrac{1}{2}\log 2. \tag{6.6}$$

Let us also pick natural numbers $n_1 < N_1 < n_2 < N_2 < n_3 < N_3 < \dots$. We let $C \subset S^1$ denote a Cantor set constructed as follows (See the construction in Example 4.6). We start with an arc of length 1 and remove an arc of length $1 - 2a$ from the middle. Next, we remove arcs of length $a(1 - 2a)$ from the middle of the two remaining arcs. We iterate this construction for n_1 steps.

After these n_1 steps, we have 2^{n_1} arcs of length a^{n_1}. At the step $n_1 + 1$, we remove arcs of relative length $1-2b$ from the middle of each of these arcs. We continue the construction with the parameter b for $N_1 - n_1$ steps. Then we use again the parameter a for $n_2 - N_1$ steps and so on. We denote by $E_{k,1} \dots, E_{k,2^k}$ the arcs remaining after k steps and denote by ℓ_k the length of these arcs. What remains at the end is the Cantor set $C = \bigcap_{k\in\mathsf{N}} \bigcup_{i=1}^{2^k} E_{k,i}$.

Let $r_0 = R_1 = 1$, $r_1 = \ell_{n_1} = a^{n_1}$, $R_2 = \ell_{N_1} = a^{n_1} b^{N_1 - n_1}$, $r_2 = \ell_{n_2} = a^{n_1+n_2-N_1} b^{N_1-n_1}$ and so on. Thus r_i (resp. R_i) is the length of a construction interval of C of level n_i (resp. N_{i-1}). We define $\rho(x) = \widetilde{\rho}(\operatorname{dist}(x, C))$ for all $x \in \mathsf{B}^2$, where $\widetilde{\rho}$ is the function defined by

$$\widetilde{\rho}(t) = \begin{cases} \left(\dfrac{R_1 R_2 \dots R_k}{r_0 r_1 \dots r_{k-1}}\right)^{\eta-\lambda} t^{\lambda}, & r_k \le t \le R_k, \\ \left(\dfrac{R_1 R_2 \dots R_k}{r_0 r_1 \dots r_k}\right)^{\eta-\lambda} t^{\eta}, & R_{k+1} \le t \le r_k. \end{cases}$$

Now, if $N_i/n_i, n_{i+1}/N_i \to \infty$ fast enough, it is easy to see that $\dim_d C = -\log 2/\log a$ and $\operatorname{Dim}_d C = -\log 2/\log b$, see e.g. [10, p. 77]. Moreover, it then follows that $k^-(x) = \lambda$ if $x \in C$ and $k^-(x) = 0$ otherwise. Next, let $h_k = \int_{t=0}^{\ell_k} \widetilde{\rho}(t)\, dt$. Since

$$\widetilde{\rho}(\ell_k)\ell_k = \xi^k \tag{6.7}$$

for all k (combine (6.5) with the definitions), it follows that

$$\tfrac{1}{2}\xi^k = \tfrac{1}{2}\widetilde{\rho}(\ell_k)\ell_k \le \int_{t=\ell_{k+1}}^{\ell_k} \widetilde{\rho}(t) \le \widetilde{\rho}(\ell_{k+1})\ell_k \le a^{\lambda}\widetilde{\rho}(\ell_k)\ell_k = a^{\lambda}\xi^k.$$

Thus

$$\tfrac{1}{2}\xi^k \le h_k = \sum_{m\ge k} \int_{t=\ell_{m+1}}^{\ell_m} \widetilde{\rho}(t)\, dt \le c_0 \xi^k. \tag{6.8}$$

From Lemma 6.1, it follows that for each $I = I_{k,i}$ we have

$$c_1 h_k \le \operatorname{diam}_{\rho}(I) \le c_2 h_k \tag{6.9}$$

for some constants $0 < c_1 < c_2 < \infty$. Let μ be the natural probability measure on C that satisfies $\mu(I_{k,i}) = 2^{-k}$. Then

$$\lim_{k\to\infty} \frac{\log \mu(I_{k,i})}{\log(\operatorname{diam}_{\rho}(I_{k,i}))} = \frac{-\log 2}{(1+\lambda)\log a},$$

using (6.8) and (6.9). But this implies $\dim_\rho(C) = \mathrm{Dim}_\rho(C) = (-\log 2)/((1+\lambda)\log a)$, see e.g. [4, Proposition 10.1] and [3, Corollary 3.20]. Thus,

$$1 < \mathrm{Dim}_\rho(C) = \mathrm{Dim}_\rho(S^1) = \frac{-\log 2}{(1+\lambda)\log a} < \frac{-\log 2}{(1+\lambda)\log b} = \frac{\mathrm{Dim}_d(C)}{1+\lambda}$$

$$= \sup_{\beta>-1} \frac{\mathrm{Dim}_d(\{x \in \partial_\rho B_d(0,1) : k^-(x) \le \beta\}}{1+\beta},$$

recall (6.6).

It remains to prove that ρ is a conformal density. The condition (5.1) is clearly satisfied so we only have to verify (5.2). We show this for $x \in C$ and $0 < r < 1$ (the general case $x \in \mathsf{B}^2$ follows easily from this). Using (5.1) we may also assume that $r = h_k$ for some $k \in \mathsf{N}$. For each $m \ge k$, we denote

$$A_m = \{y \in B_d(x, c_3\ell_k) : \ell_m \le d(y, C) \le c_3\ell_m\}.$$

Then $B_\rho(x, h_k) \subset \cup_{m\ge k} A_m$, for a suitable constant $1 < c_3 < \infty$, recall (6.9). Moreover, it follows from (5.1) and (6.7) that $c_4\xi^m/\ell_m \le \rho(y) \le c_5\xi^m/\ell_m$ for all $y \in A_m$, where $0 < c_4 < c_5 < \infty$ depend only on a, b, λ, and η. Since $\mathscr{L}^2(A_m) \le c_6 2^{m-k}\ell_m^2$, we arrive at

$$\mu_\rho(A_m) = \int_{A_m} \rho^2 \, d\mathscr{L}^2 \le c_7 2^{m-k}\xi^{2m}.$$

As $2\xi^2 < 1$ by (6.6), this yields

$$\mu_\rho(B_\rho(x, h_k)) \le \sum_{m\ge k} \mu_\rho(A_m) \le c_7 \sum_{m\ge k} 2^{m-k}\xi^{2m} \le c_8\xi^{2k} \le c_9 h_k^2,$$

where the last estimate follows from (6.8).

Below, we construct a "multifractal type" example and calculate the Hausdorff dimension of the boundary using Theorem 5.1.

EXAMPLE 6.3. We construct a domain and a conformal density that satisfies Gehring-Hayman condition (5.3) and compute the Hausdorff dimension of the boundary.

We define a density ρ on the upper half-plane $H \subset \mathsf{R}^2$ (actually we define $\rho(z)$ only for $z \in [0,1] \times (0,3]$ but the definition is easily extended to the whole of H). Let $-1 < \beta, \lambda < 0$, $\beta \ne \lambda$. We consider the triadic decomposition of $[0,1]$; Let $I_\varnothing = [0,1]$, $I_0 = [0,1/3]$, $I_1 = [1/3, 2/3]$, and $I_2 = [2/3, 1]$. If $n \in \mathsf{N}$ and, $\mathbf{i} \in \{0,1,2\}^n$, let $I_{\mathbf{i}0}, I_{\mathbf{i}1}, I_{\mathbf{i}2}$ denote its triadic subintervals enumerated from left to right. For each such triadic interval $I = I_{\mathbf{i}}$, let $Q_{\mathbf{i}} =$

$I \times [|I|, 3|I|]$. Next we define weights $\rho_{\mathbf{i}}$ inductively by the rules $\rho_{\varnothing} = 1$ and $\rho_{\mathbf{i}0} = \rho_{\mathbf{i}2} = 3^{-\lambda}\rho_{\mathbf{i}}$, $\rho_{\mathbf{i}1} = 3^{-\beta}\rho_{\mathbf{i}}$.

Let $\rho\colon [0,1] \times (0,3] \to (0,\infty)$ be a density such that $\rho(x_{\mathbf{i}}) = \rho_{\mathbf{i}}$ if $x_{\mathbf{i}}$ is the centre point of $Q_{\mathbf{i}}$. We also require that the condition (5.1) holds with some $c_0 < \infty$. This is possible because of the symmetric definition of $\rho_{\mathbf{i}}$: If $I_{\mathbf{i}}$ and $I_{\mathbf{j}}$ are neighbouring intervals of the same length, then $3^{-|\beta-\lambda|} \le |\rho_{\mathbf{i}}/\rho_{\mathbf{j}}| \le 3^{|\beta-\lambda|}$.

We will next show that the Gehring-Hayman condition (5.3) holds for the density ρ. Let $x, y \in [0,1]$ with $y - x = r > 0$. Let γ_1, γ_2, and γ_3 be the line segments joining $(x,0)$ to (x,r), (x,r) to (y,r), and (y,r) to $(y,0)$, respectively. Then a direct calculation using the definitions gives

$$\int_{\gamma_1} \rho(z)\,|dz| \le c_1 \int_{t=0}^{r} t^{\min\{\beta,\lambda\}} \frac{\rho(x,r)}{r^{\min\{\beta,\lambda\}}}\,dt \le c_2 r\rho(x,r),$$

$$\int_{\gamma_3} \rho(z)\,|dz| \le c_1 \int_{t=0}^{r} t^{\min\{\beta,\lambda\}} \frac{\rho(y,r)}{r^{\min\{\beta,\lambda\}}}\,dt \le c_2 r\rho(y,r).$$

Combining these estimates with (5.1), we obtain

$$c_3\ell_\rho(\gamma_i) \le \ell_\rho(\gamma_2) \le c_4\ell_\rho(\gamma_i) \tag{6.10}$$

for $i = 1, 3$. The condition (5.3) is satisfied if we can show that $\ell_\rho(\gamma) \ge c\ell_\rho(\gamma_2)$ for any curve joining x and y in H. Denote $h = h(\gamma) = \max_{0<t<\ell(\gamma)} d(\gamma(t))$. If $h \le r$, it follows that $\ell_\rho(\gamma) \ge c\ell_\rho(\gamma_2)$ since ρ is essentially decreasing on each vertical line segment. More precisely using (5.3) and the definitions of the weights $\rho_{\mathbf{i}}$, we get

$$\rho(a, tb) \ge c_5\rho(a,b) \tag{6.11}$$

if $(a,b) \in [0,1] \times (0,3]$ and $0 < t < 1$. Now suppose that $h > r$ and let $z = \gamma(t_0)$ where $t_0 = \min\{t > 0 : d(\gamma(t)) = r\}$. If $d(z, \gamma_2) < r$, it follows easily from (5.1) that $\ell_\rho(\gamma) \ge c\ell_\rho(\gamma_2)$. If $d(z,\gamma_2) \ge r$, let η be the line segment joining z to the closest point of γ_2. Then (6.11) implies $\ell_\rho(\gamma) \ge c_5\ell_\rho(\eta) \ge c\ell_\rho(\gamma_2)$ where the last estimate follows using (5.1). This settles the proof of (5.3).

We will next compute the Hausdorff dimension of the boundary. Let $0 \le t \le 1$ and denote $A_t = \{x \in [0,1] : k^-(x) = k^+(x) = t\beta + (1-t)\lambda\}$. Then

$$A_t = \Big\{ x = \sum_{i\in\mathbb{N}} x_i 3^{-i} : x_i \in \{0,1,2\} \text{ and } \lim_{n\to\infty} \#\{1 \le i \le n : x_i = 1\}/n = t \Big\}.$$

Using this expression, we get

$$\dim_d(A_t) = \mathrm{Dim}_d(A_t) = \frac{-t\log t + (t-1)\log((1-t)/2)}{\log 3}. \tag{6.12}$$

Indeed, if μ_t is the unique Borel probability measure on [0, 1] that satisfies $\mu_t(I_{\mathbf{i}1}) = t\mu_t(I_{\mathbf{i}})$ and $\mu_t(I_{\mathbf{i}0}) = \mu_t(I_{\mathbf{i}2})$ for all triadic intervals $I_{\mathbf{i}}$, then we have

$$\lim_{r\downarrow 0} \frac{\log \mu_t((B_d(x,r))}{\log r} = \frac{-t\log t + (t-1)\log((1-t)/2))}{\log 3}$$

and this implies (6.12). For instance, see [4, Proposition 10.4].

Thus, from Theorem 5.1 and (6.12), we get

$$\dim_\rho(A_t) = \mathrm{Dim}_\rho(A_t) = \frac{-t\log t + (t-1)\log((1-t)/2)}{(1+t\beta+(1-t)\lambda)\log 3}. \tag{6.13}$$

If $f(\beta,\lambda)$ is the maximum of (6.13) over all $0 \le t \le 1$, then we conclude that

$$\mathrm{Dim}_\rho(\partial_\rho H) \ge \dim_\rho(\partial_\rho H) \ge f(\beta,\lambda).$$

To finish this example, we show that for the Hausdorff dimension, there is an equality in the above estimate. We give the proof in the case $\beta < \lambda$, the case $\lambda < \beta$ can be handled with similar arguments. First, we observe using Theorem 5.1 (2) that

$$\dim_\rho(\{k^-(x) \ge \beta/3 + 2\lambda/3\}) \le 1/(1+\beta/3+2\lambda/3) < f(\beta,\lambda),$$

where the strict inequality is obtained via differentiating (6.13) at $t = 1/3$. On the other hand, if $t > 1/3$, and $A_t^- = \{x \in [0,1] : k^-(x) \le t\beta + (1-t)\lambda\}$, then

$$A_t^- = \left\{ x = \sum_{i\in\mathbb{N}} x_i 3^{-i} : \limsup_{n\to\infty} \#\{1 \le i \le n : x_i = 1\}/n \ge t \right\}$$

and thus $\dim_d(A_t^-) \le (-t\log t + (t-1)\log((1-t)/2))/\log 3$. To see this, observe that

$$\liminf_{r\downarrow 0} \frac{\log \mu_t(B_d(x,r))}{\log r} \le \frac{-t\log t + (t-1)\log((1-t)/2))}{\log 3}$$

for all $x \in A_t^-$ and use [4, Proposition 10.1]. Now, using the analogue of (4.6) for k^- implies $\dim_\rho(\partial_\rho H) \le f(\beta,\lambda)$, and consequently $\dim_\rho(\partial_\rho H) = f(\beta,\lambda)$.

REMARKS 6.4. a) One can estimate the numbers $f(\beta,\lambda)$ numerically. For instance, if $\beta=-1/2$ and $\lambda=-1/3$, then $f(\beta,\lambda)\approx 1.65$.

b) Inspecting (6.13), it is easy to see that

$$\max\{1/(1+\beta/3+2\lambda/3), \log 2/((1+\lambda)\log 3)\} < f(\beta,\lambda) < 1/(1+\min\{\beta,\lambda\})$$

for all choices of β and λ.

c) If above $\beta,\lambda>-1/2$, then it is not hard to see that ρ satisfies (5.2) and thus is a conformal density.

We do not know if also $\mathrm{Dim}_\rho(\partial_\rho H)\le f(\beta,\lambda)$:

QUESTION 6.5. In Example 6.3, is it true that $\mathrm{Dim}_\rho(\partial_\rho H)=f(\beta,\lambda)$?

We cannot use Theorem 5.1 to solve this question since it can be shown that $\mathrm{Dim}_d(\{x : k^-(x)=\min\{\beta,\lambda\}\})=1$.

It is true that $\dim_\rho(\partial_\rho \mathsf{B}^n)\ge n-1$ for all conformal densities ρ defined on B^n. This deep fact was proved in [1]. A straightforward estimate using Theorem 5.1 and (5.1) only implies that $\dim_\rho(\partial_\rho \mathsf{B}^n)\ge c(n,c_0)>0$, where c_0 is the constant in (5.1). See also [2, Proposition 7.1]. Next we provide an example of a density ρ on the upper half-plane H such that $\mathrm{Dim}_\rho(\partial_\rho H)=0$ and $\dim_d(\mathsf{R}\setminus\partial_\rho H)=0$.

EXAMPLE 6.6. We construct a density with $\mathrm{Dim}_\rho(\partial_\rho H)=0$ and $\dim_d(\mathsf{R}\setminus \partial_\rho H)=0$.

Given an interval $I\subset\mathsf{R}$, let T_I and U_I be the isosceles triangles with base I and heights $|I|$ and $|I|/2$ respectively. Denote $S_I=T_I\setminus U_I$.

To begin with, let $I_1, I_2,\dots$ be disjoint intervals so that $C=\mathsf{R}\setminus\cup I_i$ forms a Cantor set (a nowhere dense closed set without isolated points). Moreover, we assume that $\sum_i \mathrm{diam}_d(I_i)\le 1$. Let $\rho(x)=\exp(-1/d(x))$ if $x\in H\setminus\cup_i T_{I_i}$. We define ρ on each strip S_{I_i} so that

$$\ell_\rho(\gamma)\ge 1 \tag{6.14}$$

for any curve joining U_{I_i} to $H\setminus T_{I_i}$. We also require that ρ extends continuously to the lower boundary Γ_{I_i} of S_{I_i} (excluding the two endpoints of I_i) and that

$$\ell_\rho(\gamma)=\infty \tag{6.15}$$

if γ is a curve on S_{I_i} whose one endpoint is an endpoint of I_i. We remark that the condition (6.15) as well as the condition (6.17) below, are only used to ensure that the assumption (A2) is satisfied.

Now for each $x,y\in C$ with $d(x,y)=d>0$, we have

$$d_\rho(x,y)\le 2\int_{t=0}^{d}\exp(-1/t)\,dt + d\exp(-1/d)\le 3\exp(-1/d).$$

Thus, for each $n \in \mathbb{N}$, there is $\delta > 0$ such that $d_\rho(x, y) \le d(x, y)^n$ if $x, y \in C$ and $d(x, y) < \delta$. By Lemma 3.1, this implies $\mathrm{Dim}_\rho(C) = 0$.

We continue the construction inside the triangles U_{I_i}. We choose intervals $I_{i,j} \subset I_i$ so that $C_i = I_i \setminus \cup_j I_{i,j}$ is a Cantor set and

$$\sum_{i,j} \mathrm{diam}_d(I_{i,j})^{1/2} \le 1. \tag{6.16}$$

We define $\rho(x) = f_i(x) \exp(-1/d(x))$ on $U_i \setminus \cup_j T_{I_{i,j}}$ where $f_i(x)$ is a continuous weight that is bounded if x is bounded away from the endpoints of I_i. Close to the endpoints of I_i, we make f_i so large that

$$\ell_\rho(\gamma) = \infty \tag{6.17}$$

if γ is a curve on U_{I_i} whose one endpoint is an endpoint of I_i. Also, we define ρ on the strips S_{I_i} so that analogues of (6.14) and (6.15) hold. As above, we see that $\mathrm{Dim}_\rho(C_i) = 0$ for all i.

We continue the construction inductively inside the triangles $U_{I_{i,j}}$. At the step n, we obtain Cantor sets $C_{n,i}$ with $\mathrm{Dim}_\rho(C_{n,i}) = 0$. At the end, $\partial_\rho H$ will be the union of all these Cantor sets. Replacing the exponent $1/2$ in (6.16) by $1/n$ at the step n implies that $\dim_d(\mathbb{R} \setminus \partial_\rho H) = 0$.

It would be interesting to know, if the analogy of (5.5) for the packing dimension holds.

QUESTION 6.7. If ρ is a conformal density on $\mathbb{B}^n$, does there exist a set $A \subset S^{n-1}$ with $\mathrm{Dim}_d(A) = 0$ such that $\mathrm{Dim}_\rho(S^{n-1} \setminus A) \le n$?

ACKNOWLEDGEMENTS. The first author was supported by the Academy of Finland project #120972 and he wishes to thank professor Pekka Koskela. The second author was supported by the Academy of Finland project #126976.

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