MATH. SCAND. 113 (2013), 20–29

THE CHOW MOTIVES OF RELATIVE FULTON-MACPHERSON SPACE

FUMITOSHI SATO

Abstract

Suppose that X is a complex nonsingular projective variety and D is a smooth divisor. Compactifications of configuration spaces of distinct and non-distinct n points in X away from D were constructed by the author and B. Kim in "A generalization of Fulton-MacPherson configuration spaces" by using the method of wonderful compactification. In this paper, we give explicit presentations of Chow motives and Chow groups of these configuration spaces.

1. Introduction

Let X be a complex connected nonsingular projective algebraic variety and let D be a smooth divisor. In [4], two generalizations of Fulton-MacPherson spaces were constructed by using the method of wonderful compactification [5]. These spaces were important because they were used to give simple constructions of moduli of relative stable maps and logarithmic stable maps [1], [3].

Two spaces are defined as following:

(1) A compactification $X_D^{[n]}$ of the configuration space of n labeled points in $X \setminus D$, i.e. "not allowing those points to meets D."

(2) A compactification $X_D[n]$ of the configuration space of n distinct labeled points in $X \setminus D$, i.e. "not allowing those points to meet each other as well as D."

The goal of this paper is to give an explicit presentation of Chow motives and Chow groups of these configuration spaces. Our main theorems are:

Theorem 1.1. *We have the Chow group and motive decompositions*

$$A^*(X_D^{[n]}) = \bigoplus_{\mathscr{CH}} \bigoplus_{\vec{\mu}\in M_{\mathscr{CH}}} A^{*-\|\vec{\mu}\|}(D_{S_{\mathscr{CH}}}),$$

$$h(X_D^{[n]}) = \bigoplus_{\mathscr{CH}} \bigoplus_{\vec{\mu}\in M_{\mathscr{CH}}} h(D_{S_{\mathscr{CH}}})(\|\vec{\mu}\|),$$

Received 3 August 2011, in final form 3 February 2012.

where $\mathscr{CH}$ runs through all the chains of $\{1, 2, \ldots, n\}$, $S_{\mathscr{CH}}$ is the maximal element in $\mathscr{CH}$ and $\|\cdot\|$ is the l_1 norm.

THEOREM 1.2. *We have the Chow group and motive decompositions*

$$A^*(X_D[n]) = \bigoplus_{\mathscr{N}} \bigoplus_{\vec{\mu}\in M_{\mathscr{N}}} \left(\bigoplus_{\mathscr{CH}} \bigoplus_{\vec{\lambda}\in M_{\mathscr{CH}}} A^{*-\|\vec{\mu}\|-\|\vec{\lambda}\|}(D_{S_{\mathscr{CH}}}) \right),$$

$$h(X_D[n]) = \bigoplus_{\mathscr{N}} \bigoplus_{\vec{\mu}\in M_{\mathscr{N}}} \left(\bigoplus_{\mathscr{CH}} \bigoplus_{\vec{\lambda}\in M_{\mathscr{CH}}} h(D_{S_{\mathscr{CH}}})(\|\vec{\mu}\| + \|\vec{\lambda}\|) \right),$$

where $\mathscr{N}$ runs through all the nests of $\{1, 2, \ldots, n\}$ and $\mathscr{CH}$ runs through all the chains whose length is the number of connected components of the forest which corresponds to $\mathscr{N}$.

The paper is organized as follows. In section 2, we review theory of wonderful compactification and Chow motives after blow-up. In section 3, we review the construction of compactifications of n points in $X \setminus D$. In section 4, we compute Chow groups and motives explicitly.

1.1. Notation

- As in [2], for a subset I of $N := \{1, 2, \ldots, n\}$, let

$$I^+ := I \cup \{n + 1\}.$$

- Let $\mathrm{Bl}_Z Y$ be the blow-up of a nonsingular complex projective variety Y along a nonsingular closed subvariety Z.
- Let Y_1 be the blow-up of a nonsingular complex projective variety Y_0 along a nonsingular closed subvariety Z. If V is an irreducible subvariety of Y_0, we will use $\widetilde{V}$ or $V(Y_1)$ to denote
  - the total transform of V, if $V \subseteq Z$;
  - the proper transform of V, otherwise.

  If there is no risk to cause confusion, we will use simply V to denote $\widetilde{V}$. The space $\mathrm{Bl}_{\widetilde{V}} Y_1$ will be called the iterated blow-ups of Y_0 along centers Z, V (with the order). When we want to indicate where an iterated transform of V lives explicitly, we will write it $V(Y_n)$.
- For a partition of I of N, Δ_I denotes the polydiagonal associated to I. And consider a binary operation $I \wedge J$ on the set of all partitions satisfying

$$\Delta_I \cap \Delta_J = \Delta_{I\wedge J}.$$

  We use Δ_{I_0} instead of Δ_I when $I = \{I_0, I_1, \ldots, I_l\}$ such that $|I_i| = 1$ for all $i \geq 1$.

1.2. Acknowledgements

We would like to thank B. Kim and L. Li for many valuable discussions. Most part of this work was done during our visit at Mittag-Leffler Institute. We thanks for its hospitality. This research was supported by Japan Society for the Promotion of Science, Grant 21840056003.

2. Wonderful Compactification of Arrangements of Subvarieties

In this section, we review the theory of wonderful compactification of arrangements of subvarieties. See the detail and proofs in [5], [6].

2.1. Arrangement, building set and nest

DEFINITION 2.1 (Of clean intersection). Let Y be a complex nonsingular projective algebraic variety and let U and V be two smooth subvarieties of Y. We say that U and V intersect cleanly if $U \neq V$ and their scheme-theoretic intersection is nonsingular and the tangent bundles satisfy $T(U \cap V) = TU \cap TV$.

REMARK 2.2. If the intersection is transversal, then it is a clean intersection.

DEFINITION 2.3 (Of simple arrangement). A simple arrangement of subvarieties of Y is a finite set $\mathscr{S} = \{S_i\}$ of nonsingular closed irreducible subvarieties of Y satisfying the following conditions

(1) S_i and S_j intersect cleanly,

(2) $S_i \cap S_j$ is either empty or some S_k's.

DEFINITION 2.4 (Of building set). Let $\mathscr{S}$ be a simple arrangement of subvarieties of Y. A subset $\mathscr{G} \subseteq \mathscr{S}$ is called a building set with respect to $\mathscr{S}$, if , for any $S \in \mathscr{S}$, the minimal elements in $\mathscr{G}$ which contain S intersect transversally and their intersection is S. These minimal elements are called the $\mathscr{G}$-factors of S.

DEFINITION 2.5 (Of $\mathscr{G}$-nest). A subset $\mathscr{T} \subseteq \mathscr{G}$ is called a $\mathscr{G}$-nest if there is a flag of elements in $\mathscr{S}$: $S_1 \subset S_2 \subset \cdots \subset S_k$ such that

$$\mathscr{T} = \bigcup_{i=1}^{k} \{A : A \text{ is a } \mathscr{G}\text{-factor of } S_i\}.$$

2.2. Construction of $Y_{\mathscr{G}}$ by a sequence of blow-ups

Let Y be a complex nonsingular projective algebraic variety, $\mathscr{S}$ be a simple arrangement of subvarieties of Y and $\mathscr{G}$ be a building set with respect to $\mathscr{S}$. Order $\mathscr{G} = \{G_1, \dots, G_N\}$ such that $i < j$ if $G_i \subset G_j$. We define $(Y_k, \mathscr{S}^{(k)}, \mathscr{G}^{(k)})$ inductively, where Y_k is a blow-up of Y_{k-1} along a nonsingular variety, $\mathscr{S}^{(k)}$ is a simple arrangement of subvarieties of Y_k and $\mathscr{G}^{(k)}$ is a building set with respect to $\mathscr{S}^{(k)}$.

THEOREM 2.6. *Assume $\mathscr{S}$ is a simple arrangement of subvarieties of Y and $\mathscr{G}$ is a building set. Let G be a minimal element in $\mathscr{G}$ and consider $\pi : \widetilde{Y} := \mathrm{Bl}_G Y \to Y$. Denote the exceptional divisor by E. For any nonsingular variety V in Y, we define $\widetilde{V} \subset \mathrm{Bl}_G Y$, the $\sim$-transform of V, to be the proper transform of V if $V \nsubseteq G$, and to be $\pi^{-1}(V)$ if $V \subseteq G$.*

For simplicity of notation, for a sequence of blow-ups, we use the same notation $\widetilde{V}$ to denote the iterated one.

(1) *The collection $\mathscr{S}'$ of subvarieties in $\widetilde{Y}$ defined by*

$$\mathscr{S}' := \{\widetilde{S}\}_{S \in \mathscr{S}} \cup \{\widetilde{S} \cap E\}_{\emptyset \subset S \cap G \subset S}$$

is a simple arrangement in $\widetilde{Y}$

(2) *$\mathscr{G}' := \{\widetilde{G_i}\}_{G_i \in \mathscr{G}}$ is a building set with respect to $\mathscr{S}'$.*

(3) *Given a subset $\mathscr{T}$ of $\mathscr{G}$, define $\mathscr{T}' := \{\widetilde{A}\}_{A \in \mathscr{T}} \subseteq \mathscr{G}'$. $\mathscr{T}$ is a $\mathscr{G}$-nest if and only if $\mathscr{T}'$ is a $\mathscr{G}'$-nest.*

Let's go back to the construction of $Y_{\mathscr{G}}$.

(1) For $k = 0$, $Y_0 = Y$, $\mathscr{S}^{(0)} = \mathscr{S}$, $\mathscr{G}^{(0)} = \mathscr{G} = \{G_1, \dots, G_N\}$, $G_i^{(0)} = G_i$.

(2) Assume Y_{k-1} is already constructed. Let Y_k be the blow-up of Y_{k-1} along the nonsingular subvariety $G_k^{(k-1)}$. Define $G_i^{(k)} := \widetilde{G_i^{(k-1)}}$. Since $G_i^{(k-1)}$ for $i < k$ are all divisors, $G_k^{(k-1)}$ is minimal in $\mathscr{G}^{(k-1)}$. Thus there is a naturally induced simple arrangement $\mathscr{S}^{(k)}$ and a building set $\mathscr{G}^{(k)}$ by the Theorem 2.6.

(3) Continue the inductive construction to $k = N$, where all elements in the building set $\mathscr{G}^{(N)}$ are divisors.

THEOREM 2.7. *Denote $Y^\circ = Y \setminus \cup_{G \in \mathscr{G}} G$. There is a natural locally closed embedding*

$$Y^\circ \hookrightarrow Y \times \prod_{G \in \mathscr{G}} \mathrm{Bl}_G Y,$$

and its cloure is denoted by $Y_{\mathscr{G}}$ and called the wonderful compactification of Y with respect to $\mathscr{G}$. Then $Y_{\mathscr{G}}$ is isomorphic to Y_N which is constructed in the *above. The variety $Y_{\mathscr{G}}$ is nonsingular. For each $G \in \mathscr{G}$, there is a nonsingular divisors $D_G \subset Y_{\mathscr{G}}$ such that*

(1) *The union of these divisors is $Y_{\mathscr{G}} \setminus Y^\circ$.*
(2) *Any set of these divisors meets transversally. An intersection of divisors $D_{T_1} \cap \cdots \cap D_{T_l}$ is not empty exactly when $\{T_1, \ldots, T_l\}$ forms a $\mathscr{G}$-nest.*

THEOREM 2.8 (Order of blow-ups).
(1) *Let $\mathscr{I}_i$ be the ideal sheaf of $G_i \in \mathscr{G}$. Then*

$$Y_{\mathscr{G}} \cong \mathrm{Bl}_{\mathscr{I}_1 \ldots \mathscr{I}_N} Y.$$

(2) *If we arrange $\mathscr{G} = \{G_1, \ldots, G_N\}$ in such an order that*

$(*)$ *for any $1 \leq i \leq N$, the first i terms $G_1, \ldots, G_i$ form a building set.*

Then

$$Y_{\mathscr{G}} \cong \mathrm{Bl}_{\widetilde{G_N}} \ldots \mathrm{Bl}_{\widetilde{G_2}} \mathrm{Bl}_{G_1} Y,$$

where each blow-up is along a smooth subvariety.

2.3. Chow group and motive of $Y_{\mathscr{G}}$

Let $Y_0 := Y, Y_0\mathscr{T} := \cap_{T \in \mathscr{T}} T$ where $\mathscr{T}$ is a $\mathscr{G}$-nest. Define $r_{\mathscr{T}}(G) := \dim(\cap_{G \subset T \in \mathscr{T}} T) - \dim G$ (here we use a convention that $\cap_{G \subset T \in \mathscr{T}} T = Y$ if no T strictly contains G). Then define

$$M_{\mathscr{T}} := \{\vec{\mu} = \{\mu_G\}_{G \in \mathscr{T}} : 1 \leq \mu_G \leq r_{\mathscr{T}}(G) - 1\}.$$

Let $\|\vec{\mu}\| := \sum_{G \in \mathscr{G}} \mu_G$ for $\vec{\mu} \in M_{\mathscr{T}}$.

THEOREM 2.9. *We have the Chow group decomposition*

$$A^*(Y_{\mathscr{G}}) = A^*(Y) \oplus \bigoplus_{\mathscr{T}} \bigoplus_{\vec{\mu} \in M_{\mathscr{T}}} A^{*-\|\vec{\mu}\|}(Y_0\mathscr{T})$$

where $\mathscr{T}$ runs through all $\mathscr{G}$-nests. We also have the Chow motive decomposition

$$h(Y_{\mathscr{G}}) = h(Y) \oplus \bigoplus_{\mathscr{T}} \bigoplus_{\vec{\mu} \in M_{\mathscr{T}}} h(Y_0\mathscr{T})(\|\vec{\mu}\|)$$

where $\mathscr{T}$ runs through all $\mathscr{G}$-nests.

3. Construction of $X_D^{[n]}$ and $X_D[n]$

Fix a nonsingular divisor D of a complex nonsingular projective algebraic variety X of dimension m. In this section, we review constructions of a compactification of configuration space of n points in $X \setminus D$, $X_D^{[n]}$, and a compactification of configuration space of n distinct points in $X \setminus D$, $X_D[n]$. In this paper, we assume that D is a divisor but every thing will work in the case where D is a smooth subvariety after some adjustment. See the details in [4].

3.1. Construction

For a subset S of $N := \{1, 2, \dots, n\}$ define a nonsingular subvariety in X^n

$$D_S := \{\mathbf{x} \in X^n \mid \mathbf{x}_i \in D, \ \forall\, i \in S\}.$$

Let $\mathscr{A}$ be the collection of D_S for all $S \subseteq N := \{1, \dots, n\}$ with $|S| \geq 2$. It is clear that the collection is a simple arrangement of smooth subvarieties of X^n. Take a building set $\mathscr{G} = \mathscr{A}$. Then define $X_D^{[n]}$ to be the closure of $X^n \setminus \bigcup_S D_S$ in

$$X^n \times \prod_S \mathrm{Bl}_{D_S} X^n$$

It can be constructed by a successive blow-ups by Theorem 2.7. In particular we may order $\mathscr{G}$ as D_{12}; D_{123}; D_{13}, D_{23}; $\dots$; $D_{12\dots n}$; $D_{U\cup\{n\}}$ with $|U| = n-2$ and $U \subset N \setminus \{n\}$; $\dots$; D_{in} for $i = 1, \dots, n-1$ by Theorem 2.8.

For $I \subseteq N$ with $|I| \geq 2$, $\{\Delta_I(X_D^{[n]})\}$ forms a building set of nonsingular subvarieties of $X_D^{[n]}$ with respect to the set of $\sim$-transforms of all polydiagonals. So we define $X_D[n]$ as followings.

DEFINITION 3.1. Define $X_D[n]$ to be the closure of $X_D^{[n]} \setminus \bigcup_{|I|\geq 2} \Delta_I(X_D^{[n]})$ in

$$X_D^{[n]} \times \prod_{|I|\geq 2} \mathrm{Bl}_{\Delta_I(X_D^{[n]})} X_D^{[n]}$$

Then, it satisfies the following properties.

THEOREM 3.2.

(1) *$X_D[n]$ is a nonsingular variety. There is a natural projection from $X_D[n]$ to $X_D[|I|]$ for any subset I of N. There is a natural S_n-action on $X_D[n]$.*

(2) *The boundary is the union of divisors $\widetilde{D_S}$ with $|S| \geq 1$, and $\widetilde{\Delta_I}$ with $|I| \geq 2$ of normal crossings.*

(3) *The intersections of boundary divisors are nonempty if and only if they are nested. Here $\{D_{S_i}, \Delta_{I_j}\}$ is nested if each pair S_i and $S_{i'}$ (I_j and $I_{j'}$) is either disjoint or one is contained in the other and each pair S_i and I_j is either disjoint or I_j is contained in S_i.*

(4) *We may take a following order of blow-ups:* D_S*;* Δ_I *for* $n \notin S, I$*;* D_T *with* $n \in T$*;* Δ_J *with* $n \in J$*.*

This is a summary of Theorem 1 and 2 in [4].

4. Chow groups and motives

In this section, we will apply Theorem 2.9 to $X_D^{[n]}$ and $X_D[n]$.

4.1. Chow group and motive of $X_D^{[n]}$

In this case, our $Y = X^n$, $\mathscr{S} = \mathscr{G} = \{D_S : S \subseteq N \text{ with } |S| \geq 2\}$ where $D_S = \{\mathbf{x} \in X^n \mid \mathbf{x}_i \in D, \ \forall\, i \in S\}$. We have $\mathscr{S} = \mathscr{G}$, so a $\mathscr{G}$-nest $\mathscr{T}$ is just a chain of elements in $\mathscr{S}$, that is $\mathscr{T} = \{D_{S_1} \subset D_{S_2} \subset \cdots \subset D_{S_k}\}$. Thus $Y_0\mathscr{T} = D_{S_1}$.

A chain $\mathscr{CH}$ is a chain of proper subset of N, $S_k \subset \cdots \subset S_2 \subset S_1$, such that S_k is not a singleton. Obviously, there is one-to-one correspondence between a set of chains of $\mathscr{S}$ and a set of chains of N. We say $\emptyset$ is also a chain. We define $\max_{\mathscr{CH}(\mathscr{T})} S$ as the maximal element of $\mathscr{CH}(\mathscr{T})$ which is strictly contained in S, where $\mathscr{CH}(\mathscr{T})$ is the chain of N which corresponds to $\mathscr{T}$. If there is no such element, then we define $\max_{\mathscr{CH}(\mathscr{T})} S = \emptyset$

Now let $G = D_S$ and let's compute $r_{\mathscr{T}}(G)$;

$$\begin{aligned} r_{\mathscr{T}}(G) &= \dim\Big(\bigcap_{G \subset T \in \mathscr{T}} T\Big) - \dim G \\ &= \dim(D_{\max_{\mathscr{CH}(\mathscr{T})} S}) - \dim D_S \\ &= |S| - |\max_{\mathscr{CH}(\mathscr{T})} S|. \end{aligned}$$

REMARK 4.1 (When D is not a divisor). When D is not a divisor, then we also blow up along $D_{\{i\}}$. So we will not exclude the case such that S_k is a singleton for $\{S_k \subset \cdots \subset S_2 \subset S_1\}$. The definition of $r_{\mathscr{T}}(G)$ will be also changed. It will be multiplied by the codimension of D in X. See more details in [6].

For a chain $\mathscr{CH}(\neq \emptyset)$, define

$$M_{\mathscr{CH}} := \{\vec{\mu} = \{\mu_S\}_{S \in \mathscr{CH}} : 1 \leq \mu_S \leq |S| - |\max_{\mathscr{CH}} S| - 1\}.$$

For $\mathscr{CH} = \emptyset$, define $M_{\mathscr{CH}}$ to be the set consisting of one $\vec{\mu}$ with $\|\vec{\mu}\| = 0$ and $D_\emptyset = X^n$.

THEOREM 4.2 (Theorem 1.1). *We have the Chow group and motive decompositions*

$$A^*(X_D^{[n]}) = \bigoplus_{\mathcal{CH}} \bigoplus_{\vec{\mu}\in M_{\mathcal{CH}}} A^{*-\|\vec{\mu}\|}(D_{S_{\mathcal{CH}}}),$$

$$h(X_D^{[n]}) = \bigoplus_{\mathcal{CH}} \bigoplus_{\vec{\mu}\in M_{\mathcal{CH}}} h(D_{S_{\mathcal{CH}}})(\|\vec{\mu}\|),$$

where $\mathcal{CH}$ runs through all the chains of N and $S_{\mathcal{CH}}$ is the maximal element in $\mathcal{CH}$.

4.2. Chow group and motive of $X_D[n]$

We use the same notation as [6].

(1) We call two subsets $I, J \subseteq N$ are overlapped if $I \cap J$ is not a nonempty proper subset of both I and J. For a set $\mathcal{N}$ of subsets of N, we call I is compatible with $\mathcal{N}$, denoted by $I \sim \mathcal{N}$, if I does not overlap any elements of $\mathcal{N}$. A nest $\mathcal{N}$ is a set of subset of N such that any pair $I \neq J \in \mathcal{N}$ are not overlapped and contains all singletons. For a given nest $\mathcal{N}$, define $\mathcal{N}^\circ := \mathcal{N} \setminus \{\{1\}, \dots, \{n\}\}$. A nest $\mathcal{N}$ naturally corresponds to a tree (which may not be connected) with each node being labeled by an element of $\mathcal{N}$. Let $c(\mathcal{N})$ be the number of connected components of the forest which corresponds to $\mathcal{N}$. Denote by $c_I(\mathcal{N})$ the number of maximal elements of the set $\{J \in \mathcal{N} : J \subset I\}$, which is called the number of sons of the node I. Let $\overline{\Delta_{\mathcal{N}}} := \cap_{I\in\mathcal{N}} \Delta_I(X_D^{[n]})$ in this section.

(2) For a nest $\mathcal{N}$ ($\neq\{\{1\}, \dots, \{n\}\}$), define

$$M_{\mathcal{N}} := \{\vec{\mu} = \{\mu_I\}_{I\in\mathcal{N}} : 1 \le \mu_I \le m(c_I(\mathcal{N}) - 1) - 1\}$$

where $m = \dim X$. For $\mathcal{N} = \{\{1\}, \dots, \{n\}\}$, define $M_{\mathcal{N}} = \{\vec{\mu}\}$ with $\|\vec{\mu}\| = 0$.

As in [6], we have

PROPOSITION 4.3. *We have the Chow group and motive decompositions*

$$A^*(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\vec{\mu}\in M_{\mathcal{N}}} A^{*-\|\vec{\mu}\|}(\overline{\Delta_{\mathcal{N}}}),$$

$$h(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\vec{\mu}\in M_{\mathcal{N}}} h(\overline{\Delta_{\mathcal{N}}})(\|\vec{\mu}\|),$$

where $\mathcal{N}$ runs through all the nests of N

Now we need to simplify $A^*(\overline{\Delta_{\mathcal{N}}})$ and $h(\overline{\Delta_{\mathcal{N}}})$.

LEMMA 4.4. *D_S and Δ_I intersect cleanly.*

PROOF. We only need to prove that $TD_S \cap T\Delta_I \subset T(D_S \cap \Delta_I)$. An arc in Δ_I have a coordinate representative $(\mathbf{x}_i) \in X^n$ such that $\mathbf{x}_i = \mathbf{x}_j$ for $i, j \in I$. For an arc in Δ_I to be an arc in D_S, $\mathbf{x}_i \in D$ for all $i \in S$. Thus the arc should be an arc in $D_S \cap \Delta_I$.

PROPOSITION 4.5. *$\overline{\Delta_I}$ is isomorphic to $X_D^{[|I^c|+1]}$.*

PROOF. We need to know which blow-ups along D_S have an effect to Δ_I in a specific order of blow-ups. We can assume that $I = \{l, \cdots, n\}$ by arranging the order. Then denote $a = |I^c|$ and $b = |I|$. We will denote Δ_I by $X^a \times \Delta$ ($\cong X^{|I^c|+1}$). Then we have two different kinds of D_S. The first one is that $S \subset I^c$, which we call the first kind, the second one is that $S \not\subseteq I^c$, which we call the second kind. We will change the order of blow-ups so that we first blow up along D_S's of the first kind, and then along the second kind. More precisely, we order $D_{I^c} \times X^b, D_{1,\ldots,\hat{i},\ldots,l} \times X^b, \ldots, D_{i,j} \times X^b (i, j \in \{1, \ldots, a\})$ and then $D_{I^c} \times D^b, \ldots, D_{S'} \times D_{S''}, \ldots$ ($|S''| > 0$ and $(|S'|, |S''|)$: non-increasing in lexicographical order). This order satisfies $(*)$-condition in Theorem 2.6, so that we can blow up in this order. In this order of blow-ups, notice that $\widetilde{X^a \times \Delta}$ and $\widetilde{D_{S'} \times D_{S''}}$ for $S'' \subset I$ are separated when we blow up along $D_{S'} \times D^b$. Thus we can forget the process of blow-ups by $\widetilde{D_{S'} \times D_{S''}}$ where $S'' \subset I$ i.e. we only need to care about $D_{S'} \times D^b$ for the second kind. Under the isomorphism $X^a \times \Delta \cong X^{a+1}$, they are just $D_{S'} \times D$.

We can also apply the same technique to polydiagonals term by term. Thus we can go further from proposition 4.3.

THEOREM 4.6 (Theorem 1.2). *We have the Chow group and motive decompositions*

$$A^*(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\vec{\mu} \in M_{\mathcal{N}}} \left(\bigoplus_{\mathcal{CH}} \bigoplus_{\vec{\lambda} \in M_{\mathcal{CH}}} A^{*-\|\vec{\mu}\|-\|\vec{\lambda}\|}(D_{S_{\mathcal{CH}}}) \right),$$

$$h(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\vec{\mu} \in M_{\mathcal{N}}} \left(\bigoplus_{\mathcal{CH}} \bigoplus_{\vec{\lambda} \in M_{\mathcal{CH}}} h(D_{S_{\mathcal{CH}}})(\|\vec{\mu}\| + \|\vec{\lambda}\|) \right),$$

where $\mathcal{N}$ runs through all the nests of N and $\mathcal{CH}$ runs through all the chains whose length is $c(\mathcal{N})$.

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KAGAWA NATIONAL COLLEGE OF TECHNOLOGY
355 CHOKUSHI-CHO, TAKAMATSU
KAGAWA 761-8058
JAPAN
E-mail: fumi@math.utah.edu