MORITA EQUIVALENCE OF NEST ALGEBRAS

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Abstract

Let $\mathcal{N}_1$ (resp. $\mathcal{N}_2$) be a nest, A (resp. B) be the corresponding nest algebra, A_0 (resp. B_0) be the subalgebra of compact operators. We prove that the nests $\mathcal{N}_1$, $\mathcal{N}_2$ are isomorphic if and only if A and B are weakly-$*$ Morita equivalent if and only if A_0 and B_0 are strongly Morita equivalent. We characterize the nest isomorphisms which implement stable isomorphism between the corresponding nest algebras.

1. Introduction

Rieffel introduced the idea of Morita equivalence in Operator Theory, developing the theory of Morita equivalence for C^* and W^* algebras [17]. After the advent of the theory of operator spaces and operator algebras a parallel Morita theory for non-selfadjoint algebras was developed by Blecher, Muhly and Paulsen [6], [2]. We call this equivalence strong Morita equivalence.

Recently, two approaches have been suggested for Morita equivalence of dual operator algebras. The first one, called Δ-equivalence, was introduced [11] by the present author and is equivalent to the notion of stable isomorphism of dual operator algebras [13]. The second one, called weak-$*$ Morita equivalence was introduced by Blecher and Kashyap [3], [14] and is strictly weaker than Δ-equivalence. It is interesting that if A and B are strongly Morita equivalent approximately unital operator algebras then the second dual operator algebras A^{**}, B^{**} are weakly-$*$ Morita equivalent [3]. New results on weak-$*$ Morita equivalence and Δ-equivalence can be found in [4].

The present paper is concerned mainly about seeing what these new notions mean when specialized to the case of nest algebras. As we show the case of nest algebras is very useful to illuminate the above theories of Morita equivalence. For example, several natural questions are answered here, and the importance of weak-$*$ Morita equivalence is displayed.

More precisely, we prove that strong and weak-$*$ Morita equivalence are lattice properties: If A and B are nest algebras and A_0 and B_0 are the associated subalgebras of compact operators then A_0 and B_0 are strongly Morita equivalent if and only if A and B are weakly-$*$ Morita equivalent if and only

if the nests $\mathrm{Lat}(A)$ and $\mathrm{Lat}(B)$ are isomorphic. The main tool of the proof is that if $\theta : \mathrm{Lat}(A) \rightarrow \mathrm{Lat}(B)$ is a nest isomorphism, then we can construct a dual operator $A - B$ bimodule Y and a dual operator $B - A$ bimodule X. As we prove, the identity operator of A is the limit in the strong operator topology of a net of finite rank contractions (f_λ) where every f_λ can be approximately decomposed into the product of a row with entries in Y and a column with entries in X. Similarly, we can decompose the identity operator of the algebra B. This can be considered as a generalization of the Erdős density theorem for nest algebras [7].

In Section 3 we prove that two nest algebras are weakly-∗ Morita equivalent if and only if they are spatially Morita equivalent (Definition 3.1). We also prove that if A is a nest algebra, spatially Morita equivalent to an algebra B, then A and B are weakly-∗ Morita equivalent.

In Section 4 we present a measure theoretic result which describes when two separably acting nest algebras are stably isomorphic. As it was pointed out in [3] Example 3.7 in [12] is an example of weakly-∗ Morita equivalent algebras which are not stably isomorphic. Using the results of this paper we give a new proof of the fact that weak-∗ Morita equivalence is strictly weaker than Δ-equivalence.

In Section 5 we present a counterexample which shows that the second duals of two unital strongly Morita equivalent algebras are not necessarily stably isomorphic.

In the following paragraphs we describe the notions used in this paper; since we use extensively the basics of Operator Space Theory, we refer the reader to the monographs [5], [9], [15] and [16] for further details. A (normal) *representation* of a (dual) operator algebra A is a (w^*-continuous) completely contractive homomorphism $\alpha : A \rightarrow B(H)$ on a Hilbert space H. In case A is unital, we assume that α is unital.

Let H, K be Hilbert spaces and $A \subset B(H)$ be an algebra. A subspace $X \subset B(K, H)$ is called a left module over A if $AX \subset X$. Similarly we can define the right modules over A. A left and right module over A is called a bimodule over A. An operator space Y is an abstract left (right) operator module over an abstract operator algebra A if there exists a completely contractive bilinear map $A \times Y \rightarrow Y$ ($Y \times A \rightarrow Y$). A left and right operator module over A is called an operator bimodule over A.

Let A be a dual operator algebra and Y be a dual operator space. We say that Y is a left (right) dual operator module over A, if the above completely contractive bilinear map is separately w^*-continuous. A left and right dual operator module over A is called a dual operator bimodule over A.

Two operator bimodules Y and Z over an operator algebra A are called isomorphic as operator bimodules if there exists a completely isometric and onto

A-bimodule map $\pi : Y \to Z$. We write $Y \cong Z$ as operator A-bimodules. In case A is a dual operator algebra and Y and Z are dual operator A-bimodules we write $Y \cong Z$ as dual operator A-bimodules if the above completely isometric and onto A-bimodule map π is w^*-(bi)continuous.

If Y is a right operator module over an operator algebra A and X is a left operator module over A we denote by $Y \otimes_A^h X$ the balanced Haagerup tensor product of Y and X which linearizes the completely bounded A-balanced bilinear maps [5, 3.4]. If Y (resp. X) is a left (resp. right) operator module over an operator algebra B then $Y \otimes_A^h X$ is also a left (resp. right) operator module over B, [6, Lemma 2.4].

If Y is a dual right operator module over a dual operator algebra A and X is a left dual operator module over A we denote by $Y \otimes_A^{\sigma h} X$ the balanced normal Haagerup tensor product of Y and X which linearizes the separately w^*-continuous completely bounded A-balanced bilinear maps [13]. In the case Y (resp. X) is a left (resp. right) dual operator module over a dual operator algebra B then $Y \otimes_A^{\sigma h} X$ is also a left (resp. right) dual operator module over B, [13].

The following are the two definitions of Morita equivalence used in this paper:

DEFINITION 1.1 ([6]). The operator algebras A, B are called *strongly Morita equivalent* if there exist an $A - B$ operator module X and a $B - A$ operator module Y such that $A \cong X \otimes_B^h Y$ and $B \cong Y \otimes_A^h X$ as A and B operator bimodules respectively.

DEFINITION 1.2 ([3]). The dual operator algebras A, B are called *weakly-∗ Morita equivalent* if there exist an $A - B$ dual operator module X and a $B - A$ dual operator module Y such that $A \cong X \otimes_B^{\sigma h} Y$ and $B \cong Y \otimes_A^{\sigma h} X$ as A and B dual operator bimodules respectively.

If X is a subspace of $B(H, K)$, where H and K are Hilbert spaces, we denote by $R_\infty^{\mathrm{fin}}(X)$ (resp. $C_\infty^{\mathrm{fin}}(X)$) the space of operators $(x_1, x_2, \ldots) : H^\infty \to K$ (resp. $(x_1, x_2, \ldots)^T : H \to K^\infty$) such that $x_i \in X$ for all i and there exists $n_0 \in \mathsf{N}$ for which $x_n = 0$ for all $n \geq n_0$.

If
$$s_1 = (s_1^1, s_2^1, \ldots, s_{n_1}^1, 0, 0, \ldots), \qquad s_{n_1}^1 \neq 0$$

and
$$s_2 = (s_1^2, s_2^2, \ldots, s_{n_2}^2, 0, 0, \ldots), \qquad s_{n_2}^2 \neq 0$$

are operators in $R_\infty^{\mathrm{fin}}(X)$ we denote by (s_1, s_2) the operator

$$(s_1^1, s_2^1, \ldots, s_{n_1}^1, s_1^2, s_2^2, \ldots, s_{n_2}^2, 0, 0, \ldots)$$

which also belongs to $R_\infty^{\text{fin}}(X)$. In the same way if $s_1, s_2, \ldots, s_n \in R_\infty^{\text{fin}}(X)$ we define the operator $(s_1, s_2, \ldots, s_n) \in R_\infty^{\text{fin}}(X)$. Similarly if $t_1, t_2, \ldots, t_n \in C_\infty^{\text{fin}}(X)$ we define the operator $(t_1, t_2, \ldots, t_n)^T \in C_\infty^{\text{fin}}(X)$.

A *nest* $\mathcal{N}$ is a totally ordered set of projections of a Hilbert space H containing the zero and identity operators which is closed under arbitrary intersections and closed spans. The corresponding *nest algebra* is

$$\text{Alg}(\mathcal{N}) = \{x \in B(H) : N^\perp x N = 0 \; \forall \; N \in \mathcal{N}\}.$$

If $N \in \mathcal{N}$ we denote by N_- the projection onto the closed span of the union $\bigcup_{\substack{M < N \\ M \in \mathcal{N}}} (M(H))$. If $N_- < N$ we call the projection $N \ominus N_-$ an *atom*. If the nest has no atoms it is called a *continuous nest*. If the atoms span the identity operator the nest is called a *totally atomic nest*. An order preserving 1-1 and onto map between two nests is called a *nest isomorphism*.

If $\mathcal{N}_1$ and $\mathcal{N}_2$ are nests acting on the Hilbert spaces H_1, H_2 respectively and $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ is a nest isomorphism we denote by $\text{Op}(\theta)$ the space of operators $x \in B(H_1, H_2)$ satisfying $\theta(N)^\perp x N = 0$ for all $N \in \mathcal{N}_1$. Observe that $\text{Op}(\theta)$ is an $\text{Alg}(\mathcal{N}_2) - \text{Alg}(\mathcal{N}_1)$ bimodule.

Finally, if X is a normed space we denote by $\text{Ball}(X)$ the unit ball of X and by X^* its dual space. If Y is a vector space then $M_n(Y)$, $n \in \mathbb{N}$, is the space of $n \times n$ matrices with entries in Y. If H_1, H_2 are Hilbert spaces and $\xi \in H_2$, $\eta \in H_1$ are vectors we denote by $\xi \otimes \eta^*$ the rank 1 operator sending every $\omega \in H_1$ to $\langle \omega, \eta \rangle \xi \in H_2$, where $\langle \cdot, \cdot \rangle$ is the inner product of H_1. We also write SOT for the strong operator topology.

2. Morita equivalence of nest algebras

In this section we fix nests $\mathcal{N}_1$, $\mathcal{N}_2$ acting on the Hilbert spaces H_1, H_2 respectively, and a nest isomorphism $\theta : \mathcal{N}_1 \to \mathcal{N}_2$. We denote $A = \text{Alg} \, \mathcal{N}_1$, $B = \text{Alg} \, \mathcal{N}_2$, $X = \text{Op}(\theta)$, $Y = \text{Op}(\theta^{-1})$. If Z is a space of operators we denote its subspace of compact operators by Z_0. Observe that

$$AYB \subset Y, \qquad BXA \subset X, \qquad YX \subset A, \qquad XY \subset B,$$

$$A_0 Y_0 B_0 \subset Y_0, \qquad B_0 X_0 A_0 \subset X_0, \qquad Y_0 X_0 \subset A_0, \qquad X_0 Y_0 \subset B_0.$$

The main result of this section is Theorem 2.9. In particular we are going to prove that

$$A_0 \cong Y_0 \otimes_{B_0}^h X_0, \qquad B_0 \cong X_0 \otimes_{A_0}^h Y_0, \qquad A \cong Y \otimes_B^{\sigma h} X, \qquad B \cong X \otimes_A^{\sigma h} Y.$$

Throughout this section, p denoted the projection $\vee \{N \ominus N_- : N \in \mathcal{N}_1\}$. The following lemmas are used in Theorem 2.5, where we are going to prove a

variant of the Erdős density theorem for nest algebras: There exists a net of finite rank contractions $(f_\lambda) \subset A$ converging in the strong operator topology to the identity operator of H_1, where every f_λ is the norm limit of a sequence $(y_i^\lambda x_i^\lambda)_{i \in \mathbb{N}}$ with $y_i^\lambda \in \text{Ball}(R_\infty^{\text{fin}}(Y_0))$, $x_i^\lambda \in \text{Ball}(C_\infty^{\text{fin}}(X_0))$ for all i, λ.

LEMMA 2.1. *There exists a net (l_λ) of finite rank contractions converging in the strong operator topology to the projection p such that $l_\lambda = s_\lambda t_\lambda$ where $s_\lambda \in \text{Ball}(R_\infty^{\text{fin}}(Y_0))$, $t_\lambda \in \text{Ball}(C_\infty^{\text{fin}}(X_0))$ for all λ.*

PROOF. Suppose that $p = \vee_{k \in J} p_k$ where $p_k = N_k \ominus (N_k)_-$ for $N_k \in \mathcal{N}_1$, $k \in J$. Choose a net of finite rank contractions $(f_i)_{i \in I}$ converging in the strong operator topology to the identity operator of H_1. If $\mathscr{F} = \{F : F$ finite subset of $J\}$ the family $(g_{F,i})_{(F,i)}$ indexed by $\mathscr{F} \times I$ where $g_{F,i} = \sum_{k \in F} p_k f_i p_k$ is a net. Observe that every $g_{F,i}$ is a finite rank contraction belonging to A. We can easily check that $\text{SOT} - \lim_{(F,i)} g_{F,i} = \vee_k p_k = p$.

Let $f = p_k f_i p_k$ for some $k \in J, i \in I$ with polar decomposition $f = u|f|$. Suppose that

$$|f| = \sum_{j=1}^n \lambda_j \xi_j \otimes \xi_j^*$$

for $\lambda_j \geq 0$ and ξ_j pairwise orthogonal vectors of $p_k(H_1)$. Choose a unit vector η in $(\theta(N_k) \ominus \theta(N_k)_-)(H_2)$. Now we have

$$|f| = \sum_{j=1}^n \lambda_j \xi_j \otimes \eta^* \cdot \eta \otimes \xi_j^* = yy^*$$

where $y = (\sqrt{\lambda_1} \xi_1 \otimes \eta^*, \ldots, \sqrt{\lambda_n} \xi_n \otimes \eta^*)$. Observe that $f = uyy^*$ and $uy \in \text{Ball}(R_\infty^{\text{fin}}(Y_0))$, $y^* \in \text{Ball}(C_\infty^{\text{fin}}(X_0))$.

Suppose now that $F = \{j_1, \ldots, j_n\} \subset J, i \in I$ and $g_{F,i} = \sum_{k=1}^n p_{j_k} f_i p_{j_k}$. By the above arguments $p_{j_k} f_i p_{j_k} = s_k t_k$ where $s_k \in \text{Ball}(R_\infty^{\text{fin}}(Y_0))$, $t_k \in \text{Ball}(C_\infty^{\text{fin}}(X_0))$. So $g_{F,i} = st$ where

$$s = (s_1, \ldots, s_n) \in R_\infty^{\text{fin}}(Y_0), \qquad t = (t_1, \ldots, t_n)^T \in C_\infty^{\text{fin}}(X_0).$$

Also, since the projections $(p_k)_{k \in J}$ are pairwise orthogonal and $\|s_k s_k^*\| \leq 1$ for all k we have that

$$\|s\|^2 = \left\| \sum_{k=1}^n s_k s_k^* \right\| = \left\| \sum_{k=1}^n p_{j_k} s_k s_k^* p_{j_k} \right\| \leq 1.$$

Similarly we can prove $\|t\| \leq 1$ and this completes the proof.

LEMMA 2.2. *Suppose that $p^\perp \neq 0$, $\xi, \eta \in \text{Ball}(H_1)$ and $N \in \mathcal{N}_1$ such that $\xi = p^\perp N(\xi), \eta = p^\perp N^\perp(\eta)$. There exist rank 1 operators $(s_n)_{n \in \mathbb{N}} \subset \text{Ball}(Y)$,*

$(t_n)_{n\in\mathsf{N}} \subset \text{Ball}(X)$, *such that the operator* $\xi \otimes \eta^*$ *is the norm limit of the sequence* $(s_n t_n)_{n\in\mathsf{N}}$.

PROOF. We define the continuous order preserving map

$$\phi : p^\perp \mathscr{N}_1 \to [0, \|\xi\|^2] : p^\perp M \to \|p^\perp M(\xi)\|^2.$$

The nest $p^\perp \mathscr{N}_1$ is continuous, so ϕ is onto $[0, \|\xi\|^2]$. Suppose that (λ_n) is a strictly increasing sequence such that $\lambda_n \to \|\xi\|^2$. Choose $N_n \in \mathscr{N}_1$ such that $\phi(p^\perp N_n) = \lambda_n$. It follows that $N_n < N_{n+1} < N$ for all $n \in \mathsf{N}$ and $p^\perp N_n(\xi) \to \xi$. Similarly we can find a sequence $(M_n)_{n\in\mathsf{N}}$ such that $N < M_{n+1} < M_n$ for all $n \in \mathsf{N}$ and $p^\perp(I - M_n)(\eta) \to \eta$. For every $n \in \mathsf{N}$ we choose $\omega_n \in H_2$ such that $\|\theta(M_n) \ominus \theta(N_n)(\omega_n)\| = 1$. The operator

$$s_n = p^\perp N_n(\xi) \otimes (\theta(M_n) \ominus \theta(N_n)(\omega_n))^*$$

satisfies $s_n = N_n s_n \theta(N_n)^\perp$ and so $s_n \in \text{Ball}(Y_0)$. Similarly the operator

$$t_n = (\theta(M_n) \ominus \theta(N_n)(\omega_n)) \otimes p^\perp(I - M_n)(\eta)^*$$

satisfies $t_n = \theta(M_n) t_n M_n^\perp$ and so $t_n \in \text{Ball}(X_0)$. Now we have

$$s_n t_n = p^\perp N_n(\xi) \otimes p^\perp(I - M_n)(\eta)^*$$

which clearly converges in norm to the operator $\xi \otimes \eta^*$.

LEMMA 2.3. *Suppose that* $p^\perp \neq 0$, $\xi \in H_1$ *such that* $\|p^\perp(\xi)\| = 1$ *and* q *is the projection onto the space* $\overline{p^\perp \mathscr{N}_1'' \xi}$. *There exists a sequence of finite rank contractions* $(r_n)_{n\in\mathsf{N}} \subset A$ *converging in the strong operator topology to the projection* q *such that* $r_n = \|\cdot\| - \lim_{i\in\mathsf{N}} s_i^n t_i^n$ *where* $s_i^n \in \text{Ball}(R_\infty^{\text{fin}}(Y_0))$, $t_i^n \in \text{Ball}(C_\infty^{\text{fin}}(X_0))$ *for all* $i, n \in \mathsf{N}$.

PROOF. We define the continuous order preserving map

$$\phi : \mathscr{N}_1 p^\perp \to [0, 1], \phi(N p^\perp) = \|N p^\perp(\xi)\|^2.$$

Since the nest $\mathscr{N}_1 p^\perp$ is continuous ϕ is onto $[0, 1]$. Choose $N_{k,n} p^\perp$ the least element in $\mathscr{N}_1 p^\perp$ such that $\phi(N_{k,n} p^\perp) = \frac{k}{2^n}$, $k = 0, 1, \ldots, 2^n$.

We denote

$$E_{k,n} = (N_{k,n} \ominus N_{k-1,n}) p^\perp, \qquad \xi_{k,n} = 2^{\frac{n}{2}} E_{k,n}(\xi), \qquad r_n = \sum_{k=2}^{2^n} f_{k,n}$$

where $f_{k,n} = \xi_{k-1,n} \otimes \xi_{k,n}^*$.

As in [7, Lemma 3.9] we can prove that $\|r_n\| \leq 1$ and the sequence $(r_n)_{n\in\mathsf{N}}$ SOT-converges to the operator q.

Using Lemma 2.2 we find sequences of rank 1 operators $(s_i^{k,n})_{i \in \mathsf{N}} \subset \mathrm{Ball}(Y_0)$, $(t_i^{k,n})_{i \in \mathsf{N}} \subset \mathrm{Ball}(X_0)$, such that $s_i^{k,n} t_i^{k,n} \xrightarrow{\|\cdot\|} f_{k,n}, i \to \infty$ for all k, n. We denote

$$s_i^n = (s_i^{2,n}, s_i^{3,n}, \ldots, s_i^{2^n,n}), \qquad t_i^n = (t_i^{2,n}, t_i^{3,n}, \ldots, t_i^{2^n,n})^T$$

and we have $r_n = \|\cdot\| - \lim_i s_i^n t_i^n$. Also

$$\|s_i^n\|^2 = \left\| \sum_{k=2}^{2^n} s_i^{k,n} (s_i^{k,n})^* \right\|.$$

We may assume that $s_i^{k,n} = E_{k-1,n} s_i^{k,n}$ so

$$\|s_i^n\|^2 = \left\| \sum_{k=2}^{2^n} E_{k-1,n} s_i^{k,n} (s_i^{k,n})^* E_{k-1,n} \right\|.$$

Since $\|s_i^{k,n}\| \leq 1$ and the projections $(E_{k-1,n})_k$ are pairwise orthogonal we have $\|s_i^n\| \leq 1$. Similarly we can prove $\|t_i^n\| \leq 1$.

LEMMA 2.4. *Suppose that $p^\perp \neq 0$. There exists a net (g_λ) of finite rank contractions in A converging in the strong operator topology to $p^\perp$ such that $g_\lambda = \|\cdot\| - \lim_{i \in \mathsf{N}} s_i^\lambda t_i^\lambda$ for all λ where $s_i^\lambda \in \mathrm{Ball}(R_\infty^{\mathrm{fin}}(Y_0)), t_i^\lambda \in \mathrm{Ball}(C_\infty^{\mathrm{fin}}(X_0))$ for all $i \in \mathsf{N}$.*

PROOF. Using Zorn's Lemma we find a family of vectors $\xi_k : k \in L$ such that the projections q_k onto $\overline{p^\perp \mathcal{N}_1'' \xi_k}, k \in L$ are pairwise orthogonal and they span $p^\perp$. We assume that $\|p^\perp(\xi_k)\| = 1$ for all $k \in L$. From Lemma 2.3 there exist finite rank contractions $(r_n^k)_{n \in \mathsf{N}}$ such that $q_k = \mathrm{SOT} - \lim_{n \in \mathsf{N}} r_n^k$ and $r_n^k = \|\cdot\| - \lim_{i \in \mathsf{N}} s_i^{n,k} t_i^{n,k}$ for sequences

$$(s_i^{n,k})_{i \in \mathsf{N}} \subset \mathrm{Ball}(R_\infty^{\mathrm{fin}}(Y_0)), \qquad (t_i^{n,k})_{i \in \mathsf{N}} \subset \mathrm{Ball}(C_\infty^{\mathrm{fin}}(X_0)).$$

We define $\mathscr{F} = \{F : F \text{ finite subset of } L\}$. If $F \in \mathscr{F}$ and $n \in \mathsf{N}$ we define the finite rank contraction $g_{n,F} = \sum_{k \in F} r_n^k$. The family $(g_{n,F})_{n,F}$ indexed by $\mathsf{N} \times \mathscr{F}$ is a net. Fix $\xi \in H_1$.

Observe that for all $n \in \mathsf{N}$

$$\|r_n^k(\xi) - q_k(\xi)\|^2 = \|q_k(r_n^k - I_{H_1}) q_k(\xi)\|^2 \leq 2 \|q_k(\xi)\|^2$$

and so

$$\sum_{k \in L} \|r_n^k(\xi) - q_k(\xi)\|^2 \leq 2 \sum_{k \in L} \|q_k(\xi)\|^2 < \infty.$$

If $n \in \mathsf{N}$ and $F \in \mathscr{F}$ we have

$$\|g_{n,F}(\xi) - p^{\perp}(\xi)\|^2$$

$$= \left\| g_{n,F}(\xi) - \sum_{k \in L} q_k(\xi) \right\|^2$$

(2.1)
$$= \sum_{k \in F} \|r_n^k(\xi) - q_k(\xi)\|^2 + \|p^{\perp}(\xi)\|^2 - \sum_{k \in F} \|q_k(\xi)\|^2$$

$$\leq \sum_{k \in L} \|r_n^k(\xi) - q_k(\xi)\|^2 + \|p^{\perp}(\xi)\|^2 - \sum_{k \in F} \|q_k(\xi)\|^2$$

Since $\lim_{n \in \mathsf{N}} \|r_n^k(\xi) - q_k(\xi)\|^2 = 0$ it follows from (2.1) that

$$\lim_{(n,F)} \|g_{n,F}(\xi) - p^{\perp}(\xi)\|^2 = 0.$$

We have shown that $\mathrm{SOT} - \lim_{(n,F)} g_{n,F} = p^{\perp}$.

If $F = \{k_1, \ldots, k_r\} \subset L$ then $g_{n,F} = \sum_{m=1}^{r} r_n^{k_m}$ where $r_n^{k_m} = \|\cdot\| - \lim_{i \in \mathsf{N}} s_i^{n,k_m} t_i^{n,k_m}$. So $g_{n,F} = \|\cdot\| - \lim_{i \in \mathsf{N}} s_i^{n,F} t_i^{n,F}$ where $s_i^{n,F} = (s_i^{n,k_1}, \ldots, s_i^{n,k_r})$, $t_i^{n,F} = (t_i^{n,k_1}, \ldots, t_i^{n,k_r})^T$. Since $s_i^{n,k_j} = q_{k_j} s_i^{n,k_j}$, $t_i^{n,k_j} = t_i^{n,k_j} q_{k_j}$ and the projections (q_{k_j}) are pairwise orthogonal we conclude that $s_i^{n,F} \in \mathrm{Ball}(R_{\infty}^{\mathrm{fin}}(Y_0))$, $t_i^{n,F} \in \mathrm{Ball}(C_{\infty}^{\mathrm{fin}}(X_0))$ for all (n, F). This completes the proof.

THEOREM 2.5. *There exists a net of finite rank contractions* $(f_{\lambda})_{\lambda \in \Lambda}$ *converging in the strong operator topology to the identity operator* I_{H_1} *such that* $f_{\lambda} = \|\cdot\| - \lim_{i \in \mathsf{N}} v_i^{\lambda} u_i^{\lambda}$ *where* $(v_i^{\lambda})_{i \in \mathsf{N}} \subset \mathrm{Ball}(R_{\infty}^{\mathrm{fin}}(Y_0))$, $(u_i^{\lambda})_{i \in \mathsf{N}} \subset \mathrm{Ball}(C_{\infty}^{\mathrm{fin}}(X_0))$ *for all* $\lambda \in \Lambda$.

PROOF. If $p^{\perp} = 0$ the conclusion follows from Lemma 2.1. So we may assume that $p^{\perp} \neq 0$. From Lemmas 2.1, 2.4 there exists a net $(l_{\lambda})_{\lambda \in \Lambda}$ of finite rank contractions SOT-converging to the projection p such that $l_{\lambda} = s_{\lambda} t_{\lambda}$ where $s_{\lambda} \in \mathrm{Ball}(R_{\infty}^{\mathrm{fin}}(Y_0))$, $t_{\lambda} \in \mathrm{Ball}(C_{\infty}^{\mathrm{fin}}(X_0))$ for all $\lambda \in \Lambda$, and a net $(g_{\lambda})_{\lambda \in \Lambda}$ of finite rank contractions converging in the strong operator topology to $p^{\perp}$ such that $g_{\lambda} = \|\cdot\| - \lim_{i \in \mathsf{N}} y_i^{\lambda} x_i^{\lambda}$ for all $\lambda \in \Lambda$ where

$$y_i^{\lambda} \in \mathrm{Ball}(R_{\infty}^{\mathrm{fin}}(Y_0)), \qquad x_i^{\lambda} \in \mathrm{Ball}(C_{\infty}^{\mathrm{fin}}(X_0))$$

for all $i \in \mathsf{N}$.

We denote $f_{\lambda} = l_{\lambda} + g_{\lambda}$, $v_i^{\lambda} = (s_{\lambda}, y_i^{\lambda})$, $u_i^{\lambda} = (t_{\lambda}, x_i^{\lambda})^T$ for all $\lambda \in \Lambda, i \in \mathsf{N}$. Observe that $I_{H_1} = \mathrm{SOT} - \lim_{\lambda \in \Lambda} f_{\lambda}$ and $f_{\lambda} = \|\cdot\| - \lim_{i \in \mathsf{N}} v_i^{\lambda} u_i^{\lambda}$. Now we have

$$\|v_i^{\lambda}\|^2 = \|s_{\lambda} s_{\lambda}^* + y_i^{\lambda} (y_i^{\lambda})^*\| = \|p s_{\lambda} s_{\lambda}^* p + p^{\perp} y_i^{\lambda} (y_i^{\lambda})^* p^{\perp}\| \leq 1.$$

Similarly $\|u_i^{\lambda}\| \leq 1$ for all $\lambda \in \Lambda, i \in \mathsf{N}$.

THEOREM 2.6. *The algebras A_0, B_0 are strongly Morita equivalent. Particularly $A_0 \cong Y_0 \otimes_{B_0}^h X_0$, $B_0 \cong X_0 \otimes_{A_0}^h Y_0$ as operator modules.*

PROOF. We define the bilinear map $Y_0 \times X_0 \to A_0 : (y, x) \to yx$. This map is completely contractive and B_0-balanced. So induces a completely contractive A_0-module map $\pi : Y_0 \otimes_{B_0}^h X_0 \to A_0 : y \otimes_{B_0} x \to yx$. We shall prove that π is completely isometric. It suffices to prove that if

$$z_{i,j} = \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} x_k^{i,j}, \qquad i, j = 1, \ldots, n$$

then

$$\|(z_{i,j})_{i,j}\| \leq \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\|.$$

We recall the contractions f_λ, $(v_s^\lambda)_{s\in\mathsf{N}}$, $(u_s^\lambda)_{s\in\mathsf{N}}$, $\lambda \in \Lambda$ from Theorem 2.5.

If x is a compact operator then $x = \|\cdot\| - \lim_\lambda x f_\lambda$ ([7, Proposition 1.18]). It follows that $z_{i,j} = \|\cdot\| - \lim_\lambda \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} (x_k^{i,j} f_\lambda)$. If $\epsilon > 0$ there exists $\lambda \in \Lambda$ such that

$$\|(z_{i,j})_{i,j}\| - \epsilon < \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} (x_k^{i,j} f_\lambda) \right)_{i,j} \right\| - \frac{\epsilon}{2}.$$

Since $x_k^{i,j} f_\lambda = \|\cdot\| - \lim_{s\in\mathsf{N}} x_k^{i,j} v_s^\lambda u_s^\lambda$ there exists $s \in \mathsf{N}$ such that

$$\left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} (x_k^{i,j} f_\lambda) \right)_{i,j} \right\| - \frac{\epsilon}{2} < \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} (x_k^{i,j} v_s^\lambda u_s^\lambda) \right)_{i,j} \right\|.$$

But $x_k^{i,j} v_s^\lambda \in R_\infty^{\text{fin}}(B_0)$, so we have

$$\|(z_{i,j})_{i,j}\| - \epsilon < \left\| \left(\sum_{k=1}^{m_{i,j}} (y_k^{i,j} x_k^{i,j} v_s^\lambda) \otimes_{B_0} u_s^\lambda \right)_{i,j} \right\|$$

$$= \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \left(v_s^\lambda \otimes_{B_0} u_s^\lambda \oplus \cdots \oplus v_s^\lambda \otimes_{B_0} u_s^\lambda \right) \right\|$$

$$\leq \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\| \left\| v_s^\lambda \otimes_{B_0} u_s^\lambda \right\|$$

$$\leq \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\| \|v_s^\lambda\| \|u_s^\lambda\| \leq \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\|$$

Since ϵ was arbitrary we have

$$\|(z_{i,j})_{i,j}\| \le \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} x_k^{i,j} \right)_{i,j} \right\|.$$

We proved that π is completely isometric. It remains to prove that π is onto A_0. It suffices to prove that the space $Im\pi$ is dense in A_0.

Let $a \in \text{Ball}(A_0)$ and $\epsilon > 0$. Since $a = \|\cdot\| - \lim_\lambda f_\lambda a$ there exists $\lambda \in \Lambda$ such that $\|a - f_\lambda a\| < \frac{\epsilon}{2}$. But $f_\lambda = \|\cdot\| - \lim_s v_s^\lambda u_s^\lambda$, so there exists $s \in \mathsf{N}$ such that

$$\|f_\lambda - v_s^\lambda u_s^\lambda\| < \frac{\epsilon}{2}.$$

It follows that

$$\|a - v_s^\lambda u_s^\lambda a\| = \|a - \pi(v_s^\lambda \otimes_{B_0} (u_s^\lambda a))\| < \epsilon.$$

The proof is complete. Likewise we can prove that $B_0 \cong X_0 \otimes_{A_0}^h Y_0$.

We define the bilinear map $Y \times X \to A : (y, x) \to yx$. This map is completely contractive B-balanced and separately w^*-continuous, so induces a completely contractive w^*-continuous map $\rho : Y \otimes_B^{\sigma h} X \to A : y \otimes_B x \to yx$ which is also an A-module map. We shall prove that the restriction of ρ on the space $Y \otimes_B^h X$ is completely isometric and we shall use this fact in Theorem 2.9 to prove that $A \cong Y \otimes_B^{\sigma h} X$.

LEMMA 2.7. *The restriction of ρ on the space $Y \otimes_B^h X$ is completely isometric.*

PROOF. It suffices to prove that if

$$z_{i,j} = \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_B x_k^{i,j}, \qquad i, j = 1, \ldots, n$$

then

$$\|(z_{i,j})_{i,j}\| \le \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\|.$$

We recall the contractions f_λ, $(v_s^\lambda)_{s \in \mathsf{N}}$, $(u_s^\lambda)_{s \in \mathsf{N}}$, $\lambda \in \Lambda$ from Theorem 2.5. Fix

$\lambda \in \Lambda$. If $\epsilon > 0$ there exists $s \in \mathsf{N}$ such that

$$\left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_B (x_k^{i,j} f_\lambda) \right)_{i,j} \right\| - \epsilon$$

$$< \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_B (x_k^{i,j} v_s^\lambda u_s^\lambda) \right)_{i,j} \right\| = \left\| \left(\sum_{k=1}^{m_{i,j}} (y_k^{i,j} x_k^{i,j} v_s^\lambda) \otimes_B u_s^\lambda \right)_{i,j} \right\|$$

$$= \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} (v_s^\lambda \otimes_B u_s^\lambda \oplus \cdots \oplus v_s^\lambda \otimes_B u_s^\lambda) \right\|$$

$$\leq \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\|$$

It follows that

$$\left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_B (x_k^{i,j} f_\lambda) \right)_{i,j} \right\| \leq \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\|$$

for all $\lambda \in \Lambda$. Since

$$\left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_B x_k^{i,j} \right)_{i,j} = w^* - \lim_\lambda \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_B (x_k^{i,j} f_\lambda) \right)_{i,j}$$

we have

$$\left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_B x_k^{i,j} \right)_{i,j} \right\| \leq \left\| \left(\sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\|.$$

The second dual operator space A_0^{**} of the operator algebra A_0 is also an operator algebra with product describing in [5, Section 2.5]. The product on A_0^{**} extends the product on A_0. With this we mean that if $\iota : A_0 \to A_0^{**}$ is the canonical embedding then $\iota(ab) = \iota(a)\iota(b)$ for all $a, b \in A_0$. The following lemma is well known, so we omit the proof.

LEMMA 2.8. *The operator algebra A, (resp. B) is isomorphic as dual operator algebra with A_0^{**} (resp. B_0^{**}).*

We are now ready to present the main theorem of this paper:

THEOREM 2.9. (A) *The following are equivalent:*

 (i) *The nests $\mathcal{N}_1$, $\mathcal{N}_2$ are isomorphic.*
 (ii) *The algebras A_0, B_0 are strongly Morita equivalent.*
 (iii) *The algebras A, B are weakly-$*$ Morita equivalent.*

(B) *If $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ is a nest isomorphism, $X = \mathrm{Op}(\theta)$, $Y = \mathrm{Op}(\theta^{-1})$ then:*

  (i) $A_0 \cong Y_0 \otimes_{B_0}^h X_0$, $B_0 \cong X_0 \otimes_{A_0}^h Y_0$, *as operator bimodules,*

  (ii) $A \cong Y \otimes_B^{\sigma h} X$, $B \cong X \otimes_A^{\sigma h} Y$, *as dual operator bimodules.*

PROOF. (A) (i) $\Rightarrow$ (ii). This is Theorem 2.6.

(ii) $\Rightarrow$ (iii). If A_0 and B_0 are strongly Morita equivalent then the operator algebras A_0^{**} and B_0^{**} are weakly-$*$ Morita equivalent, [3, Section 3]. So by Lemma 2.8 A and B are weakly-$*$ Morita equivalent.

(iii) $\Rightarrow$ (i). Let (A, B, V, U) be a weak-$*$ Morita context [3]. It follows that there exist completely contractive separately w^*-continuous bilinear maps $(\cdot, \cdot) : V \times U \to A$ which is A-module and B-balanced map and $[\cdot, \cdot] : U \times V \to B$ which is B-module and A-balanced map satisfying

$$(y, x)y' = y[x, y'], \quad x'(y, x) = [x', y]x, \qquad \forall\, x, x' \in U,\ y, y' \in V,$$

and $A = \overline{\mathrm{span}}^{w^*}(\{(y, x) : x \in U, y \in V\})$, $B = \overline{\mathrm{span}}^{w^*}(\{[x, y] : x \in U, y \in V\})$.

If $N \in \mathcal{N}_1$ we define $\theta(N)$ the projection onto the space generated by vectors of the form $[xN, y](\omega)$, $x \in U$, $y \in V$, $\omega \in H_2$. Since $b[xN, y] = [bxN, y]$ for all $b \in B$ we have $\theta(N)^\perp B \theta(N) = 0$ so $\theta(N) \in \mathcal{N}_2$. Also if $N_1 \leq N_2$ then $\theta(N_1) \leq \theta(N_2)$ and so θ is an order preserving map from $\mathcal{N}_1$ into $\mathcal{N}_2$.

Similarly if $M \in \mathcal{N}_1$ we define $\sigma(M)$ the projection onto the space generated by vectors of the form $(yM, x)(\xi)$, $x \in U$, $y \in V$, $\xi \in H_1$. The map $\sigma : \mathcal{N}_2 \to \mathcal{N}_1$ is an order preserving map.

If $x, x' \in U$, $y \in V$ and $N \in \mathcal{N}_1$ then

$$\theta(N)^\perp[xN, y] = 0 \Rightarrow [\theta(N)^\perp xN, y] = 0$$
$$\Rightarrow [\theta(N)^\perp xN, y]x' = 0 \Rightarrow \theta(N)^\perp xN(y, x') = 0.$$

But the operators of the form (y, x') span the algebra A. So we have

$$(2.2) \qquad \theta(N)^\perp xN = 0 \Rightarrow xN = \theta(N)xN, \qquad \forall\, x \in U,\ N \in \mathcal{N}_1.$$

Similarly

$$(2.3) \qquad\qquad yM = \sigma(M)yM, \qquad \forall\, y \in V,\ M \in \mathcal{N}_2.$$

If $x, x' \in U$, $y, y' \in V$, $N \in \mathcal{N}_1$ we have

$$[xN^\perp, y][x'N, y'] = [[xN^\perp, y]x'N, y']$$
$$= [xN^\perp(y, x')N, y'] = 0 \quad \text{because}\ (y, x') \in \mathrm{Alg}(\mathcal{N}_1).$$

It follows that $[xN^\perp, y]\theta(N) = 0 \Rightarrow [x, N^\perp y\theta(N)] = 0$ for all $x \in U$, $y \in V$, and so

$$(2.4) \qquad N^\perp y\theta(N) = 0 \Rightarrow y\theta(N) = Ny\theta(N), \qquad \forall\, y \in V, \ N \in \mathcal{N}_1$$

Likewise we can prove

$$(2.5) \qquad\qquad x\sigma(M) = Mx\sigma(M), \qquad \forall\, x \in U, \ M \in \mathcal{N}_2$$

If $N \in \mathcal{N}_1$ and $x \in U$, $y \in V$ then $(y, x)N = (y, xN) = (y, \theta(N)xN)$ because of (2.2). The last operator is equal to $(y\theta(N), xN) = \sigma(\theta(N))(y, x)N$. It follows that $N \le \sigma(\theta(N))$.

Similarly $(y, x)^* N^\perp = (N^\perp(y, x))^* = (N^\perp y, x)^* = (N^\perp y\theta(N)^\perp, x)^*$ because of (2.4). On the other hand

$$(N^\perp y, \theta(N)^\perp x)^* = (N^\perp y, \theta(N)^\perp x\sigma(\theta(N))^\perp)^*$$
$$= \sigma(\theta(N))^\perp (N^\perp y, \theta(N)^\perp x)^*.$$

Since $I_{H_1} = w^* - \lim_i \sum_{k=1}^{n_i} (y_k^i, x_k^i)^*$ for $(y_k^i) \subset V$, $(x_k^i) \subset U$ we have $N^\perp \le \sigma(\theta(N))^\perp$ and so $N = \sigma(\theta(N))$.

Likewise we can prove that $M = \theta(\sigma(M))$ for all $M \in \mathcal{N}_2$. This completes the proof of the fact that θ is a nest isomorphism.

(B) Let $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ be a nest isomorphism and $X = \mathrm{Op}(\theta)$, $Y = \mathrm{Op}(\theta^{-1})$. Claim B(i) follows from Theorem 2.6.

Let $\rho : Y \otimes_B^{\sigma h} X \to A$ be the map which was defined above Lemma 2.7. Let $z \in \mathrm{Ball}(M_n(Y \otimes_B^{\sigma h} X))$ for a fixed $n \in \mathsf{N}$. By [3, Corollary 2.8] there exists a net $(z_i) \subset \mathrm{Ball}(M_n(Y \otimes_B^h X))$ converging in the w^* topology to z. It follows that $\rho(z_i) \xrightarrow{w^*} \rho(z)$ in $M_n(A)$. If f, g are finite rank operators in A we denote $f^n = f \oplus f \oplus \cdots \oplus f$ and similarly for g^n. We have that

$$f^n \rho(z_i) g^n \xrightarrow{\|\cdot\|} f^n \rho(z) g^n \implies \rho(f^n z_i g^n) \xrightarrow{\|\cdot\|} \rho(f^n z g^n).$$

From Lemma 2.7 it follows that $f^n z g^n \in M_n(Y \otimes_B^h X)$ and $\|\rho(f^n z g^n)\| = \|f^n z g^n\|$ for all finite rank operators f, g in A. We recall the finite rank contractions $(f_\lambda)_{\lambda \in \Lambda}$ from Theorem 2.5. For $\lambda, \mu \in \Lambda$ we have

$$\|f_\lambda^n z f_\mu^n\| = \|\rho(f_\lambda^n z f_\mu^n)\| = \|f_\lambda^n \rho(z) f_\mu^n\| \le \|\rho(z)\|.$$

Since $z f_\mu^n = w^* - \lim_\lambda f_\lambda^n z f_\mu^n$ we have $\|z f_\mu^n\| \le \|\rho(z)\|$ for all $\mu \in \Lambda$. Now taking the w^*-limit of $(z f_\mu^n)_{\mu \in \Lambda}$ we obtain $\|z\| \le \|\rho(z)\|$. We proved that

the map $\rho : Y \otimes_B^{\sigma h} X \to A$ is a complete isometry. From Theorem 2.6 and its proof we have that $A_0 = \overline{\text{span}}(Y_0 X_0)$. The equality $A = \overline{A_0}^{w^*}$ implies

$$A = \overline{\text{span}}^{w^*}(YX) = \overline{\text{span}}^{w^*}(\{\rho(y \otimes_B x) : y \in Y, x \in X\}).$$

By the Krein-Smulian Theorem the space $Im\rho$ is w^*-closed and so ρ is onto A. Similarly we can prove that $B \cong X \otimes_A^{\sigma h} Y$, as dual operator modules.

3. Spatial Morita equivalence and nest algebras

In this section we shall investigate the relation between weak-$*$ and spatial Morita equivalence for nest algebras. We give the definition of spatial Morita equivalence:

DEFINITION 3.1 (I. G. Todorov). Let C, D be w^*-closed algebras acting on the Hilbert spaces K_1, K_2 respectively. We say that C and D are *spatially Morita equivalent* if there exists a $D-C$ bimodule $V \subset B(K_1, K_2)$ and a $C-D$ bimodule $U \subset B(K_2, K_1)$ such that $C = \overline{\text{span}}^{w^*}(UV)$, $D = \overline{\text{span}}^{w^*}(VU)$.

We also need the following notions: If $\mathscr{L}$ is a set of projections acting on the Hilbert space H the set

$$\text{Alg}(\mathscr{L}) = \{x \in B(H) : p^\perp x p = 0, \forall\, p \in \mathscr{L}\}$$

is an algebra. An algebra A is called reflexive if there exists a set of projections $\mathscr{L}$ such that $A = \text{Alg}(\mathscr{L})$. In the special case where $\mathscr{L}$ is a complete lattice of commuting projections containing the zero and identity operators the algebra $\text{Alg}(\mathscr{L})$ is called a CSL algebra and the lattice $\mathscr{L}$ is called a CSL lattice. Obviously, nest algebras are CSL algebras. If A is an algebra acting on the Hilbert space H the lattice

$$\{p \in \text{pr}(B(H)) : p^\perp x p = 0, \forall\, x \in A\}$$

is called the lattice of A and we denote it by $\text{Lat}(A)$. If $\mathscr{L}$ is a CSL lattice then $\text{Lat}(\text{Alg}(\mathscr{L})) = \mathscr{L}$, [1], [8].

Two spatially Morita equivalent algebras are not always weakly-$*$ Morita equivalent even in the case one of them is a CSL algebra:

EXAMPLE 3.2. Let C be a nest algebra. We denote the algebras $A = C \oplus C$ and

$$B = \left\{ \begin{pmatrix} a & b - a \\ 0 & b \end{pmatrix} : a, b \in C \right\}.$$

Observe that A is a CSL algebra whose lattice is

$$\text{Lat}(A) = \{p \oplus q : p, q \in \text{Lat}(C)\}.$$

Since the center of C is trivial [7, Corollary 19.5] the center of A is $Z(A) = \mathsf{C} \oplus \mathsf{C}$ and the center of B is

$$Z(B) = \left\{ \begin{pmatrix} \lambda & \mu - \lambda \\ 0 & \mu \end{pmatrix} : \lambda, \mu \in \mathsf{C} \right\}.$$

We also denote the spaces

$$X = \left\{ \begin{pmatrix} a & -a \\ 0 & b \end{pmatrix} : a, b \in C \right\}, \qquad Y = \left\{ \begin{pmatrix} a & b \\ 0 & b \end{pmatrix} : a, b \in C \right\}.$$

We can check that X is an $A - B$ bimodule, Y is a $B - A$ bimodule and $XY = A$, $YX = B$. So the algebras A, B are spatially Morita equivalent. If A and B were weakly-$*$ Morita equivalent by [3, Theorem 3.7] they would have isomorphic centers through a completely isometric homomorphism. This is a contradiction because $Z(A)$ is a von Neumann algebra and $Z(B)$ is a non-selfadjoint algebra.

Despite the above example, in [10] we proved that two CSL algebras are spatially Morita equivalent if and only if their lattices are isomorphic, so by Theorem 2.9 we conclude the following theorem:

THEOREM 3.3. *Two nest algebras are spatially Morita equivalent if and only if they are weakly-$*$ Morita equivalent.*

Also despite Example 3.2 we have the following theorem:

THEOREM 3.4. *Let A be a nest algebra, B be a unital dual operator algebra and β be a completely isometric normal representation of B such that A and $\beta(B)$ are spatially Morita equivalent. It follows that A and B are weakly-$*$ Morita equivalent.*

PROOF. By [10, Theorem 4.1, Remark 4.2] $\beta(B)$ is a nest algebra whose nest is isomorphic with the nest of A. The conclusion follows from Theorems 2.9 and 3.3.

THEOREM 3.5 (Blecher-Kashyap). *If A, B are weakly-$*$ Morita equivalent unital dual operator algebras, for every completely isometric normal representation α of A there exists a completely isometric normal representation β of B such that the algebras $\alpha(A)$, $\beta(B)$ are spatially Morita equivalent.*

PROOF. Suppose that (A, B, X, Y) is a weak-$*$ Morita context [3]. We construct the proof by using arguments and ideas which appeared in [3]. If α is a completely isometric normal representation of A on the Hilbert space H the

tensor product $K = Y \otimes_A^{\sigma h} H$ with its norm is a Hilbert space on which B is represented through the w^*-continuous complete isometry β given by

$$\beta(b)(y \otimes h) = (by) \otimes h \qquad \forall\, b \in B,\, y \in Y,\, h \in H.$$

Also Blecher and Kashyap prove that the maps $\phi : Y \to B(H, K)$, $\psi : X \to B(K, H)$ given by $\phi(y)(h) = y \otimes h$ and $\psi(x)(y \otimes h) = \alpha((x, y))(h)$ are w^*-continuous complete isometries. See [3] for the properties of the bilinear map $(\cdot, \cdot) : X \times Y \to A$. We can easily check that $\psi(X)$ is an $\alpha(A) - \beta(B)$ bimodule, $\phi(Y)$ is a $\beta(B) - \alpha(A)$ bimodule and

$$\alpha(A) = \overline{\text{span}}^{w^*}(\psi(X)\phi(Y)), \qquad \beta(B) = \overline{\text{span}}^{w^*}(\phi(Y)\psi(X)).$$

COROLLARY 3.6. *If A is a nest algebra and if A and B are weakly-$*$ Morita equivalent then there exists a completely isometric normal representation β of B such that $\beta(B)$ is a nest algebra.*

PROOF. Theorem 3.5 implies that there exists a completely isometric normal representation β of B such that the algebras A and $\beta(B)$ are spatially Morita equivalent. From [10, Remark 4.1] the algebra $\beta(B)$ is reflexive and from [10, Theorem 4.2] the lattice of $\beta(B)$ is isomorphic with the nest of A. So $\beta(B)$ is a nest algebra.

COROLLARY 3.7. *If A is a CSL algebra which is not a nest algebra then A is not weakly-$*$ Morita equivalent with a nest algebra.*

PROOF. Corollary 3.6 implies that if A was weakly-$*$ Morita equivalent with a nest algebra then it would have a normal completely isometric representation α such that $\alpha(A)$ is a nest algebra. This is a contradiction since as we can easily check $\alpha(\text{Lat}(A)) = \text{Lat}(\alpha(A))$.

4. A stable isomorphism theorem for nest algebras

In this section we are going to present a new theorem which characterizes the stable isomorphism of separably acting nest algebras.

DEFINITION 4.1. Two dual operator algebras C, D are called *stably isomorphic* if there exists a Hilbert space H and a completely isometric, w^*-bicontinuous isomorphism from the algebra $C \,\bar{\otimes}\, B(H)$ onto the algebra $D \,\bar{\otimes}\, B(H)$, where $\bar{\otimes}$ is the normal spatial tensor product.

We give two relevant definitions:

DEFINITION 4.2 ([10]). Let C, D be w^* closed algebras acting on Hilbert spaces H_1 and H_2 respectively. If there exists a TRO $\mathcal{M} \subset B(H_1, H_2)$, i.e.

a subspace satisfying $\mathcal{M}\mathcal{M}^*\mathcal{M} \subset \mathcal{M}$, such that $C = \overline{\operatorname{span}}^{w^*}(\mathcal{M}^* D \mathcal{M})$ and $D = \overline{\operatorname{span}}^{w^*}(\mathcal{M} C \mathcal{M}^*)$ we write $C \overset{\mathcal{M}}{\sim} D$. We say that the algebras C, D are TRO-*equivalent* if there exists a TRO $\mathcal{M}$ such that $C \overset{\mathcal{M}}{\sim} D$.

DEFINITION 4.3 ([11]). Let C, D be abstract dual operator algebras. These algebras are called Δ-*equivalent* if they have completely isometric normal representations ϕ, ψ such that the algebras $\phi(C)$, $\psi(D)$ are TRO-equivalent.

In [13] we proved the following theorem:

THEOREM 4.4. *Two unital dual operator algebras are stably isomorphic if and only if they are Δ-equivalent.*

Blecher and Kashyap have shown that Δ-equivalence implies weak-$*$ Morita equivalence [3, Section 3]. The converse does not hold. The counterexample is [12, Example 3.7]. We shall give a new proof of this fact in Example 4.10.

Theorem 3.2 in [12] implies the following corollary:

COROLLARY 4.5. *Two nest algebras are Δ-equivalent if and only if they are TRO-equivalent.*

In what follows if X is a subset of $B(H)$ where H is a Hilbert space we denote by X' the commutant of X and by X'' the algebra $(X')'$. In [10] we proved the following criterion of TRO-equivalence for reflexive algebras:

THEOREM 4.6. *Two reflexive algebras C, D are TRO-equivalent if and only if there exists a $*$-isomorphism $\delta : (C \cap C^*)' \to (D \cap D^*)'$ such that $\delta(\operatorname{Lat} C) = \operatorname{Lat} D$.*

Comparing Theorems 4.4, 4.6 and Corollary 4.5 we get the following:

COROLLARY 4.7. *The nest algebras $\operatorname{Alg} \mathcal{N}_1$, $\operatorname{Alg} \mathcal{N}_2$ are stably isomorphic if and only if there exists a $*$-isomorphism $\delta : \mathcal{N}_1'' \to \mathcal{N}_2''$ such that $\delta(\mathcal{N}_1) = \mathcal{N}_2$.*

In the rest of this section we fix two nests $\mathcal{N}_1$, $\mathcal{N}_2$ acting on the separable Hilbert spaces H_1, H_2 respectively and we denote $A = \operatorname{Alg} \mathcal{N}_1$, $B = \operatorname{Alg} \mathcal{N}_2$. We use now extensively notions from [7, Section 7]. If ξ (resp. ω) is a unit separating vector for the algebra $\mathcal{N}_1''$ (resp. $\mathcal{N}_2''$) we define the order isomorphism ϕ_ξ (resp. ψ_ω) from $\mathcal{N}_1$ (resp. $\mathcal{N}_2$) onto a closed subset of the interval $[0, 1]$ given by $\phi_\xi(N) = \|N(\xi)\|^2$ (resp. $\psi_\omega(M) = \|M(\omega)\|^2$).

Suppose that $[0, 1] \backslash \phi_\xi(\mathcal{N}_1) = \bigcup_n (l_n, r_n)$ and $[0, 1] \backslash \psi_\omega(\mathcal{N}_2) = \bigcup_n (t_n, s_n)$.

If m is the Lebesgue measure we define the measures μ_ξ, ν_ω given by

$$\mu_\xi(S) = m(S \cap \phi_\xi(\mathcal{N}_1)) + \sum_{r_n \in S}(r_n - l_n)$$

$$\nu_\omega(S) = m(S \cap \psi_\omega(\mathcal{N}_2)) + \sum_{s_n \in S}(s_n - t_n),$$

for every Borel subset S of $[0, 1]$. We denote $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) the nest $\{M_s : 0 \leq s \leq 1\} \subset B(L^2([0, 1], \mu_\xi))$ (resp. $\{N_s : 0 \leq s \leq 1\} \subset B(L^2([0, 1], \nu_\omega)))$ where M_s (resp. N_s) is the projection onto the space $L^2([0, s], \mu_\xi)$ (resp. $L^2([0, s], \nu_\omega)$).

The algebra $\mathcal{N}_1''$ is $*$-isomorphic with the algebra $L^\infty([0, 1], \mu_\xi)$ (resp. $L^\infty([0, 1], \nu_\omega)$) acting on the Hilbert space $L^2([0, 1], \mu_\xi)$ (resp. $L^2([0, 1], \nu_\omega)$) through an isomorphism mapping the nest $\mathcal{N}_1$ (resp. $\mathcal{N}_2$) onto $\mathcal{M}_1$ (resp. $\mathcal{M}_2$).

We denote by AbsHom($[0, 1]$) the set of order homeomorphisms $\alpha : [0, 1] \to [0, 1]$ which satisfy the property $m(S) = 0 \Rightarrow m(\alpha(S)) = 0$. The theorem below describes when two separably acting nest algebras are stably isomorphic.

THEOREM 4.8. *The algebras A, B are stably isomorphic if and only if there exist separating unit vectors ξ for $\mathcal{N}_1''$, ω for $\mathcal{N}_2''$ and $\alpha \in$ AbsHom($[0, 1]$) such that $\alpha(\phi_\xi(\mathcal{N}_1)) = \psi_\omega(\mathcal{N}_2)$.*

PROOF. Suppose that the algebras A, B are stably isomorphic. From Corollary 4.7 there exists a $*$-isomorphism $\delta : \mathcal{N}_1'' \to \mathcal{N}_2''$ such that $\delta(\mathcal{N}_1) = \mathcal{N}_2$. Fix separating unit vectors ξ for $\mathcal{N}_1''$, and ω for $\mathcal{N}_2''$. Taking compositions we obtain a $*$-isomorphism

$$\widetilde{\delta} : L^\infty([0, 1], \mu_\xi) \to L^\infty([0, 1], \nu_\omega)$$

such that $\widetilde{\delta}(\mathcal{M}_1) = \mathcal{M}_2$. Every isomorphism between maximal abelian selfadjoint algebras is implementing by a unitary. So the nests $\mathcal{M}_1$, $\mathcal{M}_2$ are unitarily equivalent. By [7, Theorem 7.23] there exists $\alpha \in$ AbsHom($[0, 1]$) such that $\alpha(\phi_\xi(\mathcal{N}_1)) = \psi_\omega(\mathcal{N}_2)$.

Conversely if there exist such ξ, ω and α, by the same theorem there exists a unitary $u \in B(L^2([0, 1], \mu_\xi), L^2([0, 1], \nu_\omega))$ such that $u^*\mathcal{M}_2 u = \mathcal{M}_1$. It follows that $L^\infty([0, 1], \mu_\xi) = u^* L^\infty([0, 1], \nu_\omega)u$. Taking compositions we take a $*$-isomorphism $\delta : \mathcal{N}_1'' \to \mathcal{N}_2''$ such that $\delta(\mathcal{N}_1) = \mathcal{N}_2$. Again from Corollary 4.7 we conclude that the algebras A and B are stably isomorphic.

REMARK 4.9. If there exist separating unit vectors ξ for $\mathcal{N}_1''$, ω for $\mathcal{N}_2''$ and $\alpha \in$ AbsHom($[0, 1]$) such that $\alpha(\phi_\xi(\mathcal{N}_1)) = \psi_\omega(\mathcal{N}_2)$ then for all separating unit vectors ξ_1 for $\mathcal{N}_1''$ and ω_1 for $\mathcal{N}_2''$ there exists $\alpha_1 \in$ AbsHom($[0, 1]$) such that $\alpha_1(\phi_{\xi_1}(\mathcal{N}_1)) = \psi_{\omega_1}(\mathcal{N}_2)$. This is a consequence of [7, Proposition 7.22].

The next example shows that weak-∗ Morita equivalence is strictly weaker than Δ-equivalence.

EXAMPLE 4.10. Let C be the Cantor set, γ be an order homeomorphism of $[0, 1]$ such that $m(\gamma(C)) > 0$. Suppose that $[0, 1] \setminus C = \bigcup_n (l_n, r_n)$ and $[0, 1] \setminus \gamma(C) = \bigcup_n (t_n, s_n)$. We denote by μ the measure

$$\mu(S) = \sum_{r_n \in S} (r_n - l_n)$$

and by ν the measure

$$\nu(S) = m(S \cap \gamma(C)) + \sum_{s_n \in S} (s_n - t_n).$$

We denote $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) the nest $\{M_s : 0 \leq s \leq 1\} \subset B(L^2([0, 1], \mu))$ (resp. $\{N_s : 0 \leq s \leq 1\} \subset B(L^2([0, 1], \nu)))$ where M_s (resp. N_s) is the projection onto the space $L^2([0, s], \mu)$ (resp. $L^2([0, s], \nu)$).

The map $\theta : \mathcal{M}_1 \to \mathcal{M}_2 : M_s \to N_{\gamma(s)}$ is a nest isomorphism, so by Theorem 2.9 the algebras $A = \mathrm{Alg}\,\mathcal{M}_1$, $B = \mathrm{Alg}\,\mathcal{M}_2$ are weakly-∗ Morita equivalent. If the algebras A, B were Δ-equivalent by Theorem 4.8 there would exist unit vectors ξ for $\mathcal{M}_1''$, ω for $\mathcal{M}_2''$ and $\alpha \in Abs\,Hom([0, 1])$ such that $\alpha(\phi_\xi(\mathcal{M}_1)) = \psi_\omega(\mathcal{M}_2)$. From [7, Proposition 7.22] we have that $m(\phi_\xi(\mathcal{M}_1)) = m(C) = 0$ and since $m(\gamma(C)) > 0$ we have that $m(\psi_\omega(\mathcal{M}_2)) > 0$. This is a contradiction.

5. A counterexample in Morita equivalence

In this section we shall use the notions of TRO equivalence, of Δ-equivalence, of stable isomorphism and we shall consider nest and CSL algebras. See the appropriate definitions in Sections 1, 3 and 4. If C and D are *unital* operator algebras which are strongly Morita equivalent then for every $\epsilon > 0$ there exists a completely bounded isomorphism from $C \otimes_{\min} \mathcal{K}$ onto $D \otimes_{min} \mathcal{K}$ with $\|\rho\|_{cb} < 1 + \epsilon$ and $\|\rho^{-1}\|_{cb} < 1 + \epsilon$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space H and $\otimes_{\min}$ is the spatial tensor product [6, Corollary 7.10]. It follows that for every $\epsilon > 0$ there exists a completely bounded w^*-continuous isomorphism σ from $C^{**} \, \bar{\otimes} \, B(H)$ onto $D^{**} \, \bar{\otimes} \, B(H)$ with $\|\sigma\|_{cb} < 1 + \epsilon$ and $\|\sigma^{-1}\|_{cb} < 1 + \epsilon$, where $\bar{\otimes}$ is the normal spatial tensor product. One may wonder, if the operator algebras C^{**} and D^{**} are stably isomorphic.

In this section we give a negative answer to this question. We present a counterexample of unital strongly Morita equivalent algebras C and D whose second duals are not stably isomorphic. Moreover for these algebras there

exist normal completely isometric representations ϕ and ψ such that for every $\epsilon > 0$ there exists an invertible bounded operator T_ϵ satisfying $\|T_\epsilon\| < 1 + \epsilon$, $\|T_\epsilon^{-1}\| < 1 + \epsilon$, $\phi(C^{**}) = T_\epsilon^{-1}\psi(D^{**})T_\epsilon$ and $\phi(C) = T_\epsilon^{-1}\psi(D)T_\epsilon$.

Two nests $\mathcal{N}$ and $\mathcal{M}$ acting on the separable Hilbert spaces H and K respectively are called *similar* if there exists an order isomorphism $\theta : \mathcal{N} \to \mathcal{M}$ which preserves dimension of intervals. We say that an invertible operator $S \in B(H, K)$ implements θ if $\theta(N)$ is the projection onto $SN(H)$ for all $N \in \mathcal{N}$. In what follows if C is an operator algebra, $\Delta(C)$ is its diagonal $C \cap C^*$.

We fix similar nests $\mathcal{N}$ and $\mathcal{M}$ with corresponding nest algebras $A = \mathrm{Alg}(\mathcal{N})$ and $B = \mathrm{Alg}(\mathcal{M})$. We also assume that $\Delta(A)$ is a totally atomic maximal abelian selfadjoint algebra (*masa* in sequel) and $\Delta(B)$ is a masa with a nontrivial continuous part, [7, Example 13.15]. Suppose that $\theta : \mathcal{N} \to \mathcal{M}$ is an order isomorphism implementing similarity for $\mathcal{N}, \mathcal{M}$. We denote by A_0 (resp. B_0) the algebra of compact operators belonging to A (resp. B) and by A_1 (resp. B_1) the operator algebra $A_0 + \mathsf{C}I_H$ (resp. $B_0 + \mathsf{C}I_K$). We denote by X the space $\mathrm{Op}(\theta)$ and by Y the space $\mathrm{Op}(\theta^{-1})$.

THEOREM 5.1 (Davidson, [7, Theorem 13.20]). *For every $\epsilon > 0$ there exists an invertible bounded operator S_ϵ which implements θ such that $\|S_\epsilon\| < 1 + \epsilon$, $\|S_\epsilon^{-1}\| < 1 + \epsilon$. (Observe that $S_\epsilon \in X$ and $S_\epsilon^{-1} \in Y$ for all $\epsilon > 0$.)*

Suppose that $j : A_1 \to A_1^{**}$ is the canonical embedding. We denote by J_A the space $\overline{j(A_0)}^{w^*}$.

LEMMA 5.2.
 (i) $A_1^{**} = J_A + \mathsf{C}I$
 (ii) $J_A \cap \mathsf{C}I = 0$.

PROOF. (i) Since $|\lambda| \leq \|a + \lambda I_H\|$ for all compact operators a the functional

$$\rho : A_1 \to \mathsf{C} : a + \lambda I_H \to \lambda$$

belongs to A_1^*. If $x \in A_1^{**}$ by the Goldstine Theorem there exists a net $(a_i + \lambda_i I_H) \subset A_0 + \mathsf{C}I$ converging in the w^*-topology to x. Since (λ_i) converges to $\rho(x)$ the net (a_i) converges to $a \in J_A$ and so $x = a + \rho(x) \in J_A + \mathsf{C}I$.

 (ii) Since $\rho|_{A_0} = 0$ if $\lambda I \in J_A$ then $\lambda = 0$. So $J_A \cap \mathsf{C}I = 0$.

Suppose that $\iota : A_0 \to A_0^{**}$ is the canonical embedding. There exists a w^*-continuous completely isometric onto homomorphism $\phi : A \to A_0^{**}$ extending ι.

The map $\phi|_{A_1} : A_1 \to A_0^{**}$ extends to a w^*-continuous completely contractive map $\widehat{\phi} : A_1^{**} \to A_0^{**}$ satisfying $\widehat{\phi}(j(a)) = \phi(a)$ for all $a \in A_1$. Also

the completely contractive map $j|_{A_0} : A_0 \to A_1^{**}$ extends to a w^*-continuous completely contractive map $\widehat{\kappa} : A_0^{**} \to A_1^{**}$ such that $\widehat{\kappa}(\iota(a)) = j(a)$ for all $a \in A_0$. So the map $\widehat{\phi} \circ \widehat{\kappa} : A_0^{**} \to A_0^{**}$ satisfies

$$\widehat{\phi} \circ \widehat{\kappa}(\iota(a)) = \widehat{\phi}(j(a)) = \phi(a) = \iota(a)$$

for all $a \in A_0$. It follows that $\widehat{\phi} \circ \widehat{\kappa} = id_{A_0^{**}}$. Therefore $\widehat{\kappa}$ is a complete isometry.

We denote by θ the w^*-continuous completely isometric homomorphism $\widehat{\kappa} \circ \phi : A \to A_1^{**}$. Observe that

$$\theta(A) = \widehat{\kappa}(\phi(A)) = \widehat{\kappa}(\overline{\iota(A_0)}^{w^*}) = \overline{j(A_0)}^{w^*} = J_A.$$

Suppose that p is the projection $\theta(I_H)$. Lemma 5.2 implies that $p^\perp \neq 0$ and $A_1^{**} = J_A \oplus \mathsf{C}p^\perp$.

LEMMA 5.3. *The algebra A_1^{**} is completely isometric and w^*- continuously isomorphic with the algebra $A \oplus \mathsf{C}$ acting on the Hilbert space $H \oplus \mathsf{C}$.*

PROOF. We define the map θ and the projection p as in the above discussion. We also define the completely isometric normal representation

$$\pi : A_1^{**} = J_A \oplus \mathsf{C}p^\perp \to B(H \oplus \mathsf{C}) : a \oplus \lambda p^\perp \to \theta^{-1}(a) \oplus \lambda$$

which is onto $A \oplus \mathsf{C}$.

For every $\epsilon > 0$ we denote by T_ϵ the bounded invertible operator $S_\epsilon \oplus id_\mathsf{C} \in B(H \oplus \mathsf{C}, K \oplus \mathsf{C})$. We also denote the spaces $U = X \oplus \mathsf{C} \subset B(H \oplus \mathsf{C}, K \oplus \mathsf{C})$ and $V = Y \oplus \mathsf{C} \subset B(K \oplus \mathsf{C}, H \oplus \mathsf{C})$. Note that U is a $B \oplus \mathsf{C} - A \oplus \mathsf{C}$ bimodule and V is an $A \oplus \mathsf{C} - B \oplus \mathsf{C}$ bimodule.

By the above lemma $\pi(A_1^{**}) = A \oplus \mathsf{C}$. If $j : A_1 \to A_1^{**}$ is the canonical embedding we deduce $\pi(j(a)) = a \oplus 0$ for all $a \in A_0$ and $\pi(j(id_{A_1})) = id_{H \oplus \mathsf{C}}$. Thus,

$$\pi(j(A_1)) = \mathrm{span}\{a \oplus 0, id_{H \oplus \mathsf{C}}, a \in A_0\}.$$

Similarly if $j_2 : B_1 \to B_1^{**}$ is the canonical embedding there exists a normal completely isometric onto homomorphism $\rho : B_1^{**} \to B \oplus \mathsf{C}$ such that

$$\rho(j_2(B_1)) = \mathrm{span}\{b \oplus 0, id_{K \oplus \mathsf{C}}, b \in B_0\}.$$

Since $S_\epsilon^{-1} B_0 S_\epsilon = A_0$ and $S_\epsilon^{-1} B S_\epsilon = A$ we conclude that

$$T_\epsilon^{-1} \rho(j_2(B_1)) T_\epsilon = \pi(j(A_1)), \qquad T_\epsilon^{-1} \rho(B_1^{**}) T_\epsilon = \pi(A_1^{**})$$

for all $\epsilon > 0$.

In the following Lemmas 5.4, 5.5 we identify the algebras A_1^{**} with $A \oplus \mathsf{C}$, B_1^{**} with $B \oplus \mathsf{C}$, A_1 with $\pi(j(A_1))$ and B_1 with $\rho(j_2(B_1))$.

LEMMA 5.4. *The algebras A_1^{**} and B_1^{**} are weakly-∗ Morita equivalent.*

PROOF. Let U, V and $T_\epsilon, \epsilon > 0$ be as in the above discussion. The completely contractive bilinear map $V \times U \rightarrow A_1^{**} : (v, u) \rightarrow vu$ is separately w^*-continuous, B_1^{**}-balanced and A_1^{**}-module map. Hence, it induces the w^*-continuous completely contractive and A_1^{**}-module map

$$\tau : V \otimes_{B_1^{**}}^{\sigma h} U \rightarrow A_1^{**} : v \otimes_{B_1^{**}} u \rightarrow vu.$$

We shall prove that τ is isometric: If $(v_i) \subset V$, $(u_i) \subset U$ and $\epsilon > 0$ we have:

$$\left\| \sum_{i=1}^n v_i \otimes_{B_1^{**}} u_i \right\| = \left\| \sum_{i=1}^n (T_\epsilon^{-1} T_\epsilon v_i) \otimes_{B_1^{**}} u_i \right\|.$$

Since $T_\epsilon v_i \in UV \subset B_1^{**}$ the last norm is equal with

$$\left\| \sum_{i=1}^n T_\epsilon^{-1} \otimes_{B_1^{**}} (T_\epsilon v_i u_i) \right\| = \left\| (T_\epsilon^{-1} \otimes_{B_1^{**}} T_\epsilon) \left(\sum_{i=1}^n v_i u_i \right) \right\|$$

$$\leq \|T_\epsilon^{-1}\| \|T_\epsilon\| \left\| \sum_{i=1}^n v_i u_i \right\| \leq (1 + \epsilon)^2 \left\| \sum_{i=1}^n v_i u_i \right\|.$$

We let $\epsilon \rightarrow 0$ and we have that

$$\left\| \sum_{i=1}^n v_i \otimes_{B_1^{**}} u_i \right\| = \left\| \sum_{i=1}^n v_i u_i \right\|.$$

Similarly we can prove that τ is completely isometric. The equality $A = \overline{\operatorname{span}}^{w^*}(YX)$ implies that $A_1^{**} = \overline{\operatorname{span}}^{w^*}(VU)$. Therefore, by the Krein-Smulian Theorem τ is onto A_1^{**}. The proof of the fact $B_1^{**} \cong U \otimes_{A_1^{**}}^{\sigma h} V$ is similar.

LEMMA 5.5. *The algebras A_1 and B_1 are strongly Morita equivalent.*

PROOF. It suffices to prove that they have equivalent categories of left operator modules [2]. If C is an operator algebra we denote by $_C\operatorname{mod}$ the category of left operator modules over C. We assume that every $Z \in {_C\operatorname{mod}}$ is essential, i.e. the linear span of CZ is dense in Z. If $Z_1, Z_2 \in {_C\operatorname{mod}}$ the space of morphisms $\operatorname{Hom}_C(Z_1, Z_2)$ is the space of completely bounded maps $F : Z_1 \rightarrow Z_2$ which are C-module maps.

We fix an operator $T = T_{\epsilon_0}$ for $\epsilon_0 > 0$. If $Z \in {}_{A_1}\mathrm{mod}$ then Z^{**} is a left dual operator module over A_1^{**} in a canonical way [5, 3.8.9]. We denote by $\mathscr{F}(Z)$ the subspace of $U \otimes_{A_1^{**}}^{\sigma h} Z^{**}$

$$\mathscr{F}(Z) = \overline{\mathrm{span}}(Ta \otimes_{A_1^{**}} z : a \in A_1, z \in Z).$$

Since $U \otimes_{A_1^{**}}^{\sigma h} Z^{**}$ is a left operator module over B_1^{**} and

$$b(Ta \otimes_{A_1^{**}} z) = (bTa) \otimes_{A_1^{**}} z = T(T^{-1}bTa) \otimes_{A_1^{**}} z$$

with $T^{-1}bT \in A_1$ for all $b \in B_1$, we deduce that $\mathscr{F}(Z)$ is a left operator B_1-module.

If $W \in {}_{B_1}\mathrm{mod}$, we denote by $\mathscr{G}(W)$ the subspace of $V \otimes_{B_1^{**}}^{\sigma h} W^{**}$

$$\mathscr{G}(W) = \overline{\mathrm{span}}(aT^{-1} \otimes_{B_1^{**}} w : a \in A_1, w \in W).$$

Since $V \otimes_{B_1^{**}}^{\sigma h} W^{**}$ is a left operator module over A_1^{**}, $\mathscr{G}(W)$ clearly belongs to ${}_{\mathscr{A}_1}\mathrm{mod}$.

The space

$$\mathscr{G}(\mathscr{F}(Z)) = \overline{\mathrm{span}}(a_2T^{-1} \otimes_{B_1^{**}} Ta_1 \otimes_{A_1^{**}} z : a_1, a_2 \in A_1, z \in Z)$$

is a left operator module over A_1 and subspace of the space $V \otimes_{B_1^{**}}^{\sigma h} U \otimes_{A_1^{**}}^{\sigma h} Z^{**}$. The w^*-Morita equivalence $A_1^{**} \cong V \otimes_{B_1^{**}}^{\sigma h} U$, $B_1^{**} \cong U \otimes_{A_1^{**}}^{\sigma h} V$ induces ([3, Theorem 3.5]) a complete isometry

$$V \otimes_{B_1^{**}}^{\sigma h} U \otimes_{A_1^{**}}^{\sigma h} Z^{**} \to Z^{**} : v \otimes_{B_1^{**}} u \otimes_{A_1^{**}} z \to vuz$$

which restricts to a completely isometric map

$$R_Z : \mathscr{G}(\mathscr{F}(Z)) \to Z : a_2T^{-1} \otimes_{B_1^{**}} Ta_1 \otimes_{A_1^{**}} z \to a_2a_1z$$

for all $a_1, a_2 \in A_1, z \in Z$. This map is clearly onto Z.

Every morphism $F \in \mathrm{Hom}_{A_1}(Z_1, Z_2)$ can be extended to a morphism $\widehat{F}$ belonging to $\mathrm{Hom}_{A_1^{**}}^{\sigma}(Z_1^{**}, Z_2^{**})$, the space of w^*-continuous completely bounded A_1^{**}-module maps. (The paragraph [5, 1.4.8] can be used as a proof).

Due to weak-$*$ Morita equivalence $A_1^{**} \cong V \otimes_{B_1^{**}}^{\sigma h} U$, $B_1^{**} \cong U \otimes_{A_1^{**}}^{\sigma h} V$, there exists ([3, Theorem 3.5]) a normal completely contractive functor $\widehat{\mathscr{F}}$ between the left dual operator modules of A_1^{**} and B_1^{**} such that

$$\widehat{\mathscr{F}}(\widehat{F}) : U \otimes_{A_1^{**}}^{\sigma h} Z_1^{**} \to U \otimes_{A_1^{**}}^{\sigma h} Z_2^{**} : u \otimes_{A_1^{**}} z \to u \otimes_{A_1^{**}} \widehat{F}(z).$$

Since

$$\widehat{\mathscr{F}}(\widehat{F})(Ta \otimes_{A_1^{**}} z) = Ta \otimes_{A_1^{**}} F(z)$$

for all $a \in A_1$, $z \in Z_1$ the operator $\widehat{\mathscr{F}}(\widehat{F})$ maps $\mathscr{F}(Z_1)$ into $\mathscr{F}(Z_2)$. As a result we are enabled to define

$$\mathscr{F}(F) = \widehat{\mathscr{F}}(\widehat{F})|_{\mathscr{F}(Z_1)} : \mathscr{F}(Z_1) \to \mathscr{F}(Z_2).$$

We can easily check that $\mathscr{F}(F) \in \mathrm{Hom}_{B_1}(\mathscr{F}(Z_1), \mathscr{F}(Z_2))$.

This way we define functors $\mathscr{F} : {}_{A_1}\mathrm{mod} \to {}_{B_1}\mathrm{mod}$ and $\mathscr{G} : {}_{B_1}\mathrm{mod} \to {}_{A_1}\mathrm{mod}$. Using the above complete isometries $\{R_Z : Z \in {}_{A_1}\mathrm{mod}\}$ we prove that the functor $\mathscr{G}\mathscr{F}$ is equivalent to the identity functor $1_{{}_{A_1}\mathrm{mod}}$ and the functor $\mathscr{F}\mathscr{G}$ is equivalent to the identity functor $1_{{}_{B_1}\mathrm{mod}}$.

THEOREM 5.6. *Strong Morita equivalence of unital operator algebras doesn't imply Δ-equivalence of the second dual operator algebras.*

PROOF. We recall the unital operator algebras A_1 and B_1 which are strongly Morita equivalent by the above lemma. We shall prove that the algebras A_1^{**} and B_1^{**} are not Δ-equivalent. Suppose that they are Δ-equivalent. We define the completely isometric normal representation (see Lemma 5.3)

$$\pi : A_1^{**} \to B(H \oplus \mathsf{C}) : a \oplus \lambda p^\perp \to \theta^{-1}(a) \oplus \lambda.$$

The algebra $\pi(A_1^{**}) = A \oplus \mathsf{C}$ is a CSL algebra with lattice

$$\{N \oplus 0, N \oplus \mathsf{C} : N \in \mathcal{N}\}.$$

Suppose that $B_1^{**} = J_B \oplus \mathsf{C}q^\perp$ where q is the identity of the algebra J_B and J_B is isomorphic with the algebra B. By [12, Theorem 2.7] there exists a completely isometric normal representation σ of B_1^{**} on a Hilbert space $K_1 \oplus K_2$ of the form $\sigma(b \oplus \lambda q^\perp) = \sigma_1(b) \oplus \lambda I_{K_2}$ for all $b \in J_B$, $\lambda \in \mathsf{C}$ such that the algebras $\pi(A_1^{**})$, $\sigma(B_1^{**})$ are TRO equivalent. Since $\pi(A_1^{**})$ is a CSL algebra, $\sigma(B_1^{**})$ is also a CSL algebra, [10, Remark 5.5]. Thus the algebra $\sigma(B_1^{**})$ contains a masa and hence $\dim K_2 = 1$. Therefore we are enabled to assume that $\sigma(B_1^{**})$ is a CSL algebra acting on $K_1 \oplus \mathsf{C}$.

Since $\Delta(A)$ (resp. $\Delta(B)$) is a masa, then $\Delta(\pi(A_1^{**}))$ (resp. $\Delta(\sigma(B_1^{**}))$) is also a masa. The algebras $\Delta(\pi(A_1^{**}))$, $\Delta(\sigma(B_1^{**}))$ are TRO equivalent [10, Proposition 2.5], but TRO equivalence between masas is a unitary equivalence (use for example [10, Theorem 3.2]). This is a contradiction because $\Delta(\pi(A_1^{**})) = \Delta(A) \oplus \mathsf{C}$ is a totally atomic masa and the masa $\Delta(\sigma(B_1^{**})) \cong \Delta(B) \oplus \mathsf{C}$ has a nontrivial continuous part. As a result the algebras A_1^{**}, B_1^{**} are not Δ-equivalent.

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