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EXTENSION OF POSITIVE CURRENTS WITH SPECIAL PROPERTIES OF MONGE-AMPÈRE OPERATORS

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Abstract

In this paper we study the extension of currents across small obstacles. Our main results are: 1) Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbf{C}^n$ and T be a negative current of bidimension (p, p) on $\Omega \setminus A$ such that $dd^c T \geq -S$ on $\Omega \setminus A$ for some positive plurisubharmonic current S on Ω. Assume that the Hausdorff measure $\mathscr{H}_{2p}(A \cap \overline{\operatorname{Supp} T}) = 0$. Then $\widetilde{T}$ exists. Furthermore, the current $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is negative supported in A. 2) Let u be a positive strictly k-convex function on an open subset Ω of $\mathbf{C}^n$ and set $A = \{z \in \Omega : u(z) = 0\}$. Let T be a negative current of bidimension (p, p) on $\Omega \setminus A$ such that $dd^c T \geq -S$ on $\Omega \setminus A$ for some positive plurisubharmonic (or dd^c-negative) current S on Ω. If $p \geq k + 1$, then $\widetilde{T}$ exists. If $p \geq k + 2$, $dd^c S \leq 0$ and u of class $\mathscr{C}^2$, then $\widetilde{dd^c T}$ exists and $\widetilde{dd^c T} = dd^c \widetilde{T}$.

1. Introduction

In this paper we continue the work in [2]. So throughout this paper, we suppose that A is a closed subset of an open subset Ω of $\mathbf{C}^n$ and T is a positive (resp. negative) current of bidimension (p, p) of $\Omega \setminus A$ such that $dd^c T \leq S$ (resp. $dd^c T \geq -S$) on $\Omega \setminus A$ for some current S on Ω. Our main issue is about finding the sufficient conditions on S and A that guarantee the existence of $\widetilde{T}$ and $\widetilde{dd^c T}$, and afterword studying the features of these extensions and the relations between it. In the literature, this kind of problems have been studied before. For instance, the studies in [5], [6], [11], [13], [15], [17], [19], [20] and [21], were basically based on the case when T is a closed positive current. The case when $S = 0$ considered by Dabbek, Elkhadhra and El Mir [10]. In 2009, Dabbek and Noureddine [9] discussed the case when S is closed and positive. As you see the closedness takes its place in this kind of study, so it is natural to ask about the sharpness of the closedness of S specially in [9]. Now, you can feel the theme in this paper which is about generalizing the work in [9] by replacing the closedness of S by further conditions.

The paper is divided into three sections. In the first section we give definitions, basic properties and some facts about currents.

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In the second one, we consider the case when A is a closed complete pluripolar set and prove our first main result.

1ST MAIN THEOREM 3.3. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a negative current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT \geq -S$ on $\Omega\setminus A$ for some positive plurisubharmonic current S on Ω. Assume that $\mathscr{H}_{2p}(A\cap\overline{\operatorname{Supp} T})=0$. Then $\widetilde{T}$ exists. Furthermore, the current $R=\widetilde{dd^cT}-dd^c\widetilde{T}$ is negative supported in A. If $dd^cS\leq 0$, then $\widetilde{T}$ has the same properties of T.*

Using the above result, we obtained a version of Chern-Levine-Nirenberg inequality.

THEOREM 3.5. *Let A be a closed complete pluripolar subset of an open set Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\leq S$ on $\Omega\setminus A$ for some positive plurisubharmonic (resp. dd^c-negative) current S on Ω. Let K and L compact sets in Ω with $L\subset K^\circ$. Assume that $\mathscr{H}_{2p}(A\cap\overline{\operatorname{Supp} T})=0$, then there exists a constant $C_{K,L}>0$ such that for all u plurisubharmonic function on Ω of class $\mathscr{C}^2$ we have the following estimate*

$$\int_{L\setminus A} T\wedge dd^cu\wedge\beta^{p-1}\leq C_{K,L}\|u\|_{\mathscr{L}^\infty(K)}(\|\widetilde{T}\|_K+\|\widetilde{dd^cT}\|_K).$$

In the third section, we start with the case where A is a zero set of strictly k-convex function and include our second main result.

2ND MAIN THEOREM 4.7. *Let Ω be an open subset of $\mathbb{C}^n$ and u be a positive strictly k-convex function on Ω. Set $A=\{z\in\Omega: u(z)=0\}$ and let T be a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\leq S$ on $\Omega\setminus A$ for some positive and plurisubharmonic (or dd^c-negative) current S on Ω. If $p\geq k+1$, then $\widetilde{T}$ exists. If $p\geq k+2$, $dd^cS\leq 0$ and u is of class $\mathscr{C}^2$, then $\widetilde{dd^cT}$ exists and $\widetilde{dd^cT}=dd^c\widetilde{T}$.*

We end this paper by assuming that A is a closed set and proving the following theorem.

THEOREM 4.10. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a negative current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\geq -S$ on $\Omega\setminus A$ for some positive current S on Ω. Assume that $\mathscr{H}_{2p-2}(\overline{\operatorname{Supp} T}\cap A)$ is locally finite. Then $\widetilde{T}$ exists. If $dd^cS\leq 0$, then $\widetilde{dd^cT}$ exists and $R=\widetilde{dd^cT}-dd^c\widetilde{T}$ is negative a current supported in A.*

In 1972, Harvey [17] proved the previous result for the closed positive current T when $\mathscr{H}_{2p-1}(A)=0$. The case where $S=0$ was considered in [10].

In the inspiring article of this work [9], the authors proved the case when S is a closed positive current.

2. Preliminaries and notations

Let Ω be an open subset of $\mathbb{C}^n$. Let $\mathscr{D}_{p,q}(\Omega, k)$ be the space $\mathscr{C}^k$ compactly supported differential forms of bidegree (p, q) on Ω. A form $\varphi \in \mathscr{D}_{p,p}(\Omega, k)$ is said to be strongly positive form if φ can be written as

$$\varphi(z) = \sum_{j=1}^{N} \gamma_j(z) i\alpha_{1,j} \wedge \overline{\alpha}_{1,j} \wedge \cdots \wedge i\alpha_{p,j} \wedge \overline{\alpha}_{p,j},$$

where $\gamma_j \geq 0$ and $\alpha_{s,j} \in \mathscr{D}_{0,1}(\Omega, k)$. Then $\mathscr{D}_{p,p}(\Omega, k)$ admits a basis consisting of strongly positive forms. The dual space $\mathscr{D}'_{p,q}(\Omega, k)$ is the space of currents of bidimension (p, q) or bidegree $(n - p, n - q)$ and of order k. A current $T \in \mathscr{D}'_{p,p}(\Omega, k)$ is said to be positive if $\langle T, \varphi\rangle \geq 0$ for all forms $\varphi \in \mathscr{D}_{p,p}(\Omega, k)$ that are strongly positive. If $T \in \mathscr{D}'_{p,p}(\Omega, k)$ then it can be written as

$$T = i^{(n-p)^2} \sum_{|I|=|J|=n-p} T_{I,J} dz_I \wedge d\overline{z}_J,$$

where $T_{I,J}$ are distributions on Ω. For the positive current $T \in \mathscr{D}'_{p,p}(\Omega, k)$ the mass of T is denoted by $\|T\|$ and defined by $\sum |T_{I,J}|$ where $|T_{I,J}|$ are the total variations of the measures $T_{I,J}$. Let $\beta = dd^c|z|^2$ be the Kähler form on $\mathbb{C}^n$ (where $d = \partial + \overline{\partial}$ and $d^c = i(-\partial + \overline{\partial})$, thus $dd^c = 2i\partial\overline{\partial}$), then for each open subset $\Omega_1 \subset \Omega$ there exists a constant $C > 0$ depends only on n and p such that

$$T \wedge \frac{\beta^p}{2^p p!}(\Omega_1) \leq \|T\|_{\Omega_1} \leq C\, T \wedge \beta^p(\Omega_1).$$

Along the way a current T is said to be closed if $dT = 0$. A current T is said to be plurisubharmonic if $dd^c T$ is a positive current. Let (χ_n) be a smooth bounded sequence which vanishes on a neighborhood of closed subset $A \subset \Omega$ and χ_n converges to $\mathbb{1}_{\Omega\setminus A}$ the characteristic function of $\Omega \setminus A$, and T be a current defined on $\Omega \setminus A$. If $\chi_n T$ has a limit which does not depend on (χ_n), this limit is the trivial extension of T by zero across A noted by $\widetilde{T}$. Thus, $\widetilde{T}$ exists if and only if $\|T\|$ is locally finite across A.

A current T is said to be $\mathbb{C}$-normal if T and $dd^c T$ are of locally finite mass. We recall that T is a $\mathbb{C}$-flat current if $T = F + \partial H + \overline{\partial} S + \partial\overline{\partial} R$, where F, H, S and R are currents with locally integrable coefficients. On this class of currents, the support theorem says that for $\mathbb{C}$-flat current T of bidimension

(p, p) if $\mathscr{H}_{2p}(\operatorname{Supp} T) = 0$, then $T = 0$ (see [4], Theorem 1.13). Let $k \leq p$ and $T \in \mathscr{D}'_{p,p}(\Omega)$ with locally integrable coefficients. Set $\pi : \mathbb{C}^n \to \mathbb{C}^k$, $\pi(z', z'') = z'$ and $i_{z'} : \mathbb{C}^{n-k} \to \mathbb{C}^n$, $i_{z'}(z'') = (z', z'')$. Then the slice $\langle T, \pi, z' \rangle$, which is defined by

$$\langle T, \pi, z' \rangle(\varphi) = \int_{z'' \in \pi^{-1}(z')} i_{z'}^* T(z'') \wedge i_{z'}^* \varphi(z''), \quad \forall \varphi \in \mathscr{D}_{p-k,p-k}(\Omega),$$

is well defined $(p-k, p-k)$-current for a.e z', and supported in $\pi^{-1}(z')$. Notice that, by the pull back assumptions we obtain

$$\begin{aligned} dd^c \langle T, \pi, z' \rangle &= \langle dd^c T, \pi, z' \rangle, \\ d^c \langle T, \pi, z' \rangle &= \langle d^c T, \pi, z' \rangle, \\ \text{and} \qquad d \langle T, \pi, z' \rangle &= \langle dT, \pi, z' \rangle. \end{aligned}$$

So, we deduce that for every $\mathbb{C}$-flat current T, the slice $\langle T, \pi, z' \rangle$ is well defined for a.e z'. Moreover, we have the slicing formula

$$\int_\Omega T \wedge \varphi \wedge \pi^* \beta'^k = \int_{z' \in \pi(\Omega)} \langle T, \pi, z' \rangle(\varphi) \beta'^k,$$

where $\beta' = dd^c|z'|^2$. this formula is helpful in many cases. actually, by this formula we can prove the properties of T by testing it for its slice.[1]

We end this section by giving the following two theorems. The first one is called Chern-Levine-Nirenberg inequality and the second is a modification for that inequality proved by Al Ameer [3].

THEOREM 2.1. *Let Ω be an open subset of $\mathbb{C}^n$ and T be a closed positive current of bidimension (p, p). Let $u_1, \ldots, u_q$ are locally bounded plurisubharmonic functions on Ω. For all compact subsets K, L of Ω with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ such that*

$$\|T \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_q\|_L \leq C_{K,L} \|T\|_K \|u_1\|_{\mathscr{L}^\infty(K)} \ldots \|u_q\|_{\mathscr{L}^\infty(K)}.$$

THEOREM 2.2. *Let Ω be an open subset of $\mathbb{C}^n$. Let K and L compact sets in Ω with $L \subset K^\circ$. Assume that $T \in \mathscr{D}'_{p,p}(\Omega)$ is positive and $dd^c T$ is of order zero, then there exists a constant $C_{K,L} > 0$ such that for all plurisubharmonic function u on Ω of class $\mathscr{C}^2$ we have the following estimate*

$$\|T \wedge dd^c u\|_L \leq C_{K,L} \|u\|_{\mathscr{L}^\infty(K)} (\|T\|_K + \|dd^c T\|_K). \tag{2.1}$$

[1] More about positive currents and slice formula can be found in [11] and [16].

For convenience, we include Al Ameer's proof in our setting

PROOF. With out loss of generality we may assume that $0 \in K$ and $B(0,r) \Subset K$. Let

$$\max_{\varepsilon}(x_1,x_2) = \max(x_1,x_2) * \alpha_\varepsilon,$$

where α_ε is a regularization kernel on R^2 depending only on $\|(x_1,x_2)\|$. Fix ε_0 small enough and set

$$\phi_{\varepsilon_0} = \max_{\varepsilon_0}\left(g, H\left(|z|^2 - \frac{r^2}{3}\right)\right), \qquad \text{where} \quad H = \frac{48}{r^2}\|g\|_{\mathscr{L}^\infty(K)}.$$

Hence, on $B\big(0,\frac{r}{2}\big)$ we have $\phi_{\varepsilon_0} = g$ and $\phi_{\varepsilon_0} = H\big(|z|^2 - \frac{r^2}{3}\big)$ on $B(0,r) \setminus B\big(0,\frac{3r}{4}\big)$. This implies that

$$\int_{B(0,\frac{r}{2})} T \wedge dd^c g \wedge \beta^{p-1} \leq \int_{B(0,r)} T \wedge dd^c \phi_{\varepsilon_0} \wedge \beta^{p-1}. \tag{2.2}$$

Now, choose $0 < \delta < \frac{r}{4}$ and take a smooth function φ such that $0 \leq \varphi \leq 1$ compactly supported in $\{z \in \Omega : r - \delta < |z| < r + \delta\}$ and $\varphi = 1$ on a neighborhood of $\partial B(0,r)$. Let T_ε be a smoothing of T which is convergent weakly* to T, hence using Stokes' formula we find

$$\begin{aligned}\int_{B(0,r)} T_\varepsilon \wedge dd^c \phi_{\varepsilon_0} \wedge \beta^{p-1} &= \int_{B(0,r)} T_\varepsilon \wedge dd^c(\varphi\phi_{\varepsilon_0} + (1-\varphi)\phi_{\varepsilon_0}) \wedge \beta^{p-1} \\ &= \int_{B(0,r)} T_\varepsilon \wedge dd^c(\varphi\phi_{\varepsilon_0}) \wedge \beta^{p-1} \\ &\quad + \int_{B(0,r)} (1-\varphi)\phi_{\varepsilon_0} dd^c T_\varepsilon \wedge \beta^{p-1}.\end{aligned}$$

But on $\operatorname{Supp}\varphi \cap B(0,r)$ we have $\varphi\phi_{\varepsilon_0} = \varphi H\big(|z|^2 - \frac{r^2}{3}\big)$, thus

$$\begin{aligned}\int_{B(0,r)} T_\varepsilon \wedge dd^c \phi_{\varepsilon_0} \wedge \beta^{p-1} \leq H &\left|\int_{B(0,r)} T_\varepsilon \wedge dd^c\left(\varphi\left(|z|^2 - \frac{r^2}{3}\right)\right) \wedge \beta^{p-1}\right| \\ &+ \left|\int_{B(0,r)} (1-\varphi)\phi_{\varepsilon_0} dd^c T_\varepsilon \wedge \beta^{p-1}\right|\end{aligned} \tag{2.3}$$

using the fact that T is a positive current and $dd^c T$ is of order zero in the (2.3), we can find $C'_{K,L} > 0$ such that

$$\int_{B(0,r)} T_\varepsilon \wedge dd^c \phi_{\varepsilon_0} \wedge \beta^{p-1} \leq C'_{K,L}\|g\|_{\mathscr{L}^\infty(K)}(\|T_\varepsilon\|_K + \|dd^c T_\varepsilon\|_K) \tag{2.4}$$

and (2.1) follows from (2.2) and (2.4), after choosing an appropriate cover for the compact set L.

3. The case when A is closed pluripolar set

In this section we show our first main result. For a closed current T the result was done by El Mir and Feki [15]. The case when $S = 0$ was proved in [10] by Dabbek, Elkhadhra and El Mir. Recently, Dabbek and Noureddine [9] have shown the result when S is a positive closed current. The proof of our main result will pass through several steps. So let us first give an inequality which is very useful in our study.

LEMMA 3.1. *Let Ω be an open subset of $\mathbb{C}^n$. Let K and L compact sets in Ω with $L \subset K^\circ$. Assume that T is a positive and plurisubharmonic (resp. dd^c-negative) current of bidimension (p, p) on Ω, then there exists a constant $C_{K,L} > 0$ such that for all plurisubharmonic functions $u_1, \dots, u_q$, $1 \le q \le p$ of class $\mathscr{C}^2$ we have*

$$\|T \wedge dd^c u_1 \wedge \dots \wedge dd^c u_q\|_L \le C_{K,L} \prod_{j=1}^{q} \|u_j\|_{\mathscr{L}^\infty(K)}(\|T\|_K + \|dd^c T\|_K).$$

PROOF. By induction, the case when $q = 1$ follows from Theorem 2.2. To show the case $q = 2$, let us take L_1 compact subset such that L_1° contains L and $L_1^\circ \subset K$. Since $T \wedge dd^c u_1$ is positive and plurisubharmonic (resp. dd^c-negative), then we get

$$\begin{aligned}&\|T \wedge dd^c u_1 \wedge dd^c u_2\|_L \\ &\qquad \le C^{(1)}_{K,L} \|u_2\|_{\mathscr{L}^\infty(K)}(\|T \wedge dd^c u_1\|_{L_1} + \|dd^c T \wedge dd^c u_1\|_{L_1}).\end{aligned}$$

But $dd^c T \wedge dd^c u_1$ closed and positive (resp. negative) so by the first step and Chern-Levine-Nirenberg inequality we have

$$\begin{aligned}&\|T \wedge dd^c u_1 \wedge dd^c u_2\|_L \\ &\qquad \le C^{(1)}_{K,L} \|u_2\|_{\mathscr{L}^\infty(K)}\big[C^{(2)}_{K,L} \|u_1\|_{\mathscr{L}^\infty(K)}(\|T\|_K + 2\|dd^c T\|_K)\big] \\ &\qquad \le C_{K,L} \|u_1\|_{\mathscr{L}^\infty(K)} \|u_2\|_{\mathscr{L}^\infty(K)}(\|T\|_K + 2\|dd^c T\|_K).\end{aligned}$$

Assume that the inequality holds for $q-1$. We want to show the inequality for q. Since $T \wedge \bigwedge_{j=1}^{q} dd^c u_j$ is a positive and plurisubharmonic (resp. dd^c-negative) current, then similarly as we have done above we deduce

$$
\begin{aligned}
&\|T \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_q\|_L \\
&\quad \leq C^{(1)}_{K,L} \|u_q\|_{\mathscr{L}^\infty(K)} \left(\left\| T \wedge \bigwedge_{j=1}^{q-1} dd^c u_j \right\|_{L_1} + \left\| dd^c T \wedge \bigwedge_{j=1}^{q-1} dd^c u_j \right\|_{L_1} \right) \\
&\quad \leq C^{(2)}_{K,L} \prod_{j=1}^{q} \|u_j\|_{\mathscr{L}^\infty(K)} (\|T\|_K + (q-1)\|dd^c T\|_K) \\
&\qquad + C^{(3)}_{K,L} \prod_{j=1}^{q} \|u_j\|_{\mathscr{L}^\infty(K)} \|dd^c T\|_K \\
&\quad \leq C_{K,L} \prod_{j=1}^{q} \|u_j\|_{\mathscr{L}^\infty(K)} (\|T\|_K + q\|dd^c T\|_K).
\end{aligned}
$$

Proving our lemma.

PROPOSITION 3.2. *Let A be a closed complete pluripolar subset of an open subset $\Omega \subset \mathbb{C}^n$ and T be a positive current of bidimension (p, p) on $\Omega \setminus A$ such that $dd^c T \leq S$ on $\Omega \setminus A$ for some positive and plurisubharmonic (resp. dd^c-negative) current S on Ω. Let v be a plurisubharmonic function of class $\mathscr{C}^2$, $v \geq -1$ on Ω such that*

$$\Omega' = \{z \in \Omega : v(z) < 0\}$$

is relatively compact in Ω. Let $K \subset \Omega'$ be a compact subset and let us set

$$c_K = -\sup_{z \in K} v(z).$$

Then there exists a constant $\eta \geq 0$ such that for all integer $1 \leq s \leq p$ and for every plurisubharmonic function u on Ω' of class $\mathscr{C}^2$ satisfying that $-1 \leq u < 0$ we have,

$$\int_{K \setminus A} T \wedge (dd^c u)^p \leq c_K^{-s} \int_{\Omega' \setminus A} T \wedge (dd^c v)^s \wedge (dd^c u)^{p-s} + \eta(\|S\|_{\Omega'} + \|dd^c S\|_{\Omega'}).$$

This proposition generalizes a result in [10] where the authors considered the case of positive and dd^c-negative currents. The case when S is a closed positive done in [9].

PROOF. We follow the same techniques as in [10]. By ([13], Proposition II.2) there exists a negative plurisubharmonic function f on Ω' which is smooth on $\Omega' \setminus A$ such that

$$A \cap \Omega' = \{z \in \Omega' : f(z) = -\infty\}.$$

We choose λ, μ such that $0 < \mu < \lambda < c_K$. For $m \in \mathsf{N}$ and ε small enough we set

$$\varphi_m(z) = \mu u(z) + \frac{f(z)+m}{m+1} \quad \text{and} \quad \varphi_{m,\varepsilon}(z) = \max_\varepsilon(v(z)+1, \varphi_m(z)), \tag{3.1}$$

where $\max_\varepsilon$ is the convolution of the function $(x_1, x_2) \mapsto \max(x_1, x_2)$ by a positive regularization kernel on R^2 depending only on $\|(x_1, x_2)\|$. Thus we have $\varphi_{m,\varepsilon}(z) \in \mathrm{Psh}(\Omega') \cap C^\infty(\Omega')$. Furthermore, $\varphi_{m,\varepsilon}(z) = v(z) + 1$ in a neighborhood of $\partial\Omega' \cup (\Omega' \cap \{f \le -m\})$. Consider the open subset

$$\Omega'_m = \Omega' \cap \{f > -m\}.$$

Then by Stokes' formula we have

$$\int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1} \wedge dd^c(\varphi_{m,\varepsilon} - v - 1)$$
$$\le \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1}.$$

Hence

$$\begin{aligned} &\int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^s \\ &\quad\le \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1} \\ &\qquad + \int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1} \wedge dd^c v. \end{aligned} \tag{3.2}$$

Let us set

$$S_{k,\varepsilon} := \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1-k} \wedge (dd^c v)^k.$$

By iterating the operation in (3.2), we deduce that

$$\int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^s \le \int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c v)^s + \sum_{k=0}^{s-1} S_{k,\varepsilon}.$$

Let $R > 0$ and $K_R = \{z \in K : f(z) \ge -R\}$. For m sufficiently large, $K_R \subset \Omega'_m$ and for any $z \in K_R$,

$$\varphi_m(z) \ge -\mu + \frac{m-R}{m+1} > 1 - \lambda.$$

Moreover, $v \le -c_K$ on K_R so we get

$$v + 1 \le 1 - c_K \le 1 - \lambda$$

then $\varphi_{m,\varepsilon} = \varphi_m$ in a neighborhood of K_R. Therefore, by the above inequality we obtain

$$\int_{K_R} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_m)^s \le \int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c v)^s + \sum_{k=0}^{s-1} S_{k,\varepsilon}.$$

It is very remarkable that $(dd^c \varphi_m)^s \ge \mu^s (dd^c u)^s$ since $dd^c f \ge 0$. So

$$\mu^s \int_{K_R} T \wedge (dd^c u)^p \le \int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c v)^s + \sum_{k=0}^{s-1} S_{k,\varepsilon}. \tag{3.3}$$

Notice that each $S_{k,\varepsilon}$ is bounded independently of ε. Indeed, since S is a positive plurisubharmonic current and $\varphi_{m,\varepsilon} - v - 1 = 0$ on $\partial\Omega'_m$, then by the previous lemma there exists $\eta_k \ge 0$ such that

$$S_{k,\varepsilon} \le \eta_k \|u\|^{p-s}_{\mathscr{L}^\infty(\Omega')} \|\varphi_{m,\varepsilon}\|^{s-k-1}_{\mathscr{L}^\infty(\Omega')} \|v\|^k_{\mathscr{L}^\infty(\Omega')} (\|S\|_{\Omega'} + \|dd^c S\|_{\Omega'}). \tag{3.4}$$

Therefore there exists $\eta \ge 0$ making (3.3) as follows

$$\begin{aligned}\mu^s \int_{K_R} T \wedge (dd^c u)^p \le \int_{\Omega'_m} & T \wedge (dd^c u)^{p-s} \wedge (dd^c v)^s \\ & + \eta(\|S\|_{\Omega'} + (p-1)\|dd^c S\|_{\Omega'}).\end{aligned}$$

We finish the proof by letting first $m \to \infty$ and secondly $R \to \infty$.

Now we will prove our first main theorem using the same technique as in [10], and Proposition 3.2.

THEOREM 3.3. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a negative current of bidimension (p, p) on $\Omega \setminus A$ such that $dd^c T \ge -S$ on $\Omega \setminus A$ for some positive plurisubharmonic current S on Ω. Assume that $\mathscr{H}_{2p}(A \cap \overline{\operatorname{Supp} T}) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is negative supported in A. If $dd^c S \le 0$, then $\widetilde{T}$ has the same properties as T.*

PROOF. Let us first assume that $\widetilde{T}$ exists. Then by ([12], Theorem 1.3), the extension $\widetilde{dd^c T}$ exists and R is a negative current. If S is a dd^c-negative current then by ([10], Proposition 2) the current $-\widetilde{S}$ is negative plurisubharmonic. So $\widetilde{T}$ is negative and $dd^c \widetilde{T} \ge \widetilde{dd^c T} \ge -\widetilde{S}$. In other words, $\widetilde{T}$ and T are of the same class.

In order to show the existence of $\widetilde{T}$ we will proceed as in [10]. Since the problem is local, we will show that T is of locally finite mass near every point z_0 in A. Without loss of generality, one can assume that z_0 is the origin. Since $\mathscr{H}_{2p}(A \cap \overline{\operatorname{Supp} T}) = 0$, then by [19] there exist a system of coordinates (z', z'') of $\mathbb{C}^p \times \mathbb{C}^{n-p}$ and a polydisk $\triangle^p \times \triangle^{n-p} \subset \mathbb{C}^p \times \mathbb{C}^{n-p}$ such that $(A \cap \overline{\operatorname{Supp} T}) \cap (\triangle^p \times \partial\triangle^{n-p}) = \emptyset$. Moreover, the projection map $\pi : (A \cap \overline{\operatorname{Supp} T}) \cap (\triangle^p \times \triangle^{n-p}) \to \triangle^p$ is proper, and as $\pi(A \cap \overline{\operatorname{Supp} T})$ is closed with a zero Lebesgue measure in $\triangle^p$, one can find an open subset $O \subset \triangle^p \setminus \pi(A \cap \overline{\operatorname{Supp} T})$. Therefore the current has locally finite mass on $O \times \triangle^{n-p}$. Let $0 < \delta < 1$ such that $(A \cap \overline{\operatorname{Supp} T}) \cap (\triangle^p \times \{z'', \delta < |z''| < 1\}) = \emptyset$, and fix a and t two real numbers such that $\delta < a < t < 1$. Set

$$\rho_\varepsilon = \max_\varepsilon\left(\pi^*\rho, \frac{1}{t^2 - a^2}(|z''|^2 - t^2)\right), \tag{3.5}$$

where ρ is a smooth plurisubharmonic function on $\triangle^p$ such that $(dd^c\rho)^p$ supported in O. We have $-1 \le \rho_\varepsilon < 0$ in $t\triangle^n$ and $\rho_\varepsilon = \pi^*\rho$ on $|z''| \le a$, and we obtain

$$\begin{aligned}\int_{(t\triangle^n)\setminus A} T \wedge (dd^c\rho_\varepsilon)^p = \int_{(t\triangle^p)\times\{|z''|<a\}\setminus A} & T \wedge (dd^c(\pi^*\rho))^p \\ & + \int_{(t\triangle^p)\times\{a \le |z''| < t\}} T \wedge (dd^c\rho_\varepsilon)^p,\end{aligned}$$

since $(dd^c\pi^*\rho)^p$ supported in $O \times \triangle^{n-p}$ then both integrals of the right hand side are finite. By applying Proposition 3.2 on $-T$, we deduce that $\widetilde{T}$ exists.

Of course the condition on the Hausdorff dimension in Theorem 3.3 is sharp (see [10], Example 3). In the case when T and dd^cT have the same sign, the hypotheses in Theorem 3.3 can't insure the existence of $\widetilde{T}$. The function $u(z) = \frac{1}{|z|^2}$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ illustrates this. In fact, $dd^cu(z) = \frac{1}{|z|^4} i dz \wedge d\overline{z}$. Therefore, u and dd^cu are both positive on $\mathbb{C}^*$, and although that $\mathscr{H}_1\{0\} = 0$, the function u is non extendable on the whole of the complex plane. The following corollary gives the sufficient conditions to get the extension in this case.

Corollary 3.4. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \ge -S$ on $\Omega \setminus A$ for some positive dd^c-negative current S on Ω. Assume that $\mathscr{H}_{2p-2}(A) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $\widetilde{dd^cT} = dd^c\widetilde{T}$.*

This result has been studied before in many different cases. Actually, the authors in [10] considered the case when $S = 0$. The case when $\widetilde{dd^cT}$ exists and $\mathscr{H}_{2p}(A \cap \overline{\operatorname{Supp} T}) = 0$ done by Dabbek in [7]. Dabbek proved that in this case the residual current is positive and closed by using the same technique in [10] with the local potential of a positive closed current given in [5]. In [2], the result was proved for the positive closed current S.

PROOF. Applying ([10], Theorem 1) for the current dd^cT+S, the extension $\widetilde{dd^cT}$ exists. Now, the result follows from Theorem 5 in [10].

We end this section by the following theorem which is a version of Chern-Levine-Nirenberg inequality.

THEOREM 3.5. *Let A be a closed complete pluripolar subset of an open set Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\le S$ on $\Omega\setminus A$ for some positive plurisubharmonic (resp. dd^c-negative) current S on Ω. Let K and L compact set in Ω with $L\subset K^\circ$. Assume that $\mathscr{H}_{2p}(A\cap\overline{\operatorname{Supp}T})=0$, then there exists a constant $C_{K,L}>0$ such that for all u plurisubharmonic function on Ω of class $\mathscr{C}^2$ we have the following estimate*

$$\int_{L\setminus A} T\wedge dd^cu\wedge\beta^{p-1}\le C_{K,L}\|u\|_{\mathscr{L}^\infty(K)}(\|\widetilde{T}\|_K+\|\widetilde{dd^cT}\|_K).$$

PROOF. From Theorem 3.3, the extensions $\widetilde{T}$ and $\widetilde{dd^cT}$ exist. Moreover, $\widetilde{T}$ is positive and $\widetilde{dd^cT}$ is of order zero. Hence the result follows from Theorem 2.2.

APPLICATION OF THEOREM 3.3. Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a closed positive current of bidimension (p,p) on $\Omega\setminus A$. Assume that $\mathscr{H}_{2p-2}(A)=0$. Now, suppose that g is plurisubharmonic function on Ω which is smooth on $\Omega\setminus A$. In this case by using ([10], Theorem 1), we can find the extension $\widetilde{gT}$. But we can't use the same result to find $\widetilde{g^2T}$, since we don't know whether g^2 is plurisubharmonic or not. Despite this, we can extend g^2T over A. In fact, the current g^2T is positive. We may assume that locally $g\le 0$, so simple computation shows that

$$dd^c(g^2T)=2dg\wedge d^cg\wedge T+2g\,dd^cg\wedge T\le 2dg\wedge d^cg\wedge T$$

Now, set $S=2dg\wedge d^cg\wedge T$, then S is a positive dd^c-negative current on $\Omega\setminus A$. Applying [10], the current $\widetilde{S}$ exists and is positive dd^c-negative on Ω. Hence by Theorem 3.3, $\widetilde{g^2T}$ exists.

4. The case when A is a zero set of a strictly k-convex function

In this section we include our second main result. The result was considered before in several cases. In 1984, El Mir [13] studied the case when T is a positive closed current and A is a zero set of an exhaustion strictly plurisubharmonic function. For the positive dd^c-negative current T the result was proved in [10]. In [9] the authors obtained the result when S is a closed positive current. The case when S is positive and A is a zero set of positive exhaustion strictly 0-convex function was done in [2].

Let us start this section with the definition of k-convex functions followed by a lemma which is given in [14].

DEFINITION 4.1. Let u be a continuous real function defined on an open subset Ω of $\mathbb{C}^n$. we say that u is strictly k-convex if there exists a continuous $(1,1)$-form γ defined on Ω which admits $(n-k)$-positive eigenvalues at each point, and such that the current $dd^c u - \gamma$ is positive on Ω.

LEMMA 4.2. *Let u be a strictly k-convex function on an open subset Ω of $\mathbb{C}^n$ and let $\gamma \geq 0$ be a continuous $(1,1)$-form on Ω. Then for all $z \in \Omega$, there exist a neighborhood V_z of z and a smooth strictly plurisubharmonic function f on V_z such that*

$$dd^c u \wedge (dd^c f)^k - \gamma^{k+1} \qquad \textit{is positive on} \quad V_z.$$

PROPOSITION 4.3. *Let Ω be an open subset of $\mathbb{C}^n$ and u be a strictly k-convex function on Ω. For $c \in \mathbb{R}$, we set $\Omega_c = \{z \in \Omega : u(z) \leq c\}$. Let T be a positive current of bidimension (p,p) on $\Omega \setminus \Omega_c$ such that $dd^c T \leq S$ on $\Omega \setminus \Omega_c$ for some positive and plurisubharmonic (resp. dd^c-negative) current S on Ω. If $p \geq k+1$, then T is of finite mass near Ω_c.*

PROOF. As in [10], one can assume that $u \in \mathscr{C}^\infty(\Omega \setminus A)$. Since the problem is local, all what we need is to show that for every $z_0 \in u^{-1}\{c\}$, there exists $\omega \Subset \Omega$ contains z_0 such that

$$\int_{\omega \setminus \Omega_{c+\frac{2}{m}}} T \wedge \beta^p < \infty$$

independently of m. Since u is strictly k-convex function then there exist a system of coordinates on $\mathbb{C}^n$ and an open neighborhood V of z_0 and $\lambda > 0$ such that

$$dd^c u + \frac{\lambda}{2}\beta' - 2\beta''$$

is a positive current on V, where $\beta' = dd^c|z'|^2$, $z' \in \mathbb{C}^k$ and $\beta'' = dd^c|z''|^2$, $z'' \in \mathbb{C}^{n-k}$. Let $r > 0$ such that $B(z_0, r) \subset V$, and χ be a smooth function satisfying $\chi = 0$ on $\overline{B}\big(z_0, \frac{r}{2}\big)$ and $\chi = -1$ on $\Omega \setminus B\big(z_0, \frac{2}{3}r\big)$. For a sufficiently small $\delta > 0$, we set $v = u + \delta\chi$ and denote by φ_ε a regularization kernel on $\mathbb{C}^n$ depending only on $|z|$. Choose ε_m small enough so that $v_m = v * \varphi_{\varepsilon_m}$ satisfies $0 < v - v_m < \frac{1}{m}$ and

$$dd^c v_m + \lambda\beta' - \beta''$$

is a positive form for all m. By Lemma 4.2, if $\alpha = dd^c f$ and $m \in \mathsf{N}$, then we find that $T \wedge \beta^p \leq T \wedge dd^c v_m \wedge \alpha^{p-1}$ on $V \setminus \Omega_c$. Now let $(h_m)_m$ be a sequence of increasing convex positive functions such that

$$0 \leq \sup(t-c, 0) - h_m(t) \leq \frac{1}{m}, \qquad \forall m \in \mathsf{N},\ \forall t \in \mathsf{R}$$

and

$$h'_m(t) = 1 \qquad \text{for} \quad t \geq c + \frac{1}{m}.$$

If we set $u_m = h_m \circ v_m$, then clearly

$$dd^c u_m \wedge \alpha^{p-1} = (h'_m \circ v_m) dd^c v_m \wedge \alpha^{p-1} + (h''_m \circ v_m) i \partial v_m \wedge \overline{\partial} v_m \wedge \alpha^{p-1}$$

From the above equality and the hypotheses of $(h_m)_m$, it follows that $dd^c u_m \wedge \alpha^{p-1} \geq \beta^p$ on $B\big(z_0, \frac{r}{2}\big) \setminus \Omega_{c+\frac{2}{m}}$. Indeed, $\chi = 0$ on $B\big(z_0, \frac{r}{2}\big)$. So when $u > c + \frac{2}{m}$ we have

$$v_m \geq v - \frac{1}{m} = u - \frac{1}{m} > c + \frac{1}{m}$$

Therefore $h'_m \circ v_m = 1$ and $h''_m \circ v_m = 0$. Hence $dd^c u_m \wedge \alpha^{p-1} = dd^c v_m \wedge \alpha^{p-1}$ on $B\big(z_0, \frac{r}{2}\big) \setminus \Omega_{c+\frac{2}{m}}$. Moreover, (u_m) vanishes in a neighborhood of Ω_c, depending on m. Let g be a smooth function with compact support belonging to $\Omega \setminus \Omega_c$, $g = 1$ in a neighborhood of $\partial B(z_0, r)$, $0 \leq g \leq 1$ and vanishes on a neighborhood of $(\Omega \setminus \Omega_c) \cap B\big(z_0, \frac{2}{3} r\big)$. Let $T_{\varepsilon_k} = T * \varphi_{\varepsilon_k}$ be a smoothing of T which is of course convergent weakly to T. Let us set $B_r = B(z_0, r)$ and $\omega = B_{\frac{r}{2}}$, hence

$$\int_{\omega \setminus \Omega_{c+\frac{2}{m}}} T \wedge \beta^p \leq \lim_{\varepsilon_k \to 0} \int_{B_r} T_{\varepsilon_k} \wedge dd^c u_m \wedge \alpha^{p-1} \tag{4.1}$$

On the other hand,

$$\begin{aligned}
\int_{B_r} T_{\varepsilon_k} \wedge dd^c u_m \wedge \alpha^{p-1} &= \int_{B_r} T_{\varepsilon_k} \wedge dd^c (g u_m + (1-g) u_m) \wedge \alpha^{p-1} \\
&= \int_{B_r} T_{\varepsilon_k} \wedge dd^c (g u_m) \wedge \alpha^{p-1} \\
&\qquad + \int_{B_r} u_m (1-g) dd^c T_{\varepsilon_k} \wedge \alpha^{p-1} \\
&\leq \int_{B_r} T_{\varepsilon_k} \wedge dd^c (g u_m) \wedge \alpha^{p-1} \\
&\qquad + \int_{B_r} u_m (1-g) S_{\varepsilon_k} \wedge \alpha^{p-1}.
\end{aligned} \tag{4.2}$$

The nice choice of g makes the sequence (gu_m) converges uniformly to $(v-c)g$. Moreover, on $\operatorname{Supp} g \cap \operatorname{Supp} u_m$ the positive current T has locally finite mass. So by Lemma 3.1, we obtain that the last right hand side integrals in (4.2) are bounded independently of ε_k and m. In virtue of (4.1) we deduce that T is of finite mass on $\omega \setminus \Omega_c$.

REMARK 4.4. In the case of strictly 0-convex functions, the condition $dd^c S \geq 0$ (or $dd^c S \leq 0$) can be omitted. Indeed, in this case we can replace α by β in the proof of latter proposition. As S is positive, there exists $C > 0$ so that

$$\begin{aligned}\int_{B_r} & T_{\varepsilon_k} \wedge dd^c u_m \wedge \beta^{p-1} \\ &\leq \int_{B_r} T_{\varepsilon_k} \wedge dd^c(gu_m) \wedge \beta^{p-1} + \int_{B_r} u_m(1-g) S_{\varepsilon_k} \wedge \beta^{p-1} \\ &\leq \int_{B_r} T_{\varepsilon_k} \wedge dd^c(gu_m) \wedge \beta^{p-1} + C\|S_{\varepsilon_k}\|_{B_r}\end{aligned}$$

COROLLARY 4.5. *Let Ω be an open subset of $\mathbb{C}^n$ and let u be a positive plurisubharmonic function of class $\mathscr{C}^2$ and $0 \leq s < r$ such that $B_r\{z \in \Omega, u(z) < r\} \Subset \Omega$. Let T be a positive current of bidimension (p, p) on $\Omega \setminus B_s$ such that $dd^c T \leq S$ on $\Omega \setminus B_s$ for some positive and plurisubharmonic (or dd^c-negative) current S on Ω. Choose $\delta \in \mathbb{R}$ such that $0 < \delta < r - s$ and $B_{r+\delta} \Subset \Omega$. Then there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$\begin{aligned}\int_{B_r \setminus B_s} T \wedge (dd^c u)^p \leq C_1 \int_{C(r-\delta, r+\delta)} & T \wedge (dd^c u)^p \\ &+ C_2 \|u\|^{p-1}_{\mathscr{L}^\infty(L)} (\|S\|_L + (p-1)\|dd^c S\|_L)\end{aligned}$$

where $C(r-\delta, r+\delta) = \{z \in \Omega, r-\delta < u(z) < r+\delta\}$ and $L = \overline{B_{r+\delta}}$

PROOF. We set $\varphi_m = \max\big(u - \frac{1}{m} - s, 0\big) * \alpha_{\varepsilon_m}$. For ε_m small enough have $dd^c \varphi_m \geq \frac{1}{2} dd^c u$ on $\big\{u > \frac{2}{m} + s\big\}$, then

$$\frac{1}{2} \int_{C(\frac{2}{m}+s, r)} T \wedge (dd^c u)^p \leq \lim_{\varepsilon_k \to 0} \int_{B_r} T_{\varepsilon_k} \wedge dd^c \varphi_m \wedge (dd^c u)^{p-1} \tag{4.3}$$

Let g be a smooth function with support in $C(r-\delta, r+\delta)$ such that $0 \leq g \leq 1$ and $g = 1$ on a neighborhood of ∂B_r. The sequence $g\varphi_m$ converges toward

$g(u-s)$ in $\mathscr{C}^2$. Then by similar argument as in Proposition 4.3, we have

$$\begin{aligned}\int_{B_r} T_{\varepsilon_k} \wedge dd^c\varphi_m \wedge (dd^c u)^{p-1} &= \int_{B_r} T_{\varepsilon_k} \wedge dd^c(g\varphi_m + (1-g)\varphi_m) \wedge (dd^c u)^{p-1} \\ &= \int_{B_r} T_{\varepsilon_k} \wedge dd^c(g\varphi_m) \wedge (dd^c u)^{p-1} \\ &\quad + \int_{B_r} \varphi_m (1-g) dd^c T_{\varepsilon_k} \wedge (dd^c u)^{p-1} \\ &\le \int_{B_r} T_{\varepsilon_k} \wedge dd^c(g\varphi_m) \wedge (dd^c u)^{p-1} \\ &\quad + \int_{B_r} \varphi_m (1-g) S_{\varepsilon_k} \wedge (dd^c u)^{p-1}\end{aligned}$$

and by Lemma 3.1, there exist $C_1 > 0$ and $C_2 > 0$ independent of m and ε_k such that

$$\begin{aligned}\lim_{\varepsilon_k \to 0} \int_{B_r} & T_{\varepsilon_k} \wedge dd^c\varphi_m \wedge (dd^c u)^{p-1} \\ &\le C_1 \int_{\operatorname{Supp} g} T \wedge (dd^c u)^p + C_2 \|u\|^{p-1}_{\mathscr{L}^\infty(L)} (\|S\|_L + (p-1)\|dd^c S\|_L)\end{aligned}$$

The conclusion follows by letting m tends to ∞ in (4.3).

REMARK 4.6. If $u(z) = |z|^2$, then we don't need the plurisubharmonicity of S in Corollary 4.5.

THEOREM 4.7. *Let Ω be an open subset of $\mathbb{C}^n$ and u be a positive strictly k-convex function on Ω. Set $A = \{z \in \Omega : u(z) = 0\}$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^c T \le S$ on $\Omega \setminus A$ for some positive and plurisubharmonic (or dd^c-negative) current S on Ω. If $p \ge k+1$, then $\widetilde{T}$ exists. If $p \ge k+2$, $dd^c S \le 0$ and u is of class $\mathscr{C}^2$, then $\widetilde{dd^c T}$ exists and $\widetilde{dd^c T} = dd^c \widetilde{T}$.*

Notice that, for strictly 0-convex functions we only need the positivity of S to find $\widetilde{T}$, thanks to Remark 4.4.

PROOF. If $p \ge k+1$, then by the previous proposition $\widetilde{T}$ exists. To show the second part we first note that $S - dd^c T$ is positive dd^c-negative current of bidimension $(p-1, p-1)$ on $\Omega \setminus A$. So if $p-1 \ge k+1$, then $\widetilde{S - dd^c T}$ exists. This implies that $\widetilde{dd^c T}$ exists, and by ([10], Theorem 4) the result follows.

COROLLARY 4.8. *Let Ω be an open subset of $\mathbb{C}^n$ and A be a Cauchy-Riemann variety of class $\mathscr{C}^1$ in Ω with dimension k. Let T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^c T \le S$ on $\Omega \setminus A$ for some for some* *positive dd^c-negative current S on Ω. If $p \geq k+1$, then $\widetilde{T}$ exists. If $p \geq k+2$, then $\widetilde{dd^c T}$ exists and $dd^c\widetilde{T} = \widetilde{dd^c T}$.*

PROOF. By Theorem III.6 and Theorem III.7 in [13], locally there exists a positive strictly k-convex function u of class $\mathscr{C}^2$ such that $A = u^{-1}(\{0\})$. Then the result follows from Theorem 4.7.

As we saw in the case of pluripolar sets A, the condition on the Hausdorff dimension of A is sharp. But using Proposition 4.3, we can obtain the extension in the case of compact pluripolar sets, regardless the greatness of its Hausdorff dimension.

THEOREM 4.9. *Let A be a compact complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive (p, p) current on $\Omega \setminus A$ such that $dd^c T \leq S$ on $\Omega \setminus A$ for some positive current S on Ω. If $p \geq 1$, then $\widetilde{T}$ exists and $R = \widetilde{dd^c T} - dd^c\widetilde{T}$ is a positive current supported in A.*

PROOF. By Proposition II.2. in [13], there exists a strictly pseudoconvex open set Ω' such that $A \subset \Omega' \Subset \Omega$, and a negative plurisubharmonic function u on Ω' satisfying that $A = \{z \in \Omega', u(z) = -\infty\}$ and such that e^u is continuous. Let φ be an exhaustion continuous strictly plurisubharmonic function on Ω' and set $c = \sup\{\varphi(z), z \in A\}$. Now consider the following sequence

$$u_m = \sup\left(\varphi - c - \frac{1}{m}, e^{(\frac{1}{m})u+|z|^2} - \frac{1}{m}, 0\right)$$

Since φ is exhaustion, then there exists $\Omega'' \Subset \Omega'$ and contains A such that $u_m = \varphi - c - \frac{1}{m}$ on $\Omega' \setminus \Omega''$ for all m. Now consider $A_m = \{z \in \Omega', u_m = 0\}$ and $g \in \mathscr{C}_0^\infty(\Omega \setminus \Omega'')$, $0 \leq g \leq 1$ and $g = 1$ in a neighborhood of $\partial\Omega'$. By similar argument as in Proposition 4.3, one can show that

$$\int_{\Omega'\setminus A_m} T \wedge \beta^p < \infty$$

independently of m. Hence $\widetilde{T}$ exists, and by [12], the current R is positive and supported in A.

If T is a positive closed current, Theorem 4.9 is due to El Mir [13]. The case where T is a positive dd^c-negative current is considered in [10], they proved that $dd^c\widetilde{T} = \widetilde{dd^c T}$, if $p \geq 2$. Recently, Dabbek and Noureddine [9] studied the case when T is a quasi-plurisuperharmonic current.

In what remains in this paper we suppose that A is a closed obstacle.

THEOREM 4.10. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a negative current of bidimension (p, p) on $\Omega \setminus A$ such that $dd^c T \geq -S$ on $\Omega \setminus A$*

for some positive current S on Ω. Assume that $\mathscr{H}_{2p-2}(\overline{\operatorname{Supp} T} \cap A)$ is locally finite. Then $\widetilde{T}$ exists. If $dd^c S \leq 0$, then $\widetilde{dd^c T}$ exists and $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is a negative current supported in A.

The same result obtained by Harvey [17] when T is a closed positive current and $\mathscr{H}_{2p-1}(\overline{\operatorname{Supp} T} \cap A) = 0$. The case when $S = 0$ due to Dabbek, Elkhadhra and El Mir [10]. In [9], Dabbek and Noureddine studied the case when T is a quasi-plurisuperharmonic current.

Proof. Our problem is local. So we may assume that $0 \in \overline{\operatorname{Supp} T} \cap A$ and our aim now is studying the mass of T in a neighborhood of 0. Since $\mathscr{H}_{2p-1}(\overline{\operatorname{Supp} T} \cap A) = 0$, there exist a system of coordinates (z', z'') of $\mathbb{C}^{p-1} \times \mathbb{C}^{n-p+1}$ and plydisk $\triangle^{p-1} \times \triangle^{n-p+1} \subset \mathbb{C}^{p-1} \times \mathbb{C}^{n-p+1}$ such that $(A \cap \overline{\operatorname{Supp} T}) \cap (\triangle^{p-1} \times \partial \triangle^{n-p+1}) = \emptyset$. Moreover, for any projection $\pi_I : \mathbb{C}^n \to \mathbb{C}^{p-1}$ and any strictly multi-index $I = (i_1, \ldots, i_{p-1})$, one has $\pi_I\{0\} \cap (\overline{\operatorname{Supp} T} \cap A) = \{0\}$ (see [19], Corollary 4 (i)). Let $0 < t < 1$ such that $\triangle^{p-1} \times \{z'', t < |z''| < 1\} \cap (\overline{\operatorname{Supp} T} \cap A) = \emptyset$. For each $z' \in \triangle^{p-1}$, we set $A_{z'} = (\overline{\operatorname{Supp} T} \cap A) \cap (\{z'\} \times \triangle^{n-p+1})$. Since $\mathscr{H}_{2p-2}(\overline{\operatorname{Supp} T} \cap A)$ is locally finite, then again by ([19], Corollary 4 (ii)) we have that $\mathscr{H}_0(A_{z'})$ is finite, and we find that $A_{z'}$ is a discrete subset for a.e z'. Without lose of generality, we may assume that $A_{z'}$ is reduced to a single point $(z', 0)$. On the other hand, T is a $\mathbb{C}$-normal current on $\Omega \setminus A$, so it is $\mathbb{C}$-flat on $\Omega \setminus A$ (see [4], pp. 573–574). The slice $\langle T, \pi_I, z' \rangle$ exists for a.e z', and is a negative current of bidimension $(1, 1)$ on $\Omega \setminus A_{z'}$, supported in $\{z'\} \times \triangle^{n-p+1}$ such that $dd^c\langle T, \pi_I, z' \rangle \geq \langle -S, \pi_I, z' \rangle$ on $\Omega \setminus A_{z'}$. Let K be a compact subset of $\triangle^{p-1} \times \triangle^{n-p+1}$. Since T is negative, it is enough to show that

$$\int_{K \setminus A} -T \wedge \pi_I^* \beta'^{p-1} \wedge \beta < \infty$$

where $\beta' = dd^c |z'|^2$. Applying Remark 4.6 on the current $-T$, we obtain

$$\begin{aligned} &\int_{\triangle^{n-p+1}((z',0),1) \setminus A_{z'}} \langle -T, \pi_I, z' \rangle \wedge \beta \\ &\qquad \leq C_1 \int_{\{z'' \in \triangle^{n-p+1}, |z''| > t\}} \langle -T, \pi_I, z' \rangle \wedge \beta + C_2 \| \langle S, \pi_I, z' \rangle \|_L \end{aligned}$$

where $L = (1 + \varepsilon)\overline{\triangle^{n-p+1}}$, for small $\varepsilon > 0$. Now, by slice formula we get

$$\begin{aligned} \text{(4.4)} \qquad &\int_{K \setminus A} -T \wedge \pi_I^* \beta'^{p-1} \wedge \beta \\ &\qquad \leq C \int_{z'} \left(\int_{\triangle^{n-p+1}((z',0),1) \setminus A_{z'}} \langle -T, \pi_I, z' \rangle \wedge \beta \right) \beta'^{p-1} \end{aligned}$$

$$\leq C_1' \int_{z'} \left(\int_{\{z'' \in \triangle^{n-p+1}, |z''|>t\}} \langle -T, \pi_I, z' \rangle \wedge \beta \right) \beta'^{p-1}$$
$$+ C_2' \int_{z'} \left(\int_L \langle S, \pi_I, z' \rangle \right) \beta'^{p-1}$$
$$\leq D_1 \int_{\triangle^{p-1} \times \{z'', t<|z''|<1\}} -T \wedge \pi_I^* \beta'^{p-1} \wedge \beta$$
$$+ D_2 \int_{\triangle^{p-1} \times L} S \wedge \pi_I^* \beta'^{p-1}$$

As $-T$ is of locally finite mass outside A and S is positive, the last right hand side integrals in (4.4) are bounded. Hence, $\widetilde{T}$ exists. Now, assume that $dd^c S \leq 0$. We want to show the existence of $\widetilde{dd^c T}$. As we saw above, for almost every z', the current $\langle T, \pi_I, z' \rangle$ is negative and $dd^c \langle T, \pi_I, z' \rangle \geq \langle -S, \pi_I, z' \rangle$ apart of $A_{z'}$, which is complete pluripolar. So by Theorem 3.3, $\widetilde{\langle T, \pi_I, z' \rangle}$ exists and $\langle R, \pi_I, z' \rangle = \langle \widetilde{dd^c T}, \pi_I, z' \rangle - \langle dd^c \widetilde{T}, \pi_I, z' \rangle$ is negative for a.e z'. By similar argument as above we find that $\widetilde{dd^c T}$ exists. Indeed, for K compact subset of $\triangle^{p-1} \times \triangle^{n-p+1}$ we have

$$\int_{K \setminus A} (dd^c T + S) \wedge \pi_I^* \beta'^{p-1}$$
$$\leq B \int_{z'} \left(\int_{\triangle^{n-p+1}((z',0),1) \setminus A_{z'}} \langle (dd^c T + S), \pi_I, z' \rangle \right) \beta'^{p-1}.$$

As $\langle \widetilde{dd^c T}, \pi_I, z' \rangle$ exists, the right hand side integral in the previous inequality is bounded. So, $\widetilde{dd^c T + S}$ exists, implies that $\widetilde{dd^c T}$ exists. Remains to show that R is negative, so take a positive function $\varphi \in \mathscr{D}(\Omega)$. By slice formula, we have

$$\text{(4.5)} \qquad \int R \wedge \pi_I^* \beta'^{p-1} \wedge \varphi = \int_{z'} \langle R, \pi_I, z' \rangle (\varphi) \beta'^{p-1} \leq 0$$

Hence, $R \wedge \pi_I^* \beta'^{p-1} \leq 0$. Since (4.5) true for almost all choice of unitary coordinates (z', z''), the current R is negative and supported in A.

REMARK 4.11. In the previous theorem, the currents T and $dd^c T$ are C-normal on $\Omega \setminus A$, so the extensions $\widetilde{T}$ and $\widetilde{dd^c T}$ are C-flat (see [4]). Therefore, by the support theorem $dd^c \widetilde{T} = \widetilde{dd^c T}$ as soon as $\mathscr{H}_{2p-2}(\overline{\operatorname{Supp} T} \cap A) = 0$. Moreover, if $\mathscr{H}_{2p-4}(\overline{\operatorname{Supp} T} \cap A)$ is locally finite, then by [10], the extension $-\widetilde{S}$ is positive and plurisubharmonic. Therefore the current $\widetilde{T}$ has the same properties of T.

As usual the case when T and $dd^c T$ coincide in the sign is not the optimal case. We see in ([10], Corollary 6), to find the extension of the positive plurisubharmonic current T apart of the closed set A, the Hausdorff dimension of A need to be reduced. Actually, we guarantee the existence of $\widetilde{T}$ if $\mathcal{H}_{2p-3}(A) = 0$. In our setting we have the following version.

COROLLARY 4.12. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p, p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive and dd^c-negative current S on Ω. Assume that $\mathcal{H}_{2p-4}(A)$ is locally finite. Then $\widetilde{T}$ exists. Moreover, $\widetilde{dd^cT} = dd^c\widetilde{T}$.*

PROOF. As $dd^cT + S$ is a positive and dd^c-negative current of bidimension $(p-1, p-1)$, the extension $\widetilde{dd^cT + S}$ exists by the previous theorem. Thus $\widetilde{dd^cT}$ exists, and the results follows thanks to Theorem 5 in [10].

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