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UNIFORM DOMAINS AND UNIFORM DOMAIN DECOMPOSITION PROPERTY IN REAL NORMED VECTOR SPACES

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Abstract

Let E be a real normed vector space with $\dim(E) \geq 2$, D a proper subdomain of E. In this paper we characterize uniform domains in E in terms of the uniform domain decomposition property. In addition, we discuss the relation between quasiballs and domains with the quasiball decomposition property in R^n.

1. Introduction and Main Results

Throughout the paper, we assume that E is a real normed vector space with $\dim(E) \geq 2$ and the norm of a vector $z \in E$ is denoted by $|z|$. For any two points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$. D is always assumed to be a proper domain in E and $\mathsf{B}(x_0, r) = \{x \in E : |x - x_0| < r\}$, the open ball centered at x_0 of radius $r > 0$. Similarly, for the closed balls and spheres, we use the notations $\overline{\mathsf{B}}(x_0, r)$ and $\partial\mathsf{B}(x_0, r)$.

We now introduce two basic concepts: uniform domains and John domains.

Definition 1.1. A proper domain D in E is called *uniform* in the norm metric provided there exists a constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying (cf. [18] and [20])

(1) $\min_{j=1,2} \ell(\gamma[z_j, z]) \leq c\, d_D(z)$ for all $z \in \gamma$, and

(2) $\ell(\gamma[z_1, z_2]) \leq c|z_1 - z_2|$.

Here $\ell(\gamma)$ denotes the arclength of γ, $\gamma[z_j, z]$ the part of γ between z_j and z. The distance from z to the boundary ∂D of D in E is denoted by $d_D(z)$.

D is said to be a *John domain* if it satisfies the first condition in above but not necessarily the second one (see [16]).

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John [10], Martio and Sarvas [15] were the first who introduced John domains and uniform domains in R^2, respectively. Now, there are plenty of alternative characterizations for uniform and John domains (see [4], [5], [6], [8], [11], [14], [18]), and their importance along with some special domains throughout the function theory is well documented, see [5], [11], [16], [18]. Moreover, uniform domains in E enjoy numerous geometric and function theoretic features in many areas of modern mathematical analysis, see [1], [2], [3], [4], [8], [9], [18].

We refer to the books of Väisälä [17] and Vuorinen [21] for the definition of *K-quasiconformal* (K-qc) homeomorphism of R^n and for basic facts regarding *quasiconformal* (qc) mappings.

A Jordan curve γ in $\overline{\mathsf{R}}^2 = \mathsf{R}^2 \cup \{\infty\}$ is called a *K-qc circle* (or simply *qc circle*) if there is a K-qc mapping f of $\overline{\mathsf{R}}^2$ onto itself such that $\gamma = f(\partial \mathsf{B}^2)$, and $f(\mathsf{B}^2)$ is called a *K-quasidisk* (or simply *quasidisk*), where B^2 denotes the unit disk in R^2. We say that a domain $D \subset \overline{\mathsf{R}}^n$ is a *K-quasiball* (or simply *quasiball*) if there exists a K-qc mapping f of $\overline{\mathsf{R}}^n$ onto itself such that $D = f(\mathsf{B}^n)$, where B^n denotes the unit ball in R^n.

As a characterization of qc circles, Martio and Sarvas [15] proved that a Jordan domain in R^2 is uniform if and only if its boundary is a qc circle. After that, Gehring and Hag [7, Theorems 3.10 and 4.1] proved that a finitely connected domain D in R^2 is uniform if and only if there is a constant K such that each component of ∂D is either a point or a K-qc circle. As a further generalization, Gehring and Osgood proved

THEOREM A ([8, Theorem 5]). *A domain D in R^2 is a uniform domain if and only if it is quasiconformally decomposable.*

Here a domain D in R^2 is said to be *quasiconformally decomposable* if there exists a constant K with the following property: For each pair z_1, z_2 in D, there exists a subdomain D_0 of D such that z_1, z_2 are contained in $\overline{D_0}$ and ∂D_0 is a K-qc circle. Obviously, D_0 is a K-quasidisk.

We refer to [8] for some applications of Theorem A including a new proof of the injectivity properties of uniform domains in $\overline{\mathsf{R}}^2$. The situation is very different in R^n. The 3-dimensional analog of Theorem A fails to hold even for simply connected domains, see [13, Example 3.8]. In order to consider the generalization of Theorem A in R^n or real normed vector spaces E, we introduce the following concepts.

DEFINITION 1.2. A domain D in E is said to have the *uniform domain decomposition property* if there exists a positive constant c with the following property: For each pair of points z_1, z_2 in D, there exists a subdomain D_0 of D such that $z_1, z_2 \in D_0$ and D_0 is a *simply connected c-uniform domain.*

A domain D in R^n is said to have the *quasiball decomposition property* if there exists a positive constant K with the following property: For each pair of points z_1, z_2 in D, there exists a subdomain D_0 of D such that $z_1, z_2 \in D_0$ and D_0 is a K*-quasiball.*

By proving the Lipschitz continuous first differentiability of quasihyperbolic geodesics in R^n, Martin obtained

THEOREM B ([13, Theorem 5.1]). *Let D be a uniform domain in R^n. Then there is a constant L, depending only on the constant of uniformity for D, such that for each pair of points x_1, x_2 in D there is an L-bi-Lipschitz embedding $f : \overline{\mathsf{B}}^n(0, |x_1 - x_2|) \to D$ with $\{x_1, x_2\} \subset f(\overline{\mathsf{B}}^n(0, |x_1 - x_2|))$.*

Obviously, Theorem B shows that

COROLLARY 1.3. *A domain in R^n is uniform if and only if it has the uniform domain decomposition property.*

It easily follows from [8, Corollary 3] that

PROPOSITION 1.4. *Let D be a domain in R^n. If D has the quasiball decomposition property, then it has the uniform domain decomposition property.*

For a simply connected domain D in R^2, D is uniform if and only if it is a quasidisk [9, Lemma 6.4] if and only if it is a quasiball. In view of Theorem A, it is easy to formulate the following proposition which characterizes uniform domains.

PROPOSITION 1.5. *For any domain D in R^2, the following are equivalent.*

(1) *D is uniform;*
(2) *D is quasiconformally decomposable;*
(3) *D has the uniform domain decomposition property;*
(4) *D has the quasiball decomposition property.*

By [13, Example 3.8], it is natural to consider a suitable generalization of Proposition 1.5 which works for E or R^n. To achieve this goal, in this paper, we mainly consider the following two questions.

QUESTION 1.6. Is it true that a domain D in E is uniform if and only if it has the uniform domain decomposition property?

QUESTION 1.7. Is it true that a domain D in R^n is a quasiball if and only if it has the quasiball decomposition property?

In the proof of Theorem A, the authors [8] have utilized the Riemann mapping theorem. In the absence of the Riemann mapping theorem in E when $\dim(E) \geq 3$, it is natural that the methods used in the proof of Theorem A are no more useful in E when $\dim(E) \geq 3$. It is known that a quasihyperbolic geodesic between any two points in E exists if the dimension of E is finite, see [8, Lemma 1]. But this is not true in arbitrary spaces. A counterexample (due to Alestalo) has been given in [18, Section 3], see also [19, Section 2]. Hence the method of proof used in Theorem B is invalid either. By using a different method of proof, we obtain the following theorems and delay their proofs until a few necessary preliminaries have been developed. Moreover, our method of proof works also for the case $E = \mathsf{R}^2$.

THEOREM 1.8. *Let E be a real normed vector space with* $\dim(E) \geq 2$. *Then a domain D in E is uniform if and only if it has the uniform domain decomposition property.*

THEOREM 1.9. *Every quasiball in R^n has the quasiball decomposition property.*

We see from the following example that the converse of Theorem 1.9 is not necessarily true.

EXAMPLE 1.10. Let $e_1 = (1, 0, 0)$ denote the unit vector in the direction of x_1-axis and $D = \mathsf{B}^3 \setminus L$ in R^3, where $L = \{te_1 : \frac{1}{2} \leq t < 1\}$. Then D has the quasiball decomposition property, but D is not a quasiball.

2. Proof of Theorem 1.8

We start with some preliminary results. The proof of Theorem 1.8 is given in Subsection 2.24.

LEMMA 2.1. *For any $x_1, x_2 \in G \subset E$, if $\overline{\mathsf{B}}(x_1, r_1) \cap \overline{\mathsf{B}}(x_2, r_2) \neq \emptyset$, $\frac{1}{4}d_G(x_1) \leq r_1 \leq \frac{8}{9}d_G(x_1)$ and $\frac{1}{4}d_G(x_2) \leq r_2 \leq \frac{8}{9}d_G(x_2)$, then*

$$\frac{1}{17}d_G(x_2) \leq d_G(x_1) \leq 17d_G(x_2) \qquad \text{and} \qquad \frac{1}{68}r_1 \leq r_2 \leq 68r_1.$$

PROOF. For any $y \in \partial\mathsf{B}(x_1, r_1) \cap \overline{\mathsf{B}}(x_2, r_2)$, since

$$d_G(y) \geq d_G(x_2) - r_2, \qquad d_G(x_1) \geq d_G(y) - r_1$$

and

$$d_G(y) \geq d_G(x_1) - r_1, \qquad d_G(x_2) \geq d_G(y) - r_2,$$

we see that the lemma holds.

For any $z_1, z_2 \in D$, we assume that $\alpha \subset D$ is a rectifiable arc joining them with

(1) $\ell(\alpha[z_1, z_2]) \le c\,|z_1 - z_2|$, and

(2) $\min_{j=1,2} \ell(\alpha[z_j, z]) \le c\, d_D(z)$ for all $z \in \alpha$.

Let z_0 be a point in α which bisects α. Denote $\alpha[z_1, z_0]$ and $\alpha[z_2, z_0]$ by γ and β, respectively. And assume $M = [2^{16c}]$, where $[\cdot]$ denotes the greatest integer part.

We prove Theorem 1.8 by constructing a simply connected domain $D_1 \subset D$ containing z_1 and z_2. This construction is included in Lemma 2.14. At first, we prepare two elementary results.

LEMMA 2.2. *There exists a simply connected domain* $D_{1,0} = \bigcup_{i=1}^{k_1} B_{1,i} \subset D$ *such that*

(1) $z_1, z_0 \in D_{1,0}$;

(2) *For each* $i \in \{1, \dots, k_1\}$, $\frac{1}{3} d_D(x_{1,i}) \le r_{1,i} \le \frac{7}{8} d_D(x_{1,i})$;

(3) *If* $k_1 \ge 3$, *then for any* $i, j \in \{1, \dots, k_1\}$ *with* $|i - j| \ge 2$, *we have* $\operatorname{dist}(B_{1,i}, B_{1,j}) \ge \frac{1}{32M^2} \max\{r_{1,i}, r_{1,j}\}$;

(4) *If* $k_1 \ge 2$, *then* $r_{1,i} + r_{1,i+1} - |x_{1,i} - x_{1,i+1}| \ge \frac{1}{32M^2} \max\{r_{1,i}, r_{1,i+1}\}$ *for each* $i \in \{1, \dots, k_1 - 1\}$,

where $B_{1,i} = \mathsf{B}(x_{1,i}, r_{1,i})$, $x_{1,i} \in \gamma$, $x_{1,i} \notin B_{1,i-1}$ *and* $\operatorname{dist}(B_{1,i}, B_{1,j})$ *denotes the distance from* $B_{1,i}$ *to* $B_{1,j}$.

PROOF. Let $x_{1,1} = z_1$. Set $A_{1,1} = \mathsf{B}(x_{1,1}, r_{1,1})$ with $r_{1,1} = \frac{1}{2} d_D(x_{1,1})$.

If $z_0 \in A_{1,1}$, then we let $B_{1,1} = A_{1,1}$, and the domain $D_{1,0} = B_{1,1}$ is the desired.

If $z_0 \notin A_{1,1}$, then we let $x_{1,2}$ be the last intersection point of γ from z_1 to z_0 with $\partial A_{1,1}$. Set $A_{1,2} = \mathsf{B}(x_{1,2}, r_{1,2})$ with $r_{1,2} = \frac{1}{2} d_D(x_{1,2})$.

If $z_0 \in A_{1,2}$ and $A_{1,1}$ is contained in $A_{1,2}$, then we let $B_{1,1} = A_{1,2}$, and the domain $D_{1,0} = B_{1,1}$ is the needed. If $z_0 \in A_{1,2}$ and $A_{1,1}$ is not contained in $A_{1,2}$, then we let $B_{1,1} = A_{1,1}$, $B_{1,2} = A_{1,2}$, and the domain $D_{1,0} = B_{1,1} \cup B_{1,2}$ is the desired.

If $z_0 \notin A_{1,2}$, then we let $x_{1,3}$ be the last intersection point of γ from z_1 to z_0 with $\partial A_{1,2}$. Set $A_{1,3} = \mathsf{B}(x_{1,3}, r_{1,3})$ with $r_{1,3} = \frac{1}{2} d_D(x_{1,3})$.

We continue this procedure until there is some $i \in \{1, \dots, s-2\}$ such that $\operatorname{dist}(B_{1,i}, B_{1,s}) < \frac{1}{32M^2} \max\{r_{1,i}, r_{1,s}\}$. Obviously, $s \ge 3$.

Let $A_{1,t}$ be the first ball from $A_{1,1}$ to $A_{1,s-1}$ such that $\overline{A}_{1,i} \cap \overline{A}_{1,s} \ne \emptyset$. For the case $t = 1$ and $z_0 \in A_{1,s}$, if $A_{1,1}$ is contained in $\mathsf{B}\big(x_{1,s}, \frac{3}{4} d_D(x_{1,s})\big)$, we take $D_{1,0} = B_{1,1} = \mathsf{B}\big(x_{1,s}, \frac{3}{4} d_D(x_{1,s})\big)$. Otherwise, the similar reasoning as in Lemma 2.1 shows that we can let $D_{1,0} = B_{1,1} \cup B_{1,2}$, where $B_{1,1} = A_{1,1}$ and $B_{1,2} = \mathsf{B}\big(x_{1,s}, \frac{3}{4} d_D(x_{1,s})\big)$. When $t = 1$ and $z_0 \notin A_{1,s}$ or $t \ne 1$, we have the following claim.

CLAIM 2.3. *There are q balls $C_{1,1} = \mathsf{B}(y_{1,1}, p_{1,1}), \dots, C_{1,q} = \mathsf{B}(y_{1,q}, p_{1,q})$ (possibly, $q = 1$) in D such that*

(a) $\{y_{1,1}, \dots, y_{1,q}\} \subset \{x_{1,1}, \dots, x_{1,s}\}$;

(b) *the conditions* (2), (3) *and* (4) *in the lemma are satisfied by the balls* $C_{1,1}, \dots, C_{1,q}$.

The proof for the case $t = 1$ is obvious: If $A_{1,1}$ is contained in $\mathsf{B}\big(x_{1,s}, \frac{3}{4}\, d_D(x_{1,s})\big)$, then we let $C_{1,1} = \mathsf{B}\big(x_{1,s}, \frac{3}{4}\, d_D(x_{1,s})\big)$ and so $q = 1$. Otherwise, we let $C_{1,1} = A_{1,1}$, $C_{1,2} = \mathsf{B}\big(x_{1,s}, \frac{3}{4}\, d_D(x_{1,s})\big)$. The similar reasoning as in Lemma 2.1 implies that $C_{1,1}$ and $C_{1,2}$ satisfy Conditions (2) and (4) in the lemma, and hence $q = 2$. For the remaining case $t > 1$, we divide the proof into two cases.

CASE 2.4. $r_{1,t} + r_{1,s} - |x_{1,t} - x_{1,s}| \ge \frac{1}{8M}\, r_{1,s}$.

We let $C_{1,i} = A_{1,i}$ for each $i \in \{1, \dots, t\}$ and $C_{1,t+1} = \mathsf{B}\big(x_{1,s}, \big(1 - \frac{1}{16M}\big) r_{1,s}\big)$. Since for each $i \in \{1, \dots, t\}$, $r_{1,s} = \frac{1}{2} d_D(x_{1,s}) \ge \frac{1}{2c}\ell(\alpha[z_1, x_{1,s}]) \ge \frac{1}{2c} r_{1,i}$, we see that the balls $C_{1,1}, C_{1,2}, \dots, C_{1,t}, C_{1,t+1}$ satisfy the conditions (2) $\sim$ (4) in the lemma. Hence $q = t + 1$.

CASE 2.5. $r_{1,t} + r_{1,s} - |x_{1,t} - x_{1,s}| < \frac{1}{8M}\, r_{1,s}$.

We consider the ball $A'_{1,s} = \mathsf{B}\big(x_{1,s}, \frac{7}{4} r_{1,s}\big)$. Let A_{1,s_1} be the first ball from $A_{1,1}$ to $A_{1,t}$, whose closure $\overline{A}_{1,s_1}$ has nonempty intersection with $\overline{A}'_{1,s}$. Denote $d_i = \operatorname{dist}(A_{1,i}, A_{1,s})$ $(s_1 \le i \le t)$. Clearly, $d_t = 0$. We divide the rest argument into two parts.

SUBCASE 2.6. $d_{s_1} \le \frac{5}{16} r_{1,s}$.

In this case, we take $C_{1,i} = A_{1,i}$ $(1 \le i \le s_1)$ and $C_{1,s_1+1} = \mathsf{B}\big(x_{1,s}, \frac{23}{16} r_{1,s}\big)$. Then the balls $C_{1,1}, C_{1,2}, \dots, C_{1,s_1}, C_{1,s_1+1}$ satisfy the conditions (2) $\sim$ (4) in our lemma. This shows $q = s_1 + 1$.

SUBCASE 2.7. $d_{s_1} > \frac{5}{16} r_{1,s}$.

Let $\delta_1 = d_{s_1}$ and δ_2 be the first d_i from d_{s_1} to d_t satisfying $d_i < \delta_1$. Clearly, $\delta_1 > \delta_2$. By repeating the procedure, we get $\delta_1, \dots, \delta_m \in \{d_{s_1}, \dots, d_t\}$ such that

$$\delta_1 > \delta_2 > \cdots > \delta_m = 0.$$

Observe that $\delta_1 > \frac{5}{16} r_{1,s}$ and hence $m \ge 2$. For each $h \in \{1, \dots, m-1\}$, we denote $A_{1,i_h} = \mathsf{B}(x_{1,i_h}, r_{1,i_h})$ the first ball from $A_{1,1}$ to $A_{1,t}$ with $d_{i_h} = \delta_h$ and define $\varepsilon_h = \delta_h - \delta_{h+1}$.

SUBCLAIM 2.8. *There must exist some $j \in \{1, \dots, m-1\}$ such that $\varepsilon_j > \frac{1}{8M} r_{1,s}$.*

If $m \le M$, then the existence of $j \in \{1, \ldots, m-1\}$ with $\varepsilon_j > \frac{1}{8M} r_{1,s}$ is obvious because otherwise,

$$\frac{5}{16} r_{1,s} < \delta_1 - \delta_m \le (m-1)\frac{1}{8M} r_{1,s} < \frac{1}{8} r_{1,s},$$

which is a contradiction.

We assume that $m > M$. To prove the existence of j, we suppose on the contrary that $\varepsilon_h \le \frac{1}{8M} r_{1,s}$ for all $h \in \{1, \ldots, m-1\}$. Note that

$$\delta_{m-M} - \delta_m = \varepsilon_{m-M} + \cdots + \varepsilon_{m-1} \le \frac{1}{8} r_{1,s}.$$

Then for any $h \in \{m-M, \ldots, m-1\}$, we have

$$\delta_h \le \frac{1}{8} r_{1,s}. \tag{2.9}$$

If there exists some $h \in \{m-M, \ldots, m-1\}$ such that $A_{1,i_h} = \mathsf{B}(x_{1,i_h}, r_{1,i_h}) \not\subset (A'_{1,s} \setminus A_{1,s})$ then $(A'_{1,s} \setminus A_{1,s}) \cap A_{1,i_h}$ contains a ball, denoted by A_{0,i_h}, with radius $r_{0,i_h} = \frac{\frac{3}{4} r_{1,s} - \delta_h}{2} \ge \frac{5}{16} r_{1,s}$. Hence $r_{1,i_h} \ge \frac{5}{16} r_{1,s}$.

On the other hand, if $A_{1,i_h} = B(x_{1,i_h}, r_{1,i_h}) \subset (A'_{1,s} \setminus A_{1,s})$ for some $h \in \{m-M, \ldots, m-1\}$ then we see that $r_{1,i_h} > \frac{1}{8} r_{1,s}$. Otherwise,

$$\frac{1}{8} r_{1,s} \ge r_{1,i_h} \ge \frac{1}{3} d_D(x_{1,i_h}) \ge \frac{1}{3}\left(\frac{3}{4} r_{1,s} - \delta_h - r_{1,i_h}\right) \ge \frac{1}{6} r_{1,s},$$

which obviously is a contradiction. Thus we have proved that for each $h \in \{m-M, \ldots, m-1\}$,

$$r_{1,i_h} > \frac{1}{8} r_{1,s}. \tag{2.10}$$

It follows that

$$\begin{aligned} 3c r_{1,s} &\ge c\, d_D(x_{1,s}) \\ &\ge \ell(\gamma[z_1, x_{1,s}]) \\ &\ge \frac{M-1}{8} r_{1,s}, \end{aligned} \tag{2.11}$$

which is the desired contradiction since $M = [2^{16c}]$. The proof of Subclaim 2.8 is complete.

We come back to the proof of Claim 2.3. Let j be the least number in $\{1, \ldots, m-1\}$ satisfying Subclaim 2.8 and let $A''_{1,s} = \mathsf{B}(x_{1,s}, r''_{1,s})$, where

$$r''_{1,s} = r_{1,s} + \delta_{j+1} + \frac{1}{16M} r_{1,s}.$$

Then for all $i < i_{j+1}$, $A''_{1,s} \cap A_{1,i} = \emptyset$. We take $C_{1,i} = A_{1,i}$ for each $i \in \{1, \ldots, i_{j+1}\}$ and $C_{1,i_{j+1}+1} = A''_{1,s}$. It follows from $r''_{1,s} \le \frac{7}{4} r_{1,s}$ that the balls $C_{1,1}, \ldots, C_{1,i_{j+1}}, C_{1,i_{j+1}+1}$ satisfy the conditions (2), (3) and (4). Thus $q = i_{j+1} + 1$ in the case. The proof of Claim 2.3 is finished.

We continue the proof of our lemma.

If $z_0 \in C_{1,q}$, then by letting $B_{1,i} = C_{1,i}$ for each $i \in \{1, \ldots, q\}$, we see that the domain $D_{1,0} = \bigcup_{i=1}^{q} B_{1,i}$ is the desired.

If $z_0 \notin C_{1,q}$, then we let $x_{1,q+1}$ be the last intersection point of γ from z_1 to z_0 with $\partial C_{1,q}$. Set $C_{1,q+1} = \mathsf{B}(x_{1,q+1}, r_{1,q+1})$ with $r_{1,q+1} = \frac{1}{2}\, d_D(x_{1,q+1})$.

By repeating the procedure as above, we will get a set of points $\{x_{1,i}\}_{i=1}^{k_1}$ on γ and a set of balls $\{C_{1,i} = \mathsf{B}(x_{1,i}, r_{1,i})\}_{i=1}^{k_1}$ in D such that Conditions (2), (3) and (4) are satisfied and z_0 is contained in C_{1,k_1}. By letting $B_{1,i} = C_{1,i}$ for each $i \in \{1, \ldots, k_1\}$, we know that $D_{1,0} = \bigcup_{i=1}^{k_1} B_{1,i}$ is the needed domain. Hence we see that Lemma 2.2 holds.

By a similar argument as in the proof of Lemma 2.2, we get

COROLLARY 2.12. *There exists a simply connected domain* $D_{2,0} = \bigcup_{u=1}^{k_2} B_{2,u} \subset D$ *such that*

(1) $z_2, z_0 \in D_{2,0}$;

(2) *For each* $u \in \{1, \ldots, k_2\}$, $\frac{1}{3} d_D(x_{2,u}) \le r_{2,u} \le \frac{7}{8} d_D(x_{2,u})$;

(3) *If* $k_2 \ge 3$, *then for any* $u, v \in \{1, \ldots, k_2\}$ *with* $|u - v| \ge 2$, *we have* $\operatorname{dist}(B_{2,u}, B_{2,v}) \ge \frac{1}{32M^2} \max\{r_{2,u}, r_{2,v}\}$;

(4) *If* $k_2 \ge 2$, *then* $r_{2,u} + r_{2,u+1} - |x_{2,u} - x_{2,u+1}| \ge \frac{1}{32M^2} \max\{r_{2,u}, r_{2,u+1}\}$ *for each* $u \in \{1, \ldots, k_2 - 1\}$,

where $B_{2,u} = \mathsf{B}(x_{2,u}, r_{2,u})$, $x_{2,u} \in \beta$ *and* $x_{2,u} \notin B_{2,u-1}$.

LEMMA 2.13. $d_D(x_{2,k_2}) \ge \frac{1}{2c} \ell(\beta)$.

PROOF. If $|z_0 - x_{2,k_2}| \le \frac{1}{2} d_D(z_0)$, then $d_D(x_{2,k_2}) \ge d_D(z_0) - |z_0 - x_{2,k_2}| \ge \frac{1}{2} d_D(z_0)$. If $|z_0 - x_{2,k_2}| > \frac{1}{2} d_D(z_0)$, then $d_D(x_{2,k_2}) \ge r_{2,k_2} \ge \frac{1}{2} d_D(z_0)$. From the inequality $\ell(\beta) \le c\, d_D(z_0)$, our lemma follows.

LEMMA 2.14. *There exists a simply connected domain* $D_1 = \bigcup_{i=1}^{k} B_i \subset D$ *such that*

(1) $z_1, z_2 \in D_1$;

(2) *For each* $i \in \{1, \dots, k\}$, $\frac{1}{12} d_D(x_i) \le r_i \le d_D(x_i)$;

(3) *If* $k \ge 3$, *then for any* $i, j \in \{1, \dots, k\}$ *with* $|i - j| \ge 2$, *we have* $\operatorname{dist}(B_i, B_j) \ge \frac{1}{64M^8} \max\{r_i, r_j\}$;

(4) *If* $k \ge 2$, *then* $r_i + r_{i+1} - |x_i - x_{i+1}| \ge \frac{1}{64M^8} \max\{r_i, r_{i+1}\}$ *for each* $i \in \{1, \dots, k-1\}$,

where $B_i = \mathsf{B}(x_i, r_i)$, $x_i \in \alpha$ *and* $x_i \notin B_{i-1}$.

PROOF. We divide the proof into two cases.

CASE 2.15. For any $i \in \{1, \dots, k_1\}$ and $u \in \{1, \dots, k_2 - 1\}$, we have $r_{1,i} + r_{2,u} - |x_{1,i} - x_{2,u}| \le \frac{1}{64M^7} \max\{r_{1,i}, r_{2,u}\}$.

For each $i \in \{1, \dots, k_1 - 1\}$, we let $A_{1,i} = \mathsf{B}(x_{1,i}, R_{1,i})$ with $R_{1,i} = \big(1 - \frac{1}{64M^3}\big) r_{1,i}$ and for each $u \in \{1, \dots, k_2 - 1\}$, let $A_{2,u} = \mathsf{B}(x_{2,u}, R_{2,u})$ with $R_{2,u} = \big(1 - \frac{1}{64M^3}\big) r_{2,u}$. Let $A_{1,k_1} = \mathsf{B}(x_{1,k_1}, r_{1,k_1})$. By Lemma 2.2 and Corollary 2.12, we have

CLAIM 2.16.

(1) *For any* $i \in \{1, \dots, k_1\}$, *we have* $\frac{1}{4} d_D(x_{1,i}) \le R_{1,i} \le \frac{7}{8} d_D(x_{1,i})$, *and for each* $u \in \{1, \dots, k_2 - 1\}$, *we have* $\frac{1}{4} d_D(x_{2,u}) \le R_{2,u} \le \frac{7}{8} d_D(x_{2,u})$;

(2) *If* $k_1 \ge 3$, *then for any* $i, j \in \{1, \dots, k_1\}$ *with* $|i - j| \ge 2$, *we have* $\operatorname{dist}(A_{1,i}, A_{1,j}) \ge \frac{1}{32M^2} \max\{r_{1,i}, r_{1,j}\}$;

(3) *If* $k_2 \ge 3$, *then for any* $u, v \in \{1, \dots, k_2\}$ *with* $|u - v| \ge 2$, *we have* $\operatorname{dist}(A_{2,u}, A_{2,v}) \ge \frac{1}{32M^2} \max\{r_{2,u}, r_{2,v}\}$;

(4) *For any* $i \in \{1, \dots, k_1\}$ *and* $u \in \{1, \dots, k_2 - 1\}$, *we have* $\operatorname{dist}(A_{1,i}, A_{2,u}) \ge \frac{1}{32M^4} \max\{r_{1,i}, r_{2,u}\}$.

If $\overline{\mathsf{B}}\big(x_{2,k_2}, \big(1 + \frac{1}{64M^2}\big) r_{2,k_2}\big) \cap \bigcup_{i=1}^{k_1 - 1} \overline{A}_{1,i} = \emptyset$, then we let $A_{2,k_2} = \mathsf{B}\big(x_{2,k_2}, \big(1 + \frac{1}{128M^2}\big) r_{2,k_2}\big)$. It follows from Corollary 2.12 and Lemma 2.13 that the balls $A_{1,1}, \dots, A_{1,k_1 - 1}, A_{1,k_1}$ and $A_{2,1}, \dots, A_{2,k_2}$ satisfy the conditions (1) $\sim$ (4) in the lemma, where $k = k_1 + k_2$.

In the following, we assume that $\overline{\mathsf{B}}\big(x_{2,k_2}, \big(1 + \frac{1}{64M^2}\big) r_{2,k_2}\big) \cap \bigcup_{i=1}^{k_1 - 1} \overline{A}_{1,i} \ne \emptyset$. We let $A_{1,q}$ be the first ball from $A_{1,1}$ to $A_{1,k_1 - 1}$ such that the closure $\overline{A}_{1,q}$ has nonempty intersection with $\overline{\mathsf{B}}\big(x_{2,k_2}, \big(1 + \frac{1}{64M^2}\big) r_{2,k_2}\big)$.

Let $R'_{2,k_2} = \big(1 + \frac{1}{64M^2}\big) r_{2,k_2}$. We choose $B_i = A_{1,i}$ $(1 \le i \le q)$, $B_{q+1} = \mathsf{B}\big(x_{2,k_2}, \big(1 + \frac{7}{512M^2}\big) r_{2,k_2}\big)$, $B_{q+2} = A_{2,k_2 - 1}, \dots, B_k = A_{2,1}$ whenever

$$R'_{2,k_2} + R_{1,q} - |x_{2,k_2} - x_{1,q}| \ge \frac{1}{256M^2} R'_{2,k_2}.$$

Then Corollary 2.12 and Lemma 2.13 show that the balls $B_1, B_2, \ldots, B_k$ satisfy the conditions (1) $\sim$ (4) in our lemma, where $k = q + k_2$.

On the other hand, in the case of

$$R'_{2,k_2} + R_{1,q} - |x_{2,k_2} - x_{1,q}| < \frac{1}{256M^2} R'_{2,k_2},$$

we consider the ball $B''_{2,k_2} = \mathsf{B}(x_{2,k_2}, R''_{2,k_2})$ with $R''_{2,k_2} = \big(1 + \frac{1}{128M^2}\big) r_{2,k_2}$. Obviously, $A_{1,k_1} \cap B''_{2,k_2} \neq \emptyset$. Let A_{1,q_1} be the first ball from $A_{1,q}$ to A_{1,k_1} such that the closure $\overline{A}_{1,q_1}$ has nonempty intersection with $\overline{\mathsf{B}}\big(x_{2,k_2}, \big(1 + \frac{1}{128M^2}\big) r_{2,k_2}\big)$. For each $i \in \{q, \ldots, q_1\}$, we denote $\operatorname{dist}(A_{1,i}, B''_{2,k_2})$ by d_i. Clearly, $d_{q_1} = 0$ and $d_q > \frac{1}{512M^2} r_{2,k_2}$.

Let $\eta_1 = d_q$ and η_2 be the first d_i from d_q to d_{q_1} satisfying $d_i < \eta_1$. Clearly, $\eta_1 > \eta_2$. By repeating the procedure, we get $\eta_1, \ldots, \eta_{m_1} \in \{d_q, \ldots, d_{q_1}\}$ such that

$$\eta_1 > \eta_2 > \cdots > \eta_{m_1} = 0.$$

Observe that $\eta_1 > \frac{1}{512M^2} r_{2,k_2}$ and $m_1 \geq 2$. For each $h \in \{1, \ldots, m_1 - 1\}$, we denote the first ball from $A_{1,q}$ to A_{1,q_1} with $d_{i_h} = \eta_h$ by A_{1,i_h}, i.e. $\mathsf{B}(x_{1,i_h}, R_{1,i_h})$, and define $\varepsilon_h = \eta_h - \eta_{h+1}$.

Replacing $\frac{5}{16} r_{1,s}$ by $\frac{1}{512M^2} r_{2,k_2}$ and M by M^4, the similar reasoning as in the proof of Subclaim 2.8 shows

CLAIM 2.17. *There must exist some* $j \in \{1, \ldots, m_1 - 1\}$ *such that* $\varepsilon_j > \frac{1}{256M^7} r_{2,k_2}$.

We now consider the ball $C''_{2,k_2} = \mathsf{B}(x_{2,k_2}, r''_{2,k_2})$, where

$$r''_{2,k_2} = R''_{2,k_2} + \eta_{j+1} + \frac{1}{512M^7} r_{2,k_2}.$$

By Claim 2.17, we see that $C''_{2,k_2} \cap A_{1,i} = \emptyset$ for all $i < i_{j+1}$. We take $B_i = A_{1,i}$ for each $i \in \{1, \ldots, i_{j+1}\}$, $B_{i_{j+1}+1} = C''_{2,k_2}$, $B_{i_{j+1}+2} = A_{2,k_2-1}, \ldots, B_k = A_{1,1}$. Then Lemma 2.13 yields that the balls $B_1, \ldots, B_{i_{j+1}}, B_{i_{j+1}+1}, \ldots, B_k$ satisfy the conditions (1) $\sim$ (4) in the lemma, where $k = i_{j+1} + k_2$.

CASE 2.18. There exist $i \in \{1, \ldots, k_1\}$ and $u \in \{1, \ldots, k_2 - 1\}$ such that $r_{1,i} + r_{2,u} - |x_{1,i} - x_{2,u}| > \frac{1}{64M^7} \max\{r_{1,i}, r_{2,u}\}$.

Let $B_{2,s}$ be the first ball from $B_{2,1}$ to B_{2,k_2-1} such that there exists some $i \in \{1, \ldots, k_1\}$ satisfying $r_{1,i} + r_{2,s} - |x_{1,i} - x_{2,s}| > \frac{1}{64M^7} \max\{r_{1,i}, r_{2,s}\}$.

Let $B_{1,t}$ be the first ball from $B_{1,1}$ to B_{1,k_1} satisfying $r_{1,t} + r_{2,s} - |x_{1,t} - x_{2,s}| > \frac{1}{64M^7} \max\{r_{1,t}, r_{2,s}\}$.

For any $i \in \{1, \dots, t-1\}$, we let $C_{1,i} = \mathsf{B}\big(x_{1,i}, \big(1 - \frac{1}{64M^3}\big)r_{1,i}\big)$ and $C_{1,t} = \mathsf{B}\big(x_{1,t}, \big(1 - \frac{1}{M^8}\big)r_{1,t}\big)$, and for any $u \in \{1, \dots, s-1\}$, let $C_{2,u} = \mathsf{B}\big(x_{2,u}, \big(1 - \frac{1}{64M^3}\big)r_{2,u}\big)$ and $C_{2,s} = \mathsf{B}\big(x_{2,s}, \big(1 - \frac{1}{M^8}\big)r_{2,s}\big)$. By letting $B_1 = C_{1,1}, \dots, B_{t-1} = C_{1,t-1}$, $B_t = C_{1,t}$, $B_{t+1} = C_{2,s}$, $B_{t+2} = C_{2,s-1}, \dots$ and $B_k = C_{2,1}$, we conclude from Lemma 2.1 that the balls $B_1, \dots, B_t, B_{t+1}, \dots, B_k$ satisfy the conditions (1) $\sim$ (4) in the lemma, where $k = t + s$.

The following two lemmas are also needed in the proof of Theorem 1.8.

LEMMA 2.19. *For any $i, j \in \{1, \dots, k\}$ with $j \geq i+2$, we have $\ell(\alpha[x_i, x_j]) \leq 36c^2|x_i - x_j|$.*

PROOF. If $\{x_i, x_j\} \subset \gamma$ (resp. β), by the assumption $j \geq i + 2$ and Lemma 2.14, we get

$$\ell(\alpha[x_i, x_j]) \leq cd_D(x_j) \leq 12cr_j \leq 12c|x_i - x_j|. \tag{2.20}$$

For the rest case, without loss of generality, we may assume that $x_i \in \gamma$ and $x_j \in \beta$.

If $\max\{|z_1 - x_i|, |z_2 - x_j|\} \leq \frac{1}{3}|z_1 - z_2|$, then

$$|x_i - x_j| \geq |z_1 - z_2| - |z_1 - x_i| - |z_2 - x_j| \geq \frac{1}{3}|z_1 - z_2|.$$

Hence

$$\ell(\alpha[x_i, x_j]) \leq \ell(\alpha) \leq c|z_1 - z_2| \leq 3c|x_i - x_j|. \tag{2.21}$$

If $\max\{|z_1 - x_i|, |z_2 - x_j|\} > \frac{1}{3}|z_1 - z_2|$, we may assume that $\max\{|z_1 - x_i|, |z_2 - x_j|\} = |z_1 - x_i|$. Then by the assumption $j \geq i+2$ and Lemma 2.14 we get

$$\begin{aligned} \ell(\alpha[x_i, x_j]) &\leq \ell(\alpha) \leq c|z_1 - z_2| \\ &\leq 3c|z_1 - x_i| \leq 36c^2 r_i \leq 36c^2|x_i - x_j|. \end{aligned} \tag{2.22}$$

We conclude from (2.20) $\sim$ (2.22) that Lemma 2.19 holds.

LEMMA 2.23. *For any $w_1 \neq w_2 \in D$ and $r_1 \geq r_2 > 0$, we let $w_1 \in D \setminus \mathsf{B}(w_2, r_2)$,*

$$r_1 + r_2 - |w_1 - w_2| \geq \frac{1}{64M^8} r_2$$

and $Q = \mathsf{B}(w_1, r_1) \cup \mathsf{B}(w_2, r_2)$. Then Q is $2^{11}M^8$-uniform.

Before the proof of Lemma 2.23, we introduce the following lemma.

LEMMA C ([12, Theorem 1.2]). *Suppose that D_1 and D_2 are convex domains in E, where D_1 is bounded and D_2 is c-uniform for some $c > 1$, and that there exist $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $\mathsf{B}(z_0, r) \subset D_1 \cap D_2$. If there exist constants $R_1 > 0$ and $c_0 > 1$ such that $R_1 \leq c_0 r$ and $D_1 \subset \overline{\mathsf{B}}(z_0, R_1)$, then $D_1 \cup D_2$ is a c'-uniform domain with $c' = (c+1)(2c_0+1)+c$.*

PROOF OF LEMMA 2.23. Obviously, there exists $z_0 \in \mathsf{B}(w_2, r_2) \cap \mathsf{B}(w_1, r_1)$ such that the ball $\mathsf{B}(z_0, r)$ is contained in the intersection $\mathsf{B}(w_2, r_2) \cap \mathsf{B}(w_1, r_1)$, where $r = \frac{1}{128M^8} r_2$. Hence $\mathsf{B}(w_2, r_2) \subset \mathsf{B}(z_0, 256M^8 r)$. It follows from [20] that each ball in E is 2-uniform. Then Lemma C implies that Q is $2^{11}M^8$-uniform.

2.24 PROOF OF THEOREM 1.8. It suffices to prove the necessity since the sufficiency is obvious.

Assume that D is a c-uniform domain. Then for every pair of points z_1, $z_2 \in D$, there is a rectifiable arc $\alpha \subset D$ joining them with

$$\ell(\alpha[z_1, z_2]) \leq c\,|z_1 - z_2| \qquad \text{and} \qquad \min_{j=1,2} \ell(\alpha[z_j, z]) \leq c\, d_D(z)$$

for all $z \in \alpha$.

It follows from Lemma 2.14 that there exists a domain D_1 which is simply connected satisfying Items (1) $\sim$ (4) in Lemma 2.14. Let $c_1 = \frac{1}{64M^8}$. We come to prove that D_1 is a c_2-uniform domain, where $c_2 = 72c^2(\frac{2}{c_1}+1)$.

For any $y_1, y_2 \in D_1$, there must exist $i, j \in \{1, \ldots, k\}$ such that $y_1 \in \mathsf{B}(x_i, r_i)$ and $y_2 \in \mathsf{B}(x_j, r_j)$.

If $|j-i| \leq 1$, then it follows from Lemma 2.23 and the fact $r_i + r_{i+1} - |x_i - x_{i+1}| \geq c_1 \max\{r_i, r_{i+1}\}$ (see Lemma 2.14 (4)) that there exists a rectifiable curve α_1 joining y_1 and y_2 in $\mathsf{B}(x_i, r_i) \cup \mathsf{B}(x_{i+1}, r_{i+1})$ such that

$$\ell(\alpha_1) \leq 2^{11} M^8\, |y_1 - y_2| \tag{2.25}$$

and

$$\min_{s=1,2} \ell(\alpha_1[y_s, y]) \leq 2^{11} M^8\, d_{D_1}(y) \tag{2.26}$$

for all $y \in \alpha_1$.

The remaining case we need to consider is: There are $i, j \in \{1, \ldots, k\}$ such that $j - i \geq 2$, $y_1 \in B_i$, $y_2 \in B_j$ and $\{y_1, y_2\}$ is not contained in $B_t \cup B_{t+1}$ for any $t \in \{i, \ldots, j-1\}$. It suffices to prove the case: $y_1 \notin [x_i, x_{i+1}]$ and $y_2 \notin [x_{j-1}, x_j]$ since the discussions for other cases are similar. Set

$$\alpha_2 = [y_1, x_i] \cup [x_i, x_{i+1}] \cup \ldots \cup [x_{j-1}, x_j] \cup [x_j, y_2].$$

By Items (2) and (3) in Lemma 2.14 and Lemma 2.19, we have

$$\begin{aligned}\ell(\alpha_2) &\le |y_1 - x_i| + |x_j - y_2| + \ell(\alpha[x_i, x_j]) \\ &\le 2\,\ell(\alpha[x_i, x_j]) \\ &\le 72c^2\,|x_j - x_i| \\ &= 72c^2(r_i + r_j + \operatorname{dist}(B_i, B_j)) \\ &\le 72c^2\left(\frac{2}{c_1} + 1\right)|y_1 - y_2|,\end{aligned} \tag{2.27}$$

since $|y_1 - y_2| \ge \operatorname{dist}(B_i, B_j)$.

For any $y \in \alpha_2$, if $y \in [y_1, x_i]$ or $[x_j, y_2]$, then we easily have that

$$\min_{j=1,2} \ell(\alpha_2[y_j, y]) \le d_{D_1}(y). \tag{2.28}$$

For the case $y \in [x_i, x_{i+1}] \cup \ldots \cup [x_{j-1}, x_j]$, obviously, there exists some $m \in \{i, \ldots, j-1\}$ such that $y \in [x_m, x_{m+1}]$. Without loss of generality, we may assume that $\min\{\ell(\alpha[z_1, x_m]), \ell(\alpha[x_m, z_2])\} = \ell(\alpha[z_1, x_m])$. The proof for the case $\min\{\ell(\alpha[z_1, x_m]), \ell(\alpha[x_m, z_2])\} = \ell(\alpha[z_2, x_m])$ follows from the similar reasoning.

It follows from Lemma 2.14 (2) that

$$\ell(\alpha[z_1, x_m]) \le 12c\, d_{D_1}(x_m),$$

which in turn yields that

$$\begin{aligned}\ell(\alpha_2[y_1, x_m]) &\le |y_1 - x_i| + \ell(\alpha[x_i, x_m]) \\ &\le 24c\, d_{D_1}(x_m).\end{aligned} \tag{2.29}$$

If $\min\{\ell(\alpha[z_1, x_{m+1}]), \ell(\alpha[x_{m+1}, z_2])\} = \ell(\alpha[z_1, x_{m+1}])$, then (2.29) yields that

$$\begin{aligned}\min_{s=1,2} \ell(\alpha_2[y_s, y]) &\le \ell(\alpha_2[y_1, y]) \\ &\le 24c\, d_{D_1}(x_m) + |y - x_m| \\ &\le (24c + 1)\, d_{D_1}(x_m) + d_{D_1}(y) \\ &\le \frac{2}{c_1}\left(24c + \frac{c_1}{2} + 1\right) d_{D_1}(y).\end{aligned} \tag{2.30}$$

Now we assume that $\min\{\ell(\alpha[z_1, x_{m+1}]), \ell(\alpha[x_{m+1}, z_2])\} = \ell(\alpha[z_2, x_{m+1}])$. Then Lemma 2.14 (2) implies that $\ell(\alpha[z_2, x_{m+1}]) \le 12c\, d_{D_1}(x_{m+1})$. Hence

$$\ell(\alpha_2[y_2, x_{m+1}]) \le 24c\, d_{D_1}(x_{m+1}). \tag{2.31}$$

We infer from (2.31) that

$$\begin{aligned}\min_{s=1,2} \ell(\alpha_2[y_s, y]) &\le \ell(\alpha_2[y_2, y]) \\ &\le 24c\, d_{D_1}(x_{m+1}) + |y - x_{m+1}| \\ &\le (24c+1)\, d_{D_1}(x_{m+1}) + d_{D_1}(y) \\ &\le \frac{2}{c_1}\left(24c + \frac{c_1}{2} + 1\right) d_{D_1}(y).\end{aligned} \tag{2.32}$$

Thus the inequalities (2.25) $\sim$ (2.28), (2.30) and (2.32) show that D_1 is a c_2-uniform domain. The proof of Theorem 1.8 is complete.

3. Proofs of Theorem 1.9 and Example 1.10

3.1 Proof of Theorem 1.9. Let $f\colon D \to \mathsf{B}^n$ be a quasiconformal map of $\overline{\mathsf{R}}^n$. For any $z_1, z_2 \in D$, there exists a closed ball $\overline{B}_1^n \subset \mathsf{B}^n$ such that $f(z_1), f(z_2) \in \overline{B}_1^n$. Then $f^{-1}(B_1^n)$ is a quasiball. This shows that D has the quasiball decomposition property.

3.2 Proof of Example 1.10. A result of Väisälä [17, Theorem 17.22] implies that D is not a quasiball.

For any $z_1, z_2 \in D$, let P be the plane determined by z_1 and L. Then P divides B^3 into two parts which are denoted by B_1^3 and B_2^3, respectively. We may assume that $z_1, z_2 \in \overline{B}_1^3$. Since B_1^3 is a bounded convex domain, the result in [22] shows that B_1^3 is a quasiball. This implies that D has the quasiball decomposition property.

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