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GROWTH OF LOGARITHMIC DERIVATIVE OF MEROMORPHIC FUNCTIONS

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Abstract

In this paper, we give some estimations about the growth of logarithmic derivative of meromorphic and entire functions and their applications in the theory of differential equations. We give also some examples to explain the sharpness of our results.

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory ([8], [15]). For any nonconstant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$\lim_{r\to+\infty} \frac{S(r, f)}{T(r, f)} = 0,$$

possibly outside of a set of finite linear measure in $[0, +\infty)$, where $T(r, f)$ is the Nevanlinna characteristic function of f. In the following, we give the necessary notations and basic definitions.

Definition 1.1 ([5]). Let f be a meromorphic function. Then the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of $f(z)$ are defined respectively by

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r, f)}{\log r}.$$

Definition 1.2 ([5], [15]). Let f be a meromorphic function. Then the exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r},$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$.

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Similarly, the exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}(r, 1/f)}{\log r},$$

where $\overline{N}\big(r, \frac{1}{f}\big)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$.

The following result is very important in the theory of differential equations.

THEOREM A ([6]). *Let f be a transcendental meromorphic function with $\rho(f) = \rho < \infty$, $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geqslant 0$, for $i = 1, \ldots, q$, and let $\varepsilon > 0$ be a given constant. Then*

(i) *there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geqslant R_0$ and for all $(k, j) \in H$, we have*
$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)},$$

(ii) *there exists a set $E_2 \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$ and for all $(k, j) \in H$, we have*
$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)},$$

(iii) *there exists a set $E_3 \subset [0, +\infty)$ that has finite linear measure, such that for all z satisfying $|z| \notin E_3$ and for all $(k, j) \in H$, we have*
$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}.$$

The main purpose of this paper is to give new estimations about the growth of logarithmic derivative. We also investigate the relationship between them, the hyper-order and the exponent of convergence.

THEOREM 1.1. *Suppose that $k \geqslant 2$ is an integer and let f be a meromorphic function. Then*

$$\begin{aligned}\rho\left(\frac{f'}{f}\right) &= \max\left\{\rho\left(\frac{f^{(k)}}{f}\right), k \geqslant 2\right\}\\ &= \max\left\{\rho\left(\frac{f^{(k)}}{f}\right), \rho\left(\frac{f^{(k+1)}}{f}\right), k \geqslant 2\right\}.\end{aligned} \tag{1.1}$$

COROLLARY 1.1. *Let f be a meromorphic function. If $\frac{f'}{f}$ has finite order, then for any integer $k \geqslant 2$*

$$\rho\left(\frac{f^{(k)}}{f}\right) < \infty. \tag{1.2}$$

COROLLARY 1.2. *Let f be meromorphic function. If there exists an integer $k \geqslant 1$ such that $\rho\left(\frac{f^{(k)}}{f}\right) = \rho(f)$ and $\rho(f) > \rho_2(f)$, then*

$$\max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} = \rho(f). \tag{1.3}$$

Furthermore, if f is entire function, then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f).$$

EXAMPLE 1.1. It's clear that the entire function $f(z) = e^z - 1$ satisfies

$$\rho\left(\frac{f'}{f}\right) = \rho\left(\frac{e^z}{e^z - 1}\right) = \rho(f) = 1$$

and $\rho(f) = 1 > \rho_2(f) = 0$. Then by Corollary 1.2, we have

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = 1.$$

On the other hand, the meromorphic function $f(z) = \frac{1}{e^z - 1}$ satisfies

$$\rho\left(\frac{f'}{f}\right) = \rho\left(-\frac{e^z}{e^z - 1}\right) = \rho(f) = 1, \qquad \rho(f) = 1 > \rho_2(f) = 0$$

and

$$\overline{\lambda}\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{f}\right) = 1, \qquad \overline{\lambda}(f) = \lambda(f) = 0.$$

We see that

$$\max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} = \rho(f) = 1.$$

REMARK 1.1. The condition $\rho(f) > \rho_2(f)$ in Corollary 1.2 is necessary. For example, if we take $f(z) = \exp(\exp(\exp z))$, then f satisfies

$$\rho\left(\frac{f'}{f}\right) = \rho(f) = \rho_2(f) = \infty$$

and

$$\lambda(f) = \lambda\left(\frac{1}{f}\right) = 0.$$

COROLLARY 1.3. *Let f be a meromorphic function such that for any integer $k \geqslant 1$, we have*

$$\rho\left(\frac{f^{(2k)}}{f}\right) < \rho\left(\frac{f'}{f}\right). \tag{1.4}$$

Then

$$\rho\left(\frac{f^{(2k+1)}}{f}\right) = \rho\left(\frac{f'}{f}\right) \qquad (k \geqslant 1). \tag{1.5}$$

EXAMPLE 1.2. Let $f(z) = \sin z$, it's clear that $\frac{f^{(2k)}}{f}$ is constant, for any integer $k \geqslant 1$. Then by Corollary 1.3

$$\rho\left(\frac{f^{(2k+1)}}{f}\right) = \rho\left(\frac{f'}{f}\right) \qquad (k \geqslant 1)$$

and since $\rho\left(\frac{f'}{f}\right) = \rho(f) = 1$, then we obtain that

$$\rho\left(\frac{f^{(2k+1)}}{f}\right) = \rho(f) = 1 \qquad (k \geqslant 1).$$

THEOREM 1.2. *Let f be an entire function with finite number of zeros. Then for any integer $k \geqslant 1$*

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho_2(f). \tag{1.6}$$

COROLLARY 1.4. *Let f be an entire function and c be a nonzero constant. Then*

$$\rho\left(f' + cf^2\right) = \rho(f). \tag{1.7}$$

REMARK 1.2. Corollary 1.4 was proved by S. Bank and I. Laine in [2].

2. Applications in differential equations

THEOREM 2.1. *Let $k \geqslant 1$ be an integer and let f be a solution of the differential equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.1}$$

where A_j $(j = 0, \dots, k-1)$, $F \not\equiv 0$ are entire functions satisfying

$$\max\{\rho(F), \rho(A_j)\ (j = 0, \dots, k-1)\} < \rho(f). \tag{2.2}$$

Then

$$\rho(f) = \rho\left(\frac{f'}{f}\right) = \overline{\lambda}(f) = \lambda(f). \tag{2.3}$$

Furthermore, if $\frac{f^{(j)}}{f}$ is not constant ($j \geqslant 2$ is an integer), then

$$\rho(f) = \rho\left(\frac{f^{(j)}}{f}\right) \qquad (j \geqslant 2). \tag{2.4}$$

REMARK 2.1. In Theorem 2.1, we obtain the result due to S. A. Gao, Z. X. Chen and T. W. Chen [5], but our simple proof is quite different.

THEOREM 2.2. *Let $k \geqslant 1$ be an integer, and let f be a finite order meromorphic solution of the differential equation*

$$f^{(k)} = A_1 f + A_2 f^2 + \cdots + A_{n-1} f^{n-1} + A_n f^n, \tag{2.5}$$

where A_j $(j = 1, \dots, n)$ ($n \geqslant 2$ is an integer) are meromorphic functions satisfying

$$\max\{\rho(A_j) : j = 1, \dots, n\} < \rho(f). \tag{2.6}$$

Then

$$\rho(f) = \max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\}. \tag{2.7}$$

EXAMPLE 2.1. It's clear that the function $f(z) = \frac{1}{e^z - 1}$ satisfies the differential equation

$$f' = -f - f^2$$

and

$$\max\{\rho(A_j) : j = 1, 2\} = 0 < \rho(f) = 1.$$

Hence, by Theorem 2.2, we have

$$\rho(f) = \max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} = 1.$$

3. Some auxiliary lemmas

LEMMA 3.1 ([7]). *Let $\varphi : [0, +\infty) \to \mathsf{R}$ and $\psi : [0, +\infty) \to \mathsf{R}$ be monotone non-decreasing functions such that $\varphi(r) \leqslant \psi(r)$ for all $r \notin E_4 \cup [0, 1]$, where $E_4 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists an $r_1 = r_1(\gamma) > 0$ such that $\varphi(r) \leqslant \psi(\gamma r)$ for all $r > r_1$.*

LEMMA 3.2 ([10, pp. 36–37], [11, p. 51]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ, $\mu(r)$ be the maximum term, i.e., $\mu(r) = \max\{|a_n|r^n; n = 0, 1, \ldots\}$, and let $\nu_f(r)$ be the central index of f, i.e., $\nu_f(r) = \max\{m; \mu(r) = |a_m|r^m\}$. Then*

$$\limsup_{r\to+\infty} \frac{\log \nu_f(r)}{\log r} = \rho. \tag{3.1}$$

LEMMA 3.3 ([4]). *Let $f(z)$ be an entire function of infinite order with the hyper-order $\rho_2(f) = \sigma < +\infty$. Then*

$$\limsup_{r\to+\infty} \frac{\log\log \nu_f(r)}{\log r} = \sigma. \tag{3.2}$$

LEMMA 3.4 (Wiman-Valiron, [9], [14]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire function, and let $\nu_f(r)$ be the central index of f. Let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then the estimation*

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^k (1 + o(1)) \qquad (k \geqslant 1 \textit{ is an integer}) \tag{3.3}$$

holds for all $|z|$ outside a set E_5 of r of finite logarithmic measure.

LEMMA 3.5 ([3]). *Suppose that $k \geqslant 2$ and $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ are entire functions of finite order. If f is a solution of equation* (2.1)*, then $\rho_2(f) \leqslant \max\{\rho(A_j) : j = 0, \ldots, k-1, \rho(F)\} = \sigma$.*

4. Proof of Theorem 1.1

First, we prove the inequality

$$\max\left\{\rho\left(\frac{f^{(k)}}{f}\right), k \geqslant 2\right\} \leqslant \rho\left(\frac{f'}{f}\right).$$

We have

$$\frac{f^{(k)}}{f} = \left(\frac{f^{(k-1)}}{f}\right)' + \left(\frac{f'}{f}\right)\left(\frac{f^{(k-1)}}{f}\right) \qquad (k \geqslant 2). \tag{4.1}$$

Then

$$\rho\left(\frac{f^{(k)}}{f}\right) \leqslant \max\left\{\rho\left(\frac{f'}{f}\right), \rho\left(\frac{f^{(k-1)}}{f}\right)\right\} \qquad (k \geqslant 2). \tag{4.2}$$

By the same method, we can deduce that

$$\begin{aligned} \rho\left(\frac{f^{(k)}}{f}\right) &\leqslant \max\left\{\rho\left(\frac{f'}{f}\right), \rho\left(\frac{f^{(k-1)}}{f}\right)\right\} \\ &\leqslant \max\left\{\rho\left(\frac{f'}{f}\right), \rho\left(\frac{f^{(k-2)}}{f}\right)\right\} \\ &\leqslant \cdots \leqslant \max\left\{\rho\left(\frac{f'}{f}\right), \rho\left(\frac{f''}{f}\right)\right\} \\ &\leqslant \max\left\{\rho\left(\frac{f'}{f}\right), \rho\left(\frac{f'}{f}\right)\right\} = \rho\left(\frac{f'}{f}\right) \qquad (k \geqslant 2). \end{aligned} \tag{4.3}$$

Now we prove the equality. We divide the proof in three cases.

(i) Suppose that $\rho\left(\frac{f^{(k)}}{f}\right) < \rho\left(\frac{f^{(k+1)}}{f}\right)$. By (4.1), we have

$$\frac{f^{(k+1)}}{f} - \left(\frac{f^{(k)}}{f}\right)' = \left(\frac{f'}{f}\right)\left(\frac{f^{(k)}}{f}\right) \qquad (k \geqslant 1). \tag{4.4}$$

Since $\rho\left(\frac{f^{(k)}}{f}\right) < \rho\left(\frac{f^{(k+1)}}{f}\right) \leqslant \rho\left(\frac{f'}{f}\right)$, then by (4.4) we obtain

$$\rho\left(\frac{f^{(k+1)}}{f}\right) = \rho\left(\frac{f'}{f}\right).$$

(ii) If $\rho\left(\frac{f^{(k)}}{f}\right) > \rho\left(\frac{f^{(k+1)}}{f}\right)$, then by (4.4) we have

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho\left(\frac{f'}{f}\frac{f^{(k)}}{f}\right). \tag{4.5}$$

By (4.3) we have $\rho\big(\frac{f^{(k)}}{f}\big) \leqslant \rho\big(\frac{f'}{f}\big)$. If we suppose that $\rho\big(\frac{f^{(k)}}{f}\big) < \rho\big(\frac{f'}{f}\big)$, then

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho\left(\frac{f'}{f}\frac{f^{(k)}}{f}\right) = \rho\left(\frac{f'}{f}\right), \tag{4.6}$$

which is a contradiction. Hence

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho\left(\frac{f'}{f}\right).$$

(iii) Suppose that $\rho\big(\frac{f^{(k)}}{f}\big) = \rho\big(\frac{f^{(k+1)}}{f}\big)$. By (4.3), we have $\rho\big(\frac{f^{(k)}}{f}\big) \leqslant \rho\big(\frac{f'}{f}\big)$. If we suppose that $\rho\big(\frac{f^{(k)}}{f}\big) < \rho\big(\frac{f'}{f}\big)$, then by (4.4) we obtain

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho\left(\frac{f'}{f}\right), \tag{4.7}$$

which is a contradiction. Thus, by (i),(ii) and (iii) we deduce that

$$\max\left\{\rho\left(\frac{f^{(k)}}{f}\right), \rho\left(\frac{f^{(k+1)}}{f}\right)\right\} = \rho\left(\frac{f'}{f}\right). \tag{4.8}$$

By (4.8) we can conclude that there exists always some integer $j \geqslant 1$ such that

$$\rho\left(\frac{f^{(j)}}{f}\right) = \rho\left(\frac{f'}{f}\right).$$

Thus

$$\rho\left(\frac{f'}{f}\right) = \max\left\{\rho\left(\frac{f^{(k)}}{f}\right), k \geqslant 2\right\}. \tag{4.9}$$

5. Proof of Corollary 1.2

Since there exists an integer $k \geqslant 1$ such that

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho(f),$$

then by Theorem 1.1, we have

$$\rho\left(\frac{f'}{f}\right) = \rho(f). \tag{5.1}$$

On the other hand, for any given $\varepsilon > 0$

$$
\begin{aligned}
T\left(r, \frac{f'}{f}\right) &= m\left(r, \frac{f'}{f}\right) + N\left(r, \frac{f'}{f}\right) \\
&= m\left(r, \frac{f'}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) \\
&\leqslant m\left(r, \frac{f'}{f}\right) + r^{\lambda_1+\varepsilon} + r^{\lambda_2+\varepsilon} \\
&\leqslant m\left(r, \frac{f'}{f}\right) + 2r^{\max\{\lambda_1,\lambda_2\}+\varepsilon},
\end{aligned} \tag{5.2}
$$

where $\lambda_1 = \overline{\lambda}(f)$, $\lambda_2 = \overline{\lambda}\left(\frac{1}{f}\right)$. Then by the lemma of logarithmic derivative [8] and (5.2), we have

$$
T\left(r, \frac{f'}{f}\right) \leqslant O(\log T(r, f) + \log r) + 2r^{\max\{\lambda_1,\lambda_2\}+\varepsilon} \tag{5.3}
$$

holds for all r outside of a set $E \subset (0, +\infty)$ of finite linear measure. By the standard lemma of removing an exceptional set of finite linear measure [1] and (5.3) we obtain

$$
\begin{aligned}
\rho(f) = \rho\left(\frac{f'}{f}\right) &\leqslant \max\left\{\rho_2(f), \overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} \\
&\leqslant \max\left\{\rho_2(f), \lambda(f), \lambda\left(\frac{1}{f}\right)\right\} \leqslant \rho(f),
\end{aligned} \tag{5.4}
$$

which implies

$$
\rho(f) = \max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\}. \tag{5.5}
$$

If f is entire function, then $\overline{\lambda}\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{f}\right) = 0$, so from (5.5), we obtain

$$
\overline{\lambda}(f) = \lambda(f) = \rho(f).
$$

6. Proof of Theorem 1.2

Suppose that f is an entire function with finite number of zeros. Then f can be represented by

$$
f(z) = p(z)e^{g}, \tag{6.1}
$$

where p is a polynomial and g is an entire function, and

$$f^{(k)} = \Pi e^{g}, \tag{6.2}$$

where Π is an entire function. It's clear that f satisfies the differential equation

$$pf^{(k)} - \Pi f = 0. \tag{6.3}$$

(i) If f is an entire solution of finite order, then g and Π must be polynomials and by (6.3)

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho\left(\frac{\Pi}{p}\right) = 0 = \rho_2(f). \tag{6.4}$$

(ii) If f is an entire solution of infinite order, then g and Π must be transcendental entire functions and

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho\left(\frac{\Pi}{p}\right) = \rho(\Pi). \tag{6.5}$$

We have also by (6.3)

$$\Pi = p\frac{f^{(k)}}{f}, \tag{6.6}$$

then by (6.6) and the lemma of logarithmic derivative [8]

$$\begin{aligned} T(r, \Pi) = m(r, \Pi) &\leqslant m(r, p) + m\left(r, \frac{f^{(k)}}{f}\right) \\ &= O(\log r) + O(\log rT(r, f)) \end{aligned} \tag{6.7}$$

holds for all r outside of a set $E \subset (0, +\infty)$ of finite linear measure. By the standard lemma of removing an exceptional set of finite linear measure [1] and (6.7) we obtain

$$\rho(\Pi) \leqslant \rho_2(f). \tag{6.8}$$

On the other hand by Lemma 3.4, there exists a set $E_5 \subset (1, +\infty)$ with finite logarithmic measure $lm(E_5) < +\infty$ and we can choose z satisfying $|z| = r \notin [0, 1] \cup E_5$ and $|f(z)| = M(r, f)$, such that (3.3) holds. Substituting (3.3) into (6.6) we obtain

$$|p(z)|\left(\frac{v_f(r)}{r}\right)^k |1 + o(1)| = |\Pi(z)| \leqslant M(r, \Pi) \tag{6.9}$$

holds for all z satisfying $|z| = r \notin [0, 1] \cup E_5$ and $|f(z)| = M(r, f)$. By using Lemma 3.1 and Lemma 3.3 from (6.9) we get

$$\rho_2(f) \leqslant \rho(\Pi). \tag{6.10}$$

By (6.5), (6.8) and (6.10) we deduce that

$$\rho_2(f) = \rho(\Pi) = \rho\left(\frac{f^{(k)}}{f}\right). \tag{6.11}$$

This proves Theorem 1.2.

7. Proof of Corollary 1.4

We define the entire function G

$$G(z) = \frac{1}{c}\exp\{cF(z)\}, \tag{7.1}$$

where F is the primitive of the entire function f. We have

$$G''(z) = (f' + cf^2)\exp\{cF(z)\}. \tag{7.2}$$

Then

$$\rho\left(\frac{G''}{G}\right) = \rho(f' + cf^2). \tag{7.3}$$

Since $\rho_2(G) = \rho(F) = \rho(f)$, then by Theorem 1.2 we obtain

$$\rho(f) = \rho(f' + cf^2). \tag{7.4}$$

8. Proof of Theorem 2.1

By (2.1), we can write

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{8.1}$$

Then

$$\rho(f) \leqslant \max\left\{\rho(F), \rho(A_j)\ (j = 0, \ldots, k-1), \rho\left(\frac{f^{(i)}}{f}\right)\ (i = 1, \ldots, k)\right\}. \tag{8.2}$$

By using (2.2) and Theorem 1.1, we obtain from (8.2)

$$\rho(f) \leqslant \max\left\{\rho\left(\frac{f^{(i)}}{f}\right) : i = 1, \ldots, k\right\} = \rho\left(\frac{f'}{f}\right) \leqslant \rho(f) \tag{8.3}$$

and by Corollary 1.2 and Lemma 3.5 we can deduce easily that

$$\rho(f) = \rho\left(\frac{f'}{f}\right) = \overline{\lambda}(f) = \lambda(f). \tag{8.4}$$

Now, we denote respectively by $n(r, 0, f)$ and $\overline{n}(r, 0, f)$ the number of zeros and distinct zeros of f in the disc $\{z : |z| < r\}$. It's clear that if $\frac{f^{(j)}}{f}$ $(j \geqslant 2)$ is not a constant, then

$$\overline{n}(r, 0, f) \leqslant n\left(r, 0, \frac{f}{f^{(j)}}\right). \tag{8.5}$$

Hence

$$\overline{N}\left(r, \frac{1}{f}\right) \leqslant N\left(r, \frac{f^{(j)}}{f}\right) \leqslant T\left(r, \frac{f^{(j)}}{f}\right), \tag{8.6}$$

which implies

$$\overline{\lambda}(f) \leqslant \rho\left(\frac{f^{(j)}}{f}\right). \tag{8.7}$$

By (8.4) and (8.7), we deduce

$$\rho(f) = \overline{\lambda}(f) = \lambda(f) \leqslant \rho\left(\frac{f^{(j)}}{f}\right) \leqslant \rho(f), \tag{8.8}$$

it follows that

$$\rho(f) = \overline{\lambda}(f) = \lambda(f) = \rho\left(\frac{f^{(j)}}{f}\right) \qquad (j \geqslant 2).$$

9. Proof of Theorem 2.2

By (2.5), we can write

$$\frac{f^{(k)}}{f} = A_1 + A_2 f + \cdots + A_{n-1} f^{n-2} + A_n f^{n-1}, \tag{9.1}$$

which implies by using the theorem due to Valiron [13] and Mohon'ko [12]

$$\begin{aligned} T\left(r, \frac{f^{(k)}}{f}\right) &= T(r, A_1 + A_2 f + \cdots + A_{n-1} f^{n-2} + A_n f^{n-1}) \\ &= (n-1)T(r, f) + S(r, f). \end{aligned} \tag{9.2}$$

By using the standard lemma of removing an exceptional set of finite linear measure [1] and Corollary 1.2, we obtain from (9.2)

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho(f) = \max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\}.$$

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