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SQUARE FUNCTIONS FOR RITT OPERATORS ON NONCOMMUTATIVE L^p-SPACES

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Abstract

For any Ritt operator T acting on a noncommutative L^p-space, we define the notion of *completely* bounded functional calculus $H^\infty(B_\gamma)$ where B_γ is a Stolz domain. Moreover, we introduce the 'column square functions'

$$\|x\|_{p,T,c,\alpha} = \bigg\| \bigg(\sum_{k=1}^{+\infty} k^{2\alpha-1} |T^{k-1}(I-T)^\alpha(x)|^2 \bigg)^{\frac{1}{2}} \bigg\|_{L^p(M)}$$

and the 'row square functions'

$$\|x\|_{p,T,r,\alpha} = \bigg\| \bigg(\sum_{k=1}^{+\infty} k^{2\alpha-1} |(T^{k-1}(I-T)^\alpha(x))^*|^2 \bigg)^{\frac{1}{2}} \bigg\|_{L^p(M)}$$

for any $\alpha > 0$ and any $x \in L^p(M)$. Then, we provide an example of Ritt operator which admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$ such that the square functions $\|\cdot\|_{p,T,c,\alpha}$ (or $\|\cdot\|_{p,T,r,\alpha}$) are not equivalent to the usual norm $\|\cdot\|_{L^p(M)}$. Moreover, assuming $1 < p < 2$ and $\alpha > 0$, we prove that if $\operatorname{Ran}(I-T)$ is dense and T admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$ then there exists a positive constant C such that for any $x \in L^p(M)$, there exists $x_1, x_2 \in L^p(M)$ satisfying $x = x_1 + x_2$ and $\|x_1\|_{p,T,c,\alpha} + \|x_2\|_{p,T,r,\alpha} \leqslant C\|x\|_{L^p(M)}$. Finally, we observe that this result applies to a suitable class of selfadjoint Markov maps on noncommutative L^p-spaces.

1. Introduction

Let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace. For any $1 \leqslant p < \infty$, we let $L^p(M)$ denote the associated (noncommutative) L^p-space. Let T be a bounded operator on $L^p(M)$. Consider the following 'square function'

$$(1) \quad \|x\|_{p,T,1} = \inf\bigg\{ \bigg\| \bigg(\sum_{k=1}^{+\infty} |u_k|^2 \bigg)^{\frac{1}{2}} \bigg\|_{L^p} + \bigg\| \bigg(\sum_{k=1}^{+\infty} |v_k^*|^2 \bigg)^{\frac{1}{2}} \bigg\|_{L^p} : u_k + v_k = k^{\frac{1}{2}}\big(T^k(x) - T^{k-1}(x)\big) \text{ for any integer } k \bigg\}$$

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if $1 < p \leqslant 2$ and

$$\|x\|_{p,T,1} = \max\left\{\left\|\left(\sum_{k=1}^{+\infty} k|T^k(x) - T^{k-1}(x)|^2\right)^{\frac{1}{2}}\right\|_{L^p(M)}, \left\|\left(\sum_{k=1}^{+\infty} k\left|(T^k(x) - T^{k-1}(x))^*\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(M)}\right\} \tag{2}$$

if $2 \leqslant p < \infty$, defined for any $x \in L^p(M)$. Such quantities were introduced in [13] and studied in this paper and in [2]. Similar square functions for continuous semigroups played a key role in the recent development of H^∞-calculus and its applications. See in particular the paper [9], the survey [12] and the references therein.

For any $\gamma \in \left]0, \frac{\pi}{2}\right[$, let B_γ be the interior of the convex hull of 1 and the disc $D(0, \sin\gamma)$. Suppose $1 < p < \infty$. Let T be a Ritt operator with $\operatorname{Ran}(I - T)$ dense in $L^p(M)$ which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$, i.e. there exists an angle $\gamma \in \left]0, \frac{\pi}{2}\right[$ and a positive constant K such that $\|\varphi(T)\|_{L^p(M)\to L^p(M)} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$ for any complex polynomial φ. A result of [13] essentially says that

$$\|x\|_{L^p(M)} \approx \|x\|_{p,T,1}, \qquad x \in L^p(M) \tag{3}$$

(see also [2, Remark 6.4]). Now, consider the following 'column and row square functions'

$$\|x\|_{p,T,c,1} = \left\|\left(\sum_{k=1}^{+\infty} k\left|T^k(x) - T^{k-1}(x)\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(M)} \tag{4}$$

and

$$\|x\|_{p,T,r,1} = \left\|\left(\sum_{k=1}^{+\infty} k\left|\left(T^k(x) - T^{k-1}(x)\right)^*\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(M)} \tag{5}$$

defined for any $x \in L^p(M)$. Assume $1 < p < 2$. In this context, if $x \in L^p(M)$, it is natural to search sufficient conditions to find a decomposition $x = x_1 + x_2$ such that $\|x_1\|_{p,T,c,1}$ and $\|x_2\|_{p,T,r,1}$ are finite. The first main result of this paper is the next theorem. It strengthens the above equivalence (3) in the case where T actually admits a *completely* bounded $H^\infty(B_\gamma)$ functional calculus, i.e. there exists a positive constant K such that $\|\varphi(T)\|_{cb,L^p(M)\to L^p(M)} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$ for any complex polynomial φ.

THEOREM 1.1. *Suppose* $1 < p < 2$. *Let* T *be a Ritt operator on* $L^p(M)$ *with* $\mathrm{Ran}(I - T)$ *dense in* $L^p(M)$. *Assume that* T *admits a completely bounded* $H^\infty(B_\gamma)$ *functional calculus for some* $\gamma \in \left]0, \frac{\pi}{2}\right[$. *Then we have*

$$\|x\|_{L^p(M)} \approx \inf\big\{\|x_1\|_{p,T,c,1} + \|x_2\|_{p,T,r,1} : x = x_1 + x_2\big\}, \qquad x \in L^p(M).$$

In this context, it is natural to compare the both quantities of (4) and (5). The second principal result of this paper is the following theorem. It says that in general, 'column and row square functions' are not equivalent.

THEOREM 1.2. *Suppose* $1 < p \neq 2 < \infty$. *Then there exists a Ritt operator* T *on the Schatten space* S^p, *with* $\mathrm{Ran}(I - T)$ *dense in* S^p, *which admits a completely bounded* $H^\infty(B_\gamma)$ *functional calculus for some* $\gamma \in \left]0, \frac{\pi}{2}\right[$ *such that*

$$\sup\left\{\frac{\|x\|_{p,T,c,1}}{\|x\|_{p,T,r,1}} : x \in S^p\right\} = \infty \qquad \text{if} \quad 2 < p < \infty$$

and

$$\sup\left\{\frac{\|x\|_{p,T,r,1}}{\|x\|_{p,T,c,1}} : x \in S^p\right\} = \infty \qquad \text{if} \quad 1 < p < 2.$$

Moreover, the same result holds with $\|\cdot\|_{p,T,c,1}$ *and* $\|\cdot\|_{p,T,r,1}$ *switched.*

The paper is organized as follows. Section 2 gives a brief presentation of noncommutative L^p-spaces and Ritt operators and we introduce the notions of Col-Ritt and Row-Ritt operators and completely bounded $H^\infty(B_\gamma)$ functional calculus which are relevant to our paper. The next section 3 mostly contains preliminary results concerning Col-Ritt and Row-Ritt operators. Section 4 is devoted to prove Theorems 1.2. In section 5, we present a proof of Theorem 1.1. We end this section by giving some natural examples to which this result can be applied.

In the above presentation and later on in the paper we will use $\lesssim$ to indicate an inequality up to a constant which does not depend to the particular element to which it applies. Then $A(x) \approx B(x)$ will mean that we both have $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

2. Background and preliminaries

We start with a few preliminaries on noncommutative L^p-spaces. Let M be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let M_+ be the set of all positive elements of M and let S_+ be the set of all x in M_+ such that $\tau(x) < \infty$. Then let S be the linear span of S_+. For any $1 \leqslant p < \infty$, define

$$\|x\|_{L^p(M)} = \big(\tau(|x|^p)\big)^{\frac{1}{p}}, \qquad x \in S,$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the modulus of x. Then $(S, \|\cdot\|_{L^p(M)})$ is a normed space. The corresponding completion is the noncommutative L^p-space associated with (M, τ) and is denoted by $L^p(M)$. By convention, we set $L^\infty(M) = M$, equipped with the operator norm. The elements of $L^p(M)$ can also be described as measurable operators with respect to (M, τ). Further multiplication of measurable operators leads to contractive bilinear maps $L^p(M) \times L^q(M) \to L^r(M)$ for any $1 \leqslant p, q, r \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ (noncommutative Hölder's inequality). Using trace duality, we then have $L^p(M)^* = L^{p^*}(M)$ isometrically for any $1 \leqslant p < \infty$. Moreover, complex interpolation yields $L^p(M) = [L^\infty(M), L^1(M)]_{\frac{1}{p}}$ for any $1 \leqslant p \leqslant \infty$. We refer the reader to [25] for details and complements.

Let $1 \leqslant p < \infty$. If we equip the space $B(\ell^2)$ with the operator norm and the canonical trace tr, the space $L^p(B(\ell^2))$ identifies to the Schatten-von Neumann class S^p. This is the space of those compact operators x from ℓ^2 into ℓ^2 such that $\|x\|_{S^p} = \big(\mathrm{tr}(x^*x)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$. Elements of $B(\ell^2)$ or S^p are regarded as matrices $A = [a_{ij}]_{i,j\geqslant 1}$ in the usual way.

If the von Neumann algebra $B(\ell^2) \overline{\otimes} M$ is equipped with the semifinite normal faithful trace $\mathrm{tr} \otimes \tau$, the space $L^p(B(\ell^2) \overline{\otimes} M)$ canonically identifies to a space $S^p(L^p(M))$ of matrices with entries in $L^p(M)$. Moreover, under this identification, the algebraic tensor product $S^p \otimes L^p(M)$ is dense in $S^p(L^p(M))$. We refer to [22] for more about these spaces and complements.

If $1 \leqslant p < \infty$, we say that a linear map on $L^p(M)$ is completely bounded if $I_{S^p} \otimes T$ extends to a bounded operator on $S^p(L^p(M))$. In this case, the completely bounded norm $\|T\|_{cb,L^p(M)\to L^p(M)}$ of T is defined by

$$\|T\|_{cb,L^p(M)\to L^p(M)} = \|I_{S^p} \otimes T\|_{S^p(L^p(M))\to S^p(L^p(M))}.$$

We use the convention to define $\|T\|_{cb,L^p(M)\to L^p(M)}$ by $+\infty$ if T is not completely bounded.

We shall use various ℓ^2-valued noncommutative L^p spaces. We refer to [9, Chapter 2] for more information on these spaces. For any $\sum_{k=1}^n x_k \otimes a_k \in L^p(M) \otimes \ell^2$, we set

$$\Big\|\sum_{k=1}^n x_k \otimes a_k\Big\|_{L^p(M,\ell^2_c)} = \bigg\|\Big(\sum_{i,j=1}^n \langle a_j, a_i\rangle x_i^* x_j\Big)^{\frac{1}{2}}\bigg\|_{L^p(M)}.$$

We have for any family $(x_k)_{k\geqslant 1}$ in $L^p(M)$

$$\Big\|\sum_{k=1}^n x_k \otimes e_k\Big\|_{L^p(M,\ell^2_c)} = \bigg\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p(M)} = \Big\|\sum_{k=1}^n e_{k1} \otimes x_k\Big\|_{S^p(L^p(M))}. \tag{6}$$

The space $L^p(M,\ell^2_c)$ is the completion of $L^p(M)\otimes\ell^2$ for this norm. It identifies to the space of sequences $(x_k)_{k\geqslant 1}$ in $L^p(M)$ such that $\sum_{k=1}^{+\infty} x_k\otimes e_k$ is convergent for the above norm. We define $L^p(M,\ell^2_r)$ similarly. For any finite family $(x_k)_{1\leqslant k\leqslant n}$ in $L^p(M)$, we have

$$\Big\|\sum_{k=1}^n x_k\otimes e_k\Big\|_{L^p(M,\ell^2_r)}=\Big\|\Big(\sum_{k=1}^n |x_k^*|^2\Big)^{\frac12}\Big\|_{L^p(M)}=\Big\|\sum_{k=1}^n e_{1k}\otimes x_k\Big\|_{S^p(L^p(M))}.$$

For any $1\leqslant p<\infty$ and for any $x_1,\dots,x_n\in L^p(M)$, we have

$$\text{(7)}\quad \Big\|\sum_{k=1}^n x_k\otimes e_k\Big\|_{L^p(M,\ell^2_c)} =\sup\bigg\{\Big|\sum_{k=1}^n\langle x_k,y_k\rangle_{L^p(M),L^{p^*}(M)}\Big| : \Big\|\sum_{k=1}^n y_k\otimes e_k\Big\|_{L^{p^*}(M,\ell^2_r)}\leqslant 1\bigg\}.$$

A similar formula holds for the space $L^p(M,\ell^2_r)$. For simplicity, we write $S^p(\ell^2_c)$ for $L^p(B(\ell^2),\ell^2_c)$. If $2\leqslant p<\infty$ we define the Banach space $L^p(M,\ell^2_{\mathrm{rad}})=L^p(M,\ell^2_c)\cap L^p(M,\ell^2_r)$. For any $u\in L^p(M,\ell^2_{\mathrm{rad}})$, we have

$$\|u\|_{L^p(\ell^2_{\mathrm{rad}})}=\max\big\{\|u\|_{L^p(M,\ell^2_c)},\|u\|_{L^p(M,\ell^2_r)}\big\}.$$

If $1\leqslant p\leqslant 2$ we define the Banach space $L^p(M,\ell^2_{\mathrm{rad}})=L^p(M,\ell^2_c)+L^p(M,\ell^2_r)$. For any $u\in L^p(M,\ell^2_{\mathrm{rad}})$, we have

$$\|u\|_{L^p(M,\ell^2_{\mathrm{rad}})}=\inf\big\{\|u_1\|_{L^p(M,\ell^2_c)}+\|u_2\|_{L^p(M,\ell^2_r)}\big\}.$$

where the infimum runs over all possible decompositions $u=u_1+u_2$ with $u_1\in L^p(M,\ell^2_c)$ and $u_2\in L^p(M,\ell^2_r)$. Recall that, if $1<p<\infty$, we have an isometric identification

$$\text{(8)}\qquad L^p(M,\ell^2_{\mathrm{rad}})^*=L^{p^*}(M,\ell^2_{\mathrm{rad}}).$$

Let X be a Banach space and let $(\varepsilon_k)_{k\geqslant 1}$ be a sequence of independent Rademacher variables on some probability space Ω. Let $\mathrm{Rad}(X)\subset L^2(\Omega;X)$ be the closure of $\mathrm{Span}\{\varepsilon_k\otimes x : k\geqslant 1, x\in X\}$ in the Bochner space $L^2(\Omega;X)$. Thus for any finite family $x_1,\dots,x_n$ in X, we have

$$\Big\|\sum_{k=1}^n\varepsilon_k\otimes x_k\Big\|_{\mathrm{Rad}(X)}=\bigg(\int_\Omega\Big\|\sum_{k=1}^n\varepsilon_k(\omega)x_k\Big\|_X^2\,d\omega\bigg)^{\frac12}.$$

If $1 \leqslant p < \infty$, the noncommutative Khintchine's inequalities (see [15] and [25]) implies

$$\mathrm{Rad}(L^p(M)) \approx L^p(M, \ell^2_{\mathrm{rad}}). \tag{9}$$

We say that a set $\mathscr{F} \subset B(X)$ is R-bounded if there is a constant $C \geqslant 0$ such that for any finite families $T_1, \ldots, T_n$ in $\mathscr{F}$, and $x_1, \ldots, x_n$ in X, we have

$$\Big\| \sum_{k=1}^n \varepsilon_k \otimes T_k(x_k) \Big\|_{\mathrm{Rad}(X)} \leqslant C \Big\| \sum_{k=1}^n \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)}.$$

In this case, we let $R(\mathscr{F})$ denote the smallest possible C, which is called the R-bound of $\mathscr{F}$. R-boundedness was introduced in [3] and then developed in the fundamental paper [6]. We refer to the latter paper and to [11, Section 2] for a detailed presentation.

On noncommutative L^p-spaces, it will be convenient to consider two naturals variants of this notion, introduced in [9, Chapter 4]. Let $1 < p < \infty$. A subset $\mathscr{F}$ of $B\big(L^p(M)\big)$ is Col-bounded (resp. Row-bounded) if there exists a constant $C \geqslant 0$ such that for any finite families $T_1, \ldots, T_n$ in $\mathscr{F}$, and $x_1, \ldots, x_n$ in $L^p(M)$, we have

$$\Big\| \Big(\sum_{k=1}^n |T_k(x_k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \leqslant C \Big\| \Big(\sum_{k=1}^n |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \tag{10}$$

$$\left(\text{resp. } \Big\| \Big(\sum_{k=1}^n |T_k(x_k)^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \leqslant C \Big\| \Big(\sum_{k=1}^n |x_k^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \right). \tag{11}$$

The least constant C satisfying (10) will be denoted by $\mathrm{Col}(\mathscr{F})$. Obviously any Rad-bounded (resp. Col-bounded, resp. Row-bounded) set is bounded. It follows from (9) that if a subset $\mathscr{F}$ of $B(L^p(M))$ is both Col-bounded and Row-bounded, then it is Rad-bounded.

Note that contrary to the case of R-boundedness, a singleton $\{T\}$ is not automatically Col-bounded or Row-bounded. Indeed, $\{T\}$ is Col-bounded (resp. Row-bounded) if and only if $T \otimes I_{\ell^2}$ extends to a bounded operator on $L^p(M, \ell^2_c)$ (resp. $L^p(M, \ell^2_r)$). And it turns out that if $1 < p \neq 2 < \infty$, according to [9, Example 4.1], there exists a bounded operator T on S^p such that $T \otimes I_{\ell^2}$ does not extend to a bounded operator on $S^p(\ell^2_c)$. Moreover, $T \otimes I_{\ell^2}$ extends to a bounded operator on $S^p(\ell^2_r)$. Then, we also deduce that there are sets $\mathscr{F}$ which are Rad-bounded and Col-bounded without being Row-bounded. Similarly, one may find sets which are Rad-bounded and Row-bounded without being Col-bounded, or which are Rad-bounded without being either Row-bounded or Col-bounded.

We turn to Ritt operators, the key class of this paper, and recall some of their main features. Details and complements can be found in [2], [4], [5], [13], [16], [18], [19] and [27]. Let X be a Banach space. We say that an operator $T \in B(X)$ is a Ritt operator if the two sets

$$\{T^n : n \geqslant 0\} \qquad \text{and} \qquad \{n(T^n - T^{n-1}) : n \geqslant 1\} \tag{12}$$

are bounded. This is equivalent to the spectral inclusion

$$\sigma(T) \subset \overline{\mathbb{D}} \tag{13}$$

and the boundedness of the set

$$\{(\lambda - 1)R(\lambda, T) : |\lambda| > 1\} \tag{14}$$

where $R(\lambda, T) = (\lambda I - T)^{-1}$ denotes the resolvent operator and $\mathbb{D}$ denotes the open unit disc centered at 0. Likewise we say that T is an R-Ritt operator if the two sets in (12) are R-bounded. This is equivalent to the inclusion (13) and the R-boundedness of the set (14).

Let T be a Ritt operator. The boundedness of (14) implies the existence of a constant $K \geqslant 0$ such that $|\lambda - 1| \|R(\lambda, T)\|_{X \to X} \leqslant K$ whenever $\mathrm{Re}(\lambda) > 1$. This means that $I - T$ is a sectorial operator. Thus for any $\alpha > 0$, one can consider the fractional power $(I - T)^\alpha$. We refer to [8, Chapter 3], [11] and [17] for various definitions of these (bounded) operators and their basic properties.

We will use the following two naturals variants of the notion of R-Ritt operator.

Definition 2.1. Suppose $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. We say that T is a Col-Ritt (resp. Row-Ritt) operator if the two sets (12) are Col-bounded (resp. Row-bounded).

Remark 2.2. Assume that $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. Using (7), it is easy to see that T is Col-Ritt if and only if T^* is Row-Ritt on $L^{p^*}(M)$.

We let $\mathscr{P}$ denote the algebra of all complex polynomials. Let T be a bounded operator on a Banach space X. Let $\gamma \in \left]0, \frac{\pi}{2}\right[$. Accordingly with [13], we say that T has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if there exists a constant $K \geqslant 1$ such that

$$\|\varphi(T)\|_{X \to X} \leqslant K \|\varphi\|_{H^\infty(B_\gamma)}$$

for any $\varphi \in \mathscr{P}$. Naturally, we let:

Definition 2.3. Suppose $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. Let $\gamma \in \left]0, \frac{\pi}{2}\right[$. We say that T admits a completely bounded $H^\infty(B_\gamma)$

functional calculus if T is completely bounded and if $I_{S^p} \otimes T$ admits a bounded $H^\infty(B_\gamma)$ functional calculus on $S^p(L^p(M))$.

Let T be a bounded operator on $L^p(M)$ and $\gamma \in \left]0, \frac{\pi}{2}\right[$. Note that T admits a completely bounded $H^\infty(B_\gamma)$ functional calculus if and only if there exists a constant $K \geqslant 1$ such that

$$\|\varphi(T)\|_{cb, L^p(M) \to L^p(M)} \leqslant K \|\varphi\|_{H^\infty(B_\gamma)}$$

for any $\varphi \in \mathscr{P}$.

3. Results related to Col-Ritt or Row-Ritt operators

In the subsequent sections, we need some preliminary results on Col-Ritt or Row-Ritt operators that we present here. Some of them are analogues of existing results in the context of R-Ritt operators, for which we will omit proofs.

We start with a variant of [2, Proposition 2.8] suitable with our context. The proof is similar, using [9, Lemma 4.2] instead of [2, Lemma 2.1].

PROPOSITION 3.1. *Suppose* $1 < p < \infty$. *Let* T *be a Col-Ritt operator on* $L^p(M)$. *For any* $\alpha > 0$, *the set*

$$\left\{n^\alpha (\varrho T)^{n-1}(I - \varrho T)^\alpha : n \geqslant 1, \varrho \in \left]0, 1\right]\right\}$$

is Col-bounded. Moreover, a similar result holds for Row-Ritt operators.

Moreover, we need the following result [13].

THEOREM 3.2. *Suppose* $1 < p < \infty$. *Let* T *be a bounded operator on* $L^p(M)$ *with a bounded* $H^\infty(B_\gamma)$ *functional calculus for some* $\gamma \in \left]0, \frac{\pi}{2}\right[$. *Then* T *is R-Ritt.*

In the next statement, we establish a variant of the above result.

THEOREM 3.3. *Suppose* $1 < p < \infty$. *Let* T *be a bounded operator on* $L^p(M)$. *Assume that* T *admits a completely bounded* $H^\infty(B_\gamma)$ *functional calculus for some* $\gamma \in \left]0, \frac{\pi}{2}\right[$. *Then the operator* T *is both Col-Ritt and Row-Ritt.*

PROOF. We will only show the ‘column’ result, the proof for the ‘row’ one being the same. We wish to show that the sets

$$\mathscr{F} = \{T^m : m \geqslant 0\} \qquad \text{and} \qquad \mathscr{G} = \{m(T^m - T^{m-1}) : m \geqslant 1\}$$

are Col-bounded. We consider the operator $I \otimes T$ on the noncommutative L^p-space $S^p(L^p(M))$. Then, applying Theorem 3.2, we obtain that the sets

$$\mathscr{T} = \{I_{S^p} \otimes T^m : m \geqslant 0\} \qquad \text{and} \qquad \mathscr{K} = \{m I_{S^p} \otimes (T^m - T^{m-1}) : m \geqslant 1\}$$

are Rad-bounded. Now consider $x_1, \ldots, x_n$ in $L^p(M)$ and $T_1, \ldots, T_n$ in $\mathscr{F}$. For any finite sequence $(\varepsilon_k)_{1 \leqslant k \leqslant n}$ valued in $\{-1, 1\}$, we have

$$\Big\| \Big(\sum_{k=1}^{n} |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} = \Big\| \Big(\sum_{k=1}^{n} (\varepsilon_k x_k)^* (\varepsilon_k x_k) \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} = \Big\| \sum_{k=1}^{n} \varepsilon_k e_{k1} \otimes x_k \Big\|_{S^p(L^p(M))}.$$

Then passing to the average over all possible choices of $\varepsilon_k = \pm 1$, we obtain that

$$\Big\| \Big(\sum_{k=1}^{n} |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} = \Big\| \sum_{k=1}^{n} \varepsilon_k \otimes e_{k1} \otimes x_k \Big\|_{\mathrm{Rad}(S^p(L^p(M)))}.$$

By a similar computation, we have

$$\Big\| \Big(\sum_{k=1}^{n} |T_k(x_k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} = \Big\| \sum_{k=1}^{n} \varepsilon_k \otimes (I_{S^p} \otimes T_k)(e_{k1} \otimes x_k) \Big\|_{\mathrm{Rad}(S^p(L^p(M)))}.$$

It follows that

$$\Big\| \Big(\sum_{k=1}^{n} |T_k(x_k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \leqslant \mathrm{Rad}(\mathscr{T}) \Big\| \Big(\sum_{k=1}^{n} |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)}.$$

This concludes the proof of Col-boundedness of $\mathscr{F}$ with $\mathrm{Col}(\mathscr{F}) \leqslant \mathrm{Rad}(\mathscr{T})$. The proof for the set $\mathscr{G}$ is identical.

REMARK 3.4. Suppose $1 < p \neq 2 < \infty$. The complete boundedness assumption in Theorem 3.3 cannot be replaced by a boundedness assumption.

PROOF. We have already recalled that, there exists a bounded operator T on S^p such that $\{T\}$ is not Col-bounded. Let us fix $\gamma \in \left]0, \frac{\pi}{2}\right[$. We may clearly assume that $\sigma(T)$ is included in the open set B_γ. Using the Dunford calculus, it is easy to prove that T is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus. The set $\{T\}$ is not Col-bounded. Hence T cannot be Col-Ritt.

Now, we give a precise definition of 'square functions' which clarifies (1), (2), (4) and (5) and a few comments. Let T a Ritt operator on $L^p(M)$. For any $\alpha > 0$, let us consider

$$x_k = k^{\alpha - \frac{1}{2}} T^{k-1} (I - T)^\alpha (x)$$

for any $k \geqslant 1$. If the sequence belongs to the space $L^p(M, \ell_c^2)$, then $\|x\|_{p,T,c,\alpha}$ is defined as the norm of $(x_k)_{k\geqslant 1}$ in that space. Otherwise, we set $\|x\|_{p,T,c,\alpha} = \infty$. In particular, $\|x\|_{p,T,c,\alpha}$ can be infinite. We define the quantities $\|x\|_{p,T,r,\alpha}$ by the same way. The quantities $\|x\|_{p,T,\alpha}$ are defined similarly in [2], using the space $L^p(M, \ell_{\mathrm{rad}}^2)$ instead of $L^p(M, \ell_c^2)$.

Finally, note that, if $2 \leqslant p < \infty$, we have

$$\|x\|_{p,T,\alpha} = \max\{\|x\|_{p,T,c,\alpha}, \|x\|_{p,T,r,\alpha}\}.$$

and if $1 \leqslant p \leqslant 2$, we have

$$\|x\|_{p,T,\alpha} = \inf\Big\{\|u\|_{L^p(M,\ell_c^2)} + \|v\|_{L^p(M,\ell_r^2)} : u_k + v_k = k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^\alpha x, k \geqslant 1\Big\}.$$

In [13], the following connection between the boundedness of square functions and functional calculus is established.

THEOREM 3.5. *Suppose* $1 < p < \infty$. *Let* T *be a bounded operator on* $L^p(M)$. *The following assertions are equivalent.*

(1) *The operator* T *is R-Ritt and* T *and its adjoint* T^* *both satisfy uniform estimates*

$$\|x\|_{p,T,1} \lesssim \|x\|_{L^p(M)} \qquad \text{and} \qquad \|y\|_{p^*,T^*,1} \lesssim \|y\|_{L^{p^*}(M)}$$

for any $x \in L^p(M)$ *and* $y \in L^{p^*}(M)$.

(2) *The operator* T *admits a bounded* $H^\infty(B_\gamma)$ *functional calculus for some* $\gamma \in \left]0, \frac{\pi}{2}\right[$.

Recall a special case of the principal result of [2].

THEOREM 3.6. *Let* T *be an R-Ritt operator on* $L^p(M)$ *with* $1 < p < \infty$. *For any* $\alpha, \beta > 0$ *we have an equivalence*

$$\|x\|_{p,T,\alpha} \approx \|x\|_{p,T,\beta}, \qquad x \in L^p(M).$$

We shall now present a variant suitable to our context.

For any integer $n \geqslant 1$, we identify the algebra M_n of all $n \times n$ matrices with the space of linear maps $\ell_n^2 \to \ell_n^2$. For any infinite matrix $[c_{ij}]_{i,j\geqslant 1}$, we set

$$\|[c_{ij}]\|_{\mathrm{reg}} = \sup_{n\geqslant 1}\big\|[|c_{ij}|]_{1\leqslant i,j\leqslant n}\big\|_{B(\ell_n^2)}$$

This is the so-called 'regular norm'. We refer to [20] and [24] for more information on regular norms.

The next proposition will be useful. This result is similar to [2, Proposition 2.6].

PROPOSITION 3.7. *Suppose* $1 < p < \infty$. *Let* $[c_{ij}]_{i,j \geqslant 1}$ *be an infinite matrix with* $\|[c_{ij}]\|_{\mathrm{reg}} < \infty$. *Suppose that* $\{T_{ij} : i, j \geqslant 1\}$ *is a Col-bounded set of operators on* $L^p(M)$. *Then the linear map*

$$[c_{ij}T_{ij}] : \begin{array}{rcl} L^p\big(M, \ell^2_c\big) & \longrightarrow & L^p(M, \ell^2_c) \\ \displaystyle\sum_{j=1}^{+\infty} x_j \otimes e_j & \longmapsto & \displaystyle\sum_{i=1}^{+\infty}\bigg(\sum_{j=1}^{+\infty} c_{ij}T_{ij}(x_j)\bigg) \otimes e_i \end{array}$$

is well-defined and bounded. Moreover, we have a similar result for Row-bounded sets.

PROOF. We shall only prove the 'Col' result. We can assume that $\|[c_{ij}]\|_{\mathrm{reg}} \leqslant 1$. Let $n \geqslant 1$. By [2, Lemma 2.2], we can write $c_{ij} = a_{ij}b_{ij}$ for any $1 \leqslant i, j \leqslant n$ with

$$\sup_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|^2 \leqslant 1 \qquad \text{and} \qquad \sup_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |b_{ij}|^2 \leqslant 1.$$

Let $x_1, \dots, x_n \in L^p(M)$ and $y_1, \dots, y_n \in L^{p^*}(M)$. Since the set $\{T_{ij} \mid i, j \geqslant 1\}$ is Col-bounded, there exists a positive constant C such that

$$\begin{aligned} &\bigg|\sum_{i=1}^{n}\bigg\langle\sum_{j=1}^{n} c_{ij}T_{ij}(x_j), y_i\bigg\rangle_{L^p(M), L^{p^*}(M)}\bigg| \\ &= \bigg|\sum_{i,j=1}^{n}\big\langle a_{ij}b_{ij}T_{ij}(x_j), y_i\big\rangle_{L^p(M), L^{p^*}(M)}\bigg| \\ &= \bigg|\sum_{i,j=1}^{n}\big\langle T_{ij}(b_{ij}x_j), a_{ij}y_i\big\rangle_{L^p(M), L^{p^*}(M)}\bigg| \\ &\leqslant \bigg\|\bigg(\sum_{i,j=1}^{n} |T_{ij}(b_{ij}x_j)|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^p(M)}\bigg\|\bigg(\sum_{i,j=1}^{n} |(a_{ij}y_i)^*|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^{p^*}(M)} \\ &\leqslant C\bigg\|\bigg(\sum_{i,j=1}^{n} |b_{ij}x_j|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^p(M)}\bigg\|\bigg(\sum_{i,j=1}^{n} |a_{ij}y_i^*|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^{p^*}(M)}. \end{aligned}$$

Now, we have

$$\sum_{i,j=1}^{n} |b_{ij}x_j|^2 = \sum_{j=1}^{n} |x_j|^2\bigg(\sum_{i=1}^{n} |b_{ij}|^2\bigg) \leqslant \sum_{j=1}^{n} |x_j|^2.$$

Similarly, we have

$$\sum_{i,j=1}^{n} |a_{ij} y_i^*|^2 \leqslant \sum_{i=1}^{n} |y_i^*|^2.$$

Consequently

$$\left| \sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} c_{ij} T_{ij}(x_j),\, y_i \right\rangle_{L^p(M), L^{p^*}(M)} \right| \\ \leqslant C \left\| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \left\| \left(\sum_{i=1}^{n} |y_i^*|^2 \right)^{\frac{1}{2}} \right\|_{L^{p^*}}.$$

Taking the supremum over all $y_1, \ldots, y_n \in L^{p^*}(M)$ such that $\left\| \left(\sum_{i=1}^{n} |y_i^*|^2 \right)^{\frac{1}{2}} \right\|_{L^{p^*}(M)} \leqslant 1$, we obtain

$$\left\| \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} T_{ij}(x_j) \right) \otimes e_i \right\|_{L^p(M, \ell^2_c)} \leqslant C \left\| \sum_{j=1}^{n} x_j \otimes e_j \right\|_{L^p(M, \ell^2_c)}$$

by (7). We conclude with [9, Corollary 2.12].

Now, we state a result which allows to estimate square functions $\|\cdot\|_{p,T,c,\alpha}$ and $\|\cdot\|_{p,T,r,\alpha}$ by means of approximation processes, whose proof is similar to [2, Lemma 3.2].

LEMMA 3.8. *Suppose* $1 < p < \infty$. *Assume that* T *is a Col-Ritt operator on* $L^p(M)$. *Let* $\alpha > 0$.

(1) *Let* V *be an operator on* $L^p(M)$ *such that* $TV = VT$ *with* $\{V\}$ *Col-bounded. Then, for any* $x \in L^p(M)$, *we have*

$$\|V(x)\|_{p,T,c,\alpha} \leqslant \mathrm{Col}(\{V\}) \|x\|_{p,T,c,\alpha}.$$

(2) *Let* $\nu \geqslant \alpha + 1$ *be an integer and let* $x \in \mathrm{Ran}((I - T)^{\nu})$. *Then*

$$\|x\|_{p,\varrho T,c,\alpha} \xrightarrow{\varrho \to 1^-} \|x\|_{p,T,c,\alpha}.$$

Moreover, the same result holds with $\|\cdot\|_{p,T,c,\alpha}$ *replaced by* $\|\cdot\|_{p,T,r,\alpha}$ *for Row-Ritt operators.*

Now we state an equivalence result in our context similar to Theorem 3.6.

THEOREM 3.9. *Let* T *be a bounded operator on* $L^p(M)$ *with* $1 < p < \infty$. *Let* $\alpha, \beta > 0$.

(1) *If T is Col-Ritt, we have an equivalence*

$$\|x\|_{p,T,c,\alpha} \approx \|x\|_{p,T,c,\beta}, \qquad x \in L^p(M).$$

(2) *If T is Row-Ritt, we have an equivalence*

$$\|x\|_{p,T,r,\alpha} \approx \|x\|_{p,T,r,\beta}, \qquad x \in L^p(M).$$

PROOF. The proof is similar to the one of [2, Theorem 3.3], using Proposition 3.1, Proposition 3.7, Lemma 3.8 and [9, Corollary 2.12].

4. Comparison between squares functions and the usual norm

We aim at showing Theorem 1.2. We will provide an example on the Schatten space S^p. This example also prove that in general, row and column square functions are not equivalent (Theorem 4.3).

Let a a bounded operator on ℓ^2. Assume $1 < p < \infty$. We let $\mathscr{L}_a : S^p \to S^p$ the left multiplication by a on S^p defined by $\mathscr{L}_a(x) = ax$ and we denote $\mathscr{R}_a : S^p \to S^p$ the right multiplication. It is clear that $\mathscr{L}_a^*$ and $\mathscr{R}_a^*$ are the right multiplication and the left multiplication by a on S^{p^*}. Note that, by [9, Proposition 8.4 (4)], if $I - a$ has dense range then $\mathrm{Ran}(I - \mathscr{L}_a)$ is dense in S^p. The next statement gives a link between properties of a and its associated multiplication operators.

PROPOSITION 4.1. *Suppose $1 < p < \infty$. Assume that a is a bounded operator on ℓ^2.*

(1) *If a is a Ritt operator then the left multiplication $\mathscr{L}_a$ is a Ritt operator on S^p.*

(2) *Let $\gamma \in \left]0, \frac{\pi}{2}\right[$. Then $\mathscr{L}_a$ has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if a has one. In that case, $\mathscr{L}_a$ actually has a completely bounded $H^\infty(B_\gamma)$ functional calculus.*

Moreover, we have a similar result for right multiplication.

PROOF. We have $\sigma(\mathscr{L}_a) \subset \sigma(a)$. Moreover, if $\lambda \in \rho(a)$ we have $R(\lambda, \mathscr{L}_a) = \mathscr{L}_{R(\lambda,a)}$. The first assertion clearly follows. The statement (2) is a straightforward consequence of

$$I_{S^p} \otimes \mathscr{L}_a = \mathscr{L}_{I_{\ell^2} \otimes a} \qquad \text{and} \qquad f(\mathscr{L}_a) = \mathscr{L}_{f(a)}, \quad f \in \mathscr{P}.$$

The proof of the ‘right’ result is identical.

We denote by $(e_k)_{k\geqslant 1}$ the canonical basis of ℓ^2. Now, for any integer $k \geqslant 1$, we fix $a_k = 1 - \frac{1}{2^k}$. We consider the selfadjoint bounded diagonal operator a on ℓ^2 defined by

$$a\left(\sum_{k=1}^{+\infty} x_k e_k\right) = \sum_{k=1}^{+\infty} a_k x_k e_k. \tag{15}$$

It follows from the Spectral Theorem for normal operators, that the operator a admits a bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma \in \left]0, \frac{\pi}{2}\right[$. Thus $\mathscr{L}_a$ and $\mathscr{R}_a$ admit a completely bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma \in \left]0, \frac{\pi}{2}\right[$ (hence $\mathscr{L}_a$ and $\mathscr{R}_a$ are Ritt operators).

LEMMA 4.2. *Assume that* $2 \leqslant p < \infty$. *Let* a *be the bounded operator on* ℓ^2 *defined by* (15). *If* $\mathscr{L}_a : S^p \to S^p$ *and* $\mathscr{R}_a : S^p \to S^p$ *are the left and right multiplication operators associated to* a, *we have*

$$\|x\|_{p,\mathscr{L}_a,c,1} \approx \|x\|_{S^p} \quad \textit{and} \quad \|x\|_{p,\mathscr{R}_a,r,1} \approx \|x\|_{S^p}, \qquad x \in S^p. \tag{16}$$

PROOF. We will only show the result for the operator $\mathscr{L}_a$, the proof for $\mathscr{R}_a$ being the same. For any $x \in S^p$ and any $\varrho \in]0,1[$, we have

$$\begin{aligned}
&k\big((\varrho\mathscr{L}_a)^{k-1}(I-\varrho\mathscr{L}_a)(x)\big)^*\big((\varrho\mathscr{L}_a)^{k-1}(I-\varrho\mathscr{L}_a)(x)\big)\\
&\quad = k\big((\varrho a)^{k-1}(I-\varrho a)x\big)^*\big((\varrho a)^{k-1}(I-\varrho a)x\big)\\
&\quad = kx^*(I-\varrho a)(\varrho a)^{2(k-1)}(I-\varrho a)x\\
&\quad = kx^*(I-\varrho\mathscr{L}_a)^2(\varrho\mathscr{L}_a)^{2(k-1)}(x).
\end{aligned}$$

Now, for any $z \in \mathsf{D}$, we have

$$\sum_{k=1}^{+\infty} kz^{k-1} = (1-z)^{-2}. \tag{17}$$

Since the operator $\mathscr{L}_a$ is a contraction, we deduce that, for every $\varrho \in]0,1[$, the operator $I - (\varrho\mathscr{L}_a)^2$ is invertible and that we have

$$\sum_{k=1}^{+\infty} k(\varrho\mathscr{L}_a)^{2(k-1)} = \big(I - (\varrho\mathscr{L}_a)^2\big)^{-2}, \tag{18}$$

the series being absolutely convergent. Then we deduce that the series

$$\sum_{k=1}^{+\infty} k\big((\varrho\mathscr{L}_a)^{k-1}(I-\varrho\mathscr{L}_a)(x)\big)^*\big((\varrho\mathscr{L}_a)^{k-1}(I-\varrho\mathscr{L}_a)(x)\big)$$

is convergent in the Banach space $S^{\frac{p}{2}}$ and that

$$\begin{aligned}\sum_{k=1}^{+\infty} k\big((\varrho\mathscr{L}_a)^{k-1}(I-\varrho\mathscr{L}_a)(x)\big)^*\big((\varrho\mathscr{L}_a)^{k-1}(I-\varrho\mathscr{L}_a)(x)\big)\\ = x^*(I-\varrho\mathscr{L}_a)^2\big(I-(\varrho\mathscr{L}_a)^2\big)^{-2}x\\ = x^*(I+\varrho a)^{-2}x.\end{aligned}$$

We deduce that

$$\|x\|_{\varrho\mathscr{L}_a,c,1} = \big\|\big(x^*(I+\varrho a)^{-2}x\big)^{\frac{1}{2}}\big\|_{S^p} = \big\|(I+\varrho a)^{-1}x\big\|_{S^p}.$$

Then, for any $x \in S^p$, we obtain the estimate

$$\|x\|_{p,\varrho\mathscr{L}_a,c,1} \leqslant \|(I+\varrho a)^{-1}\|_{B(\ell^2)}\|x\|_{S^p} \leqslant \|x\|_{S^p}.$$

By a similar computation, for any $x \in S^p$, we have

$$\tfrac{1}{2}\|x\|_{S^p} \leqslant \|x\|_{p,\varrho\mathscr{L}_a,c,1}.$$

Applying Lemma 3.8 (2), we deduce an equivalence

$$\tfrac{1}{2}\|x\|_{S^p} \leqslant \|x\|_{p,\mathscr{L}_a,c,1} \leqslant \|x\|_{S^p}, \qquad x \in \operatorname{Ran}((I-\mathscr{L}_a)^2).$$

For any integer $n \geqslant 1$, we let d_n the bounded diagonal operator on ℓ^2 defined by the matrix $\operatorname{diag}(1,\dots,1,0,\dots)$. It is not difficult to see that, for any integer $n \geqslant 1$, the range of $\mathscr{L}_{d_n}$ is a subspace of $\operatorname{Ran}\big((I-\mathscr{L}_a)^2\big)$. Hence we actually have

$$\tfrac{1}{2}\|\mathscr{L}_{d_n}(x)\|_{S^p} \leqslant \|\mathscr{L}_{d_n}(x)\|_{p,\mathscr{L}_a,c,1} \leqslant \|\mathscr{L}_{d_n}(x)\|_{S^p}, \qquad x \in S^p, n \geqslant 1.$$

Then, on the one hand, we obtain

$$\|\mathscr{L}_{d_n}(x)\|_{p,\mathscr{L}_a,c,1} \leqslant \|x\|_{S^p}, \qquad x \in S^p, n \geqslant 1.$$

By [9, Corollary 2.12] and (6), this latter inequality is equivalent to

$$\left\|\sum_{k=1}^{l} e_{k1} \otimes k^{\frac{1}{2}}\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(\mathscr{L}_{d_n}(x))\right\|_{S^p(S^p)} \lesssim \|x\|_{S^p}, \quad x \in S^p, n \geqslant 1, l \geqslant 1.$$

Passing to the limit in the above inequality, we infer that

$$\left\|\sum_{k=1}^{l} e_{k1} \otimes k^{\frac{1}{2}}\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(x)\right\|_{S^p(S^p)} \lesssim \|x\|_{S^p}, \qquad x \in S^p, l \geqslant 1.$$

Using again [9, Corollary 2.12], we obtain that

$$\|x\|_{p,\mathscr{L}_a,c,1} \leqslant \|x\|_{S^p}, \qquad x \in S^p.$$

Note, in particular that, for any $x \in S^p$, we have $\|x\|_{p,\mathscr{L}_a,c,1} < \infty$. On the other hand, note that, for any integer $n \geqslant 1$, the operators $\mathscr{L}_a$ and $\mathscr{L}_{d_n}$ commute. Hence, for any $x \in S^p$ and any integer $n \geqslant 1$, we have

$$\begin{aligned}\|\mathscr{L}_{d_n}(x)\|_{S^p} &\lesssim \left\|\mathscr{L}_{d_n}(x)\right\|_{p,\mathscr{L}_a,c,1} \\ &= \left\|\sum_{k=1}^{+\infty} e_{k1} \otimes k^{\frac{1}{2}} \mathscr{L}_a^{k-1}(I - \mathscr{L}_a)(\mathscr{L}_{d_n}(x))\right\|_{S^p(S^p)} \\ &= \left\|(I_{S^p} \otimes \mathscr{L}_{d_n})\left(\sum_{k=1}^{+\infty} e_{k1} \otimes k^{\frac{1}{2}} \mathscr{L}_a^{k-1}(I - \mathscr{L}_a)(x)\right)\right\|_{S^p(S^p)}.\end{aligned}$$

Letting n to the infinity, we deduce that

$$\|x\|_{S^p} \lesssim \|x\|_{p,\mathscr{L}_a,c,1}, \qquad x \in S^p.$$

The proof is complete.

THEOREM 4.3. *Let $\alpha > 0$. Let a be the bounded operator on ℓ^2 defined by* (15). *Let $\mathscr{L}_a : S^p \to S^p$ and $\mathscr{R}_a : S^p \to S^p$ be the left and right multiplication operators associated to a. Assume that $2 < p < \infty$. Then*

$$\sup\left\{\frac{\|x\|_{p,\mathscr{L}_a,c,\alpha}}{\|x\|_{p,\mathscr{L}_a,r,\alpha}} : x \in S^p\right\} = \infty \quad \text{and} \quad \sup\left\{\frac{\|x\|_{p,\mathscr{R}_a,r,\alpha}}{\|x\|_{p,\mathscr{R}_a,c,\alpha}} : x \in S^p\right\} = \infty. \tag{19}$$

Assume that $1 < p < 2$. Then

$$\sup\left\{\frac{\|x\|_{p,\mathscr{L}_a,r,\alpha}}{\|x\|_{p,\mathscr{L}_a,c,\alpha}} : x \in S^p\right\} = \infty \quad \text{and} \quad \sup\left\{\frac{\|x\|_{p,\mathscr{R}_a,c,\alpha}}{\|x\|_{p,\mathscr{R}_a,r,\alpha}} : x \in S^p\right\} = \infty. \tag{20}$$

PROOF. By Theorem 3.9, it suffices to prove the result for one specific real α. Throughout the proof, we will use $\alpha = 1$. We first assume that $2 < p < \infty$. Given an integer $n \geqslant 1$, we consider $e = e_1 + \cdots + e_n \in \ell_n^2$ and $x = \frac{1}{\sqrt{n}} e \otimes e \in S^p$. Clearly, we have

$$xx^* = \sum_{i,j=1}^{n} e_{ij}.$$

Now, we have

$$\begin{aligned} k\big(\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(x)\big)\big(\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(x)\big)^* &= k\big(a^{k-1}(I-a)x\big)\big(a^{k-1}(I-a)x\big)^* \\ &= ka^{k-1}(I-a)xx^*(I-a)a^{k-1} \\ &= \sum_{i,j=1}^{n} ka^{k-1}(I-a)e_{ij}(I-a)a^{k-1} \\ &= \sum_{i,j=1}^{n} (1-a_i)(1-a_j)k(a_ia_j)^{k-1}e_{ij}. \end{aligned}$$

Using the equality (17), we obtain that the series

$$\sum_{k=1}^{+\infty} k\big(\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(x)\big)\big(\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(x)\big)^*$$

is convergent in $S^{\frac{p}{2}}$ and that

$$\sum_{k=1}^{+\infty} k\big(\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(x)\big)\big(\mathscr{L}_a^{k-1}(I-\mathscr{L}_a)(x)\big)^* = \sum_{i,j=1}^{n} (1-a_i)(1-a_j)(1-a_ia_j)^{-2}e_{ij}.$$

Now, note that

$$(1-a_i)(1-a_j)(1-a_ia_j)^{-2} = \frac{2^{i+j}}{(2^i+2^j-1)^2}.$$

We deduce that

$$\begin{aligned} \|x\|_{p,\mathscr{L}_a,r,1} &= \left\| \left(\sum_{i,j=1}^{n} \frac{2^{i+j}}{(2^i+2^j-1)^2} e_{ij} \right)^{\frac{1}{2}} \right\|_{S^p} \\ &= \left\| \sum_{i,j=1}^{n} \frac{2^{i+j}}{(2^i+2^j-1)^2} e_{ij} \right\|_{S^{\frac{p}{2}}}^{\frac{1}{2}}. \end{aligned}$$

We let $A = \big[\frac{2^{i+j}}{(2^i+2^j-1)^2}\big]_{1\leqslant i,j\leqslant n}$ be the $n\times n$ matrix in the last right member of the above equations. We have

$$\|A\|_{S_n^2}^2 = \sum_{i,j=1}^{n} \left(\frac{2^{i+j}}{(2^i+2^j-1)^2}\right)^2 = \sum_{i,j=1}^{n} \frac{4^{i+j}}{(2^i+2^j-1)^4}.$$

Moreover, note that

$$\frac{4^{i+j}}{(2^i+2^j-1)^4} \leqslant 16\frac{4^{i+j}}{(2^i+2^j)^4} = 16\left(\frac{1}{2^{i-j}+2^{j-i}+2}\right)^2 \leqslant \frac{16}{4^{|i-j|}}.$$

Thus we have

$$\|A\|_{S_n^2}^2 \leqslant 32\left(\sum_{k\in\mathbb{Z}} \frac{1}{4^{|k|}}\right) n \approx n.$$

If $4 \leqslant p < \infty$, we obtain

$$\|x\|_{p,\mathscr{L}_a,r,1} = \|A\|_{S_n^{\frac{p}{2}}}^{\frac{1}{2}} \leqslant \|A\|_{S_n^2}^{\frac{1}{2}} \lesssim n^{\frac{1}{4}}.$$

Since $x = \frac{1}{\sqrt{n}} e \otimes e$ is rank one, its norm in S^p does not depend on p, and it is equal to $\frac{1}{\sqrt{n}}\|e\|_{\ell_n^2}^2 = \sqrt{n}$. Then, by Lemma 4.2, we have $\|x\|_{p,\mathscr{L}_a,c,1} \approx \sqrt{n}$. We obtain the first equality of (19) in that case.

If $2 < p \leqslant 4$, we can write $\frac{1}{\frac{p}{2}} = \frac{1-\theta}{1} + \frac{\theta}{2}$ with $0 < \theta \leqslant 1$. Then

$$\|x\|_{p,\mathscr{L}_a,r,1}^2 = \|A\|_{S_n^{\frac{p}{2}}} \leqslant \|A\|_{S_n^1}^{1-\theta} \|A\|_{S_n^2}^{\theta}.$$

By construction, we have $A \geqslant 0$, hence we have

$$\|A\|_{S_n^1} = \operatorname{tr}\left(\sum_{i,j=1}^{n} \frac{2^{i+j}}{(2^i+2^j-1)^2} e_{ij}\right) = \sum_{i=1}^{n} \frac{4^i}{(2^{i+1}-1)^2} \leqslant \sum_{i=1}^{n} \frac{4^i}{(2^i)^2} = n.$$

Thus

$$\|x\|_{p,\mathscr{L}_a,r,1}^2 \lesssim n^{1-\theta} n^{\frac{\theta}{2}} = n^{1-\frac{\theta}{2}}.$$

Recall that $\|x\|_{p,\mathscr{L}_a,c,1} \approx \sqrt{n}$. We obtain that

$$\frac{\|x\|_{p,\mathscr{L}_a,c,1}}{\|x\|_{p,\mathscr{L}_a,r,1}} \gtrsim \frac{n^{\frac{1}{2}}}{n^{\frac{1}{2}-\frac{\theta}{4}}} = n^{\frac{\theta}{4}}.$$

Since n was arbitrary and $\theta > 0$, we obtain the first part of (19) in this case. Likewise, the above proof has a ‘right analog’ which proves the second equality of (19).

We now turn to the proof of (20). We assume that $1 < p < 2$. The second part of (19) says

$$\sup\left\{\frac{\|y\|_{p^*,\mathscr{L}_a^*,r,1}}{\|y\|_{p^*,\mathscr{L}_a^*,c,1}} : y \in S^{p^*}\right\} = \infty. \tag{21}$$

To prove the first equality of (20), assume on the contrary that there is a constant $K > 0$ such that for any $x \in S^p$

$$\|x\|_{p,\mathscr{L}_a,r,1} \leqslant K\|x\|_{p,\mathscr{L}_a,c,1}. \tag{22}$$

We begin by showing a duality relation between $\|\cdot\|_{p^*,\mathscr{L}_a^*,c,1}$ and $\|\cdot\|_{p,\mathscr{L}_a,r,1}$. Let $y \in S^{p^*}$ and $x \in S^p$. For any integer $n \geqslant 1$, recall that d_n is the bounded diagonal operator on ℓ^2 defined by the matrix $\operatorname{diag}(1, \ldots, 1, 0, \ldots)$. By (18), for any $0 < \varrho < 1$ and any integer $n \geqslant 1$, we have

$$\begin{aligned}
&\left|\langle y, \mathscr{L}_{d_n}(x)\rangle_{S^{p^*},S^p}\right| \\
&\quad= \left|\left\langle y, \sum_{k=1}^{+\infty} k(\varrho\mathscr{L}_a)^{2(k-1)}(I-(\varrho\mathscr{L}_a)^2)^2\mathscr{L}_{d_n}(x)\right\rangle_{S^{p^*},S^p}\right| \\
&\quad= \left|\sum_{k=1}^{+\infty}\big\langle y, k(\varrho\mathscr{L}_a)^{2(k-1)}(I-(\varrho\mathscr{L}_a)^2)^2\mathscr{L}_{d_n}(x)\big\rangle_{S^{p^*},S^p}\right| \\
&\quad= \Bigg|\sum_{k=1}^{+\infty}\Big\langle k^{\frac{1}{2}}(\varrho\mathscr{L}_a^*)^{k-1}(I-\varrho\mathscr{L}_a^*)(I+\varrho\mathscr{L}_a^*)^2 y, \\
&\qquad\qquad\qquad k^{\frac{1}{2}}(\varrho\mathscr{L}_a)^{k-1}(I-\varrho\mathscr{L}_a)\mathscr{L}_{d_n}(x)\Big\rangle\Bigg| \\
&\quad\leqslant \left\|\left(k^{\frac{1}{2}}(\varrho\mathscr{L}_a^*)^{k-1}(I-\varrho\mathscr{L}_a^*)(I+\varrho\mathscr{L}_a^*)^2 y\right)_{k\geqslant 1}\right\|_{S^p(\ell^2_c)}\|\mathscr{L}_{d_n}(x)\|_{p,\varrho\mathscr{L}_a,r,1}.
\end{aligned}$$

Now, it is easy to see that $\{\mathscr{L}_a^*\}$ is Col-bounded. We infer that

$$\begin{aligned}
&\left|\langle y, \mathscr{L}_{d_n}(x)\rangle_{S^{p^*},S^p}\right| \\
&\qquad\lesssim \left\|\left(k^{\frac{1}{2}}(\varrho\mathscr{L}_a^*)^{k-1}(I-\varrho\mathscr{L}_a^*)y\right)_{k\geqslant 1}\right\|_{S^p(\ell^2_c)}\|\mathscr{L}_{d_n}(x)\|_{p,\varrho\mathscr{L}_a,r,1} \\
&\qquad= \|y\|_{p^*,\varrho\mathscr{L}_a^*,c,1}\|\mathscr{L}_{d_n}(x)\|_{p,\varrho\mathscr{L}_a,r,1}.
\end{aligned}$$

Assume for a while that $y \in \operatorname{Ran}((I-\mathscr{L}_a^*)^2)$. By Lemma 3.8 (2), letting ϱ to 1, we obtain

$$\left|\langle y, \mathscr{L}_{d_n}(x)\rangle_{S^{p^*},S^p}\right| \lesssim \|y\|_{p^*,\mathscr{L}_a^*,c,1}\|\mathscr{L}_{d_n}(x)\|_{p,\mathscr{L}_a,r,1}.$$

Letting n to the infinity, we obtain

$$\left|\langle y, x\rangle_{S^{p^*},S^p}\right| \lesssim \|y\|_{p^*,\mathscr{L}_a^*,c,1}\|x\|_{p,\mathscr{L}_a,r,1}.$$

According to (22) and the first part of (16), we deduce that

$$\left|\langle y, x\rangle_{S^{p^*},S^p}\right| \lesssim \|y\|_{p^*,\mathscr{L}_a^*,c,1}\|x\|_{p,\mathscr{L}_a,c,1} \lesssim \|y\|_{p^*,\mathscr{L}_a^*,c,1}\|x\|_{S^p}.$$

By duality, we finally obtain that

$$\|y\|_{S^{p^*}} \lesssim \|y\|_{p^*,\mathscr{L}_a^*,c,1}. \tag{23}$$

For an arbitrary $y \in S^{p^*}$, we also obtain (23) by applying it to $\mathscr{L}_{d_n}^*(y)$ and then passing to the limit. The second equivalence of (16) says that $\|y\|_{p^*,\mathscr{L}_a^*,r,1} \approx \|y\|_{S^{p^*}}$ for any $y \in S^{p^*}$. This contradicts (21) and completes the proof of the first part of (20). The proof of the second part is similar.

For an operator admitting a completely bounded $H^\infty(B_\gamma)$ functional calculus, it also seems interesting, in view of the equivalence (3), to compare the column and row square functions with the usual norm $\|\cdot\|_{L^p(M)}$. If T is a operator with $\operatorname{Ran}(I-T)$ dense in $L^p(M)$ which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$, the equivalence (3) and Theorems 3.5 and 3.6 implies that

$$\|x\|_{L^p(M)} \lesssim \|x\|_{p,T,c,1} \qquad \text{and} \qquad \|x\|_{L^p(M)} \lesssim \|x\|_{p,T,r,1}$$

if $1 < p \leqslant 2$ and

$$\|x\|_{p,T,c,1} \lesssim \|x\|_{L^p(M)} \qquad \text{and} \qquad \|x\|_{p,T,r,1} \lesssim \|x\|_{L^p(M)}$$

if $2 \leqslant p < \infty$, for any $x \in L^p(M)$. The following result says that except for $p = 2$, these estimates cannot be reversed:

COROLLARY 4.4. *Suppose that* $2 < p < \infty$ (*resp.* $1 < p < 2$). *Let* $\alpha > 0$. *There exists a Ritt operator* T *on the Schatten space* S^p, *with* $\operatorname{Ran}(I-T)$ *dense in* S^p, *which admits a completely bounded* $H^\infty(B_\gamma)$ *functional calculus with* $\gamma \in \left]0, \frac{\pi}{2}\right[$ *such that*

$$\sup\left\{\frac{\|x\|_{S^p}}{\|x\|_{p,T,c,\alpha}} : x \in S^p\right\} = \infty \quad \left(\textit{resp. } \sup\left\{\frac{\|x\|_{p,T,c,\alpha}}{\|x\|_{S^p}} : x \in S^p\right\} = \infty\right).$$

Moreover, the same result holds with $\|\cdot\|_{p,T,c,\alpha}$ *replaced by* $\|\cdot\|_{p,T,r,\alpha}$.

PROOF. One more time, we only need to prove this result for $\alpha = 1$. Then, this follows from Lemma 4.2 and Theorem 4.3.

5. An alternative square function for $1 < p < 2$

Let T be a Ritt operator on $L^p(M)$, with $1 < p < 2$. For any $\alpha > 0$, we may consider an alternative square function by letting

$$\|x\|_{p,T,0,\alpha} = \inf\left\{\|x_1\|_{p,T,c,\alpha} + \|x_2\|_{p,T,r,\alpha} : x = x_1 + x_2\right\}$$

for any $x \in L^p(M)$.

Note that if T is both Col-Ritt and Row-Ritt, by Theorem 3.9, the square functions $\|x\|_{p,T,0,\alpha}$ and $\|x\|_{p,T,0,\beta}$ are equivalent for any $\alpha, \beta > 0$.

Suppose that $\|x\|_{p,T,0,\alpha}$ is finite and that we have a decomposition $x = x_1 + x_2$ with $\|x_1\|_{p,T,c,\alpha} < \infty$ and $\|x_2\|_{p,T,r,\alpha} < \infty$. Letting $u_k = k^{\alpha-\frac{1}{2}} T^{k-1}(I - T)^{\alpha} x_1$ and $v_k = k^{\alpha-\frac{1}{2}} T^{k-1}(I - T)^{\alpha} x_2$, we have

$$k^{\alpha-\frac{1}{2}} T^{k-1}(I - T)^{\alpha} x = u_k + v_k, \qquad k \geqslant 1.$$

Moreover, the sequences u and v belong to $L^p(M, \ell^2_c)$ and $L^p(M, \ell^2_r)$ respectively. We deduce that

$$\|x\|_{p,T,\alpha} \leqslant \|x\|_{p,T,0,\alpha}, \qquad x \in L^p(M).$$

We do not know if the two square functions $\|\cdot\|_{p,T,\alpha}$ and $\|\cdot\|_{p,T,0,\alpha}$ are equivalent in general. In the next statement, we give a sufficient condition for an such equivalence to hold true.

THEOREM 5.1. *Suppose* $1 < p < 2$. *Let* T *be a bounded operator on* $L^p(M)$ *with* $\mathrm{Ran}(I - T)$ *dense in* $L^p(M)$. *Assume that* T *is both Col-Ritt and Row-Ritt. Let* $\alpha, \eta > 0$. *Suppose that* T *satisfies a 'dual square function estimate'*

$$\|y\|_{p^*,T^*,\eta} \lesssim \|y\|_{L^{p^*}(M)}, \qquad y \in L^{p^*}(M). \tag{24}$$

Then we have an equivalence

$$\|x\|_{p,T,\alpha} \approx \|x\|_{p,T,0,\alpha}, \qquad x \in L^p(M).$$

Indeed, there is a positive constant C *such that whenever* $x \in L^p(M)$ *satisfies* $\|x\|_{p,T,\alpha} < \infty$, *then there exists* $x_1, x_2 \in L^p(M)$ *such that*

$$x = x_1 + x_2 \qquad \textit{and} \qquad \|x_1\|_{p,T,c,\alpha} + \|x_2\|_{p,T,r,\alpha} \leqslant C\|x\|_{p,T,\alpha}.$$

PROOF. Since T is both Col-Ritt and Row-Ritt, it is also an R-Ritt operator. Then, by Theorem 3.6 and Theorem 3.9, we only need to prove this result for $\alpha = 1$ and $\eta = 1$. Observe that, for any $y \in L^{p^*}(M)$, we have

$$\begin{aligned}
&\left\| \left(k^{\frac{1}{2}} (T^*)^{k-1}(I + T^*)^2(I - T^*) y\right)_{k \geqslant 1} \right\|_{L^{p^*}(M, \ell^2_{\mathrm{rad}})} \\
&\qquad \lesssim \left\| (I + T^*)^2 \right\|_{L^{p^*}(M) \to L^{p^*}(M)} \left\| \left(k^{\frac{1}{2}} (T^*)^{k-1}(I - T^*) y\right)_{k \geqslant 1} \right\|_{L^{p^*}(M, \ell^2_{\mathrm{rad}})} \\
&\qquad \lesssim \|y\|_{L^{p^*}(M)}
\end{aligned}$$

by (24). We let

$$Z : \quad \begin{aligned} L^{p^*}(M) &\longrightarrow L^{p^*}\big(M, \ell^2_{\mathrm{rad}}\big) \\ y &\longmapsto \big(k^{\frac{1}{2}}(T^*)^{k-1}(I+T^*)^2(I-T^*)y\big)_{k\geqslant 1} \end{aligned}$$

denote the resulting bounded map. Let $x \in L^p(M)$ such that $\|x\|_{p,T,1} < \infty$. There exists two elements $u \in L^p(M, \ell^2_c)$ and $v \in L^p(M, \ell^2_r)$ such that for any positive integer k

$$u_k + v_k = k^{\frac{1}{2}} T^{k-1}(I-T)x \tag{25}$$

and such that

$$\|u\|_{L^p(M,\ell^2_c)} + \|v\|_{L^p(M,\ell^2_r)} \leqslant 2\|x\|_{p,T,1}.$$

Recall that we have contractive inclusions $L^p(M, \ell^2_c) \subset L^p(M, \ell^2_{\mathrm{rad}})$ and $L^p(M, \ell^2_r) \subset L^p(M, \ell^2_{\mathrm{rad}})$. Thus, by (8), we can define x_1 and x_2 of $L^p(M)$ by

$$x_1 = Z^* u \qquad \text{and} \qquad x_2 = Z^* v.$$

We will show that $x = x_1 + x_2$. Since T is a Col-Ritt-operator, by Proposition 3.1 (or by [2, Proposition 2.8]), we infer that there exists a positive constant C such that

$$\begin{aligned} \sum_{k=1}^{+\infty} \big\|k^{\frac{1}{2}} T^{k-1}(I-T)^2\big\|^2_{L^p(M)\to L^p(M)} &= \sum_{k=1}^{+\infty} k\big\|T^{k-1}(I-T)^2\big\|^2_{L^p(M)\to L^p(M)} \\ &\leqslant C^2 \sum_{k=1}^{+\infty} \frac{1}{k^3} < \infty. \end{aligned}$$

For any $1 < p < 2$, by [9, Proposition 2.5], we have the contractive inclusion $L^p(M, \ell^2_c) \subset \ell^2(L^p(M))$. We deduce that $\sum_{k=1}^{+\infty} \|u_k\|^2_{L^p(M)} < \infty$. According to the Cauchy-Schwarz inequality, we deduce that the series

$$\sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 u_k = (I+T)^2 \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T)^2 u_k$$

converges absolutely in $L^p(M)$. Now, for any $y \in L^{p^*}(M)$, we have

$$\begin{aligned} &\big\langle (I-T)x_1, y\big\rangle_{L^p(M),L^{p^*}(M)} \\ &\quad= \big\langle (I-T)Z^* u, y\big\rangle_{L^p(M),L^{p^*}(M)} \\ &\quad= \big\langle u, Z(I-T^*)y\big\rangle_{L^p(M,\ell^2_{\mathrm{rad}}),L^{p^*}(M,\ell^2_{\mathrm{rad}})} \end{aligned}$$

$$
\begin{aligned}
&= \Big\langle u, \big(k^{\frac{1}{2}}(T^*)^{k-1}(I+T^*)^2(I-T^*)^2 y\big)_{k\geqslant 1}\Big\rangle_{L^p(M,\ell^2_{\mathrm{rad}}),L^{p^*}(M,\ell^2_{\mathrm{rad}})}\\
&= \sum_{k=1}^{+\infty}\big\langle u_k, k^{\frac{1}{2}}(T^*)^{k-1}(I-(T^*)^2)^2 y\big\rangle_{L^p(M),L^{p^*}(M)}\\
&= \Big\langle \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 u_k, y\Big\rangle_{L^p(M),L^{p^*}(M)}.
\end{aligned}
$$

Thus, we deduce that

$$
(I-T)x_1 = \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 u_k. \tag{26}
$$

Similarly we have

$$
(I-T)x_2 = \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 v_k.
$$

Now, we infer that

$$
\begin{aligned}
(I-T)(x_1+x_2) &= \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 u_k + \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 v_k\\
&= \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 (u_k+v_k)\\
&= \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T^2)^2 k^{\frac{1}{2}} T^{k-1}(I-T)x \qquad \text{by (25)}\\
&= \sum_{k=1}^{+\infty} k T^{2k-2}(I+T)^2(I-T)^3 x.
\end{aligned}
$$

By (17), for any $z \in \mathsf{D}$, we have

$$
\sum_{k=1}^{+\infty} k z^{2k-2}(1-z^2)^2 = 1.
$$

Since the operator T is power bounded, we note that for every $\varrho \in \,]0,1[$ we have

$$
I = \sum_{k=1}^{+\infty} k(\varrho T)^{2k-2}(I-(\varrho T)^2)^2, \tag{27}
$$

the series being absolutely convergent. Hence, for any $\varrho \in]0, 1[$, we have

$$(I - \varrho T)x = (I - \varrho T)\sum_{k=1}^{+\infty} k(\varrho T)^{2k-2}(I - (\varrho T)^2)^2 x$$
$$= \sum_{k=1}^{+\infty} k(\varrho T)^{2k-2}(I + \varrho T)^2(I - \varrho T)^3 x.$$

It is not difficult to see that the latter series is normally convergent on [0, 1]. Hence, letting ϱ to 1, we deduce that

$$(I - T)x = \sum_{k=1}^{+\infty} kT^{2k-2}(I + T)^2(I - T)^3 x.$$

Then we obtain

$$(I - T)x = (I - T)(x_1 + x_2).$$

Since the space $\mathrm{Ran}(I - T)$ is dense in $L^p(M)$, by the Mean Ergodic Theorem (see [10, Section 2.1]), the operator $I - T$ is injective. Consequently, we have $x = x_1 + x_2$. Now, it remains to estimate $\|x_1\|_{p,T,1,c}$ and $\|x_2\|_{p,T,1,r}$. According to (26), we have

$$m^{\frac{1}{2}}T^{m-1}(I - T)x_1 = \sum_{k=1}^{+\infty} k^{\frac{1}{2}} m^{\frac{1}{2}} T^{k+m-2}(I - T^2)^2 u_k$$

for any integer $m \geqslant 1$. It is convenient to write this as $m^{\frac{1}{2}}T^{m-1}(I - T)x_1 = (I + T)^2 y_m$ with

$$y_m = \sum_{k=1}^{+\infty} k^{\frac{1}{2}} m^{\frac{1}{2}} T^{k+m-2}(I - T)^2 u_k. \tag{28}$$

Now, observe that

$$k^{\frac{1}{2}} m^{\frac{1}{2}} T^{k+m-2}(I - T)^2 = \frac{k^{\frac{1}{2}} m^{\frac{1}{2}}}{(k + m - 1)^2} \cdot (k + m - 1)^2 T^{k+m-2}(I - T)^2.$$

According to [2, Proposition 2.3] and [2, Lemma 2.4], the matrix

$$\left[\frac{k^{\frac{1}{2}} m^{\frac{1}{2}}}{(k + m - 1)^2}\right]_{k,m\geqslant 1}$$

represents an element of $B(\ell^2)$. Moreover, by Proposition 3.1, the set

$$\left\{(k+m-1)^2 T^{k+m-2}(I-T)^2 : k, m \geqslant 1\right\}$$

is Col-bounded. By Proposition 3.7, we deduce that $(y_m)_{m\geqslant 1} \in L^p(M, \ell^2_c)$ and that

$$\|(y_m)_{m\geqslant 1}\|_{L^p(M,\ell^2_c)} \lesssim \|u\|_{L^p(M,\ell^2_c)}.$$

Since $\{T\}$ is Col-bounded, we have

$$\begin{aligned}
\|x_1\|_{p,T,c,1} &= \left\| \left(m^{\frac{1}{2}} T^{m-1}(I-T)x_1\right)_{m\geqslant 1} \right\|_{L^p(M,\ell^2_c)} \\
&= \left\| \left((I+T)^2 y_m\right)_{m\geqslant 1} \right\|_{L^p(M,\ell^2_c)} \qquad \text{by (28)} \\
&\lesssim \left\| (y_m)_{m\geqslant 1} \right\|_{L^p(M,\ell^2_c)}.
\end{aligned}$$

Finally, we deduce that there exists a positive constant C such that

$$\|x_1\|_{p,T,c,1} \leqslant C\|u\|_{L^p(M,\ell^2_c)}.$$

Moreover, we have a similar result for x_2. Finally, we have

$$\|x_1\|_{p,T,c,1} + \|x_2\|_{p,T,r,1} \leqslant C\|u\|_{L^p(M,\ell^2_c)} + C\|v\|_{L^p(M,\ell^2_r)} \leqslant C\|x\|_{p,T,1}.$$

COROLLARY 5.2. *Suppose* $1 < p < 2$. *Let* T *be a bounded operator on* $L^p(M)$ *with* $\mathrm{Ran}(I-T)$ *dense in* $L^p(M)$ *and let* $\alpha > 0$. *Assume that* T *admits a completely bounded* $H^\infty(B_\gamma)$ *functional calculus for some* $\gamma \in \left]0, \frac{\pi}{2}\right[$. *Then we have an equivalence*

$$\inf\left\{\|x_1\|_{p,T,c,\alpha} + \|x_2\|_{p,T,r,\alpha} : x = x_1 + x_2\right\} \approx \|x\|_{L^p(M)}, \qquad x \in L^p(M).$$

PROOF. By Theorem 3.3, the operator T is both Col-Ritt and Row-Ritt (hence R-Ritt). Moreover, by Theorem 3.5, it satisfies a 'dual square estimate'

$$\|y\|_{p^*,T^*,1} \lesssim \|y\|_{L^{p^*}(M)}, \qquad y \in L^{p^*}(M).$$

Then, by Theorem 5.1 above, the norms $\|\cdot\|_{p,T,\alpha}$ and $\|\cdot\|_{p,T,0,\alpha}$ are equivalent. Furthermore, by Theorem 3.6 and (3), $\|\cdot\|_{p,T,\alpha}$ is equivalent to the usual norm $\|\cdot\|_{L^p(M)}$, which proves the result.

Assume now that τ is finite and normalized, that is, $\tau(1) = 1$. Following [7] and [26] (see also [1]), we say that a linear map T on M is a Markov map if T is unital, completely positive and trace preserving. As is well known, such a map is necessarily normal and for any $1 \leqslant p < \infty$, it extends to a contraction T_p on $L^p(M)$. We say that T is selfadjoint if, for any $x, x' \in M$, we have

$$\tau(T(x)x') = \tau(xT(x')).$$

This is equivalent to T_2 being selfadjoint in the Hilbertian sense. We also consider the operator

$$A_p = I - T_p.$$

The following result is proved in the proof of [13, Proposition 8.7] with *bounded* instead of *completely bounded*. But a careful reading of the proof shows that we have this stronger result. We refer to [8], [9], [12] and [13] for information on $H^\infty(\Sigma_\theta)$ functional calculus.

PROPOSITION 5.3. *Suppose* $1 < p < \infty$. *Let T be a selfadjoint Markov map on M. Then the operator A_p is sectorial and admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in \left]0, \frac{\pi}{2}\right[$.*

Assume $1 < p < \infty$. At this point, it is crucial to recall that L^p-realizations T_p of Markov maps T on M such that $-1 \notin \sigma(T_2)$ are Ritt operators, as noticed by C. Le Merdy in [13]. Let T be a selfadjoint Markov map on M. According to [13] and Proposition 5.3, we obtain that T_p admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$. Hence, by Corollary 5.2, we deduce the following result which strengthens a result of [13].

COROLLARY 5.4. *Suppose* $1 < p < 2$. *Let T be a selfadjoint Markov map on M such that $-1 \notin \sigma(T_2)$ with $\operatorname{Ran}(I - T_p)$ dense in $L^p(M)$. Then, for any $\alpha > 0$ there exists a positive constant C such that for any $x \in L^p(M)$, there exists $x_1, x_2 \in L^p(M)$ satisfying $x = x_1 + x_2$ and*

$$\Bigg\| \Bigg(\sum_{k=1}^{+\infty} k^{2\alpha-1} \big| T^{k-1}(I-T)^\alpha(x_1) \big|^2 \Bigg)^{\frac{1}{2}} \Bigg\|_{L^p(M)} + \Bigg\| \Bigg(\sum_{k=1}^{+\infty} k^{2\alpha-1} \big| \big(T^{k-1}(I-T)^\alpha(x_2) \big)^* \big|^2 \Bigg)^{\frac{1}{2}} \Bigg\|_{L^p(M)} \leqslant C \|x\|_{L^p(M)}.$$

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REFERENCES

1. Anantharaman-Delaroche, C., *On ergodic theorems for free group actions on noncommutative spaces*, Probab. Theory Related Fields 135 (2006), 520–546.
2. Arhancet, C., and Le Merdy, C., *Dilation of Ritt operators on L^p-spaces*, arXiv:1106.1513.
3. Berkson, E., and Gillespie, T. A., *Spectral decompositions and harmonic analysis on UMD spaces*, Studia Math. 112 (1994), 13–49.
4. Blunck, S., *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. 146 (2001), 157–176.
5. Blunck, S., *Analyticity and discrete maximal regularity on L_p-spaces*, J. Funct. Anal. 183 (2001), 211–230.
6. Clement, P., de Pagter, B., Sukochev, F. A., and Witvliet, H., *Schauder decomposition and multiplier theorems*, Studia Math. 138 (2000), 135–163.
7. Haagerup, U., and Musat, M., *Factorization and dilation problems for completely positive maps on von Neumann algebras*, Comm. Math. Phys. 303 (2011), 555–594.
8. Haase, M., *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications 169. Birkhõuser, Basel 2006.
9. Junge, M., Le Merdy, C., and Xu, Q., *H^∞ functional calculus and square functions on noncommutative L^p-spaces*, Astérisque 305 (2006), vi+138.
10. Krengel U., *Ergodic theorems. With a supplement by Antoine Brunel*, Gruyter Stud. Math. 6, Gruyter, Berlin 1985.
11. Kunstmann, P., and Weis, L., *L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. Functional analytic methods for evolution equations*, Lect. Notes Math. 1855, Springer, Berlin 2004.
12. Le Merdy, C., *Square functions, bounded analytic semigroups, and applications*, pp. 191–220 in: Perspectives in operator theory, Banach Center Publ., 75, Polish Acad. Sci., Warsaw 2007.
13. Le Merdy, C., *H^∞ functional calculus and square function estimates for Ritt operators*, preprint.
14. Le Merdy, C., and Xu, Q., *Maximal theorems and square functions for analytic operators on L_p-spaces*, J. Lond. Mat. Soc 86 (2012), 343–365.
15. Lust-Piquard, F., and Pisier, G., *Noncommutative Khintchine and Paley inequalities*, Ark. Mat. 29 (1991), 241–260.
16. Lyubich, Y., *Spectral localization, power boundedness and invariant subspaces under Ritt's type condition*, Studia Math. 134 (1999), 153–167.
17. Martinez Carracedo, C., and Sanz Alix, M., *The theory of fractional powers of operators*, North-Holland Math. Stud. 187. North-Holland, Amsterdam 2001.
18. Nagy, B., and Zemanek, J., *A resolvent condition implying power boundedness*, Studia Math. 134 (1999), 143–151.
19. Nevanlinna, O., *Convergence of iterations for linear equations* Lect. Math. ETH Zürich, Birkhõuser, Basel 1993.
20. Pisier, G., *Complex interpolation and regular operators between Banach lattices*, Arch. Math. 62 (1994), 261–269.
21. Pisier, G., *Regular operators between non-commutative L_p-spaces*, Bull. Sci. Math. 119 (1995), 95–118.
22. Pisier, G., *Non-commutative vector valued L_p-spaces and completely p-summing maps*, Astérisque 247 (1998), vi+131.
23. Pisier, G., *Introduction to operator space theory*, London Math. Soc Lect. Note Ser. 294, Cambr. Univ. Press, Cambridge 2003.
24. Pisier, G., *Complex interpolation between Hilbert, Banach and operator spaces*, Mem. Amer. Math. Soc. 208 (2010), no. 978.

25. Pisier, G., and Xu, Q., *Non-commutative L_p-spaces*, pp. 1459–1517 in: Handbook of the Geometry of Banach Spaces 2, North-Holland, Amsterdam 2003.
26. Ricard, E., *A Markov dilation for self-adjoint Schur multipliers*, Proc. Amer. Math. Soc. 136 (2008), 4365–4372.
27. Vitse, P., *A band limited and Besov class functional calculus for Tadmor-Ritt operators*, Arch. Math. (Basel) 85 (2005), 374–385.

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