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ON α-SHORT MODULES

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Abstract

We introduce and study the concept of α-short modules (a 0-short module is just a short module, i.e., for each submodule N of a module M, either N or $\frac{M}{N}$ is Noetherian). Using this concept we extend some of the basic results of short modules to α-short modules. In particular, we show that if M is an α-short module, where α is a countable ordinal, then every submodule of M is countably generated. We observe that if M is an α-short module then the Noetherian dimension of M is either α or $\alpha + 1$. In particular, if R is a semiprime ring, then R is α-short as an R-module if and only if its Noetherian dimension is α.

1. α-short modules and α-almost Noetherian modules

Lemonnier [21], introduced the concept of deviation and codeviation of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module M_R give the concepts of Krull dimension (in the sense of Rentschler and Gabriel, see [19], [10]) and dual Krull dimension of M, respectively. Later, Chambless in [8] undertook a systematic study of the notion of dual Krull dimension and called it N-dimension. The second author also extensively studied the latter dimension in his Ph.D. thesis [13] and called it Noetherian dimension. Kirby in [20] calls it Noetherian dimension too, but Roberts in [22] calls this dual dimension again Krull-dimension. The latter dimension is also called dual Krull dimension in some other articles, see for example , [1], [2], [3] and [4]. In this article, all rings are associative with $1 \neq 0$, and all modules are unital right modules. If M is an R-module, by $n\text{-dim}\, M$, $k\text{-dim}\, M$ we mean the Noetherian dimension and the Krull dimension of M over R, respectively. It is convenient, when we are dealing with the latter dimensions, to begin our list of ordinals with -1.

Bilhan and Smith in [7], introduced short modules. They show that short modules are countably generated. We shall call an R-module M to be α-short, if for each submodule N of M, either $n\text{-dim}\, N \leqslant \alpha$ or $n\text{-dim}\, \frac{M}{N} \leqslant \alpha$ and α is the least ordinal number with this property. Using this concept, we observe that each α-short module M is either with $n\text{-dim}\, M = \alpha$ or $n\text{-dim}\, M = \alpha+1$. Consequently, if M is a short module, then either M is Noetherian or $n\text{-dim}\, M = 1$,

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a fact which seems to have been overlooked in [7]. By applying the previous facts we prove more general results and obtain every single result in [7] as a consequence of our results. For example, we show that every submodule of an α-short module M, where α is countable, is countably generated, which is much stronger than the fact that every short module is countably generated, see [18, Corollary 1.2]. If an R-module M has Noetherian dimension and α is an ordinal number, then M is called α-conotable if $n\text{-dim}\, M = \alpha$ and $n\text{-dim}\, N < \alpha$ for all proper submodules N of M. An R-module M is called conotable if M is α-conotable for some ordinal α (note, conotable modules are also called atomic, dual critical and N-critical in some other articles, see for example [16], [20], [3] and [8]). For all concepts and basic properties of rings and modules which are not defined in this paper, we refer the reader to [6], [10], [18].

We recall that an R-module M is called a short module if for each submodule N of M, either N or $\frac{M}{N}$ is Noetherian, see [7]. In this section we introduce and study α-short and α-almost Noetherian modules. We extend some of the basic results of short (resp. almost Noetherian) modules to α-short (resp. α-almost Noethereian) modules.

Next, we give our definition of α-short modules.

DEFINITION 1.1. An R-module M is called α-short, if for each submodule N of M, either $n\text{-dim}\, N \leqslant \alpha$ or $n\text{-dim}\, \frac{M}{N} \leqslant \alpha$ and α is the least ordinal number with this property.

Clearly each 0-short module is just a short module.

REMARK 1.2. If M is an R-module with $n\text{-dim}\, M = \alpha$, then M is β-short for some $\beta \leqslant \alpha$.

REMARK 1.3. If M is an α-short module, then each submodule and each factor module of M is β-short for some $\beta \leqslant \alpha$.

We need the following result which is also in [14].

LEMMA 1.4. *If M is an R-module and for each submodule N of M, either N or $\frac{M}{N}$ has Noetherian dimension, then so does M.*

PROOF. Let $M_1 \subseteq M_2 \subseteq \cdots$ be any ascending chain of submodules of M. If there exists some i such that $\frac{M}{M_i}$ has Noetherian dimension, then each $\frac{M_{k+1}}{M_k}$ has Noetherian dimension for $k \geqslant i$. Otherwise M_i has Noetherian dimension for each i. Thus there exists some integer k such that in any case each $\frac{M_{i+1}}{M_i}$ has Noetherian dimension for each $i \geqslant k$. Consequently M has Noetherian dimension.

The previous result and Remark 1.2, immediately yield the next result.

COROLLARY 1.5. *Let M be an α-short module. Then M has Noetherian dimension and n-dim $M \geqslant \alpha$.*

We recall that an R-module M is called almost Noetherian if every proper submodule of M is finitely generated, see [7]. It is trivial to see that every almost Noetherian R-module is either Noetherian or 1-conotable. In the following definition we consider a related concept.

DEFINITION 1.6. An R-module M is called α-almost Noetherian, if for each proper submodule N of M, n-dim $N < \alpha$ and α is the least ordinal number with this property.

Clearly each α-almost Noetherian module M, where $\alpha = 0, 1$, is almost Noetherian (note, in fact if $\alpha = 0$ then M is simple, i.e., it is 0-conotable and if $\alpha = 1$ then it is either Noetherian or 1-conotable). It is also manifest that if M is an α-almost Noetherian module, then each submodule and each factor module of M is β-almost Noetherian for some $\beta \leqslant \alpha$.

The next three trivial, but useful facts, are needed.

LEMMA 1.7. *If M is an α-almost Noetherian module, then M has Noetherian dimension and n-dim $M \leqslant \alpha$. In particular, n-dim $M = \alpha$ if and only if M is α-conotable.*

LEMMA 1.8. *If M is a module with n-dim $M = \alpha$, then either M is α-conotable, in which case it is α-almost Noetherian, or it is $\alpha + 1$-almost Noetherian.*

LEMMA 1.9. *If M is an α-almost Noetherian module, then either M is α-conotable or $\alpha = n$-dim $M + 1$. In particular, if M is an α-almost Noetherian module, where α is a limit ordinal, then M is α-conotable.*

The following is now immediate.

PROPOSITION 1.10. *An R-module M has Noetherian dimension if and only if M is α-short (resp. α-almost Noetherian) for some ordinal α.*

The following, which is also evident, is stronger than [7, Lemma 1.9].

COROLLARY 1.11. *Every α-short (resp. α-almost Noetherian) module has finite uniform dimension.*

PROPOSITION 1.12. *If M is an α-short R-module, then either n-dim $M = \alpha$ or n-dim $M = \alpha + 1$.*

PROOF. Clearly in view of Remark 1.2, Corollary 1.5, we have n-dim $M \geqslant \alpha$. If n-dim $M \neq \alpha$, then n-dim $M \geqslant \alpha + 1$. Now let $M_1 \subseteq M_2 \subseteq \cdots$ be any ascending chain of submodules of M. If there exists some k such that

n-dim $\frac{M}{M_k} \leqslant \alpha$, then n-dim $\frac{M_{i+1}}{M_i} \leqslant n$-dim $\frac{M}{M_i} = n$-dim $\frac{M/M_k}{M_i/M_k} \leqslant n$-dim $\frac{M}{M_k} \leqslant \alpha$ for each $i \geqslant k$. Otherwise n-dim $M_i \leqslant \alpha$ (note, M is α-short) for each i, hence n-dim $\frac{M_{i+1}}{M_i} \leqslant \alpha$ for each i. Thus in any case there exists an integer k such that for each $i \geqslant k$, n-dim $\frac{M_{i+1}}{M_i} \leqslant \alpha$. This shows that n-dim $M \leqslant \alpha + 1$, i.e., n-dim $M = \alpha + 1$.

COROLLARY 1.13. *If M is a short module, then either n-dim $M = 1$ or M is Noetherian.*

In view of Proposition 1.12, the following remark is now evident.

REMARK 1.14. If M is a β-short R-module, then it is an α-almost Noetherian module such that $\beta \leqslant \alpha \leqslant \beta + 2$. We claim that all the cases in the latter inequality can occur. To see this, we note that every 1-conotable module is 0-short which is also 1-almost Noetherian and every α-conotable module, where α is a limit ordinal, is an α-short module which is also α-almost Noetherian (note, for every ordinal α, there exists an α-conotable module, see the comment at the end of this section). Finally, there exists a 2-almost Noetherian module which is 0-short, see Example 2.11.

REMARK 1.15. An R-module M is -1-short if and only if it is simple. Thus any -1-short module is 0-conotable and 0-critical (note, an R-module M is called α-critical, if k-dim $M = \alpha$ and k-dim $\frac{M}{N} < \alpha$ for all nonzero submodules N of M).

PROPOSITION 1.16. *Let M be an R-module, with n-dim $M = \alpha$, where α is a limit ordinal. Then M is α-short.*

PROOF. We know that M is β-short for some $\beta \leqslant \alpha$. If $\beta < \alpha$, then by Proposition 1.12, n-dim $M \leqslant \beta + 1 < \alpha$, which is a contradiction. Thus M is α-short.

PROPOSITION 1.17. *Let M be an R-module and n-dim $M = \alpha = \beta + 1$. Then M is either α-short or it is β-short.*

PROOF. We know that M is γ-short for some $\gamma \leqslant \alpha$. If $\gamma < \beta$ then by Proposition 1.12, we have n-dim $M \leqslant \gamma + 1 < \beta + 1$, which is impossible. Hence we are done.

For the conotable modules we have the following proposition.

PROPOSITION 1.18. *Let M be an α-conotable R-module, where $\alpha = \beta + 1$, then M is a β-short module.*

PROOF. Let $N \subsetneq M$, therefore n-dim $N < \alpha$. Thus n-dim $N \leqslant \beta$. This shows that for some $\beta' \leqslant \beta$, M is β'-short. If $\beta' < \beta$, then $\beta' + 1 \leqslant \beta < \alpha$.

But n-dim $M \leqslant \beta'+1 \leqslant \beta < \alpha$, by Proposition 1.12, which is a contradiction. Thus $\beta' = \beta$ and we are done.

The following remark, which is a trivial consequence of the previous fact, shows that the converse of Proposition 1.16, is not true in general.

REMARK 1.19. Let M be an $\alpha + 1$-conotable R-module, where α is a limit ordinal. Then M is an α-short module but n-dim $M \neq \alpha$.

PROPOSITION 1.20. *Let M be an R-module such that n-dim $M = \alpha + 1$. Then M is either an α-short R-module or there exists a submodule N of M such that n-dim $N = n$-dim $\frac{M}{N} = \alpha + 1$.*

PROOF. We know that M is α-short or an $\alpha + 1$-short R-module, by Proposition 1.17. Let us assume that M is not an α-short R-module, hence there exists a submodule N of M such that n-dim $N \geqslant \alpha + 1$ and n-dim $\frac{M}{N} \geqslant \alpha + 1$. This shows that n-dim $N = \alpha + 1$ and n-dim $\frac{M}{N} = \alpha + 1$ and we are through.

The next proposition is a generalization of [7, Proposion 1.8].

PROPOSITION 1.21. *Let M be a nonzero α-short R-module. Then either M is β-almost Noetherian for some ordinal $\beta \leqslant \alpha + 1$ or there exists a submodule N of M with n-dim $\frac{M}{N} \leqslant \alpha$.*

PROOF. Suppose that M is not β-almost Noetherian for any $\beta \leqslant \alpha + 1$. This means that there must exist a submodule N of M such that n-dim $N \nleqslant \alpha$. Inasmuch as M is α-short, we infer that n-dim $\frac{M}{N} \leqslant \alpha$ and we are done.

Finally we conclude this section by providing some examples of α-almost Noetherian (resp. α-short) modules, where α is any ordinal.

First, we recall that if M is an Artinian R-module with n-dim $M = \alpha$, then for any ordinal $\beta \leqslant \alpha$ there exists a β-conotable R-submodule of M, see the comment which follows [18, Proposition 1.11]. We should remind the reader that the latter fact is much stronger than [7, Proposition 1.1]. We also recall that given any ordinal α there exists an Artinian module M such that n-dim $M = \alpha$, see [17, Example 1] and [9]. Consequently, we may take M to be an Artinan module with n-dim $M = \alpha$ and for any ordinal $\beta \leqslant \alpha$, we take N to be its β-conotable submodule, then by Lemma 1.8, N is β-almost Noetherian module. We recall that the only α-almost Noetherian modules, where α is a limit ordinal, are α-conotable modules, see Lemma 1.9. Therefore to see an example of an α-almost Noetherian module which is not α-conotable, the ordinal α must be a non-limit ordinal. Thus we may take M to be a non-conotable module with n-dim $M = \beta$, where $\alpha = \beta + 1$, see [17, Example 1], hence it follows trivially that M is an α-almost Noetherian module. As for examples of α-short modules, one can similarly use the facts that there are Artinian modules

M with Noetherian dimension equal to α and for each $\beta \leqslant \alpha$ there are β-conotable submodules of M and then apply Propositions 1.16, 1.17, 1.18, to give various examples of α-short modules (for example, by Proposition 1.18, every $\alpha + 1$-conotable module is α-short).

2. Properties of α-short modules and α-almost Noetherian modules

In this section some properties of α-short modules, α-almost Noetherian modules over an arbitrary ring R are investigated.

The following is an extension of [7, Proposition 2.4] in the case $\alpha = 0$.

PROPOSITION 2.1. *If M is an α-short* (*resp. α-almost Noetherian*) *module, where α is a countable ordinal, then every submodule of M is countably generated.*

PROOF. Clearly $n\text{-dim}\, M = \alpha$ or $n\text{-dim}\, M = \alpha + 1$ (resp. $n\text{-dim}\, M \leqslant \alpha$), by Proposition 1.12 (resp. Lemma 1.7). But we know that every module with countable Noetherian dimension is countably generated, see [18, Corollary 1.8], hence we are through.

COROLLARY 2.2. *Short modules are countably generated.*

The following lemma is an extension of [7, Lemma 1.4].

LEMMA 2.3. *Let R be a ring, if K is a submodule of an R-module M such that $n\text{-dim}\, K \leqslant \alpha$ and $\frac{M}{K}$ is an α-short R-module. Then M is α-short.*

PROOF. Let N be a submodule of M, then $n\text{-dim}\, N \cap K \leqslant \alpha$. If $n\text{-dim}\, \frac{N}{N\cap K} \leqslant \alpha$, then $n\text{-dim}\, N \leqslant \alpha$. Now suppose that $n\text{-dim}\, \frac{N}{N\cap K} > \alpha$, then $\frac{N+K}{K}$ is a submodule of the α-short module $\frac{M}{K}$ such that $n\text{-dim}\, \frac{N+K}{K} > \alpha$. Therefore we must have $n\text{-dim}\, \frac{M/K}{N+K/K} = n\text{-dim}\, \frac{M}{N+K} \leqslant \alpha$. But $n\text{-dim}\, \frac{N+K}{N} = n\text{-dim}\, \frac{K}{N\cap K} \leqslant n\text{-dim}\, K \leqslant \alpha$, hence $n\text{-dim}\, \frac{M}{N} = \sup\{n\text{-dim}\, \frac{N+K}{N}, n\text{-dim}\, \frac{M}{N+K}\} \leqslant \alpha$. This implies that M is β-short for some $\beta \leqslant \alpha$. But $\frac{M}{K}$ is α-short, hence by Remark 1.3, we must also have $\alpha \leqslant \beta$ and we are done.

The following is an extension of [7, Lemma 1.6]. It is also the dual of the previous lemma.

LEMMA 2.4. *Let R be a ring, if K is a submodule of an R-module M such that K is an α-short R-module and $n\text{-dim}\, \frac{M}{K} \leqslant \alpha$. Then M is α-short.*

PROOF. Let N be any submodule of M. Then $n\text{-dim}\, \frac{N+K}{K} \leqslant n\text{-dim}\, \frac{M}{K} \leqslant \alpha$. Hence $n\text{-dim}\, \frac{N}{N\cap K} \leqslant \alpha$. If $n\text{-dim}\, N\cap K \leqslant \alpha$, then $n\text{-dim}\, N \leqslant \alpha$. Now suppose that n-dim $N \cap K > \alpha$. Since K is α-short, we infer that n-dim $\frac{K}{K \cap N} \leqslant \alpha$ and hence n-dim $\frac{M}{N \cap K} = \sup\{n\text{-dim } \frac{K}{N \cap K}, n\text{-dim } \frac{M}{K}\} \leqslant \alpha$. But

$$n\text{-dim } \frac{M}{N \cap K} = \sup\left\{n\text{-dim } \frac{N}{N \cap K}, n\text{-dim } \frac{M}{N}\right\} \leqslant \alpha.$$

Therefore n-dim $\frac{M}{N} \leqslant \alpha$. This shows that M is β-short for some $\beta \leqslant \alpha$. But K is α-short, hence $\beta \not< \alpha$, i.e., $\beta = \alpha$ and we are done.

COROLLARY 2.5. *Let R be a ring and M be an R-module. If $M = M_1 \bigoplus M_2$ such that M_1 is an α-short module and n-dim $M_2 \leqslant \alpha$, then M is α-short.*

We note that the $\mathbb{Z}$-module $\mathbb{Z}$ is Noetherian and the $\mathbb{Z}$-module $\mathbb{Z}_{P^\infty}$ is a 0-short module. By the previous corollary , $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ is a 0-short module. It is also clear that $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ is not Noetherian.

The following proposition is an extension of [7, Theorem 1.11].

PROPOSITION 2.6. *Let R be a ring and M be an R-module containing submodules $L \subseteq N$ such that $\frac{N}{L}$ is α-short, n-dim $\frac{M}{N} \leqslant \alpha$, and n-dim $L \leqslant \alpha$. Then M is α-short.*

PROOF. Since $\frac{N}{L}$ is α-short and n-dim $L \leqslant \alpha$, then N is α-short, by Lemma 2.3. But n-dim $\frac{M}{N} \leqslant \alpha$ and since N is α-short, M is α-short, by Lemma 2.4.

The next two results are now in order.

PROPOSITION 2.7. *Let R be a ring and M be a nonzero α-short module, which is not a conotable module, then M contains a submodule L such that n-dim $\frac{M}{L} \leqslant \alpha$.*

PROOF. Since M is not conotable, we infer that there exists a submodule $L \subsetneq M$, such that n-dim $L = n$-dim M. We know that n-dim $M = \alpha$ or n-dim $M = \alpha + 1$, by Proposition1.12. If n-dim $M = \alpha$ it is clear that n-dim $\frac{M}{L} \leqslant \alpha$. Hence we may suppose that n-dim $L = n$-dim $M = \alpha + 1$. Consequently, n-dim $\frac{M}{L} \leqslant \alpha$ and we are done.

PROPOSITION 2.8. *Let N be a submodule of an R-module M such that N is α-short and $\frac{M}{N}$ is β-short. Let $\mu = \sup\{\alpha, \beta\}$, then M is γ-short such that $\mu \leqslant \gamma \leqslant \mu + 1$.*

PROOF. Since N is α-short, thus by Proposition1.12, n-dim $N = \alpha$ or n-dim $N = \alpha + 1$. Similarly since $\frac{M}{N}$ is β-short, n-dim $\frac{M}{N} = \beta$ or n-dim $\frac{M}{N} = \beta + 1$. We infer that M has Noetherian dimension and n-dim $M = \sup\{n\text{-dim } N, n\text{-dim } \frac{M}{N}\}$. Therefore $\mu \leqslant n$-dim $M \leqslant \mu + 1$. But by Remark 1.2, M is γ-short for some ordinal number γ and by Proposition 1.12, $\gamma \leqslant n$-dim $M \leqslant \gamma + 1$.

This shows that $\gamma = \mu$, or $\gamma = \mu + 1$ (note, we always have $\mu \leqslant \gamma$) and we are done.

Using Lemma 1.9, we give the next immediate result which is the counterpart of the previous proposition for α-almost Noetherian modules.

PROPOSITION 2.9. *Let N be a submodule of an R-module M such that N is α-almost Noetherian and $\frac{M}{N}$ is β-almost Noetherian. Let $\mu = \sup\{\alpha, \beta\}$, then M is γ-almost Noetherian such that $\mu \leqslant \gamma \leqslant \mu + 1$.*

COROLLARY 2.10. *Let R be a ring. If M_1 is an α_1-short (resp. α_1-almost Noetherian) R-module and M_2 is an α_2-short (resp. α_2- almost Noetherian) R-module and let $\alpha = \sup\{\alpha_1, \alpha_2\}$. Then $M_1 \oplus M_2$ is μ-short (resp. μ- almost Noetherian) for some ordinal number μ such that $\alpha \leqslant \mu \leqslant \alpha + 1$.*

The next example shows that in the previous corollary we may have all the cases for μ.

EXAMPLE 2.11. If $M_1 = M_2 = \mathbb{Z}$, then M_1 and M_2 are 0-short (resp. 1-almost Noetherian) $\mathbb{Z}$-modules such that $M_1 \oplus M_2$ is also 0-short (resp. 1-almost Noetherian). Now let $M_1 = M_2 = \mathbb{Z}_{p^\infty}$. In this case the $\mathbb{Z}$-module $\mathbb{Z}_{P^\infty}$ is 0-short (resp. 1-almost Noetherian), but the $\mathbb{Z}$-module $\mathbb{Z}_{P^\infty} \oplus \mathbb{Z}_{P^\infty}$ is 1-short (resp. 2-almost Noetherian). We should also note that $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ is a 0-short $\mathbb{Z}$-module which is 2-almost Noetherian.

THEOREM 2.12. *Let M be a nonzero R-module and α be an ordinal number. Let every proper factor module of M be γ-short for some ordinal number $\gamma \leqslant \alpha$. If $\alpha = -1$, then M is also μ-short for some $\mu \leqslant 0$. If not, then M is μ-short where $\mu \leqslant \alpha$. Moreover, $n\text{-dim}\, M \leqslant \alpha + 1$.*

PROOF. If $\alpha = -1$, then each proper nonzero submodule of M is both a maximal and a simple submodule of M, i.e., $n\text{-dim}\, M = 0$. Hence let us assume that $\alpha \geqslant 0$. Now let $0 \neq N \subseteq M$ be any submodule such that $\frac{M}{N}$ is γ-short for some ordinal number γ with $\gamma \leqslant \alpha$. We infer that $n\text{-dim}\, \frac{M}{N} \leqslant \gamma + 1 \leqslant \alpha + 1$, by Proposition 1.12. But we know that $n\text{-dim}\, M = \sup\{n\text{-dim}\, \frac{M}{N} : N \neq 0\}$, see [16, Proposition 1.4]. This shows that $n\text{-dim}\, M \leqslant \alpha + 1$. If $n\text{-dim}\, M \leqslant \alpha$, then it is clear that M is μ-short for some $\mu \leqslant \alpha$. Hence we may suppose that $n\text{-dim}\, M = \alpha + 1$. If $0 \neq N \subsetneq M$ is a submodule of M, then we are to show that either that $n\text{-dim}\, \frac{M}{N} \leqslant \alpha$ or $n\text{-dim}\, N \leqslant \alpha$. To this end, let us suppose that $n\text{-dim}\, \frac{M}{N} = \alpha + 1$ and show that $n\text{-dim}\, N \leqslant \alpha$. Now let $0 \neq N' \subsetneq N \subsetneq M$. Since $\frac{M}{N'}$ is γ-short for some $\gamma \leqslant \alpha$, and $n\text{-dim}\, \frac{M/N'}{N/N'} = n\text{-dim}\, \frac{M}{N} = \alpha + 1$, we must have $n\text{-dim}\, \frac{N}{N'} \leqslant \alpha$. But $n\text{-dim}\, N = \sup\{n\text{-dim}\, \frac{N}{N'} :\ 0 \neq N' \subseteq N\} \leqslant \alpha$ and we are through. The final part has already been proved.

COROLLARY 2.13. *Let every proper factor module of M be* 0*-short* (*i.e., every proper factor module of M is a short module*)*, then so is M.*

REMARK 2.14. If every proper factor module of an R-module M is -1-short, then every proper submodule of M is both a maximal and a minimal submodule of M, and vice versa.

The next result is the dual of Theorem 2.12.

THEOREM 2.15. *Let α be an ordinal number and M be an R-module. If every proper submodule of M is γ-short for some ordinal number $\gamma \leqslant \alpha$. Then either $n\text{-dim}\, M = \alpha + 1$ or M is μ-short for some ordinal number $\mu \leqslant \alpha$. In particular, M is μ-short for some ordinal $\mu \leqslant \alpha + 1$.*

PROOF. Let $N \subsetneq M$ be any submodule. Since N is γ-short for some ordinal number $\gamma \leqslant \alpha$, we infer that $n\text{-dim}\, N \leqslant \gamma + 1 \leqslant \alpha + 1$, by Proposition 1.12. This immediately implies that $n\text{-dim}\, M \leqslant \alpha + 2$, see [16, Proposition 1.4]. If $n\text{-dim}\, M \leqslant \alpha+1$ then we are through. Hence we may suppose that $n\text{-dim}\, M = \alpha + 2$ and M is not μ-short for any $\mu \leqslant \alpha$ and seek a contradiction. Since M is not μ-short for any $\mu \leqslant \alpha$, we infer that there must exist a submodule N of M such that $n\text{-dim}\, N \geqslant \alpha + 1$. But we have already observed that $n\text{-dim}\, N \leqslant \alpha+1$, hence $n\text{-dim}\, N = \alpha+1$. We now claim that $n\text{-dim}\, \frac{M}{N} \leqslant \alpha+1$ which trivially implies that $n\text{-dim}\, M = \alpha + 1$ and this is the contradiction that we were looking for (note, $n\text{-dim}\, M = \alpha+2$). To see this, we note that for any proper submodule P of M containing N we must have $n\text{-dim}\, \frac{P}{N} \leqslant \alpha$, for P is γ-short for some $\gamma \leqslant \alpha$ and $n\text{-dim}\, P = \alpha+1$. But $n\text{-dim}\, \frac{M}{N} \leqslant \sup\{n\text{-dim}\, \frac{P}{N} : \frac{P}{N} \subsetneq \frac{M}{N}\} + 1 \leqslant \alpha + 1$, see [16, Proposition 1.4] and we are done. The final part is now evident.

The following example shows that in the previous theorem we may have $\mu = \alpha + 1$.

EXAMPLE 2.16. Let $M = A \oplus B$, where A and B are simple R-modules. Clearly M is 0-short. We claim that every proper submodule P of M is -1-short (i.e., P is simple). Since $P \subsetneq M$ and M is semisimple, there exists a maximal submodule Q of M such that $P \subseteq Q \subsetneq M$. Now we can not have $Q \cap A \neq 0 \neq Q \cap B$, for otherwise $Q \supseteq A$ and $Q \supseteq B$, hence $Q = M$, which is absurd. Hence we may suppose that, $Q \cap A = 0$, consequently $M = Q \oplus A$, which means that $\frac{M}{A} \simeq Q$. But $\frac{M}{A} \simeq B$, i.e., Q is simple, thus $P = Q$ or $P = 0$, and we are done.

The next immediate result is the counterparts of Theorems 2.12, 2.15, for α-almost Noetherian modules.

PROPOSITION 2.17. *Let M be an R-module and α be an ordinal number. If each proper submodule N of M (resp. each proper factor module of M) is γ-almost Noetherian with $\gamma \leqslant \alpha$, then M is a μ-almost Noetherian module with $\mu \leqslant \alpha + 1$, n-dim $M \leqslant \alpha + 1$ (resp. with $\mu \leqslant \alpha + 1$, n-dim $M \leqslant \alpha$).*

The following proposition will raise the natural question, namely, for which rings R, R is α-short if and only if n-dim $R = \alpha$, or more generally, for which R-modules M, M is α-short if and only if n-dim $M = \alpha$.

PROPOSITION 2.18. *Let R be a semiprime ring. Then the right R-module R is α-short if and only if n-dim $R = \alpha$.*

PROOF. Let R be α-short as an R-module. We are to show that n-dim $R = \alpha$. If for each essential right ideal E of R, $n\text{-dim}\,\frac{R}{E} \leqslant \alpha$ then $n\text{-dim}\,R = \sup\{n\text{-dim}\,\frac{R}{E} : E \subseteq_e R\} \leqslant \alpha$, see [16, Proposition 1.5]. Since R is α-short we have n-dim $R = \alpha$, by Proposition 1.12. Now suppose that there exists an essential right ideal E' of R such that $n\text{-dim}\,\frac{R}{E'} \nleqslant \alpha$. Since R is α-short, we infer that n-dim $E' \leqslant \alpha$. But R is a right Goldie ring, by [10, Corollary 3.4]. Hence there exists a regular element c in E', which implies that $n\text{-dim}\,R = n\text{-dim}\,cR \leqslant n\text{-dim}\,E'_R \leqslant \alpha$. Consequently, we must have n-dim $R = \alpha$, by Proposition 1.12. Conversely, by Remark 1.2, R is β-short for some $\beta \leqslant \alpha$. But by the first part of the proof, we must have n-dim $R = \beta$, i.e., $\beta = \alpha$ and we are through.

Clearly every α-almost Noetherian (resp. α-short) module has Noetherian dimension (i.e., it has Krull dimension, for by a nice result due to Lemonnier, every module has Noetherian dimension if and only if it has Krull dimension, see [21, Corollary 6]). Consequently, we have the following immediate result, which is the counterpart of [7, Proposition 1.2].

PROPOSITION 2.19. *The following statements are equivalent for a ring R.*

(1) *Every R-module with Krull dimension is Noetherian.*
(2) *Every α-short R-module is Noetherian for all α.*
(3) *Every α-almost Noetherian R-module is Noetherian for all α.*

We should remind the reader that the comment which follows [7, Proposition 1.2], trivially remains valid if we replace short modules in that comment by α-short modules. Moreover, if R is a right perfect ring (i.e., every R-module is a Loewy module) then every α-short (resp. α-almost Noetherian) R-module is both Artinian and Noetherian, see [17, Proposition 2.1], which is stronger than the fact that short modules are Noetherian over right perfect rings, see the aforementioned comment in [7].

Before concluding this section with our last observation, let us cite the next result which is in [17, Theorem 2.9], see also [11, Theorem 3.2].

THEOREM 2.20. *For a commutative ring R the following statements are equivalent.*

1. *Every R-module with finite Noetherian dimension is Noetherian.*
2. *Every Artinian R-module is Noetherian.*
3. *Every R-module with Noetherian dimension is both Artinian and Noetherian.*

Now in view of the above theorem and the well-known fact that each domain with Krull dimension 1 is Noetherian, see [10, Proposition 6.1] and also [18, Corollary 2.15], we observe the following result which is much stronger than [7, Proposition 1.3].

PROPOSITION 2.21. *The following statements are equivalent for a commutative ring R.*

1. *Every Artinian R-module is Noetherian.*
2. *Every m-short module is both Artinian and Noetherian for all integers $m \geqslant -1$.*
3. *Every α-short module is both Artinian and Noetherian for all ordinals α.*
4. *Every m-almost Noetherian R-module is both Artinian and Noetherian for all non-negative integers m.*
5. *Every α-almost Noetherian R-module is both Artinian and Noetherian for all ordinals α.*
6. *No homomorphic image of R can be isomorphic to a dense subring of a complete local domain of Krull dimension* 1.

PROOF. Only the proof of (5) $\to$ (6) $\to$ (1), which is an easy consequence of [7, Proposition 1.3], is needed.

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