PERTURBATIONS OF PLANAR ALGEBRAS

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(Dedicated to Sunder on the occasion of his 60th birthday)

Abstract

We analyze the effect of pivotal structures (on a 2-category) on the planar algebra associated to a 1-cell as in [8] and come up with the notion of *perturbations of planar algebras by weights* (a concept that appeared earlier in Michael Burns' thesis [6]); we establish a one-to-one correspondence between weights and pivotal structures. Using the construction of [8], to each bifinite bimodule over II_1-factors, we associate a *bimodule planar algebra* in such a way that extremality of the bimodule corresponds to sphericality of the planar algebra. As a consequence of this, we reproduce an extension of Jones' theorem ([13]) (of associating 'subfactor planar algebras' to extremal subfactors). Conversely, given a bimodule planar algebra, we construct a bifinite bimodule whose associated bimodule planar algebra is the one which we start with, using perturbations and Jones-Walker-Shlyakhtenko-Kodiyalam-Sunder method of reconstructing an extremal subfactor from a subfactor planar algebra. The perturbation technique helps us to construct an example of a family of non-spherical planar algebras starting from a particular spherical one; we also show that this family is associated to a known family of subfactors constructed by Jones.

1. Introduction

In the pioneering and celebrated work [12] of Jones, the theory of subfactors saw a new opening with, among other ideas and results, the introduction of the concepts of index and the tower of basic construction for subfactors, which, over the years, had various applications in the understanding of II_1-factors, knot theory, quantum groups, TQFTs and other fields. There have been a lot of pathbreaking works in this theory since its initiation – see, for instance, [12], [24], [25], [26], [28], [29].

Further, an important aspect of this paper of Jones was the evolution of an invariant called the standard invariant of the subfactor, which basically consists of a grid of finite dimensional C^*-algebras with some rich structure. It has turned out that, among other invariants, the standard invariant of a finite index subfactor is its most important invariant. An instance to justify this claim is that, for certain 'good' family of subfactors (namely, the amenable ones, see [28]), their standard invariants turn out to be complete invariants. As such, it

was motivating enough for people to work on obtaining a better understanding of this invariant. Sorin Popa (in [29]) gave an algebraic axiomatization of the standard invariant as a grid of finite dimensional C^*-algebras, which he called a *standard λ-lattice*. Conversely, given such a λ-lattice, he constructed an *extremal* subfactor whose standard invariant is the λ-lattice which he started with. Ocneanu also came up with a 'group-like' structure on the standard invariant and called them *paragroups*. Subsequently, Vaughan Jones (in [13]) developed a very effective pictorial reformulation of the standard invariant which he called *planar algebra*, and associated a planar algebra satisfying certain natural conditions (referred as *subfactor planar algebra*) to any extremal subfactor. In the converse direction, starting from a subfactor planar algebra Jones reconstructed a subfactor whose associated planar algebra is isomorphic to the given one, using Popa's characterization of λ-lattices. Later, in [30], Popa extended his correspondence to *generalized λ-lattices* on one hand and finite index subfactors (not necessarily extremal) on the other. In 2003, Michael Burns in his thesis (see [6]), established a similar correspondence replacing generalized λ-lattices with planar algebras satisfying appropriate conditions (without any assumption of sphericality). Very recently, reconstruction of extremal subfactor from a subfactor planar algebra has also been performed using random matrix and free probability techniques in [10] followed by simpler treatments using planar algebra machinery in [19] and [22]. Planar algebra techniques have recently found applications in developing new methods of constructing certain class of subfactors as well.

We now present a brief outline of the motivation and the results that brought this article into existence.

(I) Our motivation stemmed solely from the investigation of the following:

  (a) A construction of a planar algebra starting from a 1-cell in a pivotal 2-category was given by the second named author in his thesis (see [8]). This construction was purely algebraic, with the description of the action of tangles being given in terms of *graphical calculus of morphisms*, analogous to the ones used in [20]. However, the actual manifestation of the pivotal structure in the planar picture remained unclear and required further analysis.

  (b) In the operator algebra context, a nice prototype is the 2-category of bifinite bimodules over II_1-factors. So, one would like to investigate the planar algebras obtained using the method in [8] from a bifinite bimodule ${}_A\mathscr{H}_B$ where A and B are II_1 factors.

  (c) Another question in this context is whether the Jones' planar algebra associated to an extremal finite index subfactor $N \subset M$, is isomorphic to the planar algebra coming from the bimodule ${}_N L^2(M)_M$.

In this paper, we make an attempt to answer these and other natural questions, the answers of which we list below:

(II) Summary of the main results:

(a) We answer the question (I)(c), that is, we show that one does not always get back Jones' subfactor planar algebra from the construction in [8] unless the right pivotal structure is chosen.

(b) In order to find out the exact dependence of the planar algebra from the construction in [8] on the pivotal structure, we come up with the concept of weights of a planar algebra and perturbations of planar algebras by weights; later, we realized that such objects also appeared in Michael Burns' thesis to prove Jones' theorem for non-extremal subfactors. Finally, we establish a one-to-one correspondence between weights on a planar algebra and pivotal structures on the associated 2-category.

(c) To each bifinite bimodule over II_1-factors, using the construction in [8], we associate a *bimodule planar algebra* in such a way that extremality of the bimodule corresponds to sphericality of the planar algebra. Moreover, this also shows that bimodules with different left and right dimensions, gives the right platform to investigate planar algebras with different modulii coming from shaded and unshaded loops.

(d) Conversely, given a bimodule planar algebra, we construct a bifinite bimodule whose associated bimodule planar algebra is the one that we start with, using perturbations and Jones-Walker-Shlyakhtenko-Kodiyalam-Sunder method of reconstructing an extremal subfactor from a subfactor planar algebra.

(e) We give explicit construction of examples of non-spherical planar algebras; more precisely we show that the perturbation class of the diagonal planar algebra with respect to the free group F_2, generated by two free generators and trivial cocycle contains a continuum of non-spherical unimodular bimodule planar algebras with index greater than 4; we also prove that this family is associated to a known family of subfactors constructed by Jones.

(III) Some nice consequences:

(a) As a consequence of (II)(c), we reproduce an extension of Jones' theorem (of associating 'subfactor planar algebras' to extremal subfactors). This was proved earlier by Michael Burns in his thesis [6]; the reconstruction of a non-extremal subfactor had appeared in [30].

(b) We show that the perturbation class of a bimodule planar algebra contains a unique spherical unimodular bimodule planar algebra which can also

be characterized by the minimality of its index. Subfactor version of such results had appeared in the works of Hiai and Popa.

All results in this article are derived using standard facts on bimodules, subfactors and planar algebras, which can be found, for instance, in [1], [13], [18], [24], [25], [26], [28], [29].

We now briefly describe the organization of this paper. Section 2 serves as a quick recollection of various definitions, standard facts and basic aspects of planar algebras, pivotal bicategories and the bicategory of bifinite bimodules.

In Section 3, we define *weight* of a planar algebra P and *perturbation* of P by the weight. A planar algebra with modulus (δ_-, δ_+) can be *normalized*, that is, perturbed with an appropriate scalar weight to get a *unimodular* planar algebra (that is, having $\delta_- = \delta_+$) although the *index* (:= the product of the δ's) remains unchanged. If the actions of the 0-tangles in the normalization are invariant under spherical isotopy, then the planar algebra is called spherical; this is a slight modification of Jones' definition of sphericality in order to accommodate non-unimodular planar algebras. At the very end of this section, we make few immediate observations involving perturbations, ∗-structures and positivity in planar algebras.

In the first part of Section 4, we associate a strict 2-category to a planar algebra and show that weights of the planar algebra are in one-to-one correspondence with pivotal structures on the associated 2-category. Conversely, if we start with a bicategory with two pivotal structures, then the planar algebras obtained from any 1-cell using [8] method, are perturbations of each other. This section ends with a quick recollection (from [8]) of the method of associating a planar algebra to an 1-cell in a strict 2-category.

Section 5 is an omnibus section and is the crux of this paper. In this section, we first formalize what we mean by a bimodule planar algebra; then, following the above-mentioned method of [8], we associate a bimodule planar algebra to each bifinite bimodule such that the extremality of the bimodule exactly corresponds to the sphericality of its associated planar algebra. This, in turn, provides an extension to Jones' theorem [13, Theorem 4.2.1], that is, we associate a unimodular bimodule planar algebra to an arbitrary finite index subfactor. Such extension was also obtained by Burns; however, our techniques are completely independent and rely on simple graphical calculus of morphisms in the pivotal 2 category. In the converse direction, given any bimodule planar algebra, we obtain a bifinite bimodule whose associated bimodule planar algebra is isomorphic to the one that we started with, through the application of perturbations and following the strategy of [19], [22].

In the first part of Section 6, we show that the perturbation class of every bimodule planar algebra contains a unique spherical unimodular bimodule

planar algebra which can also be characterized by the property of having the minimal index in the perturbation class. Minimizing indices of conditional expectations onto a subfactor already appeared in the literature in the work of Hiai (in [11]) and then Popa (in [28]); we are now able to connect this circle of ideas with our notion of perturbation of planar algebra. In the second part, we construct concrete examples of nonspherical planar algebras purely algebraically; more precisely we show that the perturbation class of the diagonal planar algebra with respect to the free group F_2, generated by two free generators and trivial cocycle contains a continuum of non-spherical unimodular bimodule planar algebras with index greater than 4. As suggested by Jones, we prove that these planar algebras are isomorphic to the ones associated to the (non-extremal) subfactors that he constructed in [12] in order to prove that every index greater than 4 is realized.

In the final section, we discuss some questions pertaining to perturbations and weights of a planar algebra.

2. Preliminaries

This section is mainly a recollection of various definitions, standard facts and setting up of notations which will be used in the subsequent sections.

2.1. *Planar algebras*

Since its inception in [13], the formalism of planar algebras has undergone gradual modifications – see, for instance, [13], [14], [21], [8]. The starting ingredient for defining a planar algebra is the operad of tangles. A *tangular diagram T* consists of a subset D_0 (referred as the *external disc*) of R^2, homeomorphic to the unit disc along with: (a) finitely many (possibly none) non-intersecting subsets $D_1, \ldots, D_b$ (referred as *internal discs*) in the interior of D_0, each of which is also homeomorphic to the unit disc, (b) the boundary of each disc (internal or external) having even number of marked points numbered clockwise, (c) non-intersecting paths (called *strings*) in $D_0 \setminus \left[\bigcup_{i=1}^{b} \mathrm{Int}(D_i) \right]$, which are either loops or meet the boundaries of the discs exactly at two distinct marked points in such a way that every marked point is an endpoint of a string, and (d) a checker-board shading on the connected components of $\mathrm{Int}(D_0) \setminus \left[\left(\bigcup_{i=1}^{b} D_i \right) \cup \{\text{strings}\} \right]$. We will usually indicate the checker-board shading and the numbering of the marked points simply by putting a $-$ (resp., $+$) sign in the shaded (resp., unshaded) connected component near the boundary segment between the last and first marked points. Such a sign along with half the number of marked points on a disc is called its *color*. A *tangle* is the class of a tangular diagram under the equivalence of *planar isotopy* (preserving the shading and the distinguished boundary components). If a tangle

T has $b > 0$ (resp., no) internal disc(s) and the color of the disc D_i is $\varepsilon_i k_i$ for $0 \leq i \leq b$, then the tangle is usually expressed as $T : (\varepsilon_1 k_1, \ldots, \varepsilon_b k_b) \to \varepsilon_0 k_0$ (resp., $T : \emptyset \to \varepsilon_0 k_0$). See Figure 2.1 for illustrations. Before we proceed further, we will set up some notations about planar algebras which will be used in the forthcoming sections. Although at first, these notations may seem to be a hindrance towards the readability to some extent, they will greatly reduce the need for drawing elaborate planar pictures again and again. Understanding and decoding these notations correctly will definitely facilitate smooth reading. We will not give the definition of planar algebra which can be found in [13]; however, we will be consistent with the notation described in [8].

(1) We will consider the natural binary operation on $\{-, +\}$ given by $++ := +, +- := -, -+ := -$ and $-- := +$. Notations such as $(-)^l$ have to be understood in this context.

(2) We will denote the set of all possible colors of discs in tangles by $\mathrm{Col} := \{\varepsilon k : \varepsilon \in \{+, -\}, k \in \mathsf{N}_0\}$ where $\mathsf{N}_0 := \mathsf{N} \cup \{0\}$.

(3) In a tangle, we will replace (isotopically) parallel strings by a single strand labelled by the number of strings, and an internal disc with color εk will be replaced by a bold dot with the sign ε placed at the angle corresponding to the distinguished boundary component of the disc. For example, $\left(\varepsilon\ \bigcirc\ \overset{\varepsilon}{}\right)$ will be replaced by $\left(\varepsilon\ \overset{2}{\bullet}_4\ \varepsilon\right)$. In a similar token, if P is a planar algebra, we will replace a P-labelled internal disc by a bold dot with the label being placed at the angle corresponding to the distinguished boundary component of the disc; for instance, $\left(\varepsilon\ x\ \overset{\varepsilon}{}\right)$ will be replaced by $\left(x\ \overset{2}{\bullet}_4\ \varepsilon\right)$ where $x \in P_{\varepsilon 3}$.

We will reserve alphabets like x, y, z to denote elements of P; ε, η, ν to denote a sign, and k, l, m to denote a natural number to avoid confusion. It should be clear from the context what a bold dot or a string in a picture is labelled by.

(4) We set some notation for a set of 'generating tangles' in Figure 2.1.

(5) $\mathscr{T}_{\varepsilon k}$ (resp., $\mathscr{T}_{\varepsilon k}(P)$) will denote the set of tangles (resp., P-labelled tangles) which has εk as the color of the external disc; $\mathscr{P}_{\varepsilon k}(P)$ will be the vector space $\mathscr{T}_{\varepsilon k}(P)$ as a basis. The action of P induces a linear map $\mathscr{P}_{\varepsilon k}(P) \ni T \overset{P}{\longmapsto} P_T \in P_{\varepsilon k}$.

A *planar algebra* P is a 'representation' of the operad of tangles, that is, it consists of complex vector spaces $\{P_{\varepsilon k} : \varepsilon k \in \mathrm{Col}\}$ and for every tangle

$$M_{\varepsilon k} = \boxed{\begin{matrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{matrix}\ k} : (\varepsilon k, \varepsilon k) \to \varepsilon k \qquad\qquad 1_{\varepsilon k} = \boxed{\ k\ }^{\varepsilon} : \emptyset \to \varepsilon k$$

Multiplication tangle   Unit tangle

$$I_{\varepsilon k} = \boxed{\begin{matrix} 2k \\ \varepsilon \\ \varepsilon \end{matrix}} : \varepsilon k \to \varepsilon k \qquad\qquad E_{\varepsilon(k+1)} = \boxed{\ k\ }^{\varepsilon} : \emptyset \to \varepsilon(k+2)$$

Identity tangle   Jones projection tangle

$$RI_{\varepsilon k} = \boxed{\begin{matrix} \varepsilon\ k \\ \varepsilon \\ k \end{matrix}} : \varepsilon k \to \varepsilon(k+1) \qquad LI_{\varepsilon k} = \boxed{\begin{matrix} k \\ \varepsilon \\ k \end{matrix}}^{-\varepsilon} : \varepsilon k \to -\varepsilon(k+1)$$

Right inclusion tangle   Left inclusion tangle

$$RE_{\varepsilon(k+1)} = \boxed{\begin{matrix} \varepsilon\ k \\ \varepsilon\ \bigcirc \\ k \end{matrix}} : \varepsilon(k+1) \to \varepsilon k \qquad LE_{\varepsilon(k+1)} = \boxed{\begin{matrix} k \\ \bigcirc\ \varepsilon \\ k \end{matrix}}^{-\varepsilon} : \varepsilon(k+1) \to -\varepsilon k$$

Right conditional expectation tangle   Left conditional expectation tangle

FIGURE 2.1. Generating tangles

$T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_b k_b) \to \varepsilon_0 k_0$ (resp., $T : \emptyset \to \varepsilon_0 k_0$), there exists an action of T given by a multi-linear map $P_T : \times_{i=1}^{b} P_{\varepsilon_i k_i} \to P_{\varepsilon_0 k_0}$ (resp., a vector $P_T \in P_{\varepsilon_0 k_0}$) such that the action preserves (i) composition and (ii) identity (that is, $P_{I_{\varepsilon k}} = \mathrm{id}_{P_{\varepsilon k}}$). Note that $\{P_{\varepsilon k}\}_{k \in \mathbb{N}_0}$ has a unital filtered algebra structure with multiplication, unit and inclusion given by the actions of $M_{\varepsilon k}$, $1_{\varepsilon k}$ and $RI_{\varepsilon k}$, respectively.

DEFINITION 2.1. A planar algebra P is said to

(1) be connected (resp., finite dimensional) if $\dim(P_{\pm 0}) = 1$ (resp., $\dim(P_{\varepsilon k})$ $< \infty$ for all $\varepsilon k \in \mathrm{Col}$).

(2) have modulus $(\delta_-, \delta_+) \in \mathbb{C}^2$ if $P \boxed{\overset{\pm}{\bigcirc}} = \delta_{\pm} P_{1_{\pm 0}} = \delta_{\pm} 1_{P_{\pm 0}}$ and in this case, the scalar $\delta_+ \delta_-$ is called the index of P.

(3) be unimodular if it has modulus (δ_-, δ_+) such that $\delta_- = \delta_+$.

(4) be a $*$-planar algebra if there exists a conjugate-linear involution $* : P_{\varepsilon k} \to P_{\varepsilon k}$ for all $\varepsilon k \in \mathrm{Col}$ satisfying the $*$-condition:

(2.1) $P_T(x_1, \ldots, x_b)^* = P_{T^*}(x_1^*, \ldots, x_b^*)$ (resp., $(P_T)^* = P_{T^*}$)

for each tangle $T : (\varepsilon_1 k_1 \times \cdots \times \varepsilon_b k_b) \to \varepsilon_0 k_0$ (resp., $T : \emptyset \to \varepsilon_0 k_0$) and

$x_i \in P_{\varepsilon_i k_i}$, $1 \le i \le b$, where the adjoint T^* of a tangle T is obtained by reflecting it about a horizontal line keeping the shading and distinguished boundary components intact.

(5) be C^*-planar algebra if each $P_{\varepsilon k}$ is a C^*-algebra such that its multiplication is the same as that induced by $M_{\varepsilon k}$ and P forms a $*$-planar algebra with respect to the $*$.

(6) be irreducible if $\dim(P_{+1}) = 1$ (equivalently, $\dim(P_{-1}) = 1$).

Note that a connected planar algebra P has modulus, and two canonical *picture traces* $P_{TR^r_{\varepsilon k}} : P_{\varepsilon k} \to P_{\varepsilon 0} \cong \mathsf{C}$ and $P_{TR^l_{\varepsilon k}} : P_{\varepsilon k} \to P_{(-)^k \varepsilon 0} \cong \mathsf{C}$ induced by the trace tangles $TR^r_{\varepsilon k} :=$ and $TR^l_{\varepsilon k} :=$. A connected C^*-planar algebra P is said to be *positive* if the canonical picture traces are positive definite.

We now recall the definition of the *n-th dual* of a planar algebra P with modulus (δ_-, δ_+), denoted by $\lambda_n(P)$. For a tangle T, let $\lambda_n(T)$ be the tangle obtained from T by attaching n parallel strings on the disc D (internal or external) enclosing closely the sign ε_D near the distinguished boundary component of the disc.

Vectors spaces: For all colors εk, $\lambda_n(P)_{\varepsilon k} := \mathrm{Range}(P_{\lambda_n(I_{\varepsilon k})})$.

Action of tangles: For all tangles T,

$$\lambda_n(P)_T := \left[\prod_{D \in \{\text{internal disc(s) of } T\}} \left(\prod_{l=1}^{n} \delta_{(-)^l \varepsilon_D} \right)^{-1} \right] P_{\lambda_n(T)}.$$

Note that $\lambda_0(P) = P$, $\lambda_n(\lambda_m(P)) = \lambda_{m+n}(P)$ and $\lambda_n(P)$ has modulus $(\delta_{(-)^{n+1}}, \delta_{(-)^n})$ for all $m, n \in \mathsf{N}_0$.

For any mathematical category, studying the morphisms in that category is very crucial. A *morphism* $\varphi : P \to Q$ for two planar algebras P and Q, is a collection of linear maps $\varphi_{\varepsilon k} : P_{\varepsilon k} \to Q_{\varepsilon k}$, $\varepsilon k \in \mathrm{Col}$ which are equivariant with the action of tangles (that is, $\varphi \circ P_T = Q_T \circ \varphi$ or $\varphi \circ P_T = Q_T$ according as T has at least one internal disc or none). Given such a double sequence of linear maps, it is not always necessary to verify its equivariance with every tangle. For instance, it suffices to do the same only for the actions of a generating set of tangles mentioned in Figure 2.1.

REMARK 2.2. Let P and Q be connected planar algebras with non-zero modulii. Then, a linear map $\varphi : P \to Q$ is a planar algebra morphism if it is equivariant with the actions of any of the following sets of tangles:

(1) $\bigcup_{k \in \mathsf{N}_0} \{M_{\pm k}, RI_{\pm k}, LI_{\pm k}, E_{\pm 1}\}$

(2) $\bigcup_{k \in \mathbb{N}_0} \{M_{+k}, RI_{+k}, LI_{-k}, E_{+(k+1)}, LE_{+(k+1)}\}$

Sometimes, in order to obtain morphisms between planar algebras, it is even enough to obtain a morphism on the 'positive parts'. More precisely, we have the following lemma.

LEMMA 2.3. *Let P and Q be planar algebras having the same pair of non-zero moduli, say* (δ_-, δ_+), *and suppose there exist linear maps* $\varphi_{+k} : P_{+k} \to Q_{+k}$, $k \geq 0$, *equivariant with the action of tangles with discs of only positive colors. Then there exists a unique planar algebra morphism* $\tilde{\varphi} : P \xrightarrow{\sim} Q$ *such that* $\varphi_{+k} = \tilde{\varphi}_{+k}$ *for all* $k \geq 0$. *Moreover, if* φ *is an isomorphism, so is* $\tilde{\varphi}$.

PROOF. Consider $\tilde{\varphi} : P \to Q$ given by $\tilde{\varphi}_{+k} = \varphi_{+k}$ and

$$P_{-k} \ni x \xmapsto{\tilde{\varphi}_{-k}} \delta_-^{-1} Q_{LE_{+(k+1)}} \circ \varphi_{+(k+1)} \circ P_{LI_{-k}}(x) \in Q_{-k}$$

for all $k \geq 0$. It is straight forward to check that $\tilde{\varphi}$ is equivariant with the action of tangles of all types (not necessarily having positive colors for all discs). The uniqueness follows from the definition.

For the second part, since φ is equivariant with the action of $E'_{+(k+1)} := LI_{-k} \circ LE_{+(k+1)}$, we have $\varphi_{+(k+1)}(\text{Ran } P_{E'_{+(k+1)}}) \subseteq \text{Ran } Q_{E'_{+(k+1)}}$, with equality if φ is surjective; also, φ_{-k} is injective if so is φ because $Q_{LE_{+(k+1)}}$ restricted to Ran $Q_{E'_{+(k+1)}}$ is injective. Thus, $\tilde{\varphi}$ is an isomorphism if φ is so.

Let P be a planar algebra. Then, by a *P-labelled* (resp., *semi-labelled*) tangle, we mean a tangle whose all (resp., some) internal discs are labelled by elements of P such that an internal disc of color εk is labelled by an element of $P_{\varepsilon k}$. For simplicity, we will replace a *P*-labelled internal disc by a bold dot as before with the label being placed at the angle corresponding to the distinguished boundary component of the disc.

2.2. *Bicategories*

In this subsection, for the sake of completeness, we recall the notion of bicategories and structures of rigidity and pivotality on them which will be used later. Most of the material in this subsection can be found in any standard textbook on bicategories.

DEFINITION 2.4. A bicategory $\mathscr{B}$ consists of:

- a class $\mathscr{B}_0$ whose elements are called 0-cells;
- for $\alpha, \beta \in \mathscr{B}_0$, a category $\mathscr{B}(\alpha, \beta)$ whose object X shall be called 1-cells and denoted by $\alpha \xrightarrow{X} \beta$, and whose morphism $X_1 \xrightarrow{f} X_2$ from $\alpha \xrightarrow{X_1} \beta$ to $\alpha \xrightarrow{X_2} \beta$ shall be called 2-cells;

- for $\alpha, \beta, \gamma \in \mathscr{B}_0$, there exists a functor $\otimes : \mathscr{B}(\beta, \gamma) \times \mathscr{B}(\alpha, \beta) \to \mathscr{B}(\alpha, \gamma)$;
- Associativity constraint: For each triple $\alpha \xrightarrow{X} \beta, \beta \xrightarrow{Y} \gamma, \gamma \xrightarrow{Z} \delta$ of 1-cells, there exists an isomorphism $(Z \otimes Y) \otimes X \xrightarrow{\alpha_{Z,Y,X}} Z \otimes (Y \otimes X)$ in $\mathrm{Mor}(\mathscr{B}(\alpha, \delta))$;
- Identity object: for each 0-cell α, there exists a 1-cell $\alpha \xrightarrow{1_\alpha} \alpha$ (called the identity on α);
- Unit Constraint: for each 1-cell $\alpha \xrightarrow{X} \beta$, there exist isomorphisms $1_\beta \otimes X \xrightarrow{\lambda_X} X$ and $X \otimes 1_\alpha \xrightarrow{\rho_X} X$ in $\mathrm{Mor}(\mathscr{B}(\alpha, \beta))$

such that $\alpha_{Z,Y,X}, \lambda_X$ and ρ_X are natural in Z, Y and X, and satisfy the pentagon and the triangle axioms (which are exactly similar to the ones in the definition of a tensor category).

On a bicategory $\mathscr{B}$, one can perform the operation op (resp., co) and obtain a new bicategory $\mathscr{B}^{\mathrm{op}}$ (resp., $\mathscr{B}^{\mathrm{co}}$) by setting (i) $\mathscr{B}_0^{\mathrm{op}} = \mathscr{B}_0 = \mathscr{B}_0^{\mathrm{co}}$, (ii) $\mathscr{B}^{\mathrm{op}}(\beta, \alpha) = \mathscr{B}(\alpha, \beta) = (\mathscr{B}^{\mathrm{co}}(\alpha, \beta))^{\mathrm{op}}$ as categories (where *op* of a category is basically reversing the directions of the morphisms).

A bicategory will be called a *strict 2-category* if the associativity and the unit constraints are identities. A C-*linear* bicategory $\mathscr{B}$ is a bicategory such that $\mathscr{B}(\alpha, \beta)$ is a C-linear category for every $\alpha, \beta \in \mathscr{B}_0$ and the functor $\otimes$ is additive.

DEFINITION 2.5. Let $\mathscr{B}, \mathscr{B}'$ be bicategories. A *weak functor* $F = (F, \varphi) : \mathscr{B} \to \mathscr{B}'$ consists of:

- a function $F : \mathscr{B}_0 \to \mathscr{B}_0'$,
- for all $\alpha, \beta \in \mathscr{B}_0$, there exists a functor $F^{\alpha,\beta} : \mathscr{B}(\alpha, \beta) \to \mathscr{B}'(F(\alpha), F(\beta))$ written simply as F,
- for all $\alpha, \beta, \gamma \in \mathscr{B}_0$, there exists a natural isomorphism $\varphi^{\alpha,\beta,\gamma} : \otimes' \circ (F^{\beta,\gamma} \times F^{\alpha,\beta}) \to F^{\alpha,\gamma} \circ \otimes$ written simply as φ (where $\otimes$ and $\otimes'$ are the tensor functors of $\mathscr{B}$ and $\mathscr{B}'$ respectively),
- for all $\alpha \in \mathscr{B}_0$, there exists an invertible (with respect to composition) 2-cell $\varphi_\alpha : 1_{F(\alpha)} \to F(1_\alpha)$,

satisfying commutativity of certain diagrams (consisting of 2-cells) which are analogous to the hexagonal and rectangular diagrams appearing in the definition of a tensor functor.

DEFINITION 2.6. Let $F = (F, \varphi), G = (G, \psi) : \mathscr{B} \to \mathscr{B}'$ be weak functors. A *weak transformation* $\sigma : F \to G$ consists of:

- for all $\alpha \in \mathscr{B}_0$, there exists a 1-cell $\sigma_\alpha \in \mathrm{ob}(\mathscr{B}'(F(\alpha), G(\alpha)))$,

- for all α, $\beta \in \mathscr{B}_0$, there exists a natural transformation $\sigma^{\alpha,\beta} : (\sigma_\beta \otimes' F^{\alpha,\beta}) \to G^{\alpha,\beta} \otimes' \sigma_\alpha$ written simply as σ (where $(\sigma_\beta \otimes' F^{\alpha,\beta})$, $G^{\alpha,\beta} \otimes' \sigma_\alpha : \mathscr{B}(\alpha, \beta) \to \mathscr{B}(F(\alpha), G(\beta))$ are functors defined in the obvious way), satisfying the following: For all $X \in \mathrm{ob}(\mathscr{B}(\beta, \gamma))$, $Y \in \mathrm{ob}(\mathscr{B}(\alpha, \beta))$ where $\alpha, \beta, \gamma \in \mathscr{B}_0$, the following two diagrams commute:

$$\begin{array}{ccccc}
\sigma_\gamma \otimes' F(X) \otimes' F(Y) & \xrightarrow{\sigma_X \otimes' \mathrm{id}_{F(Y)}} & G(X) \otimes' \sigma_\beta \otimes' F(Y) & \xrightarrow{\mathrm{id}_{G(X)} \otimes' \sigma_Y} & G(X) \otimes' G(Y) \otimes' \sigma_\alpha \\
{\scriptstyle \mathrm{id}_{\sigma_\gamma} \otimes' \varphi_{X,Y}} \downarrow & & & & \downarrow {\scriptstyle \psi_{X,Y} \otimes' \mathrm{id}_{\sigma_\alpha}} \\
\sigma_\gamma \otimes' F(X \otimes Y) & & \xrightarrow{\quad\quad\quad \sigma_{X \otimes Y} \quad\quad\quad} & & G(X \otimes Y) \otimes' \sigma_\alpha
\end{array}$$

$$\begin{array}{ccccc}
\sigma_\alpha \otimes' \mathrm{id}_{F(\alpha)} & \xrightarrow{\rho'_{\sigma_\alpha}} & \sigma_\alpha & \xleftarrow{\lambda'_{\sigma_\alpha}} & \mathrm{id}_{G(\alpha)} \otimes' \sigma_\alpha \\
{\scriptstyle \mathrm{id}_{\sigma_\alpha} \otimes' \varphi_\alpha} \downarrow & & & & \downarrow {\scriptstyle \psi_\alpha \otimes' \mathrm{id}_{\sigma_\alpha}} \\
\sigma_\alpha \otimes' F(1_\alpha) & & \xrightarrow{\quad\quad \sigma_{1_\alpha} \quad\quad} & & G(1_\alpha) \otimes' \sigma_\alpha
\end{array}$$

where λ' and ρ' are the left and right unit constraints of $\mathscr{B}'$ respectively.

When such a weak transformation exists, we say that F and G are *weakly isomorphic*. We have the following useful *Coherence Theorem* for bicategories. See [23] for a proof.

THEOREM 2.7. *Every bicategory $\mathscr{B}$ is biequivalent to some strict 2-category $\mathscr{B}'$, i.e., there exist weak functors $F : \mathscr{B} \to \mathscr{B}'$ and $G : \mathscr{B}' \to \mathscr{B}$ such that $\mathrm{id}_{\mathscr{B}}$ (resp., $\mathrm{id}_{\mathscr{B}'}$) is weakly isomorphic to $G \circ F$ (resp., $F \circ G$).*

In view of this, time and again we will suppress (and will not mention about it) the associativity and unit constraints to give a simpler look to expressions involving these constraints.

Rigid Structure on a bicategory.   Let $\alpha \xrightarrow{X} \beta$ be a 1-cell in a bicategory $\mathscr{B}$. A *right dual of X* is a 1-cell $\beta \xrightarrow{X^\#} \alpha$ such that there exist 2-cells $X^\# \otimes X \xrightarrow{e_X} 1_\alpha$ and $1_\beta \xrightarrow{c_X} X \otimes X^\#$ satisfying

$$(\mathrm{id}_X \otimes e_X) \circ (c_X \otimes \mathrm{id}_X) = \mathrm{id}_X \quad \text{and} \quad (e_X \otimes \mathrm{id}_{X^\#}) \circ (\mathrm{id}_{X^\#} \otimes c_X) = \mathrm{id}_{X^\#}.$$

A bicategory is said to be (*right*) *rigid* if right dual exists for every 1-cell. Further, in a rigid bicategory $\mathscr{B}$, one can consider right dual as an invertible weak functor $\# = (\#, s) : \mathscr{B} \to \mathscr{B}^{\mathrm{op\,co}}$ in the following way:

- for each 1-cell X, we fix a triplet $(X^\#, e_X, c_X)$ so that when $X = 1_\alpha$ for a 0-cell α, then $X^\# = 1_\alpha$, $e_X = \lambda_{1_\alpha} (= \rho_{1_\alpha}$, see [20] for a proof) and

$$c_X = \lambda_{1_\alpha}^{-1} = \rho_{1_\alpha}^{-1};$$

- # induces identity map on $\mathscr{B}_0$;

- for each pair of 0-cells α and β, define the contravariant functor # : $\mathscr{B}(\alpha, \beta) \to \mathscr{B}(\beta, \alpha)$ as follows: for each $X, Y \in \mathrm{ob}(\mathscr{B}(\alpha, \beta))$ and 2-cell $f : X \to Y$, set $\#(X) = X^\#$ and $\#(f)$, denoted by $f^\#$, be given by the following composition

$$Y^\# \xrightarrow{\rho_{Y^\#}^{-1}} Y^\# \otimes 1_\alpha \xrightarrow{\mathrm{id}_{Y^\#} \otimes c_X} Y^\# \otimes X \otimes X^\#$$
$$\xrightarrow{\mathrm{id}_{Y^\#} \otimes f \otimes \mathrm{id}_{X^\#}} Y^\# \otimes Y \otimes X^\# \xrightarrow{e_Y \otimes \mathrm{id}_{X^\#}} 1_\beta \otimes X^\# \xrightarrow{\lambda_{X^\#}} X^\#;$$

- for all $\alpha, \beta, \gamma \in \mathscr{B}_0$, the natural isomorphism $s : \otimes \circ (\mathrm{flip}) \circ (\#^{\beta,\gamma} \times \#^{\alpha,\beta}) \to \#^{\alpha,\gamma} \circ \otimes$ is defined by: For $X \in \mathrm{ob}(\mathscr{B}(\alpha, \beta))$, $Y \in \mathrm{ob}(\mathscr{B}(\beta, \gamma))$, the invertible 2-cell $s_{X,Y}$ is given by the composition

$$X^\# \otimes Y^\# \xrightarrow{\mathrm{id}_{(X^\# \otimes Y^\#)} \otimes c_{(Y \otimes X)}} X^\# \otimes Y^\# \otimes (Y \otimes X) \otimes (Y \otimes X)^\#$$
$$\xrightarrow{\mathrm{id}_{X^\#} \otimes e_Y \otimes \mathrm{id}_X \otimes \mathrm{id}_{(Y \otimes X)^\#}} (X^\# \otimes X) \otimes (Y \otimes X)^\# \xrightarrow{e_X \otimes \mathrm{id}_{(Y \otimes X)^\#}} (Y \otimes X)^\#;$$

- for all $\alpha \in \mathscr{B}_0$, the invertible 2-cell $s_\alpha : 1_\alpha \to 1_\alpha$ is given by identity morphism on 1_α.

Note that the above prescription of the dual functor $(\#, s)$ carries forward almost verbatim to another weak functor $(\tilde{\#}, \tilde{s}) : \mathscr{B}^{\mathrm{op\,co}} \to (\mathscr{B}^{\mathrm{op\,co}})^{\mathrm{op\,co}} = \mathscr{B}$. This allows us to consider the composition $(\tilde{\#}, \tilde{s}) \circ (\#, s) : \mathscr{B} \to \mathscr{B}$. This is again a weak functor and we abuse notation to denote it by $(\#\#, t)$ and call it the *bi-dual functor*.

DEFINITION 2.8. A bicategory $\mathscr{B}$ is said to be pivotal if $\mathscr{B}$ is (right) rigid and there exists a weak transformation $a : \mathrm{id}_{\mathscr{B}} \to \#\#$ such that $a_\varepsilon = 1_\varepsilon$ for all $\varepsilon \in \mathscr{B}_0$.

We now recall some useful standard properties of a pivotal bicategory.

PROPOSITION 2.9. *Let $\mathscr{B}$ be a pivotal bicategory with pivotality given by the weak transformation $a : \mathrm{id}_{\mathscr{B}} \to \#\#$. Then, for all 1-cells $\alpha \xrightarrow{Y} \beta$ and $\beta \xrightarrow{X} \gamma$, we have*

(1) $a_{X \otimes Y} = t_{X,Y} \circ (a_X \otimes a_Y)$, *and*
(2) $a_X^\# = a_{X^\#}^{-1}$.

Let $\alpha_i \xrightarrow{X_i} \alpha_{i+1}$, $1 \le i \le n$ be 1-cells in $\mathscr{B}$. Suppressing associativity, consider the morphism $t_{X_1,\dots,X_n} := (t_{X_1 \otimes \cdots \otimes X_{n-1}, X_n}) \circ \cdots \circ (t_{X_1 \otimes X_2, X_3} \otimes$

$\mathrm{id}_{X_4^{\#\#} \otimes \cdots \otimes X_n^{\#\#}}) \circ (t_{X_1, X_2} \otimes \mathrm{id}_{X_3^{\#\#} \otimes \cdots \otimes X_n^{\#\#}}) \in \mathrm{Mor}(X_1^{\#\#} \otimes \cdots \otimes X_n^{\#\#}, (X_1 \otimes \cdots \otimes X_n)^{\#\#})$. Then, a simple iterative application of Proposition 2.9 (1) gives:

COROLLARY 2.10. $a_{X_1 \otimes \cdots \otimes X_n} = t_{X_1, \ldots, X_n} \circ (a_{X_1} \otimes \cdots \otimes a_{X_n})$ for all 1-cells $\alpha_i \xrightarrow{X_i} \alpha_{i+1}$, $1 \leq i \leq n$.

2.3. Bicategory of bifinite bimodules

In this subsection, for the sake of completeness, we first recall certain standard facts about subfactors and modules over II_1-factors (which can be found, for instance, in [1], [7], [9], [12], [18], [24], [25], [28], [29], [31]). And while doing so, we also illustrate how the collection of all bimodules inherit a canonical structure of a pivotal bicategory. We make a stand-in assumption that all Hilbert spaces are separable and their inner products are linear in second and conjugate-linear in first variable.

The following proposition gives a characterization of the basic construction.

PROPOSITION 2.11. *Let $A \subset B \subset C$ be unital inclusions of II_1-factors and suppose there is a projection $e \in \mathscr{P}(C)$ satisfying*

(1) $exe = E_A(x)e$ *for all $x \in B$, and*

(2) $Be = Ce$.

Then the above tower is an instance of basic construction.

Given a finite index subfactor $N \subset M$, a *left* (resp., *right*) *basis of M over N* is a finite subset B of M satisfying any of the following equivalent conditions:

(i) $x = \sum_{b \in B} E_N(xb^*)b$ (resp., $x = \sum_{b \in B} bE_N(b^*x)$) for all $x \in M$,

(ii) $x = \sum_{b \in B} b^*E_N(bx)$ (resp., $x = \sum_{b \in B} E_N(xb)b^*$) for all $x \in M$,

(iii) $\sum_{b \in B} b^*eb = 1$ (resp., $\sum_{b \in B} beb^* = 1$) where e is a Jones projection in a basic construction of $N \subset M$.

Proof of existence of such basis can be found in [24].

Let A and B be II_1-factors and ${}_A\mathscr{H}$ (resp., $\mathscr{K}_B$) be a left A-module (resp., right B-module) such that $\dim({}_A\mathscr{H}) < \infty$ (resp., $\dim(\mathscr{K}_B) < \infty$), equivalently, $A' := {}_A\mathscr{L}(\mathscr{H})$ (resp., $B' := \mathscr{L}_B(\mathscr{K})$) is a II_1-factor. Consider the set of bounded vectors $({}_A\mathscr{H})^o := \{\xi \in \mathscr{H} : \text{there exists } k > 0 \text{ s.t. } \|a\xi\|^2 \leq k \, \mathrm{tr}_A(aa^*) \text{ for all } a \in A\}$ (resp., $(\mathscr{K}_B)^o := \{\eta \in \mathscr{K} : \text{there exists } k > 0 \text{ s.t. } \|\eta b\|^2 \leq k \, \mathrm{tr}_B(bb^*) \text{ for all } b \in B\}$) which forms a dense subspace of $\mathscr{H}$ (resp., $\mathscr{K}$) and is closed under the action of A' (resp., B'). Using the Radon-Nikodym derivative with respect to the faithful trace, one can obtain the A-valued (resp., B-valued) inner product ${}_A\langle \cdot, \cdot \rangle : ({}_A\mathscr{H})^o \times ({}_A\mathscr{H})^o \to A$

(resp., $\langle \cdot, \cdot \rangle_B : (\mathcal{K}_B)^o \times (\mathcal{K}_B)^o \to B$) defined by the equation $\mathrm{tr}_A(a_A\langle \xi, \xi' \rangle) = \langle \xi, a\xi' \rangle$ (resp., $\mathrm{tr}_B(\langle \eta, \eta' \rangle_B b) = \langle \eta, \eta'b \rangle$) for all $\xi, \xi' \in (_A\mathcal{H})^o, a \in A$ (resp., $\eta, \eta' \in (\mathcal{K}_B)^o, b \in B$). It is easy to check that the inner product has the following properties:

(1) $_A\langle \xi, \xi \rangle$ (resp., $\langle \eta, \eta \rangle_B) \geq 0$,

(2) $_A\langle \xi, \xi' \rangle^* = {_A\langle \xi', \xi \rangle}$ (resp., $\langle \eta, \eta' \rangle_B^* = \langle \eta', \eta \rangle_B$),

(3) $_A\langle \xi, a\xi' \rangle = a_A\langle \xi, \xi' \rangle$ and $_A\langle a\xi, \xi' \rangle = {_A\langle \xi, \xi' \rangle}a^*$ (resp., $\langle \eta, \eta'b \rangle_B = \langle \eta, \eta' \rangle_B b$ and $\langle \eta b, \eta' \rangle_B = b^*\langle \eta, \eta' \rangle_B$),

(4) $_A\langle \xi, x\xi' \rangle = {_A\langle x^*\xi, \xi' \rangle}$ (resp., $\langle \eta, y\eta' \rangle = \langle y^*\eta, \eta' \rangle_B$)

for all $a \in A$, $x \in A'$, $\xi, \xi' \in (_A\mathcal{H})^o$ (resp., $b \in B$, $y \in B'$, $\eta, \eta' \in (\mathcal{K}_B)^o$). Also, given a finite index subfactor C of A (resp., B), one can use a basis for $C \subset A$ (resp., $C \subset B$) to obtain $(_A\mathcal{H})^o = (_C\mathcal{H})^o$ (resp., $(\mathcal{K}_B)^o = (\mathcal{K}_C)^o$) where the C-valued inner product is given by the A-valued (resp., B-valued) inner product composed with the trace preserving conditional expectation onto C. Further, there exists a finite subset $\{\xi_i\}_i$ (resp., $\{\eta_j\}_j$) of $(_A\mathcal{H})^o$ (resp., $(\mathcal{K}_B)^o$) satisfying $\mathrm{id}_{(_A\mathcal{H})^o} = \sum_i {_A\langle \xi_i, \cdot \rangle}\xi_i$ (resp., $\mathrm{id}_{(\mathcal{K}_B)^o} = \sum_j \eta_j\langle \eta_j, \cdot \rangle_B$); such a subset is called *basis* for the module. Such a basis also satisfies the following conditions which are completely straight forward to verify.

PROPOSITION 2.12. (i) $\mathrm{tr}_{A'}(x) = [\dim(_A\mathcal{H})]^{-1} \sum_i \langle \xi_i, x\xi_i \rangle$ (resp., $\mathrm{tr}_{B'}(y) = [\dim(\mathcal{K}_B)]^{-1} \sum_j \langle \eta_j, \eta_j y \rangle)$ for all $x \in A'$ (resp., $y \in B'$) which implies $(_A\mathcal{H})^o = (_{A'}\mathcal{H})^o$ (resp., $(\mathcal{K}_B)^o = (_{B'}\mathcal{H})^o$),
(ii) $\sum_i {_{A'}\langle \xi_i, \xi_i \rangle} = \dim(_A\mathcal{H})1_{A'}$ (resp., $\sum_j {_{B'}\langle \eta_j, \eta_j \rangle} = \dim(\mathcal{K}_B)1_{B'}$).

The dual Hilbert space or the contragredient $\overline{\mathcal{H}} := \{\overline{\xi} : \xi \in \mathcal{H}\}$ (resp., $\overline{\mathcal{K}} := \{\overline{\eta} : \eta \in \mathcal{K}\}$) with bar being a conjugate linear unitary, can be equipped with a right A-module (resp., left B-module) structure given by $\overline{\xi}a = \overline{a^*\xi}$ for $a \in A, \xi \in \mathcal{H}$ (resp., $b\overline{\eta} = \overline{\eta b^*}$ for $b \in B, \eta \in \mathcal{K}$). Note that (i) $(\overline{\mathcal{H}}_A)^o = \overline{(_A\mathcal{H})^o}$ (resp., $(_B\overline{\mathcal{K}})^o = \overline{(\mathcal{K}_B)^o}$), (ii) $\langle \overline{\xi}, \overline{\xi'} \rangle_A = {_A\langle \xi', \xi \rangle}$ for $\xi, \xi' \in (_A\mathcal{H})^o$ (resp., $_B\langle \overline{\eta}, \overline{\eta'} \rangle = \langle \eta', \eta \rangle_B$ for $\eta, \eta' \in (\mathcal{K}_B)^o$), (iii) $\{\overline{\xi_i}\}_i$ (resp., $\{\overline{\eta_j}\}_j$) is a basis for $\overline{\mathcal{H}}_A$ (resp., $_B\overline{\mathcal{K}}$) and (iv) $\dim(_A\mathcal{H}) = \dim(\overline{\mathcal{H}}_A)$ (resp., $\dim(\mathcal{K}_B) = \dim(_B\overline{\mathcal{K}})$).

Next, we briefly recall few aspects of bimodules over II_1-factors A and B. Let $\mathcal{H}$ be an A-B-bimodule. If $\dim(\mathcal{H}_B) < \infty$, then the Jones index of the subfactor $A \subset B'$ turns out to be $[B' : A] = \dim(_A\mathcal{H}) \dim(\mathcal{H}_B) =: \mathrm{index}(_A\mathcal{H}_B)$. A bimodule $_A\mathcal{H}_B$ is called *bifinite* if $\mathrm{index}(_A\mathcal{H}_B) < \infty$, and a bifinite bimodule $_A\mathcal{H}_B$ is called *extremal* if the canonical traces of the II_1-factors A' and B' coincide on the intertwiner space $_A\mathcal{L}_B(\mathcal{H})$ (which has finite complex dimension due to the finiteness of the index). Note that if $\mathrm{index}(_A\mathcal{H}_B) = 1$, then $_A\mathcal{H}_B$ is an

irreducible A-B-bimodule. Also, if index$(_A\mathcal{H}_B) < \infty$, then $(_A\mathcal{H})^o = (\mathcal{H}_B)^o$; we will write $\mathcal{H}^o$ for this space of bounded vectors.

Given two bifinite bimodules $_A\mathcal{H}_B$ and $_B\mathcal{K}_C$, one can consider the tensor product $\mathcal{H} \otimes_B \mathcal{K}$ defined as the completion of the space

$$\mathcal{H}^o \otimes_B \mathcal{K}^o := \frac{\mathcal{H}^o \otimes_{\mathrm{alg}} \mathcal{K}^o}{\mathrm{span}\{\xi a \otimes \eta - \xi \otimes a\eta \mid a \in A, \xi \in \mathcal{H}^o, \eta \in \mathcal{K}^o\}}$$

with respect to the inner product $\langle \xi \otimes_B \eta, \xi' \otimes_B \eta' \rangle = \langle \xi, \xi'_B\langle \eta, \eta' \rangle \rangle = \langle \eta, \langle \xi, \xi' \rangle_B \eta' \rangle$ for $\xi, \xi' \in \mathcal{H}^o, \eta, \eta' \in \mathcal{K}^o$, which is equipped with the obvious A-C bimodule structure. The following is a list of very useful properties of this tensor product.

(1) $(\mathcal{H} \otimes_B \mathcal{K})^o = \mathcal{H}^o \otimes_B \mathcal{K}^o$.

(2) The A-valued (resp., C-valued) inner product is given by $_A\langle \xi \otimes_B \eta, \xi' \otimes_B \eta' \rangle = {}_A\langle \xi, \xi'_B\langle \eta, \eta' \rangle \rangle$ (resp., $\langle \xi \otimes_B \eta, \xi' \otimes_B \eta' \rangle_C = \langle \eta, \langle \xi, \xi' \rangle_B \eta' \rangle_C$) for $\xi, \xi' \in \mathcal{H}^o, \eta, \eta' \in \mathcal{K}^o$.

(3) If $\{\xi_i\}_i$ and $\{\eta_j\}_j$ are basis for $_A\mathcal{H}$ and $_B\mathcal{K}$ (resp., $\mathcal{H}_B$ and $\mathcal{K}_C$) respectively, then $\{\xi_i \otimes_B \eta_j\}_{i,j}$ forms a basis for $_A\mathcal{H} \otimes_B \mathcal{K}$ (resp., $\mathcal{H} \otimes_B \mathcal{K}_C$).

(4) The left dimension, the right dimension and the index of the bifinite bimodules are multiplicative with respect to this tensor product.

(5) The map $\mathcal{L}_B(\mathcal{H}) \ni x \mapsto x \otimes_B \mathrm{id}_{\mathcal{K}} \in \mathcal{L}_C(\mathcal{H} \otimes_B \mathcal{K})$ (resp., $_B\mathcal{L}(\mathcal{K}) \ni y \mapsto id_{\mathcal{H}} \otimes_B y \in {}_A\mathcal{L}(\mathcal{H} \otimes_B \mathcal{K})$) is an inclusion of unital $*$-algebras.

Let $\mathcal{H}$ be an A-B-bimodule with $\dim(_A\mathcal{H}) < \infty$ (resp., $\dim(\mathcal{H}_B) < \infty$) and $\{\xi_i\}_i$ (resp., $\{\eta_j\}_j$) be a basis for $_A\mathcal{H}$ (resp., $\mathcal{H}_B$). Then, it is easy to see that the bounded vector $\sum_j \eta_j \otimes_B \overline{\eta_j}$ (resp., $\sum_i \overline{\xi_i} \otimes_A \xi_i$) is independent of the basis and is A-A-central, that is, $a(\sum_j \eta_j \otimes_B \overline{\eta_j}) = (\sum_j \eta_j \otimes_B \overline{\eta_j})a$ for all $a \in A$ (resp., B-B-central).

LEMMA 2.13. *Let $\mathcal{K}$ be a right B-module with $\dim(\mathcal{K}_B) < \infty$ and $\mathcal{H}$ be a bifinite A-B bimodule for II_1 factors A and B. Then, the inclusion of II_1 factors*

$$\mathcal{L}_B(\mathcal{K}) \hookrightarrow \mathcal{L}_A(\mathcal{K} \otimes_B \overline{\mathcal{H}}) \hookrightarrow \mathcal{L}_B(\mathcal{K} \otimes_B \overline{\mathcal{H}} \otimes_A \mathcal{H})$$

$$x \mapsto x \otimes_B \mathrm{id}_{\overline{\mathcal{H}}}, \quad y \mapsto y \otimes_A \mathrm{id}_{\mathcal{H}}$$

is an instance of basic construction with Jones projection e given (on bounded vectors) by

$$\mathcal{K} \otimes_B \overline{\mathcal{H}} \otimes_A \mathcal{H} \ni \xi \otimes_B \bar{\eta} \otimes_A \zeta$$

$$\overset{e}{\longmapsto} \frac{1}{\dim_A \mathcal{H}} \sum_i (\xi \otimes_B \bar{\bar{\xi_i}} \otimes_A \xi_i)\langle \eta, \zeta \rangle_B \in \mathcal{K} \otimes_B \overline{\mathcal{H}} \otimes_A \mathcal{H},$$

where $\{\xi_i\}_i$ is a basis for $_A\mathcal{H}$.

PROOF. Set $N = \mathscr{L}_B(\mathscr{K})$, $M = \mathscr{L}_A(\mathscr{K} \otimes_B \overline{\mathscr{H}})$ and $M_1 = \mathscr{L}_B(\mathscr{K} \otimes_B \overline{\mathscr{H}} \otimes_A \mathscr{H})$. Let $\{\eta_j\}_j$ (resp., $\{\sigma_k\}_k$) be a basis for $\mathscr{H}_B$ (resp., $\mathscr{K}_B$). Then, using Proposition 2.12, it is completely routine to check that the map $E_N : M \to N$ given by $E_N(x)(\xi) = [\dim(_A\mathscr{H})]^{-1} \sum_{i,i',k} \sigma_k \langle \xi_{i'}, \langle \sigma_k \otimes_B \bar{\xi}_{i'}, x(\xi \otimes_B \bar{\xi}_i) \rangle_A \xi_i \rangle_B$ for all $\xi \in \mathscr{K}^o$, $x \in M$, is the unique tr_M preserving conditional expectation from M onto N. Further, it can also be readily shown that $e \in \mathscr{P}(N' \cap M_1)$ and $e(x \otimes_A \text{id}_{\mathscr{H}})e = e(E_N(x) \otimes_B \text{id}_{\overline{\mathscr{H} \otimes_A \mathscr{H}}})$ for all $x \in M$. In view of Proposition 2.11, it just remains to show that $M_1 e = Me$. Let $x_1 \in M_1$ and consider the map $x : \mathscr{K} \otimes_B \overline{\mathscr{H}} \to \mathscr{K} \otimes_B \overline{\mathscr{H}}$ given (on bounded vectors) by $x(\xi \otimes_B \bar{\eta}) = \sigma_k \otimes_B \bar{\xi}_{i'} {}_A \langle \eta, \eta_j \langle \sigma_k \otimes_B \bar{\xi}_{i'} \otimes_A \eta_j, x_1(\xi \otimes_B \bar{\xi}_i \otimes_A \xi_i) \rangle_B$ for all $\xi \in (\mathscr{K}_B)^o$, $\eta \in (\mathscr{H}_B)^o$. Clearly, $x \in M$, and it involves nothing more than straightforward verification to show that $(x \otimes_A \text{id}_{\mathscr{H}})e = x_1 e$.

We denote the bicategory of bifinite bimodules with $\mathscr{B}$ whose 0-cells are II_1-factors; for II_1-factors A and B, the objects of the category $\mathscr{B}(B, A)$ are bifinite A-B bimodules and morphisms or 2-cells are A-B linear maps between such bimodules (which are automatically bounded). The tensor functor is given by the usual relative tensor product of bimodules and for each II_1-factor A the identity object in $\mathscr{B}(A, A)$ is the canonical A-A-bimodule $L^2(A)$. There is a natural associativity constraint for relative tensor product of bimodules. Further, for an A-B-bimodule $\mathscr{H}$, the unit constraints are given by the canonical isomorphisms $L^2(A) \otimes_A \mathscr{H} \overset{A\text{-}B}{\cong} \mathscr{H}$ and $\mathscr{H} \otimes_B L^2(B) \overset{A\text{-}B}{\cong} \mathscr{H}$. Thus, $\mathscr{B}$ has a natural bicategory structure. For the (right) rigid structure on $\mathscr{B}$, for each A-B-bimodule $\mathscr{H}$, we set $(_A\mathscr{H}_B)^{\#} = {}_B\overline{\mathscr{H}}_A$ and define the evaluation and coevaluation maps $e_{\mathscr{H}} \in {}_B\mathscr{L}(\overline{\mathscr{H}} \otimes_A \mathscr{H}, L^2(B))$ and $c_{\mathscr{H}} \in {}_A\mathscr{L}_A(L^2(A), \mathscr{H} \otimes_B \overline{\mathscr{H}})$ respectively, (on bounded vectors) by

$$e_{\mathscr{H}}(\bar{\xi} \otimes_A \eta) = \langle \xi, \eta \rangle_B \quad \text{and} \quad c_{\mathscr{H}}(\hat{a}) = \sum_i a(\eta_j \otimes_B \bar{\eta}_j)$$
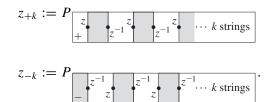
for all $\xi, \eta \in \mathscr{H}^o$, $a \in A$, $b \in B$, where $\{\eta_j\}$ is a basis for the right B-module $\mathscr{H}_B$. Thus, $\mathscr{B}$ indeed inherits a canonical rigid structure. Finally, the canonical isomorphism $_A\mathscr{H}_B \overset{A\text{-}B}{\cong} {}_A\overline{\overline{\mathscr{H}}}_B$ for any bifinite A-B-bimodule $_A\mathscr{H}_B$, equips $\mathscr{B}$ with a pivotal structure. Note that, for $\theta \in {}_A\mathscr{L}_B(\mathscr{H}, \mathscr{K})$, it can be easily shown that $\theta^{\#}(\bar{\xi}) = \overline{\theta^*(\xi)}$ and, hence, $\theta^{\#\#}(\bar{\bar{\xi}}) = \overline{\overline{\theta(\xi)}}$ for all $\xi \in \mathscr{H}^o$, where θ^* is the usual adjoint of the intertwiner θ.

3. Perturbations of planar algebras

In this section, we define perturbation of a planar algebra to obtain a new one which has the same filtered algebra structure but the action of Jones projections and conditional expectation tangles differ. As we will see, this will turn out to be crucial in the following sections. We must mention here that such objects already appeared in the work of Michael Burns [6] while extending JonesÆ theorem (of associating a spherical planar algebra to an extremal subfactor) in the non-extremal case; however, for us, these arose in a purely different context, namely, while detecting the effect of different pivotal structures on the planar algebra associated to a 1-cell in a strict 2-category (as in [8]).

DEFINITION 3.1. Let P be a planar algebra. An invertible element $z \in P_{+1}$ is said to be a weight of P if $z_{\varepsilon k} \in \mathscr{Z}(P_{\varepsilon k})$ for all $\varepsilon k \in \mathrm{Col}$, where

$$z_{+k} := P \quad \boxed{\begin{array}{c} z \\ + \end{array} \bullet \begin{array}{c} z \\ z^{-1} \end{array} \bullet \begin{array}{c} z \\ z^{-1} \end{array} \bullet \cdots k \text{ strings}}$$

and

$$z_{-k} := P \quad \boxed{\begin{array}{c} z^{-1} \\ - \quad z \end{array} \bullet \begin{array}{c} z^{-1} \\ z \end{array} \bullet \begin{array}{c} z^{-1} \\ \end{array} \bullet \cdots k \text{ strings}} \, .$$

Given a weight z of a planar algebra P and an invertible decomposition $z = ab$ for a, b invertible in P_{+1}, we now construct a new planar algebra $P^{(a,b)}$ as follows:

(i) *Vector spaces*: $P^{(a,b)}_{\varepsilon k} := P_{\varepsilon k}$ for all $\varepsilon k \in \mathrm{Col}$.

(ii) *Actions of tangles*: Let T be a tangle and $\hat{T}$ be a standard form representative (see [8, § 4]) of the isotopy class of T. We replace each local maximum and minimum appearing in $\hat{T}$ as in Figure 3.1 and call the resulting semi-labelled tangular diagram $\hat{T}^{(a,b)}$.
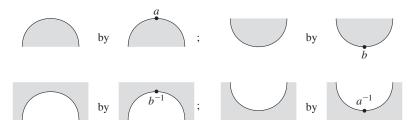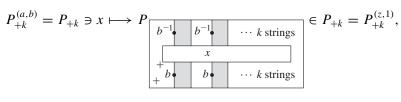

FIGURE 3.1. Perturbing a planar algebra

Define $P^{(a,b)}_T := P_{\hat{T}^{(a,b)}}$. The immediate thing to check is the well-definedness of $P^{(a,b)}_T$. Note that the above prescription is invariant under the sliding, wiggling and 360°-rotation moves, applying a finite sequence of which takes one
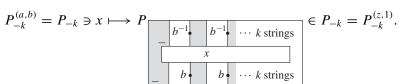
standard form representative to another. Hence, $P_T^{(a,b)}$ is well defined and $P^{(a,b)}$ is a planar algebra. As mentioned before, note that $P^{(a,b)}$ has same filtered algebra structure as P whereas the action of Jones projection tangles and conditional expectation tangles differ.

We will refer $P^{(a,b)}$ as the *perturbation of P by the decomposition $z = ab$ of the weight z*.

REMARK 3.2. For an invertible decomposition $z = ab$ of a weight z of a planar algebra P and any $\lambda \in \mathsf{C} \setminus \{0\}$, $P^{(a,b)} = P^{(\lambda a, \lambda^{-1} b)}$. Further, the planar algebras $P^{(a,b)}$, $P^{(b,a)}$, $P^{(z,1)}$, and $P^{(1,z)}$ are all isomorphic. Hence, up to isomorphism, the perturbation of P only depends upon the weight z.

To see the first part, observe that the number of local maxima is the same as the number of local minima in a standard form representative of a tangle, which results in cancellation of the scalars appearing in tangle maps due to λ in the latter perturbation. In the second part, for instance, the isomorphism $P^{(a,b)} \cong P^{(z,1)}$ is obtained by the maps

$$P_{+k}^{(a,b)} = P_{+k} \ni x \longmapsto P_{\boxed{}} \in P_{+k} = P_{+k}^{(z,1)},$$

$$P_{-k}^{(a,b)} = P_{-k} \ni x \longmapsto P_{\boxed{}} \in P_{-k} = P_{-k}^{(z,1)}.$$

It is straight forward to verify the equivariance of this isomorphism with the actions of the generating tangles in Figure 2.1 and hence all tangles.

A trivial example of a weight of a planar algebra P is a non-zero scalar $\lambda \in \mathsf{C}$. By the above remark, $P^{(\lambda,\mu)} = P^{(\lambda\mu,1)} = P^{(1,\lambda\mu)} = P^{(\mu,\lambda)}$ for all non-zero scalars λ and μ. We will usually refer to such perturbations as *scalar perturbations*.

A perturbation of a planar algebra with modulus need not have modulus (except for perturbations by scalar weights). However, if the planar algebra is connected then so are its perturbations, but the moduli of the perturbations might vary. For instance, for a planar algebra P with modulus (δ_-, δ_+), the scalar perturbation $P^{(\lambda,1)}$ has modulus $(\lambda^{-1}\delta_-, \lambda\delta_+)$.

DEFINITION 3.3. The normalized planar algebra associated to a planar algebra P with modulus (δ_-, δ_+) is its scalar perturbation by the weight $\sqrt{\frac{\delta_-}{\delta_+}}$.

Note that the normalization of P is a unimodular planar algebra. Although scalar perturbations change the modulus, the index however remains the same; in the last section, we will come across an example of a perturbation class whose normalized planar algebras realize all indices greater than or equal to 4.

DEFINITION 3.4. A connected planar algebra P is said to be spherical if the actions of 0-tangles in its normalization are invariant under spherical isotopy.

Note that this property is equivalent to demanding that the normalized left and right picture traces on P_{+1} are identical. The relevance of the above definition will become clear in the section on bimodule planar algebras, where we establish a correspondence between sphericality and extremality. We must also point out that a non-unimodular planar algebra could be spherical according to the above definition which is not allowed in Jones' original definition in [13].

In general, a perturbation of a $*$-planar algebra need not be a $*$-planar algebra. However, for certain specific weights, the perturbations also turn out to be $*$-planar algebras. For instance, if P is a $*$- (resp., positive) planar algebra, it is routine to verify that a perturbation of the type $P^{(a, \lambda a^*)}$ for non-zero real (resp., positive) scalar λ becomes a $*$- (resp., positive) planar algebra with $*$-structure coming from the original one.

4. Weights and Pivotality

In this section, we mention how one can canonically associate a pivotal C-linear strict 2-category to a planar algebra; conversely, from [8], we recall how to associate a planar algebra to a 1-cell in a pivotal C-linear strict 2-category. We then establish a relation between weights and perturbations of planar algebras and pivotal structures on bicategories.

4.1. Planar algebras to bicategories

Let P be a planar algebra. From P, we first describe a C-linear strict 2-category $\mathscr{B}$ and see that it inherits canonical rigid and pivotal structures. Set $\mathscr{B}_0 = \{+, -\}$; for $\varepsilon, \eta \in \mathscr{B}_0$, set $\mathrm{ob}(\mathscr{B}(\varepsilon, \eta)) = 2\mathsf{N}_0 + \delta_{\varepsilon \neq \eta}$. To avoid confusion, we will write an object $k \in \mathrm{ob}(\mathscr{B}(\varepsilon, \eta))$ as εk (whence $\eta = (-)^k \varepsilon$). For two objects $\varepsilon k, \varepsilon l$, set $\mathrm{Mor}(\varepsilon k, \varepsilon l) = P_{(-)^k \varepsilon (\frac{k+l}{2})} = P_{(-)^l \varepsilon (\frac{k+l}{2})}$. Composition of morphisms be given by the bilinear map

$$\mathrm{Mor}(\varepsilon l, \varepsilon m) \times \mathrm{Mor}(\varepsilon k, \varepsilon l) \ni (x, y) \mapsto x \circ y := P_{\boxed{\begin{smallmatrix} & x \mid m \\ (-)^k \varepsilon & l \\ & y \mid k \end{smallmatrix}}} \in \mathrm{Mor}(\varepsilon k, \varepsilon m).$$

For each 1-cell εk, the identity morphism is given by $P_{1_{\varepsilon k}}$. The tensor functor $\otimes : \mathscr{B}(\eta, \sigma) \times \mathscr{B}(\varepsilon, \eta) \to \mathscr{B}(\varepsilon, \sigma)$ is defined by $\eta l \otimes \varepsilon k := \varepsilon(k + l)$ and

$$\mathrm{Mor}(\eta m, \eta n) \times \mathrm{Mor}(\varepsilon k, \varepsilon l) \ni (x, y)$$

$$\mapsto x \otimes y := P_{\boxed{\begin{smallmatrix} & n & & l \\ x\bullet & & y\bullet & \\ \sigma & m & & k \end{smallmatrix}}} \in \mathrm{Mor}(\varepsilon(k + m), \varepsilon(l + n)).$$

For each 0-cell ε, set the identity object $1_\varepsilon = \varepsilon 0 \in \mathrm{ob}(\mathscr{B}(\varepsilon, \varepsilon))$. With the above structure, it is easily seen that $\mathscr{B}$ is a strict 2-category. We now describe a (right) rigid structure # on $\mathscr{B}$ as follows:

0-cells: $\varepsilon^{\#} := \varepsilon$.

1-cells: $(\varepsilon k)^{\#} = (-)^k \varepsilon k$.

The evaluation map $e_{\varepsilon k} : (\varepsilon k)^{\#} \otimes \varepsilon k = \varepsilon 2k \to \varepsilon 0$ (resp., coevaluation map $c_{\varepsilon k} : (-)^k \varepsilon 0 \to \varepsilon k \otimes (\varepsilon k)^{\#} = (-)^k \varepsilon 2k)$ will be given by $e_{\varepsilon k} := P_{\boxed{\begin{smallmatrix} \varepsilon & \\ & k \end{smallmatrix}}}$

(resp., $c_{\varepsilon k} := P_{\boxed{\begin{smallmatrix} & k \\ (-)^k \varepsilon & \end{smallmatrix}}}$). Thus, we obtain a weak functor $\# : \mathscr{B} \to \mathscr{B}^{\mathrm{op\,co}}$

which yields $x^{\#} = P_{\boxed{\begin{smallmatrix} k \\ \varepsilon\bullet x \\ l \end{smallmatrix}}}$ for all $x \in \mathrm{Mor}(\varepsilon k, \varepsilon l)$, and satisfies $\# \circ \# = \mathrm{id}_{\mathscr{B}}$.

This strict 2-category $\mathscr{B}$ also inherits a canonical pivotal structure $a : \mathrm{id}_{\mathscr{B}} \to \# \circ \# = \mathrm{id}_{\mathscr{B}}$, which is identity on $\mathscr{B}_0$, the objects $a_\varepsilon := \varepsilon 0 \in \mathrm{ob}(\mathscr{B}(\varepsilon, \varepsilon))$ and the morphisms $a_{\varepsilon k} := \mathrm{id}_{\varepsilon k} : \varepsilon k \to \varepsilon k$. In fact, we have a correspondence between the pivotal structures on $\mathscr{B}$ and the weights of P.

PROPOSITION 4.1. *There is a one-to-one correspondence between weights of a planar algebra P and pivotal structures on the strict 2-category $\mathscr{B}$ associated to P as above.*

PROOF. Given a weight z of P, define $a_{\varepsilon k} := z_{(-)^k \varepsilon k} \in \mathrm{Mor}(\varepsilon k, \varepsilon k)$. For naturality of a, consider $f \in \mathrm{Mor}(\varepsilon k, \varepsilon l)$. Note that $a_{\varepsilon l}^{-1} \circ f \circ a_{\varepsilon k} = z_{(-)^k \varepsilon \frac{k+l}{2}}^{-1} \circ f \circ z_{(-)^k \varepsilon \frac{k+l}{2}} = f$ where the first equality readily follows from pictures and the second one holds because z's are central. Also, the tensor condition $a_{(-)^k \varepsilon l} \otimes a_{\varepsilon k} = a_{\varepsilon(k+l)}$ is easy to verify. Setting $a_\varepsilon = 1_\varepsilon$, we have a weak transformation $a : \mathrm{id}_{\mathscr{B}} \to \# \circ \# = \mathrm{id}_{\mathscr{B}}$. This proves that a gives a pivotal structure.

Conversely, if $a : \mathrm{id}_{\mathscr{B}} \to \# \circ \# = \mathrm{id}_{\mathscr{B}}$ is a pivotal structure, set $z = a_{-1} \in \mathrm{Mor}(-1, -1) = P_{+1}$. Then, the tensor property of a, namely $a_{(-)^k \varepsilon l} \otimes a_{\varepsilon k} = a_{\varepsilon(k+l)}$, along with naturality of a and Proposition 2.9 (2) implies that $z_{\varepsilon k} \in \mathscr{Z}(P_{\varepsilon k})$ and hence z is a weight of P.

4.2. Bicategories to planar algebras

We first briefly recall from [8] the planar algebra associated to any 1-cell in a pivotal C-linear strict 2-category $\mathscr{B}$. Let $X \in \mathrm{ob}(\mathscr{B}(-, +))$ for $\{+, -\} \subset \mathscr{B}_0$, $\# : \mathscr{B} \to \mathscr{B}^{\mathrm{op\, co}}$ be a right rigid structure with respect to evaluation and coevaluation e and c respectively, and $a : \mathrm{id}_B \to \# \circ \#$ be a pivotal structure. The ingredients of the planar algebra P associated to $(\mathscr{B}, \#, a, X)$ are as follows:

Vectors spaces: $P_{\varepsilon k} := \mathrm{End}(X_{\varepsilon k})$, where

$$X_{\varepsilon k} := \begin{cases} X \otimes X^\# \otimes X \otimes X^\# \otimes X \otimes \cdots k \text{ many tensor factors if } \varepsilon = +, \\ X^\# \otimes X \otimes X^\# \otimes X \otimes X^\# \otimes \cdots k \text{ many tensor factors if } \varepsilon = -, \end{cases}$$

if $k \geq 1$ and $X_{\varepsilon 0} := 1_\varepsilon \in \mathrm{ob}(\mathscr{B}(\varepsilon, \varepsilon))$.

Actions of tangles: Given a tangle $T : (\varepsilon_1 k_1, \ldots \varepsilon_b k_b) \to \varepsilon_0 k_0$ and elements $x_i \in P_{\varepsilon_i k_i}$, $1 \leq i \leq b$, the action $P_T(x_1, \ldots, x_b)$ is given by (i) choosing a standard form representative T_1 in the isotopy class of T labelled with x_i in the i-th internal box, (ii) cutting T_1 into horizontal stripes so that each stripe should have at most one local maximum, minimum or (labelled) internal rectangle, (iii) assigning a 2-cell to each horizontal stripe as prescribed in Figure 4.1 and (iv) successively composing these 2-cells with the one coming from the bottom stripe being the rightmost in the composition.

$$\rightsquigarrow \mathrm{id}_{X_{(-)^k k}} \otimes (e_{X^\#} \circ (a_X \otimes id_{X^\#})) \otimes \mathrm{id}_{X_{+l}} \in \mathrm{Mor}(X_{(-)^k (k+l+2)}, X_{(-)^k (k+l)})$$

$$\rightsquigarrow \mathrm{id}_{X_{(-)^{k+1} k}} \otimes e_X \otimes \mathrm{id}_{X_{-l}} \in \mathrm{Mor}(X_{(-)^{k+1}(k+l+2)}, X_{(-)^{k+1}(k+l)})$$

$$\rightsquigarrow \mathrm{id}_{X_{(-)^k k}} \otimes c_X \otimes \mathrm{id}_{X_{+l}} \in \mathrm{Mor}(X_{(-)^k (k+l)}, X_{(-)^k (k+l+2)})$$

$$\rightsquigarrow \mathrm{id}_{X_{(-)^{k+1} k}} \otimes ((\mathrm{id}_{X^\#} \otimes a_X^{-1}) \circ c_{X^\#}) \otimes \mathrm{id}_{X_{-l}} \in \mathrm{Mor}(X_{(-)^{k+1}(k+l)}, X_{(-)^{k+1}(k+l+2)})$$

$$\rightsquigarrow \mathrm{id}_{X_{(-)^k \varepsilon k}} \otimes x \otimes \mathrm{id}_{X_{(-)^m \varepsilon l}} \in P_{(-)^k \varepsilon (k+l+m)} \quad \text{for all } x \in P_{\varepsilon m}$$

FIGURE 4.1. Prescription for bicategory planar algebra

PROPOSITION 4.2. *Let a and $\tilde{a}$ be two pivotal structures on $\mathscr{B}$ as above, P and $\tilde{P}$ be the planar algebras associated to a 1-cell X with respect to a and $\tilde{a}$ respectively. Then, P and $\tilde{P}$ are perturbations of each other.*

PROOF. Set $z = a_X^{-1} \circ \tilde{a}_X \in P_{+1} = \text{End}(X)$. We assert that z is a weight of P and $P^{(z,1)} = \tilde{P}$. It follows from the definition of action of tangles and [8, Lemma 4.2] that $z^{\#} = P_{\boxed{z}}$. Note that z_{+k} (as in Definition 3.1) is given by

$$z_{+k} = z \otimes (z^{-1})^{\#} \otimes z \otimes \cdots \ k\text{-tensors}$$
$$= (a_X \otimes a_{X^{\#}} \otimes a_X \otimes \cdots)^{-1} \circ (\tilde{a}_X \otimes \tilde{a}_{X^{\#}} \otimes \tilde{a}_X \otimes \cdots)$$
$$= a_{X_{+k}}^{-1} \circ \tilde{a}_{X_{+k}} \in \mathscr{L}(P_{+k}),$$

where in the second (resp., last) equality, we have used the pivotal property of a and $\tilde{a}$ as in Proposition 2.9 (resp., Corollay 2.10). And on similar lines one establishes that $z_{-k} \in \mathscr{L}(P_{-k})$. This shows that z is indeed a weight of P. Now, it is a matter of routine verification that $P_T^{(z,1)} = \tilde{P}_T$ for a set of generating tangles as in Figure 2.1. Thus, we conclude that $P^{(z,1)} = \tilde{P}$.

5. Bimodule planar algebras

5.1. Planar algebra associated to a bimodule

In this subsection, we associate a 'bimodule planar algebra' to a bifinite bimodule with a natural correspondence between extremality and sphericality. This will pave the way for us to associate a unimodular bimodule planar algebra to a finite index subfactor, giving an extension of Jones' Theorem [13, Theorem 4.2.1] to an arbitrary finite index subfactor (not necessarily extremal).

DEFINITION 5.1. A finite dimensional, connected, positive C^*-planar algebra is called bimodule planar algebra.

Let $_A\mathscr{H}_B$ be a finite index bimodule for II_1 factors A and B. Before stating the next theorem, we set up some notations that will be used throughout this section: $\mathscr{H}_{+0} := L^2(A)$, $\mathscr{H}_{-0} := L^2(B)$ and for $k \geq 1$, $\mathscr{H}_{\varepsilon k}$ is the tensor product (over A or B) of k-many modules $\mathscr{H}$ and $\overline{\mathscr{H}}$ alternately with $\mathscr{H}$ (resp., $\overline{\mathscr{H}}$) being the left-most module if $\varepsilon = +$ (resp., $-$).

THEOREM 5.2. Let $_A\mathscr{H}_B$ be a finite index bimodule for II_1-factors A and B. Then,

(i) P defined by

$$P_{\varepsilon k} = \begin{cases} _A\mathscr{L}_A(\mathscr{H}_{+k}) \ \text{or} \ _A\mathscr{L}_B(\mathscr{H}_{+k}), & \text{if } \varepsilon = +, \\ _B\mathscr{L}_B(\mathscr{H}_{-k}) \ \text{or} \ _B\mathscr{L}_A(\mathscr{H}_{-k}), & \text{if } \varepsilon = -, \end{cases}$$

according as k is even or odd, has a unique bimodule planar algebra structure with $*$-structure coming from the usual adjoints of intertwiners, satisfying:

(a) *action of multiplication tangles matches with the composition of operators in the intertwiner spaces,*

(b) $P_{RI_{\varepsilon k}} = \begin{cases} P_{\varepsilon k} \ni T \mapsto T \otimes_A \mathrm{id}_{\mathscr{H}} \in P_{\varepsilon(k+1)}, & \text{if either } \varepsilon = + \text{ and } k \text{ is} \\ & \text{even, or } \varepsilon = - \text{ and } k \text{ is} \\ & \text{odd,} \\[1em] P_{\varepsilon k} \ni T \mapsto T \otimes_B \mathrm{id}_{\overline{\mathscr{H}}} \in P_{\varepsilon(k+1)}, & \text{otherwise,} \end{cases}$

(c) $P_{LI_{\varepsilon k}} = \begin{cases} P_{\varepsilon k} \ni T \mapsto \mathrm{id}_{\overline{\mathscr{H}}} \otimes_A T \in P_{-\varepsilon(k+1)}, & \text{if } \varepsilon = +, \\[0.7em] P_{\varepsilon k} \ni T \mapsto \mathrm{id}_{\mathscr{H}} \otimes_B T \in P_{-\varepsilon(k+1)}, & \text{if } \varepsilon = - \text{ and} \end{cases}$

(d) $P_{E_{+1}}$ *(resp.,* $P_{E_{-1}}$*) is given by*

$$\mathscr{H}^o \otimes_B \overline{\mathscr{H}}^o \ni \xi \otimes_B \overline{\eta} \overset{A-A}{\longmapsto} \sum_j {}_A\langle \eta, \xi \rangle (\eta_j \otimes_B \overline{\eta}_j) \in \mathscr{H}^o \otimes_B \overline{\mathscr{H}}^o$$

$$(\text{resp.,} \ \overline{\mathscr{H}}^o \otimes_A \mathscr{H}^o \ni \overline{\eta} \otimes_A \xi \overset{B-B}{\longmapsto} \sum_i \langle \eta, \xi \rangle_B (\overline{\xi}_i \otimes_A \xi_i) \in \overline{\mathscr{H}}^o \otimes_A \mathscr{H}^o)$$

where $\{\xi_i\}_i$ (resp., $\{\eta_j\}_j$) is any basis for ${}_A\mathscr{H}$ (resp., $\mathscr{H}_B$). (We will refer P (resp., normalized P) as the bimodule (resp., normalized or unimodular) planar algebra associated to ${}_A\mathscr{H}_B$).

(ii) *P (as in (i)) is spherical if and only if ${}_A\mathscr{H}_B$ is extremal (as in § 2.3).*

(iii) *If $B' := \mathscr{L}_B(\mathscr{H})$, then the normalized planar algebras associated to ${}_A\mathscr{H}_B$ and ${}_A L^2(B')_{B'}$ are isomorphic as $*$-planar algebras.*

PROOF. (i) The planar algebra (in [8]) associated to the 1-cell ${}_A\mathscr{H}_B$ in the pivotal bicategory of bifinite bimodules (§ 2.3) has same vector spaces as of P; we provide P with the same planar algebra structure, the prescription for the actions of tangles for which is given in Figure 4.1. It is clear from this prescription that P satisfies the conditions (a)–(d). Since P is connected, the fact that P with the above mentioned $*$-structure is a C^*-planar algebra, can be verified readily by checking the $*$-condition (2.1) for the tangles $M_{\varepsilon k}$, $RI_{\varepsilon k}$, $LI_{\varepsilon k}$, $E_{\varepsilon k}$. It now remains to show that the C^*-planar algebra P is positive. For this, it is enough to show that the tangle maps $P_{RE_{\varepsilon k}}$ and $P_{LE_{\varepsilon k}}$ are positive definite for all $\varepsilon k \in \mathrm{Col}$. We prove this only for $P_{RE_{+2k}}$; the others can be verified using similar arguments. For each $x \in P_{+2k} = {}_A\mathscr{L}_A(\mathscr{H}_{+2k})$ and $\zeta \in \mathscr{H}^o_{+(2k-1)}$, we

have

$$
\begin{aligned}
P_{RE_{+2k}}&(x)(\zeta) \\
&= \sum_i \big((\mathrm{id}_{\mathcal{H}_{+(2k-1)}} \otimes_B e_{\mathcal{H}}) \circ (x \otimes_A \mathrm{id}_{\mathcal{H}})\big)(\zeta \otimes_B \bar{\xi}_i \otimes_A \xi_i) \\
&= \sum_{i,i',k} (\mathrm{id}_{\mathcal{H}_{+(2k-1)}} \otimes_B e_{\mathcal{H}})\big(\gamma_k \otimes_B \bar{\xi}_i \langle \gamma_k \otimes_B \bar{\xi}_{i'}, x(\zeta \otimes_B \bar{\xi}_i)\rangle_A \otimes_A \xi_i\big) \\
&= \sum_{i,i',k} \gamma_k \big\langle \xi_{i'}, \langle \gamma_k \otimes_B \bar{\xi}_{i'}, x(\zeta \otimes_B \bar{\xi}_i)\rangle_A \xi_i\big\rangle_B,
\end{aligned}
$$

where $\{\gamma_k\}_k$ is a basis for $(\mathcal{H}_{+(2k-1)})_B$. This gives

$$
\begin{aligned}
\langle \zeta, P_{RE_{+2k}}(x)(\zeta)\rangle &= \sum_{k,i,i'} \big\langle \zeta \otimes_B \overline{\langle \gamma_k \otimes_B \bar{\xi}_{i'}, x(\zeta \otimes_B \bar{\xi}_i)\rangle_A \xi_i}, \gamma_k \otimes_B \bar{\xi}_{i'}\big\rangle \\
&= \sum_i \langle \zeta \otimes_B \bar{\xi}_i, x(\zeta \otimes_B \bar{\xi}_i)\rangle.
\end{aligned}
$$

Thus, $P_{RE_{+2k}}(x)$ is positive semi-definite. In addition, the above also gives

$$
\sum_k \langle \gamma_k, P_{RE_{+2k}}(x)(\gamma_k)\rangle = \sum_{k,i} \langle \gamma_k \otimes_B \bar{\xi}_i, x(\gamma_k \otimes_B \bar{\xi}_i)\rangle = \dim(\mathcal{H}_{+2k})_A \operatorname{tr}_C(x),
$$

where $C := \mathcal{L}_A(\mathcal{H}_{+2k})$ and the last equality is a consequence of Proposition 2.12. This proves that $P_{RE_{+2k}}$ is also faithful. Thus, P is a bimodule planar algebra; and for uniqueness, we appeal again to Remark 2.2.

(ii) It is enough to prove that the normalized left and the right picture traces on P_{+1} are given by the unique traces on the II_1-factors $A' := {}_A\mathcal{L}(\mathcal{H})$ and $B' := \mathcal{L}_B(\mathcal{H})$ respectively. We will only exhibit a proof for the left one. First, note that

$$
P_{\boxed{\bigcirc}} = e_{\mathcal{H}} \circ (\mathrm{id}_{\overline{\mathcal{H}}} \otimes a_{\mathcal{H}}^{-1}) \circ c_{\overline{\mathcal{H}}} = \sum_i \langle \xi_i, \xi_i \rangle_B \hat{} = \dim({}_A\mathcal{H}) 1_{P_{-0}}
$$

and

$$
P_{\boxed{+\bigcirc}} = e_{\overline{\mathcal{H}}} \circ (a_{\mathcal{H}} \otimes \mathrm{id}_{\overline{\mathcal{H}}}) \circ c_{\mathcal{H}} = \sum_j {}_A\langle \eta_j, \eta_j \rangle \hat{} = \dim(\mathcal{H}_B) 1_{P_{+0}}
$$

where $\{\xi_i\}_i$ (resp., $\{\eta_j\}_j$) is a basis for ${}_A\mathcal{H}$ (resp., $\mathcal{H}_B$). So, the modulus of the

planar algebra P is $(\dim {}_A\mathcal{H}, \dim \mathcal{H}_B)$. Further, for each $x \in P_{+1} = {}_A\mathcal{L}_B(\mathcal{H})$,

$$\mathcal{H}_{-0} := L^2(B) \ni \hat{1} \overset{P_{LE_{+1}}(x)}{\longmapsto} \sum_i \langle \xi_i, x\xi_i \rangle_B \hat{} = \sum_i \langle \xi_i, x\xi_i \rangle \hat{1}$$

$$= \dim({}_A\mathcal{H}) \operatorname{tr}_{A'}(x)\hat{1} \in L^2(B) =: \mathcal{H}_{-0}.$$

The first equality follows from the fact that $\sum_i \langle \xi_i, x\xi_i \rangle_B$ commutes with B and the second one comes again from Proposition 2.12.

(iii) Note that the planar algebra associated to ${}_A\mathcal{H}_B$ may very well be non-unimodular but the index of P is same as that of ${}_A\mathcal{H}_B$. Set $\delta_- = \dim({}_A\mathcal{H})$, $\delta_+ = \dim(\mathcal{H}_B)$ and $\delta = \sqrt{\delta_+ \delta_-}$. Let Q denote the bimodule planar algebra associated to ${}_A L^2(B')_{B'}$. So, $\tilde{P} := P^{(\sqrt{\frac{\delta_-}{\delta_+}}, 1)}$ and $\tilde{Q} := Q^{(\delta, 1)}$ are the normalizations of P and Q respectively. To establish an isomorphism between $\tilde{P}$ and $\tilde{Q}$, we consider the following B'-B' linear maps $u : \mathcal{H} \otimes_B \overline{\mathcal{H}} \to \mathcal{K} := L^2(B')$ and $w : \mathcal{H} \otimes_B \overline{\mathcal{H}} \to \overline{\mathcal{K}}$ given by

$$\mathcal{H}^o \otimes_B \overline{\mathcal{H}}^o \ni \xi \otimes_B \overline{\eta} \overset{u}{\mapsto} {}_{B'}\langle \eta, \xi \rangle \hat{1} \in \mathcal{K}$$

implying

$$\mathcal{K} \ni \hat{1} \overset{u^*}{\mapsto} \delta_+^{-1} \sum_j (\eta_j \otimes_B \overline{\eta}_j) \in \mathcal{H}^o \otimes_B \overline{\mathcal{H}}^o$$

and

$$\mathcal{H}^o \otimes_B \overline{\mathcal{H}}^o \ni \xi \otimes_B \overline{\eta} \overset{w}{\mapsto} {}_{B'}\langle \eta, \xi \rangle \overline{\hat{1}} \in \overline{\mathcal{K}}$$

implying

$$\overline{\mathcal{K}} \ni \overline{\hat{1}} \overset{w^*}{\mapsto} \delta_+^{-1} \sum_j (\eta_j \otimes_B \overline{\eta}_j) \in \mathcal{H}^o \otimes_B \overline{\mathcal{H}}^o.$$

Clearly, u and w are co-isometries; further, $\operatorname{index}({}_{B'}\mathcal{H} \otimes_B \overline{\mathcal{H}}_{B'}) = 1$ implies u and w are unitaries as well. Let $v : \mathcal{H} \otimes_B \overline{\mathcal{H}} \to \mathcal{K} \otimes_{B'} \overline{\mathcal{K}}$ be the B'-B' linear unitary obtained from u after composing with the unitary taking $\hat{1} \in \mathcal{K}$ to $(\hat{1} \otimes_{B'} \overline{\hat{1}}) \in \mathcal{K} \otimes_{B'} \overline{\mathcal{K}}$. Define $\varphi : \tilde{P} \to \tilde{Q}$ in the following way:

$$\tilde{P}_{+2l} = {}_A\mathcal{L}_A(\mathcal{H}_{+2l}) \ni x \overset{\varphi_{+2l}}{\longmapsto} \operatorname{Ad}_{v_l}(x) \in {}_A\mathcal{L}_A(\mathcal{K}_{+2l}) = \tilde{Q}_{+2l}$$

for $l \geq 1$, where $v_l := (v \otimes_A v \otimes_A \cdots l \text{ tensors}) \in {}_{B'}\mathcal{L}_{B'}(\mathcal{H}_{+2l}, \mathcal{K}_{+2l})$, and for $l \geq 0$,

$$\tilde{P}_{+(2l+1)} = {}_A\mathcal{L}_B(\mathcal{H}_{+(2l+1)}) \ni x$$

$$\overset{\varphi_{+(2l+1)}}{\longmapsto} \operatorname{Ad}_{(v_l \otimes_A u)}(x \otimes_B \operatorname{id}_{\overline{\mathcal{H}}}) \in {}_A\mathcal{L}_{B'}(\mathcal{K}_{+(2l+1)}) = \tilde{Q}_{+(2l+1)},$$

$$\tilde{P}_{-(2l+1)} = {}_B\mathscr{L}_A(\mathscr{H}_{-(2l+1)}) \ni x$$
$$\overset{\varphi_{-(2l+1)}}{\longmapsto} \mathrm{Ad}_{(w\otimes_A v_l)}(\mathrm{id}_{\mathscr{H}} \otimes_B x) \in {}_{B'}\mathscr{L}_A(\mathscr{K}_{-(2l+1)}) = \tilde{Q}_{-(2l+1)},$$
$$\tilde{P}_{-(2l+2)} = {}_B\mathscr{L}_B(\mathscr{H}_{-(2l+2)}) \ni x$$
$$\overset{\varphi_{-(2l+2)}}{\longmapsto} \mathrm{Ad}_{(w\otimes_A v_l\otimes_A u)}(\mathrm{id}_{\mathscr{H}} \otimes_B x \otimes_A \mathrm{id}_{\overline{\mathscr{H}}}) \in {}_{B'}\mathscr{L}_{B'}(\mathscr{K}_{-(2l+2)}) = \tilde{Q}_{-(2l+2)}.$$

Clearly, each $\varphi_{\varepsilon k}$ is an injective $*$-algebra homomorphism and each φ_{+2l} is surjective. To see that $\varphi_{+(2l+1)}$ (resp., $\varphi_{-(2l+1)}$) is surjective, note that

$$\mathrm{Ad}_{(v_l\otimes_A u)^*}({}_A\mathscr{L}_{B'}(\mathscr{K}_{+(2l+1)})) = {}_A\mathscr{L}_{B'}(\mathscr{H}_{+(2l+1)} \otimes_B \overline{\mathscr{H}})$$
$$= \{x \otimes_B \mathrm{id}_{\overline{\mathscr{H}}} : x \in {}_A\mathscr{L}_B(\mathscr{H}_{+(2l+1)})\}$$

(resp.,

$$\mathrm{Ad}_{(w\otimes_A v_l)^*}({}_{B'}\mathscr{L}_A(\mathscr{K}_{-(2l+1)})) = {}_{B'}\mathscr{L}_A(\mathscr{H} \otimes_B \mathscr{H}_{-(2l+1)})$$
$$= \{\mathrm{id}_{\mathscr{H}} \otimes_B x : x \in {}_B\mathscr{L}_A(\mathscr{H}_{-(2l+1)})\})$$

where the second equality follows from Lemma 2.13 since ${}_B\overline{\mathscr{H}}_{B'}$ (resp., ${}_{B'}\mathscr{H}_B$) has index 1. Same arguments also imply $\varphi_{-(2l+2)}$ is surjective. Thus, φ is an isomorphism preserving actions of multiplication tangles.

Actions of inclusion tangles: We only show that φ is equivariant with respect to the action of RI_{+k}; the same for RI_{-k} and $LI_{\varepsilon k}$ can be verified along similar lines. For $x \in \tilde{P}_{+(2l+1)}$, note that

$$[\mathrm{Ad}_{v_l\otimes_A u}(x \otimes_B \mathrm{id}_{\overline{\mathscr{H}}})] \otimes_{B'} \mathrm{id}_{\overline{\mathscr{K}}} = \mathrm{Ad}_{v_l\otimes_A u\otimes_{B'}\mathrm{id}_{\overline{\mathscr{K}}}}(x \otimes_B \mathrm{id}_{\overline{\mathscr{H}}} \otimes_{B'} \mathrm{id}_{\overline{\mathscr{K}}})$$
$$= \mathrm{Ad}_{v_l\otimes_A v\circ\theta}(x \otimes_B \mathrm{id}_{\overline{\mathscr{H}}} \otimes_{B'} \mathrm{id}_{\overline{\mathscr{K}}})$$
$$= \mathrm{Ad}_{v_l\otimes_A v}(x \otimes_B \mathrm{id}_{\overline{\mathscr{H}}}),$$

where $\mathscr{H} \otimes_B \overline{\mathscr{H}} \otimes_{B'} \overline{\mathscr{K}} \ni \xi \otimes_B \overline{\eta} \otimes_{B'} \overline{\tilde{1}} \overset{\theta}{\mapsto} \xi \otimes_B \overline{\eta} \in \mathscr{H} \otimes_B \overline{\mathscr{H}}$ is a B'-B' unitary intertwiner. This implies $\varphi(\tilde{P}_{RI_{+(2l+1)}}(x)) = \tilde{Q}_{RI_{+(2l+1)}}(\varphi(x))$. Equivariance of φ with respect to the action of RI_{+2l} is even more easier.

Action of $E_{\pm 1}$: From condition (d) of part (i) and definition of perturbation, we obtain

$$\varphi(\tilde{P}_{E_{+1}})(\hat{x} \otimes_{B'} \overline{\hat{y}}) = \frac{1}{\delta_+} \sum_j v \tilde{P}_{E_{+1}}(xy^*\eta_j \otimes_B \overline{\eta}_j)$$

$$= \frac{1}{\delta_+}\sqrt{\frac{\delta_-}{\delta_+}} \sum_{j,j'} v({}_A\langle xy^*\eta_j, \eta_j\rangle\eta_{j'} \otimes \overline{\eta_{j'}})$$

$$= \frac{1}{\delta_+} \sqrt{\frac{\delta_-}{\delta_+}} \sum_{j,j'} E_A({}_{B'}\langle xy^* \eta_j, \eta_j\rangle)\eta_{j'} \otimes \overline{\eta_{j'}}$$

$$= \frac{1}{\delta_+} \sqrt{\frac{\delta_-}{\delta_+}} E_A(xy^*)(\dim \mathscr{H}_B)^2 \hat{1} \otimes_{B'} \bar{\hat{1}}$$

$$= \tilde{Q}_{E_{+1}}(\hat{x} \otimes_{B'} \hat{y})$$

for all $x, y \in B'$. This gives $\varphi(\tilde{P}_{E_{+1}}) = \tilde{Q}_{E_{+1}}$. On the other hand, since $\varphi(\tilde{P}_{E_{-1}})$ and $\tilde{Q}_{E_{-1}}$ are both B'-B' linear, in order to show their equality, it suffices to show that $\varphi(\tilde{P}_{E_{-1}})(\bar{\hat{1}} \otimes_A \hat{1}) = \tilde{Q}_{E_{-1}}(\bar{\hat{1}} \otimes_A \hat{1})$. Again, from condition (d) of part (i) and definition of perturbation, we have

$$\varphi(\tilde{P}_{E_{-1}})(\bar{\hat{1}} \otimes_A \hat{1})$$

$$= \delta_+^{-2} \sum_{j,j'} \big((w \otimes_A u) \circ (\mathrm{id}_{\mathscr{H}} \otimes_B \tilde{P}_{E_{-1}} \otimes_A \mathrm{id}_{\overline{\mathscr{H}}})\big)(\eta_j \otimes_B \overline{\eta}_j \otimes_A \eta_{j'} \otimes_B \overline{\eta}_{j'})$$

$$= (\delta\delta_+)^{-1} \sum_{j,j',i} (w \otimes_A u)(\eta_j \otimes_B \langle \eta_j, \eta_{j'} \rangle_B \overline{\xi}_i \otimes_A \xi_i \otimes_B \overline{\eta}_{j'})$$

$$= (\delta\delta_+)^{-1} \sum_{j',i} \overline{{}_{B'}\langle \eta_{j'}, \xi_i \rangle\hat{}} \otimes_A {}_{B'}\langle \eta_{j'}, \xi_i \rangle\hat{}$$

$$= (\delta\delta_+)^{-1} \sum_{k,j',i} \overline{E_A({}_{B'}\langle \eta_{j'}, \xi_i \rangle c_k^*)\hat{c}_k} \otimes_A {}_{B'}\langle \eta_{j'}, \xi_i \rangle\hat{}$$

$$= (\delta\delta_+)^{-1} \sum_{k,j',i} \overline{\hat{c}}_k \otimes_A {}_{B'}\langle \eta_{j'}, {}_A\langle \xi_i, c_k \eta_{j'} \rangle \xi_i \rangle\hat{}$$

$$= \delta^{-1} \sum_k \overline{\hat{c}}_k \otimes_A \hat{c}_k = \tilde{Q}_{E_{-1}}(\bar{\hat{1}} \otimes_A \hat{1}),$$

where $\{c_k\}$ is a left basis for the subfactor $A \subset B'$. Thus, in view or Remark 2.2, φ is an isomorphism of $*$-planar algebras.

We now proceed to associate a unimodular bimodule planar algebar to a finite index type II_1 subfactor $N \subset M$. For this, the obvious thing to consider, is the unimodular bimodule planar algebra associated to ${}_N L^2(M)_M$. However, we would like to find out the actions of tangles on the relative commutants (instead of intertwiner spaces) and a set of conditions which uniquely determines the action exactly the way [13, Theorem 4.2.1] states for extremal subfactors. Before doing so, we first set up some notations and recall certain standard subfactor theory facts – see [1], [12], [18], [25], [24].

Let $N \subset M$ be a subfactor with $\delta^2 := [M : N] < \infty$ ($\delta > 0$) and $\{M_k\}_{k \geq 1}$ be a tower of basic constructions with $\{e_k \in \mathscr{P}(M_k)\}_{k \geq 1}$ being a set of Jones projections. We will have instances to apply the following useful fact.

LEMMA 5.3 ([24]). *For each $x_1 \in M_1$, there is a unique $x \in M$ satisfying $x_1 e_1 = x e_1$; this unique element is given by $x = [M : N]E_M(x_1 e_1)$.*

For each $k \geq 1$, set

$$e_{[-1,k]} = \delta^{k(k+1)}(e_{k+1} e_k \cdots e_1)(e_{k+2} e_{k+1} \cdots e_2)$$
$$\cdots (e_{2k+1} e_{2k} \cdots e_{k+1}) \in N' \cap M_{2k+1},$$

$$e_{[0,k]} = \delta^{k(k-1)}(e_{k+1} e_k \cdots e_2)(e_{k+2} e_{k+1} \cdots e_2)$$
$$\cdots (e_{2k} e_{2k-1} \cdots e_{k+1}) \in M' \cap M_{2k},$$

and

$$v_k = \delta^k e_k e_{k-1} \cdots e_1 \in N' \cap M_k.$$

Then, the tower of II_1 factors $N \subset M_k \subset M_{2k+1}$ (resp., $M \subset M_k \subset M_{2k}$) is an instance of basic construction with $e_{[-1,k]}$ (resp., $e_{[0,k]}$) as Jones projection, that is, there exists an isomorphism $\varphi_{-1,k} : M_{2k+1} \longrightarrow \mathscr{L}_N(L^2(M_k))$ (resp., $\varphi_{0,k} : M_{2k} \longrightarrow \mathscr{L}_M(L^2(M_k)))$ given by

$$\varphi_{-1,k}(x_{2k+1})\hat{x}_k = \delta^{2(k+1)} E_{M_k}(x_{2k+1} x_k e_{[-1,k]}\widehat{}$$

(resp.,

$$\varphi_{0,k}(x_{2k})\hat{x}_k = \delta^{2k} E_{M_k}(x_{2k} x_k e_{[0,k]}\widehat{}$$

for all $x_i \in M_i, i = k, 2k, 2k + 1$, which is identity restricted to M_k and sends $e_{[-1,k]}$ (resp., $e_{[0,k]}$) to the projection with range $L^2(N)$ (resp., $L^2(M)$). Also, $\varphi_{-1,k}(M_i' \cap M_{2k+1}) = {}_{M_i}\mathscr{L}_N(L^2(M_k))$ (resp., $\varphi_{0,k}(M_i' \cap M_{2k}) = {}_{M_i}\mathscr{L}_M(L^2(M_k)))$ and $\varphi_{0,k} = \varphi_{-1,k}|_{M_{2k}}$ for all $k \geq 0, -1 \leq i \leq k$. Further, for each $k \geq 0$, we have an M_k-M linear unitary given (on bounded vectors) by

$$L^2(M_k) \otimes_N L^2(M) \ni \hat{y} \otimes_N \hat{z} \xmapsto{u_k} (y v_{k+1} z)\widehat{} \in L^2(M_{k+1})$$

for all $y \in M_k$ and $z \in M$. This unitary also satisfies the equation $\varphi_{0,k+1}(x) = u_k(\varphi_{-1,k}(x) \otimes_N \mathrm{id}_{L^2(M)})u_k^*$ for all $x \in M_{2k+1}$.

Apart from these, if ${}_N\mathscr{H}_M$ denotes the bimodule ${}_N L^2(M)_M$, then for each $k \geq 1$, we have a bunch of unitary intertwiners (determined by the following actions on the bounded vectors)

$$\mathscr{H}_{+2k} \ni \hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_M \bar{\hat{x}}_{2k}$$
$$\xmapsto[N\text{-}N]{w_{+2k}} (x_1 x_2^* v_1 x_3 x_4^* v_2 x_5 \cdots x_{2k-2}^* v_{k-1} x_{2k-1} x_{2k}^*)\widehat{} \in L^2(M_{k-1}),$$

$$\mathscr{H}_{-2k} \ni \bar{\hat{x}}_1 \otimes_N \hat{x}_2 \otimes_M \cdots \otimes_N \hat{x}_{2k}$$
$$\xmapsto[M\text{-}M]{w_{-2k}} (x_1^* v_1 x_2 x_3^* v_2 x_4 \cdots x_{2k-3}^* v_{k-1} x_{2k-2} x_{2k-1}^* v_k x_{2k})\widehat{} \in L^2(M_k),$$

$$\mathscr{H}_{+(2k-1)} \ni \hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_N \hat{x}_{2k-1}$$
$$\xmapsto[N\text{-}M]{w_{+(2k-1)}} (x_1 x_2^* v_1 x_3 x_4^* v_2 x_5 \cdots x_{2k-2}^* v_{k-1} x_{2k-1})\widehat{} \in L^2(M_{k-1}),$$

$$\mathscr{H}_{-(2k-1)} \ni \bar{\hat{x}}_1 \otimes_N \hat{x}_2 \otimes_M \cdots \otimes_M \bar{\hat{x}}_{2k-1}$$
$$\xmapsto[M\text{-}N]{w_{-(2k-1)}} (x_1^* v_1 x_2 x_3^* v_2 x_4 \cdots x_{2k-3}^* v_{k-1} x_{2k-2} x_{2k-1}^*)\widehat{} \in L^2(M_{k-1})$$

and a very useful formula (see [13]) $x_1 v_1 x_2 v_2 \cdots v_k x_{k+1} = x_1 v_k^* x_2 v_{k-1}^* \cdots v_1^* x_{k+1}$ for all $x_i \in M$, $1 \le i \le 2k$.

We are now ready to present the extension of Jones' Theorem [13, Theorem 4.2.1], which associates a unimodular bimodule planar algebra to a finite index subfactor and gives a natural correspondence between extremality and sphericality.

THEOREM 5.4. *Let $M_{-1} := N \subset M =: M_0$ be a subfactor with $\delta^2 := [M : N] < \infty$ ($\delta > 0$) and $\{M_k\}_{k \ge 1}$ be a tower of basic constructions with $\{e_k \in \mathscr{P}(M_k)\}_{k \ge 1}$ being a set of Jones projections. Then, P defined by $P_{\varepsilon k} = N' \cap M_{k-1}$ or $M' \cap M_k$ according as $\varepsilon = +$ or $-$, has a unique unimodular bimodule planar algebra structure with the $*$-structure given by the usual $*$ of the relative commutants such that for each $k \in \mathsf{N}_0$,*

(1) *the action of multiplication tangles is given by the usual multiplication in the relative commutants,*

(2) *the action of the left inclusion tangle LI_{-k} is given by the usual inclusion $M' \cap M_k \subset N' \cap M_k$,*

(3) *the action of the right inclusion tangle RI_{+k} is given by the usual inclusion $M_{k-1} \subset M_k$,*

(4) $P_{E_{+(k+1)}} = \delta e_{k+1}$,

(5) $P_{LE_{+(k+1)}} = \delta^{-1} \sum_i b_i^* x b_i$ *for all $x \in P_{+(k+1)}$,*

where $\{b_i\}_i$ is a left basis for the subfactor $N \subset M$. In particular, P is spherical if and only if the subfactor $N \subset M$ is extremal. (P will be referred as the planar algebra associated to the tower $\{M_k\}_{k \ge -1}$ with Jones projections $\{e_k\}_{k \ge 1}$.)

PROOF. Let Q denote the bimodule planar algebra associated to the bifinite bimodule $_N\mathscr{H}_M := {_N}L^2(M)_M$ (as in Theorem 5.2) and $\tilde{Q} = Q^{(\delta,1)}$ be its normalization. With notations as above, we have the $*$-algebra isomorphisms

$$P_{+2k} = N' \cap M_{2k-1} \ni x$$
$$\overset{\chi_{+2k}}{\longmapsto} \mathrm{Ad}_{w^*_{+2k}}(\varphi_{-1,k-1}(x)) \in {}_N\mathcal{L}_N(\mathcal{H}_{+2k}) = Q_{+2k} = \tilde{Q}_{+2k},$$

$$P_{-2k} = M' \cap M_{2k} \ni x$$
$$\overset{\chi_{-2k}}{\longmapsto} \mathrm{Ad}_{w^*_{-2k}}(\varphi_{0,k}(x)) \in {}_M\mathcal{L}_M(\mathcal{H}_{-2k}) = Q_{-2k} = \tilde{Q}_{-2k},$$

$$P_{+(2k+1)} = N' \cap M_{2k} \ni x$$
$$\overset{\chi_{+(2k+1)}}{\longmapsto} \mathrm{Ad}_{w^*_{+(2k+1)}}(\varphi_{0,k}(x)) \in {}_N\mathcal{L}_M(\mathcal{H}_{+(2k+1)}) = Q_{+(2k+1)} = \tilde{Q}_{+(2k+1)},$$

$$P_{-(2k+1)} = M' \cap M_{2k+1} \ni x$$
$$\overset{\chi_{-(2k+1)}}{\longmapsto} \mathrm{Ad}_{w^*_{-(2k+1)}}(\varphi_{-1,k}(x)) \in {}_M\mathcal{L}_N(\mathcal{H}_{-(2k+1)}) = Q_{-(2k+1)} = \tilde{Q}_{-(2k+1)}$$

for $k \in \mathsf{N}_0$. We provide P with a unimodular bimodule planar algebra structure from that of $\tilde{Q}$ as follows: For each tangle $T : (\varepsilon_1 k_1 \times \cdots \times \varepsilon_b k_b) \to \varepsilon_0 k_0$ (resp., $T : \emptyset \to \varepsilon_0 k_0$), we define $P_T = \chi^{-1}_{\varepsilon_0 k_0} \circ \tilde{Q}_T \circ (\times^b_{i=1} \chi_{\varepsilon_i, k_i})$ (resp., $P_T = \chi^{-1}_{\varepsilon_0 k_0}(\tilde{Q}_T)$). Thus, P inherits a unimodular bimodule planar algebra structure with modulus (δ, δ). We now show that P satisfies all the conditions in the statement, which forces the planar algebra structure on P to be unique by Remark 2.2 (2). To begin with, note that, (1) needs no further verification. We will establish the relations in (2)–(5) only for even k's because the proofs for odd k's, are completely analogous to those in the even case.

(2) Suppose $k = 2l$ and $y \in P_{-2l} = M' \cap M_{2l}$. Unravelling the definitions, we just need to show that $\varphi_{0,l}(y) = \mathrm{Ad}_{w_{+(2l+1)}}(\mathrm{id}_{\mathcal{H}} \otimes_M \chi_{-2l}(y))$. For $\xi = \hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_N \hat{x}_{2l-1} \otimes_M \bar{\hat{x}}_{2l} \otimes_N \hat{x}_{2l+1} \in \mathcal{H}^o_{+(2l+1)}$ where $x_i \in M$ for all $1 \le i \le 2l + 1$, we have

$$w_{+(2l+1)} \circ (\mathrm{id}_{L^2(M)} \otimes_M \chi_{-2l}(y))(\xi)$$
$$= w_{+(2l+1)}\big(\hat{x}_1 \otimes_M (w^*_{-2l}\varphi_{0,l}(y)w_{-2l}$$
$$\qquad \cdot (\bar{\hat{x}}_2 \otimes_N \cdots \otimes_N \hat{x}_{2l-1} \otimes_M \bar{\hat{x}}_{2l} \otimes_N \hat{x}_{2l+1})\big)$$
$$= \sum_i \widehat{(x_1 z^*_{1,i} v_1 z_{2,i} z^*_{3,i} v_2 \cdots v_l z_{2l,i})}$$
$$= x_1 \varphi_{0,l}(y) w_{-2l}(\bar{\hat{x}}_2 \otimes_N \cdots \otimes_N \hat{x}_{2l-1} \otimes_M \bar{\hat{x}}_{2l} \otimes_N \hat{x}_{2l+1})$$
$$= \varphi_{0,l}(y) w_{-2l}\big(\widehat{(x_2 x^*_1)} \otimes_N \cdots \otimes_N \hat{x}_{2l-1} \otimes_M \bar{\hat{x}}_{2l} \otimes_N \hat{x}_{2l+1}\big)$$
$$= \varphi_{0,l}(y) w_{+(2l+1)}(\xi),$$

where $(w^*_{-2l}\varphi_{0,l}(y)w_{-2l})(\bar{\hat{x}}_2 \otimes_N \cdots \otimes_N \hat{x}_{2l-1} \otimes_M \bar{\hat{x}}_{2l} \otimes_N \hat{x}_{2l+1}) = \sum_i \bar{\hat{z}}_{1,i} \otimes_N \hat{z}_{2,i} \otimes_M \cdots \otimes_N \hat{z}_{2l,i} \in \mathcal{H}^o_{-2l}$ for some $z_{j,i} \in M$ and $\{i\}$ finite, $1 \le j \le 2l$; as was desired.

(3) First consider $k = 2l$, $y \in P_{+2l} = N' \cap M_{2l-1}$ and $\xi = \hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N$

$\cdots \otimes_M \bar{\hat{x}}_{2l} \otimes_N \hat{x}_{2l+1} \in \mathscr{H}^o_{+(2l+1)}$ with x_i's as above, we have

$$
\begin{aligned}
\chi_{+(2l+1)}&(y)(\xi) \\
&= \delta^{2l} w^*_{+(2l+1)} \big(E_{M_l}(yx_1 x_2^* v_1 \cdots x_{2l}^* v_l x_{2l+1} e_{[0,l]} \widehat{)} \big) \\
&= w^*_{+(2l+1)} \circ \varphi_{0,l}(y) \circ u_{l-1} \big(w_{+2l}(\hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \\
&\qquad \cdots \otimes_N x_{2l-1} \otimes_M \bar{\hat{x}}_{2l}) \otimes_N \hat{x}_{2l+1} \big) \\
&= w^*_{+(2l+1)} \circ u_{l-1,0} \circ (\varphi_{-1,l-1}(y) \otimes_N \mathrm{id}_{L^2(M)}) \\
&\qquad \cdot (w_{+2l}(\hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_N x_{2l-1} \otimes_M \bar{\hat{x}}_{2l}) \otimes_N \hat{x}_{2l+1}) \\
&= w^*_{+(2l+1)} \big(\varphi_{-1,l-1}(y) \circ w_{+2l}(\hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \\
&\qquad \cdots \otimes_N x_{2l-1} \otimes_M \bar{\hat{x}}_{2l}) v_l x_{2l+1} \big) \\
&= \big(\mathrm{Ad}_{w^*_{+2l}}(\varphi_{-1,l-1}(y))(\hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_M \bar{\hat{x}}_{2l}) \big) \otimes_N \hat{x}_{2l+1} \\
&= (\chi_{+2l}(y) \otimes_N \mathrm{id}_{L^2(M)})(\xi) = \tilde{Q}_{RI_{+2l}}(\chi_{+2l}(y))(\xi).
\end{aligned}
$$

On the other hand, if k is odd, say $2l+1$, $y \in P_{+(2l+1)} = N' \cap M_{2l}$ and $\xi = \hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_N \hat{x}_{2l+1} \otimes_M \bar{\hat{x}}_{2l+2} \in \mathscr{H}^o_{+(2l+2)}$, then

$$
\begin{aligned}
\tilde{Q}_{RI_{+(2l+1)}}&(\chi_{+(2l+1)}(y))(\xi) \\
&= (\chi_{+(2l+1)}(y) \otimes_M \mathrm{id}_{\overline{\mathscr{H}}})(\xi) \\
&= w^*_{+(2l+1)} \big(\varphi_{0,l}(y)(x_1 x_2^* v_1 \cdots v_{l-1} x_{2l-1} x_{2l}^* v_l x_{2l+1}) \widehat{} \big) \otimes_M \bar{\hat{x}}_{2l+2} \\
&= w^*_{+(2l+2)} \big(\varphi_{0,l}(y)((x_1 x_2^* v_1 \cdots v_l x_{2l+1}) \widehat{}) x_{2l+2}^* \big) \\
&= w^*_{+(2l+2)} \big(\varphi_{0,l}(y)(x_1 x_2^* v_1 \cdots v_l x_{2l+1} x_{2l+2}^*) \widehat{} \big) \\
&= w^*_{+(2l+2)} \big(\varphi_{-1,l}(y)(x_1 x_2^* v_1 \cdots v_l x_{2l+1} x_{2l+2}^*) \widehat{} \big) \\
&= \chi_{+(2l+2)}(y)(\xi).
\end{aligned}
$$

(4) The initial case $k = 1$ is trivial. Let $k = 2l$. For $\xi = \hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_M \bar{\hat{x}}_{2l} \otimes_N \hat{x}_{2l+1} \in \mathscr{H}^o_{+(2l+1)}$ with x_i's in M, we have

$$
\begin{aligned}
\chi_{2l+1}(\delta e_{2l})(\xi) &= \delta^{2l+1} w^*_{2l+1} \big(E_{M_l}(e_{2l} x_1 x_2^* v_1 \cdots x_{2l}^* v_l x_{2l+1} e_{[0,l]} \widehat{)} \big) \\
&= \delta^{2l+1} w^*_{2l+1} \big(x_1 x_2^* v_1 \cdots v_l x_{2l+1} E_{M_l}(e_{2l} e_{[0,l]} \widehat{)} \big) \\
&= \delta w^*_{2l+1} \big((x_1 x_2^* v_1 \cdots v_l x_{2l+1} e_2) \widehat{} \big) \\
&= \delta w^*_{2l+1} \big((x_1 x_2^* v_l^* \cdots v_1^* x_{2l+1} e_2) \widehat{} \big) \\
&= w^*_{2l+1} \big((x_1 x_2^* v_l^* \cdots x_{2l-2}^* v_2^* x_{2l-1} x_{2l}^* x_{2l+1}) \widehat{} \big)
\end{aligned}
$$

where, in the third equality, we have used the fact that $E_{M_l}(e_{2l}e_{[0,l]}) = \delta^{-2l}e_2$, which holds by the uniqueness condition in Lemma 5.3 applied to the equality $e_{2l}e_{[0,l]} = e_2e_{[0,l]}$ (a routine verification involving Temperley-Lieb relations satisfied by the Jones' projections $\{e_l\}$). On the other hand, by the definition of $\tilde{Q}$ and Theorem 5.2 (i)(d), we have

$$\tilde{Q}_{E_{+2l}}(\xi) = \delta^{-1}\sum_i \hat{x}_1 \otimes_M \bar{\hat{x}}_2 \otimes_N \cdots \otimes_N \hat{x}_{2l-1} \otimes_M (b_i x_{2l+1}^* x_{2l})^{\bar{\hat{\,}}} \otimes_N \hat{b}_i$$

$$= \delta^{-1} w_{+(2l+1)}^*\big((x_1 x_2^* v_1 \cdots x_{2l-2}^* v_{l-1} x_{2l-1} x_{2l}^* x_{2l+1} b_i^* v_l b_i)^{\hat{\,}}\big)$$

$$= w_{+(2l+1)}^*\big((x_1 x_2^* v_l^* \cdots x_{2l-2}^* v_2^* x_{2l-1} x_{2l}^* x_{2l+1})^{\hat{\,}}\big).$$

(5) Let $k = 2l$, $y \in P_{+(2l+1)} = N' \cap M_{2l}$ and $\xi = \bar{\hat{x}}_1 \otimes_N \hat{x}_2 \otimes_M \cdots \otimes_N \hat{x}_{2l} \in \mathcal{H}_{-2l}^o$ where $x_i \in M$ for all $1 \le i \le 2l$. Then, using the precription of the action of tangles in the planar algebra Q (described in Figure 4.1, we get

$$\tilde{Q}_{LE_{+(2l+1)}}(\chi_{+(2l+1)}(y))(\xi)$$

$$= \delta^{-1}\sum_i (e_{\mathcal{H}} \otimes_M \mathrm{id}_{\mathcal{H}_{-2l}})(\bar{\hat{b}}_i \otimes_N \chi_{+(2l+1)}(y)(\hat{b}_i \otimes_M \bar{\hat{x}}_1 \otimes_N \cdots \otimes_N \hat{x}_{2l}))$$

$$= \delta^{2l-1}\sum_i (e_{\mathcal{H}} \otimes_M \mathrm{id}_{\mathcal{H}_{-2l}})$$
$$\cdot (\bar{\hat{b}}_i \otimes_N w_{+(2l+1)}^*\big(E_{M_l}(b_i x_1^* v_l^* x_2 \cdots x_{2l-1}^* v_1^* x_{2l} e_{[0,l]})^{\hat{\,}}\big))$$

$$= \delta^{2l-1}\sum_i (e_{\mathcal{H}} \otimes_M \mathrm{id}_{\mathcal{H}_{-2l}})(w_{-(2l+2)}^*\big((b_i^* v_{l+1}^* z_l)^{\hat{\,}}\big))$$

$$= \delta^{2l-1}\sum_{i,i'} (e_{\mathcal{H}} \otimes_M \mathrm{id}_{\mathcal{H}_{-2l}})(w_{-(2l+2)}^*\big((b_i^* v_{l+1}^* b_{i'}^* v_l^* E_{M_{l-1}}(v_l b_{i'} z_l))^{\hat{\,}}\big))$$

$$= \delta^{2l-1}\sum_{i,i'} (e_{\mathcal{H}} \otimes_M \mathrm{id}_{\mathcal{H}_{-2l}})$$
$$\cdot (\bar{\hat{b}}_i \otimes_N (b_{i'}^*)^{\hat{\,}} \otimes_M \bar{\hat{1}} \otimes_N w_{+(2l-1)}^*\big((E_{M_{l-1}}(v_l b_{i'} z_l))^{\hat{\,}}\big))$$

$$= \delta^{2l-1}\sum_{i,i'} b_i^* b_{i'}^* \bar{\hat{1}} \otimes_N w_{+(2l-1)}^*\big((E_{M_{l-1}}(v_l b_{i'} z_l))^{\hat{\,}}\big)$$

$$= \delta^{2l-1}\sum_{i,i'} w_{-2l}^*\big((b_i^* b_{i'}^* v_l^* E_{M_{l-1}}(v_l b_{i'} z_l))^{\hat{\,}}\big)$$

$$= \delta^{2l-1}\sum_i w_{-2l}^*\big((b_i^* z_l)^{\hat{\,}}\big)$$

$$= \delta^{2l-1}\sum_i w_{-2l}^*\big(E_{M_l}(b_i^* x b_i x_1^* v_l^* x_2 \cdots x_{2l-1}^* v_1^* x_{2l} e_{[0,l]})^{\hat{\,}}\big)$$

$$= \delta^{-1}\sum_i \chi_{-2l}(b_i^* x b_i)(\xi),$$

where $z_l := E_{M_l}(x b_i x_1^* v_l^* x_2 \cdots x_{2l-1}^* v_1^* x_{2l} e_{[0,l]})$ and in the fourth and eighth equalities we have used the fact that $\{v_l b_i\}$ is a left Pimsner-Popa basis for $M_{l-1} \subset M_l$ – see [18].

REMARK 5.5. Starting with an extremal subfactor, the associated planar algebra in Theorem 5.4 is indeed the same as that in [13, Theorem 4.2.1] since the map $_N\mathscr{L}(L^2(M_k)) =: N' \ni x \longmapsto \delta^{-2} \sum_i b_i^* x b_i \in M' := {_M\mathscr{L}(L^2(M_k))}$ is the unique $tr_{N'}$ preserving conditional expectation from N' onto M'. Such extension of [13, Theorem 4.2.1] was first established by Michael Burns in his thesis. However, the techniques used in [6] are different from our proof which is built up using graphical calculus of morphisms in a pivotal bicategory. Further, Jones and Penneys, in [17], also obtain an extension of [13, Theorem 4.2.1] in a slightly general set up.

REMARK 5.6. Apart from the action of the tangles given in conditions (1)–(5), we also mention below the action of few other useful tangles.

(a) $P_{RE_{+k}} = \delta E_{M_{k-2}}^{M_{k-1}}\big|_{P_{+k}}$, following from conditions (3) and (4).

(b) $P_{TR_{+k}^r} = \delta^k \, tr_{M_{k-1}}\big|_{P_{+k}}$, following from the action of right conditional expectation tangle in (a).

(c) $\delta^{-k} P_{TR_{+2l}^l}$ (resp., $\delta^{-k} P_{TR_{+(2l-1)}^l}$) is given by the trace on $P_{+2l} = N' \cap M_{2l-1}$ (resp., $P_{+(2l-1)} = N' \cap M_{2l-2}$) induced by the canonical trace on $_N\mathscr{L}(L^2(M_{l-1}))$ via the map $\varphi_{-1,l-1}$ (resp., $\varphi_{0,l-1}$); this could be derived from the precription of the action. It also turns out that this trace on P_{+2l} (resp., $P_{+(2l-1)}$) matches with the one induced by the canonical trace on $_N\mathscr{L}(L^2(M_{2l-1}))$ (resp., $_N\mathscr{L}(L^2(M_{2l-2}))$) via the usual inclusion.

COROLLARY 5.7. *If P is the planar algebra associated to the tower $\{M_k\}_{k\geq-1}$ with Jones projections $\{e_k\}_{k\geq1}$ (as in Theorem 5.4), then*

(a) $P_{E'_{-k}}(y) = \delta \sum_i b_i^* e_1 y e_1 b_i$ *for all* $y \in P_{-k} = M' \cap M_k$, *where* $E'_{-k} = LI_{+(k-1)} \circ LE_{-k}$ *and* $\{b_i\}_i$ *is a left basis for* $N \subset M$,

(b) $\lambda_n(P) =$ *the planar algebra associated to the tower* $\{M_{k+n}\}_{k\geq-1}$ *with Jones projections* $\{e_{k+n}\}_{k\geq1}$.

PROOF. (a) We follow the same notations as in Theorem 5.4 and prove this only for the case $k = 2l \in 2\mathbb{N}$ because the case for odd k's can be proved using similar arguments. Let $a_{\mathscr{H}}$ (resp., $e_{\overline{\mathscr{H}}}$) denote the isomorphism $\mathscr{H} \ni \alpha \xmapsto{a_{\mathscr{H}}} \overline{\overline{\alpha}} \in \overline{\overline{\mathscr{H}}}$ (resp., the usual evaluation map from $\overline{\overline{\mathscr{H}}} \otimes_M \overline{\mathscr{H}}$ to $L^2(N)$), and $\xi = \hat{\overline{x}}_1 \otimes_N \hat{x}_2 \otimes_M \cdots \otimes_N \hat{x}_{2l} \in \mathscr{H}_{-2l}$ where $x_i \in M$ for all $1 \leq i \leq 2l$. Then, using the prescription of the actions of tangles, we get

$$\tilde{Q}_{E'_{-2l}}\big(\chi_{-2l}(y)\big)(\xi)$$

$$= \delta\big(\mathrm{id}_{\overline{\overline{\mathscr{H}}}} \otimes_N \big[e_{\overline{\mathscr{H}}} \circ (a_{\mathscr{H}} \otimes_M \mathrm{id}_{\overline{\mathscr{H}}})\big] \otimes_N \mathrm{id}_{\mathscr{H}_{+(2l-1)}}\big)$$

$$\cdot \big(\hat{\overline{x}}_1 \otimes_N \hat{1}_M \otimes_M \big[w_{-2l}^* \circ \varphi_{0,l}(y) \circ w_{-2l}\big(\hat{\overline{1}}_M \otimes_N \hat{x}_2 \otimes_M \cdots \otimes_N \hat{x}_{2l}\big)\big]\big)$$

$$= \delta \sum_i \left(\mathrm{id}_{\overline{\mathscr{H}}} \otimes_N \left[e_{\overline{\mathscr{H}}} \circ (a_{\mathscr{H}} \otimes_M \mathrm{id}_{\overline{\mathscr{H}}}) \right] \otimes_N \mathrm{id}_{\mathscr{H}_{+(2l-1)}} \right)$$
$$\cdot \left(\widehat{\overline{x}}_1 \otimes_N \hat{1}_M \otimes_M w^*_{-2l} (b_i^* v_l^* E_{M_{l-1}} (v_l b_i z)) \widehat{} \right)$$

(where $\hat{z} := \varphi_{0,l}(y) \circ w_{-2l} \left(\overline{\hat{1}}_M \otimes_N \hat{x}_2 \otimes_M \cdots \otimes_N \hat{x}_{2l} \right)$ and since $\{v_l b_i\}_i$ forms a left basis for $M_{l-1} \subset M_l$)

$$= \delta \sum_i \left(\mathrm{id}_{\overline{\mathscr{H}}} \otimes_N \left[e_{\overline{\mathscr{H}}} \circ (a_{\mathscr{H}} \otimes_M id_{\overline{\mathscr{H}}}) \right] \otimes_N \mathrm{id}_{\mathscr{H}_{+(2l-1)}} \right)$$
$$\cdot \left(\widehat{\overline{x}}_1 \otimes_N \hat{1}_M \otimes_M \overline{b}_i \otimes_N w^*_{+(2l-1)} (E_{M_{l-1}} (v_l b_i z)) \widehat{} \right)$$

$$= \delta \sum_i \left(\widehat{\overline{x}}_1 \otimes_N E_N(b_i^*) w^*_{+(2l-1)} (E_{M_{l-1}} (v_l b_i z)) \widehat{} \right)$$

$$= \delta \widehat{\overline{x}}_1 \otimes_N w^*_{+(2l-1)} (E_{M_{l-1}} (v_l z)) \widehat{} = \delta w^*_{-2l} (x_1^* v_l^* E_{M_{l-1}} (v_l z)) \widehat{}$$

$$= \delta^3 w^*_{-2l} (x_1^* v_{l-1}^* e_l E_{M_{l-1}} (e_l v_{l-1} z)) \widehat{} = \delta w^*_{-2l} (x_1^* v_{l-1}^* e_l v_{l-1} z) \widehat{}$$

$$= w^*_{-2l} (x_1^* v_1 \varphi_{0,l}(y) (v_1 x_2 x_3^* v_2 \cdots v_l x_{2l})) \widehat{}$$

On the other hand,

$$\delta \sum_i \chi_{-2l} (b_i^* e_1 y e_1 b_i) (\xi)$$

$$= \delta^{2l+1} \sum_i w^*_{-2l} (E_{M_l} (b_i^* e_1 y e_1 b_i x_1^* v_1 x_2 x_3^* v_2 \cdots v_l x_{2l} e_{[0,l]})) \widehat{}$$

$$= \delta^{2l+1} \sum_i w^*_{-2l} (b_i^* E_N (b_i x_1^*) e_1 E_{M_l} (y v_1 x_2 x_3^* v_2 \cdots v_l x_{2l} e_{[0,l]})) \widehat{}$$

$$= w^*_{-2l} (x_1^* v_1 \varphi_{0,l}(y) (v_1 x_2 x_3^* v_2 \cdots v_l x_{2l})) \widehat{}.$$

(b) Since λ_n is additive with respect to n, it is enough to prove the statement for $n = 1$. Note that $\lambda_1(I_{\varepsilon k}) = E'_{\varepsilon(k+1)}$ for all colors εk. By part (a) (resp., conditions (2) and (5) of Theorem 5.4), we get $\lambda_1(P)_{-k} = M_1' \cap M_{k+1}$ (resp., $\lambda_1(P)_{+k} = M' \cap M_k$). So, by the uniqueness part of Theorem 5.4, it remains to check whether the conditions (1) to (5) therein hold in this case; these verifications are completely straight forward and we will skip them.

5.2. Reconstruction of bimodule

Starting with a bimodule planar algebra, we will construct a bifinite bimodule whose associated bimodule planar algebra is isomorphic to the one which we started with. We extensively use the techniques of constructing an extremal subfactor from a spherical unimodular bimodule planar algebra in [19] and

[22]; in fact, we first show that their construction with necessary modifications, works without the assumption of sphericality.

Let P be a unimodular bimodule planar algebra with modulus (δ, δ) such that $\delta > 1$. We will work only with the positive part of P; so, for the time being, we will write P_k in place of P_{+k}. Henceforth, we will exactly follow the set-up of [19] and [22] and not mention this fact at every step; whenever some modifications become necessary, we will explicitly mention them. Set $P^k = \oplus_{l \in N_0} P_l^k$ where $P_l^k := P_{k+l}$ for $k, l \in N_0$. Define the following structures on P^k:

(1) *Multiplication*:

$$P^k \times P^k \supset P_l^k \times P_m^k \ni (x, y)$$

$$\mapsto x \cdot y := \sum_{i=0}^{2(l \wedge m)} P \begin{array}{|c|} \hline \end{array} \in \bigoplus_{i=0}^{2(l \wedge m)} P_{l+m-i}^k \subset P^k$$

We will also denote the element in the sum when $i = 0$, by $x \odot y$; the i-th element will be denoted by $x \odot_i y$.

(2) *∗-structure*:

$$P^k \supset P_l^k \ni x \overset{\dagger}{\mapsto} x^\dagger := P \begin{array}{|c|} \hline \end{array} \in P_l^k \subset P^k$$

(3) *Trace*:

$$P^k \supset P_l^k \ni x \overset{t_k}{\mapsto} t_k(x) := \delta_{l=0} \delta^{-k} P_{TR_k^l}(x) \in C$$

P^k becomes an associative ∗-algebra with respect to (1) and (2).

REMARK 5.8. The trace-functional on P^k which was defined in [19], [22], does not satisfy tracial property in the absence of sphericality. So, we had to take this specific spherical isotopy of the trace in [19], [22] so that the functional defined in (3), is indeed a trace on P^k.

To see the tracial property, note that for $x \in P_l^k$, $y \in P_m^k$,

$$t_k(x \cdot y) = \delta_{l=m} \delta^{-k} P \left(\begin{array}{c} x \\ 2l \\ y \end{array} \right) = t_k(y \cdot x).$$

We define a sesquilinear form $\langle \cdot, \cdot \rangle$ on P^k by $\langle x, y \rangle = t_k(x^\dagger \cdot y)$ for $x, y \in P^k$. It is straight forward to check that (i) t_k is positive definite and hence the sesquilinear form gives an inner product, (ii) $\{P_l^k\}_{l \in N_0}$ are mutually orthogonal subspaces of P^k with respect to this inner product. Let $L : P^k \to \mathrm{End}_C(P^k)$

and $R : P^k \to \mathrm{End}_{\mathsf{C}}(P^k)$ denote the left and right multiplication operators respectively. The proof of the boundedness of these operators in [19], [22] needs a little modification in the non-spherical case because the inner product is slightly different; however, the main idea of the proof will remain the same.

LEMMA 5.9. L_a and R_a are bounded for $a \in P^k$.

PROOF. Without loss of generality, let $a \in P_l^k$ for $l \in \mathsf{N}_0$. Now, $L_a = \sum_{i=0}^{2l} L_a^i$ where $L_a^i : P^k \to P^k$ is defined by

$$P^k \supset P_m^k \ni x \xmapsto{L_a^i} \delta_{i \leq 2m} \, (a \odot_i x) \in P_{l+m-i}^k \subset P^k.$$

Since $\{P_l^k\}_{l \in \mathsf{N}_0}$ are mutually orthogonal, therefore it is enough to show that there exists an $M > 0$ such that $L_a^i(x) \leq M\|x\|$ for all $x \in P_m^k$, $m \in \mathsf{N}_0$. Without loss of generality, let $m > 2l$. Note that

$$\|L_a^i(x)\|^2 = \delta^{-k} P \left(\begin{array}{c} a \,|\, k+i \quad\quad y \\ k \;\; 2l-i \quad k+i \\ a^* |\, k+i \quad\quad y \end{array} \right)$$

where $y \in P_{(-)^i(k+i)}$ is the positive square root of

$$P \left[\begin{array}{c} i \,\backslash x / k \\ (-)^i \,|\, k+2m-i \\ i / x^* \backslash k \end{array} \right]$$

To see that this is indeed a positive element, one needs to take an appropriate rotation of x and use positivity of the action of the right conditional expectation tangle from $(-)^i(k+m)$ to $(-)^i(k+i)$. Thus, $\|L_a^i(x)\|^2$ can also be expressed as $\|P_{T_a}(y)\|^2$ where T_a is the semi-labelled tangle $\left[\begin{array}{c} + \,|\, k \\ a \bullet \, k+i \quad\bullet\;\; k+i \\ 2l-i \quad (-)^i \end{array} \right]$. We equip the domain $P_{(-)^i(k+i)}$ of P_{T_a}, with the norm $\|\cdot\|'$ coming from $\delta^{-k} P_{TR_{(-)^i(k+i)}^l}$ so that $\|y\|' = \|x\|$. Hence, the desired inequality is satisfied by setting $M :=$ the operator norm of P_{T_a} which is independent of $m > 2m$ and $x \in P_m^k$. Boundedness of R_a will follow using the same kind of arguments.

Let $\mathscr{H}_k$ be the completion of P^k and $M_k := (LP^k)'' \subset \mathscr{L}(\mathscr{H}_k)$. Note that $\hat{1}_{P^k} \in \mathscr{H}_k$ is a cyclic, separating, trace vector for M_k. The unital $*$-algebra inclusion $P^k \supset P_l^k \ni x P \left[\begin{array}{c} k \; x \; k \\ + \;\; 2l \end{array} \right] \in P_l^{k+1} \subset P^{k+1}$ induces an inclusion $M_k \hookrightarrow M_{k+1}$. Note that $LP_0^k \subset M_0' \cap M_k$. One can also prove that $P_k \ni x \mapsto$

$L_x \in M_0' \cap M_k$ is a C^*-algebra isomorphism and M_k's are II_1-factors; for proofs, see [19], [22] where $\delta > 1$ is crucially used.

REMARK 5.10.   The above mentioned isomorphism takes the normalized left picture trace on P_k to the M_k-trace on $M_0' \cap M_k$. So, even if we assume that $\{M_k\}_{k \geq 2}$ is a tower of basic constructions of $M_0 \subset M_1$, there is no hope of the unimodular bimodule planar algebra associated to $M_0 \subset M_1$ being isomorphic to P (unless P is spherical) because of the condition in Remark 5.6 (c). We will fix this issue in the following theorem using a certain perturbation.

If Q is a unimodular bimodule planar algebra, then there exists a unique invertible positive central element $z \in Q_{+1}$ such that $Q_{TR_{+1}^l}(\cdot) = Q_{TR_{+1}^r}(\cdot z)$. It is easy to check that z is a weight of Q and $Q_{TR_{\varepsilon k}^l}(\cdot) = Q_{TR_{\varepsilon k}^r}(\cdot z_{\varepsilon k})$. We will refer this as the *trace intertwiner weight of Q*.

THEOREM 5.11. *If Q is a unimodular bimodule planar algebra with modulus (δ, δ), then there exists a finite index subfactor $M_0 \subset M_1$ of type II_1 whose associated planar algebra is isomorphic to Q.*

PROOF.   Without loss of generality, we may assume $\delta > 1$ because index of a bimodule planar algebra is at least 1 (which follows from positivity of the action of the trace tangles) and the index 1 case is a triviality. Let P be the perturbation of Q by the decomposition $z = z^{1/2} \cdot z^{1/2}$ of the trace intertwiner weight z of Q. Clearly, P is a unimodular bimodule planar algebra. Consider the tower of II_1-factors $\{M_k\}_{k \geq 0}$ constructed from P right before Remark 5.10 and the isomorphism $Q_k = P_k \ni x \overset{\phi}{\mapsto} L_x \in M_0' \cap M_k$. Note that the map $Q_{RI_k} = P_{RI_k}$ under ϕ, is given by the inclusion $M_k \subset M_{k+1}$. So, by Theorem 5.4, it remains to show that (i) $\{M_k\}_{k \geq 2}$ is a tower of basic construction of $M_0 \subset M_1$ with Jones projections $\{e_k := \delta^{-1}\phi(Q_{E_{+k}})\}_{k \geq 1}$ and (ii) $\phi \circ Q_{E_{+k}'} = \delta^{-1}\sum_{b \in B} b\phi(\cdot)b^*$ for all $k \geq 1$, where B is a right Pimsner-Popa basis for $M_0 \subset M_1$.

PROOF OF (i).   First note that the unique trace preserving conditional expectation from M_k to M_{k-1} is induced (via L) by the map

$$P^k \supset P_l^k \ni x \overset{E_{k-1}^k}{\longmapsto} \begin{cases} \delta^{-1}P \; \boxed{\begin{smallmatrix} z \\ \\ k-1 \quad x \quad k-1 \\ + \quad\quad 2l \end{smallmatrix}} \in P_l^{k+1} \subset P^{k+1} \\[4ex] \delta^{-1}P \; \boxed{\begin{smallmatrix} z^{-1} \\ \\ k-1 \quad\quad k-1 \\ + \quad x \quad 2l \end{smallmatrix}} \in P_l^{k+1} \subset P^{k+1} \end{cases}$$

Using the definition of $P = Q^{(z^{1/2}, z^{1/2})}$, it immediately follows that $Q_{E_{+k}} x Q_{E_{+k}} = \delta E_{k-1}^k(x) Q_{E_{+k}}$ for all $x \in P^k$. Thus, for all $y \in M_k$, $e_k y e_k = E_{M_{k-1}}^{M_k}(y)e_k$.

Moreover, it is straight forward to check that for all $x \in P^{k+1}$, $Q_{E_{+k}} x = \delta Q_{E_{+k}} E^k_{k-1}(Q_{E_{+k}} x)$ which implies $e_k M_{k+1} = e_k M_k$. Thus, by Proposition 2.11, M_{k+1} is a basic construction of $M_{k-1} \subset M_k$ with Jones projection e_k and $[M_k : M_{k-1}] = \delta^2$.

PROOF OF (ii). Since $L^2(M_1) \cong \mathscr{H}_1 \cong \oplus_{k \in \mathsf{N}_0} P^1_k$, therefore for each $b \in B$ and $k \in \mathsf{N}_0$, there exists $b_k \in P^1_k$ such that $\hat{b} = \sum_{k \in \mathsf{N}_0} \hat{L}_{b_k}$. Also, $\hat{b}^* = \sum_{k \in \mathsf{N}_0} \hat{L}_{b^\dagger_k}$. Since B is a basis for $M_0 \subset M_1$, we have

$$\hat{L}_{1_{P^2}} = \hat{1}_{M_2} = \sum_{b \in B} (be_1 b^*)\hat{\ } = \delta^{-1} \sum_{b \in B, k, l \in \mathsf{N}_0} (L_{b_k \cdot Q_{E_{+1}} \cdot b^\dagger_l})\hat{\ }$$

$$= \delta^{-1} \sum_{\substack{m \in \mathsf{N}_0}} \sum_{\substack{b \in B, \, i,k,l \in \mathsf{N}_0 \text{ s.t.} \\ l+k=m+i, i \leq 2(k \wedge l)}} (L_{(b_k \odot Q_{E_{+1}}) \odot_i b^\dagger_l})\hat{\ }.$$

Using the relation $Q_{E_{+1}} = P_{E^{(z^{-1/2}, z^{-1/2})}_{+1}}$, the above gives the following formula in terms of pictures:

$$\sum_{\substack{b \in B, \, i,k,l \in \mathsf{N}_0 \text{ s.t.} \\ l+k=m+i, i \leq 2(k \wedge l)}} P \; \boxed{\begin{array}{c} b_k \;\; \overset{i}{} \;\; b^*_l \\ + \;\; 2k-i \;\; 2l-i \end{array}} = \delta \delta_{m=0} P \; \boxed{\begin{array}{c} z \\ + \end{array}}$$

for all $m \in \mathsf{N}_0$ where the sum in the left hand side is with respect to the Hilbert space norm in P^2. Now, let $L_x \in M'_0 \cap M_k = LP^k_0$ for $x \in P^k_0 = Q^k_0$. So, $\sum_{b \in B} (bL_x b^*)\hat{\ } = \sum_{m \in \mathsf{N}_0} \sum_{\substack{b \in B, \, i,k,l \in \mathsf{N}_0 \text{ s.t.} \\ l+k=m+i, i \leq 2(k \wedge l)}} (L_{(b_k \odot x) \odot_i b^\dagger_l})\hat{\ }$. Note that

$$(b_k \odot x) \odot_i b^\dagger_l = P \; \boxed{\begin{array}{c} k-1 \qquad\qquad k-1 \\ b_k \;\; \overset{i}{} \;\; b^*_l \\ + \;\; 2k-i \;\; 2l-i \end{array}}$$

Using the isomrphism $\mathscr{H}_k \supset (P^k)\hat{\ } \ni \hat{x} \mapsto \hat{L}_x \in (M_k)\hat{\ } \subset L^2(M_k)$ and replacing the sum (in the expansion of $\sum_{b \in B} (bL_x b^*)\hat{\ }$) of the stuff inside the dotted box by the right side of the above formula, we obtain

$$\sum_{b \in B} b\phi(x)b^* = \sum_{b \in B} bL_x b^* = \delta L_P \boxed{\begin{array}{c} k-1 \qquad\qquad k-1 \\ \overset{x}{\underset{z}{\bigcirc}} \\ + \end{array}}$$

$$= \delta L_{Q_{E'_{+k}}(x)} = \delta\phi(Q_{E'_{+k}}(x)).$$

REMARK 5.12. We must mention here that Popa (in [30]) started with his set up of 'generalized λ-lattice' (which, we believe, is analogous to a unimodular bimodule planar algebra in our case) and gave a general method of constructing a (non-extremal) subfactor whose standard invariant corresponds to the generalized λ-lattice.

Using Theorem 5.11, we now construct a bimodule from a bimodule planar algebra in the next theorem.

THEOREM 5.13. *Let P be a bimodule planar algebra with modulus (δ_-, δ_+) such that $(\delta_+ \delta_-) > 1$. Then there exists a bifinite bimodule whose associated bimodule planar algebra is isomorphic to P.*

PROOF. Let Q be the normalization of P and $\delta := \sqrt{\delta_+ \delta_-}$. By Theorem 5.11, there exist a subfactor $N \subset M$ of type II_1, whose associated planar algebra is isomorphic to Q. Without loss of generality, let $\delta_+ > 1$. Choose $p \in \mathcal{P}(M)$ such that $\mathrm{tr}_M(p) = \delta_+^{-1}$. Set $p' := JpJ \in M' := {}_M\mathcal{L}(L^2(M))$, $\mathcal{H} := L^2(M)p = \mathrm{Range}(p')$, $A := Np'$ and $B := M_p$ where J is the canonical anti-unitary involution on $L^2(M)$. Consider the bimodule ${}_A\mathcal{H}_B$. Note that (i) $\dim({}_A\mathcal{H}) = \mathrm{tr}_{M'}(p') \dim({}_N L^2(M)) = \delta_+^{-1}\delta^2 = \delta_-$, (ii) $\dim(\mathcal{H}_B) = (\mathrm{tr}_M(p))^{-1} \dim(L^2(M)_M) = \delta_+$ and (iii) $B' := \mathcal{L}_B(\mathcal{H}) = Mp'$. Thus, the subfactors $A \subset B'$ and $N \subset M$ are isomorphic, and so are their associated planar algebras. The proof of Theorem 5.11 implies that Q (= normalized P) is isomorphic to the normalized bimodule planar algebra associated to the bimodule ${}_A L^2(B')_{B'}$ which, by Theorem 5.2 (iii), is isomorphic to that of ${}_A\mathcal{H}_B$. So, P is isomorphic to the bimodule planar algebra associated to ${}_A\mathcal{H}_B$.

REMARK 5.14. Since we have an extension of Jones' theorem [13, Theorem 4.2.1] as well as its converse for any finite index subfactor, we could have very well referred a unimodular bimodule planar algebra as a 'subfactor-planar algebra', but we abstain from doing so as the term subfactor planar algebra has already been in use for a spherical unimodular bimodule planar algebra.

6. Examples of perturbations

In this section, we will provide two examples of perturbations of bimodule planar algebras by non-scalar weights. The first holds for any non-spherical planar algebras whereas the second example is a more concrete one which involves *diagonal subfactors*.

6.1. *Non-spherical to spherical*

In Section 3, we saw that the perturbation class of every bimodule planar algebra contains a unimodular one, namely, its normalization. In view of this,

we analyze the natural question whether there is more than one unimodular bimodule planar algebra (upto isomorphism) in the perturbation class of a bimodule planar algebra. The answer is negative for irreducible ones because the weights have to be scalars.

Let P be a unimodular bimodule planar algebra (not necessarily spherical) with modulus (δ, δ). Suppose $z \in P_{+1}$ denotes the trace intertwiner weight of P (discussed in Section 5.2). Note that $z^{1/2}$ is also a weight of P. Consider the planar algebra $Q := P^{(z^{1/4}, z^{1/4})}$. It is easy to check that Q is also a bimodule planar algebra. Now, for $x \in Q_{+1} = P_{+1}$, we have

$$Q_{TR^l_{+1}}(x) = P_{TR^l_{+1}}{}^{(z^{1/4}, z^{1/4})}(x) = P_{TR^l_{+1}}(z^{-1/2}x) = P_{TR^r_{+1}}(z^{1/2}x)$$
$$= P_{TR^r_{+1}}{}^{(z^{1/4}, z^{1/4})}(x) = Q_{TR^r_{+1}}(x),$$

where the third equality follows from the trace intertwining property of z and the rest follow directly from the definition of perturbation. Setting $x = 1_{P_{+1}}$ in the above equation, we conclude that Q is unimodular with modulus $P_{TR^l_{+1}}(z^{-1/2}) = P_{TR^r_{+1}}(z^{1/2})$ and thereby, is the same as its normalization; thus, Q is spherical. We include this observation in the following proposition.

PROPOSITION 6.1. *The perturbation class of every bimodule planar algebra contains a unique spherical unimodular bimodule planar algebra which can also be characterized by the unimodular one having the minimal index value. In other words, any bimodule planar algebra assuming the minimal index in its perturbation class must be spherical.*

PROOF. To show minimality of the index, consider a spherical unimodular bimodule planar algebra P with modulus δ and a positive weight z of P. Set $Q := P^{(z^{1/2}, z^{1/2})}$. Let $\{p_i\}_{i=1}^n$ be the set of minimal central projections of $P_{+1} = Q_{+1}$. Note that $\delta = \sum_i \omega_i$ where $\omega_i = P_{TR^r_{+1}}(p_i) = P_{TR^l_{+1}}(p_i)$ for all i. Also, for all i, there exists $\lambda_i > 0$ such that $z = \sum_i \lambda_i p_i$. If (δ_-, δ_+) denotes the modulus of Q, then $\delta_\pm = \sum_i \lambda_i^{\pm 1} \omega_i$. Thus,

$$\text{index}(Q) = \sum_{i,j=1}^n \frac{\lambda_i}{\lambda_j} \omega_i \omega_j = \left(\sum_{i=1}^n \omega_i^2 \right) + \sum_{1 \le j < k \le n} \left(\frac{\lambda_j}{\lambda_k} + \frac{\lambda_k}{\lambda_j} \right) \omega_i \omega_j$$
$$\ge \left(\sum_i^n \omega_i^2 \right) + \sum_{1 \le j < k \le n} 2\omega_i \omega_j = \left(\sum_i \omega_i \right)^2 = \text{index}(P)$$

where the equality occurs if and only if $\lambda_i = \lambda_j$ for $1 \le i, j \le n$, that is, z is a scalar weight. Now, if $\tilde{P}$ is any unimodular bimodule planar algebra assuming minimal index in its perturbation class, then there exists a weight

w such that $\tilde{Q} := \tilde{P}^{(w^{1/2}, w^{1/2})}$ is unimodular spherical. Again, by the above argument, index($\tilde{Q}$) being minimal, is same as index($\tilde{P}$) and w is a scalar weight. Since both $\tilde{P}$ and $\tilde{Q}$ are unimodular, therefore $w = 1_{P_{+1}}$ and hence, $\tilde{P} = \tilde{Q}$ is spherical. This also shows the uniqueness of a spherical unimodular bimodule planar algebra in the perturbation class of a bimodule planar algebra.

REMARK 6.2. Instance of minimizing index of a conditional expectation onto a subfactor already appeared in the work of Hiai (see [11]) and then Popa (see [28]). Proposition 6.1 gives a nice way of minimizing index using perturbation of planar algebra.

REMARK 6.3. From the proof of Proposition 6.1, it is clear that if the perturbation of a spherical bimodule planar algebra by a positive weight, is spherical, then the weight must be a scalar one.

REMARK 6.4. It is easy to check that a finite depth bimodule planar algebra is always spherical because by Perron-Frobenius theorem, the index must be equal to the norm-square of the pricipal graph and perturbation does not change the principal graphs.

6.2. Spherical to non-spherical

Here we try to find whether we can perturb one of the known spherical unimodular planar algebras and get a non-spherical one. For this, we study the case of diagonal subfactors.

In Section 6.1, we found that the perturbation of a bimodule planar algebra P by a weight z, has modulus $(P_{TR^l_{+1}}(z^{-1/2}), P_{TR^r_{+1}}(z^{1/2}))$; so, for the perturbation to be unimodular, we need to find z satisfying $P_{TR^l_{+1}}(z^{-1/2}) = P_{TR^r_{+1}}(z^{1/2})$. Consider the diagonal planar algebra P constructed in [3] with respect to the free group F_2 generated by two free generators a_η for $\eta \in I := \{-, +\}$ and the trivial cocycle. We briefly recall (from [3]) few aspects of this planar algebra P which will be needed for further analysis.

Vector spaces: $P_{\eta 0} := \mathbb{C}$ and $P_{\eta k} = \mathbb{C}\{\underline{\varepsilon} \in I^{2k} : \text{alt}_\eta(\underline{\varepsilon}) = e\}$ for all $k \geq 1$, where e denotes the identity of F_2 and alt_η is given by

$$I^n \ni (\varepsilon_1, \cdots, \varepsilon_n) \overset{\text{alt}_\eta}{\longmapsto} \begin{cases} a_{\varepsilon_1}^{-1} a_{\varepsilon_2} a_{\varepsilon_3}^{-1} \cdots a_{\varepsilon_n}^{(-1)^n} \in F_2 & \text{if } \eta = +, \\ a_{\varepsilon_1} a_{\varepsilon_2}^{-1} a_{\varepsilon_3} \cdots a_{\varepsilon_n}^{(-1)^{n-1}} \in F_2 & \text{if } \eta = -. \end{cases}$$

Action of tangles: For a tangle $T : (\eta_1 k_1, \ldots, \eta_b k_b) \to \eta_0 k_0$, the action $P_T : P_{\eta_1 k_1} \times \cdots \times P_{\eta_b k_b} \to P_{\eta_0 k_0}$ is given by

$$\langle P_T(\underline{\varepsilon}^1, \cdots, \underline{\varepsilon}^b), \underline{\varepsilon}^0 \rangle := \#\{f : \{\text{strings in } T\}$$
$$\to I \mid f|_{\text{marked points of } i\text{-th disc in } T} = \underline{\varepsilon}^i \text{ for } 0 \leq i \leq b\}$$

where $\underline{\varepsilon}^i$ belongs to the distinguished basis of $P_{\eta_i k_i}$ for $0 \leq i \leq b$.

Fix a positive scalar $\lambda_- = \lambda_+^{-1}$. Consider $z := [\lambda_-(-, -) + \lambda_+(+, +)] \in P_{+1}$ and $\lambda_{\eta\underline{\varepsilon}} := \prod_{i=1}^k \lambda_{\varepsilon_i}^{\eta(-1)^{i-1}}$ for $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k) \in I^k$. Using the relation $z^{-1} = [\lambda_-^{-1}(-, -) + \lambda_+^{-1}(+, +)] \in P_{+1}$ and the action of tangles in Definition 3.1, one can derive that $z_{\eta k} = \sum_{\underline{\varepsilon} \in I^k} \lambda_{\eta\underline{\varepsilon}}(\underline{\varepsilon}, \tilde{\underline{\varepsilon}}) \in P_{\eta k}$, where $\tilde{\underline{\varepsilon}}$ is the sequence obtained by reversing the order in $\underline{\varepsilon}$. From the action of the multiplication tangle $M_{\eta k}$, we have $z_{\eta k}(\underline{\varepsilon}, \tilde{\underline{\nu}}) z_{\eta k}^{-1} = \lambda_{\eta\underline{\varepsilon}} \lambda_{\eta\underline{\nu}}^{-1}(\underline{\varepsilon}, \tilde{\underline{\nu}}) = \lambda_{\eta(\underline{\varepsilon}, \tilde{\underline{\nu}})}(\underline{\varepsilon}, \tilde{\underline{\nu}})$ for $\underline{\varepsilon}, \underline{\nu} \in I^k$ such that $\mathrm{alt}_\eta(\underline{\varepsilon}, \tilde{\underline{\nu}}) = e$. Freeness of a_- and a_+ implies that there exists a configuration of non-crossing pairings of matching signs (abbreviated as 'NC-pairing') in the sequence $(\underline{\varepsilon}, \tilde{\underline{\nu}})$, which implies that $\lambda_{\eta(\underline{\varepsilon}, \tilde{\underline{\nu}})} = 1$. Thus, $z_{\eta k}$ is central and hence, z is a weight of P. Further, positivity of z implies that $Q := P^{(z^{1/2}, z^{1/2})}$ is a bimodule planar algebra. To check unimodularity of Q, note that $P_{TR_{+1}^l}(z^{-1}) = (\lambda_-)^{-1} + (\lambda_+)^{-1} = \lambda_- + \lambda_+ = P_{TR_{+1}^r}(z)$. Now, Q is spherical if and only if, for all $s, t \in \mathsf{C}$,

$$s\lambda_-^{-1} + t\lambda_+^{-1} = Q_{TR_{+1}^l}(s(-, -) + t(+, +)) = Q_{TR_{+1}^r}(s(-, -) + t(+, +))$$
$$= s\lambda_- + t\lambda_+$$

if and only if $\lambda_- = 1 = \lambda_+$, that is, $z = 1_{P_{+1}}$. Clearly, the range of the index values of these perturbed planar algebras is $[4, \infty)$. We gather the above observation in the following proposition.

PROPOSITION 6.5. *Let P denote the diagonal planar algebra constructed in [3, § 3] with respect to the free group F_2 generated by two free generators and the trivial cocycle. Then one can perturb P to obtain a continuum of unimodular bimodule planar algebras $\{Q^\gamma\}_{\gamma \geq 4}$ such that Q^γ has (i) index γ, (ii) is equal to P if $\gamma = 4$ and (iii) is non-spherical if and only if $\gamma \neq 4$.*

REMARK 6.6. These perturbed planar algebras turn out to be the ones associated to the subfactors with index greater than 4 constructed in [12] (which are all non-extremal), as was pointed out by Vaughan Jones.

We give an explicit proof of the above remark in the following proposition.

PROPOSITION 6.7. *Let $N \subset M$ be a subfactor of type II_1 with a partition of unity $\{p_\varepsilon \in \mathscr{P}(N' \cap M) \mid \varepsilon \in I\}$ satisfying (i) $Np_\varepsilon = M_{p_\varepsilon}$ and (ii) $c_- := \mathrm{tr}_M(p_-) \neq \mathrm{tr}_M(p_+) =: c_+$. (Such subfactors were constructed by Jones in [12] with M having full fundamental group.) Then, the planar algebra associated to $N \subset M$ is isomorphic to $Q^{(c_-c_+)^{-1}}$ (as in Proposition 6.5).*

PROOF. Condtions (i) and (ii) imply $p_\varepsilon \in \mathscr{P}_{\mathrm{minl}}(N' \cap M)$, $N' \cap M \cong \mathsf{C}p_- \oplus \mathsf{C}p_+$ and $c_\varepsilon \mathrm{tr}_{N'}(p_\varepsilon)[M : N] = [M_{p_\varepsilon} : Np_\varepsilon] = 1$. So, $[M : N] = c_-^{-1} + c_+^{-1} = (c_-c_+)^{-1}$ and $\mathrm{tr}_{N'}(p_\varepsilon) = c_{-\varepsilon}$ and hence, $N \subset M$ is non-extremal.

Suppose $\{M_k\}_{k \geq 1}$ is a tower of basic constructions of $N \subset M$ with Jones projections $\{e_k\}_{k \geq 1}$ and $\tilde{P}$ be the planar algebra associated to this tower. Let $p_\eta^{(n)}$ denote the element $P_{\boxed{\begin{smallmatrix} + \\ n-1 \end{smallmatrix} \,\bullet p_\eta}}$ or $P_{\boxed{\begin{smallmatrix} + \\ n-1 \end{smallmatrix} \, p_\eta \bullet}}$ (lying in $\mathscr{P}(M'_{n-2} \cap M_{n-1})$ by Corollary 5.7 (b)) according as n is even or odd, $c_\eta^{(n)} := \operatorname{tr}_{M_{n-1}}(p_\eta^{(n)}) = c_{(-)^{n-1}\eta}$, $p_{\underline{\varepsilon}} := p_{\varepsilon_1}^{(1)} \cdots p_{\varepsilon_n}^{(n)} \in \mathscr{P}(N' \cap M_{n-1})$ and $c_{\underline{\varepsilon}} := \prod_{i=1}^n c_{\varepsilon_i}^{(i)} = \operatorname{tr}_{M_{n-1}}(p_{\underline{\varepsilon}})$ for $\eta \in I$, $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in I^n$. Clearly, $\{p_{\underline{\varepsilon}} : \underline{\varepsilon} \in I^n\}$ forms a partition of unity. Further, the formula involving indices of cut-down subfactors implies $N p_{\underline{\varepsilon}} = (M_{n-1})_{p_{\underline{\varepsilon}}}$ and hence $p_{\underline{\varepsilon}}$ is minimal in $\tilde{P}_n$. The set of $\operatorname{tr}_{M_{n-1}}$-values of these minimal projections is given by $\{c(n, k) := c_-^k c_+^{n-k} \mid 0 \leq k \leq n\}$.

In order to establish an isomorphism between $\tilde{P}$ and $Q^{(c_- c_+)^{-1}}$, we first prove that for $\underline{\varepsilon}, \underline{\eta} \in I^n$, $c_{\underline{\varepsilon}} = c_{\underline{\eta}}$ if and only if there exists an NC-pairing for the sequence $(\underline{\varepsilon}, \tilde{\underline{\eta}})$.

'if' part: The pairings give the equation $c_{(\underline{\varepsilon}, \tilde{\underline{\eta}})} = (c_- c_+)^n$. On the other hand, $c_{(\underline{\varepsilon}, \tilde{\underline{\eta}})} = c_{\underline{\varepsilon}} c_{-\underline{\eta}} = c_{\underline{\varepsilon}} \left[\frac{(c_- c_+)^n}{c_{\underline{\eta}}} \right]$ where $-(\eta_1, \ldots, \eta_n) := (-\eta_1, \ldots, -\eta_n)$.

'only if' part: We use induction on the length n. The intial case of $n = 1$ is a triviality since $c_- \neq c_+$. In the induction step, let $n > 1$ and $0 \leq k \leq n$ such that $c_{\underline{\varepsilon}} = c(n, k) = c_{\underline{\eta}}$. We may also assume $0 < k < n$ without loss of generality because there is exactly one sequence in each of the cases $k = 0$ and $k = n$. Note that if both $(\underline{\varepsilon}, \tilde{\underline{\eta}})$ and $(\underline{\eta}, \tilde{\underline{\nu}})$ have NC-pairings, then so does $(\underline{\varepsilon}, \tilde{\underline{\nu}})$ (resp., $(\underline{\eta}, \tilde{\underline{\varepsilon}})$) which can easily be obtained by taking the product (resp., $*$) of the Temperley-Lieb diagrams associated to the NC-pairings. Consider $\underline{\varepsilon}(n, k) \in I^n$ given by

$$\varepsilon_i(n, k) = \begin{cases} (-)^i & \text{if } 1 \leq i \leq k, \\ (-)^{i+1} & \text{if } k+1 \leq i \leq n. \end{cases}$$

Clearly, $c_{\underline{\varepsilon}(n,k)} = c(n, k)$; so, it is enough to show that $(\underline{\varepsilon}, \tilde{\underline{\varepsilon}}(n, k))$ has an NC-pairing. We begin by showing that there exists a $\underline{\nu} \in I^n$ such that $\nu_1 = - = \varepsilon_1(n, k)$ and $(\underline{\varepsilon}, \tilde{\underline{\nu}})$ has an NC-pairing (and hence $c_{\underline{\varepsilon}} = c_{\underline{\nu}}$ by the 'if' part). Suppose first that $\varepsilon_1 = +$. Since $0 < k < n$, $\underline{\varepsilon}$ must have a pair of consecutive matching signs. Choose the left most consecutive matching pair in $\underline{\varepsilon}$ and change both the signs in the pair to obtain $\underline{\gamma} \in I^n$. Observe that there exists an NC-pairing for $(\underline{\varepsilon}, \tilde{\underline{\gamma}})$. Also, the first consecutive matching pair in $\underline{\gamma}$ gets closer to the left side than the one in $\underline{\varepsilon}$. Apply this method succesively to obtain the desired $\underline{\nu}$. And, if $\varepsilon_1 = -$, then simply take $\underline{\nu} = \underline{\varepsilon}$. Let $\underline{\nu}'$ (resp., $\underline{\varepsilon}'(n, k)$) be the sequence obtained from $\underline{\nu}$ (resp., $\underline{\varepsilon}(n, k)$) after removing the minus sign in the first entry. Since $c_{\underline{\nu}'} = c_{\underline{\varepsilon}'(n,k)}$, by induction assumption, there exists an NC-pairing for $(\underline{\varepsilon}'(n, k), \tilde{\underline{\nu}}')$ and hence for $(\underline{\varepsilon}(n, k), \tilde{\underline{\nu}})$.

It is also true that for $\underline{\varepsilon}, \underline{\eta} \in I^n$, $c_{\underline{\varepsilon}} = c_{\underline{\eta}}$ if and only if $p_{\underline{\varepsilon}}$ and $p_{\underline{\eta}}$ are equivalent in $\tilde{P}_n$. Suppose $c_{\underline{\varepsilon}} = c_{\underline{\eta}}$. So, there exists an NC-pairing for $(\underline{\varepsilon}, \underline{\tilde{\eta}})$. Let T be the Temperley-Lieb diagram associated to this NC-pairing. Since $p_{\underline{\varepsilon}}$ and $p_{\underline{\eta}}$ are minimal in $\tilde{P}_n$, therefore it is enough to check $p_{\underline{\varepsilon}}(\tilde{P}_T)p_{\underline{\eta}} \neq 0$ which easily follows by considering the scalar $\mathrm{tr}_{M_{n-1}}\left(p_{\underline{\varepsilon}}(\tilde{P}_T)p_{\underline{\eta}}(\tilde{P}_{T^*})p_{\underline{\varepsilon}}\right)$ and then using the action of tangles to show it is non-zero (in fact, positive). In particular,

$$p(\underline{\varepsilon}, T, \underline{\tilde{\eta}}) := \sqrt{\frac{c_{\underline{\varepsilon}}}{\mathrm{tr}_{M_{n-1}}\left(p_{\underline{\varepsilon}}(\tilde{P}_T)p_{\underline{\eta}}(\tilde{P}_{T^*})\right)}}\, p_{\underline{\varepsilon}}(\tilde{P}_T)p_{\underline{\eta}}$$

is a partial isometry with $p_{\underline{\varepsilon}}$ (resp., $p_{\underline{\eta}}$) as its final (resp., initial) projection. To show that $p(\underline{\varepsilon}, T, \underline{\tilde{\eta}})$ is independent of T, consider two Temperley-Lieb elements T_1 and T_2 for two NC-pairings of $(\underline{\varepsilon}, \underline{\tilde{\eta}})$. It immediately follows from the action of tangles that $p(\underline{\varepsilon}, T_1, \underline{\tilde{\eta}})(p(\underline{\varepsilon}, T_2, \underline{\tilde{\eta}}))^* \in \mathsf{R}_+\left[p_{\underline{\varepsilon}}(\tilde{P}_{T_1})p_{\underline{\eta}}(\tilde{P}_{T_2^*})p_{\underline{\varepsilon}}\right] = \mathsf{R}_+\left[p_{\underline{\varepsilon}}(\tilde{P}_{T_1 \cdot T_2^*})p_{\underline{\varepsilon}}\right] = \mathsf{R}_+ p_{\underline{\varepsilon}}$. Hence, $p(\underline{\varepsilon}, T_1, \underline{\tilde{\eta}}) = p(\underline{\varepsilon}, T_2, \underline{\tilde{\eta}})$. So, we will denote the partial isometry simply by $p(\underline{\varepsilon}, \underline{\tilde{\eta}})$.

Consider the diagonal planar algebra P in Proposition 6.5 and its perturbation $Q = Q^{(c_- c_+)^{-1}} = P^{(z^{1/2}, z^{1/2})}$ with respect to the weight $z := \sqrt{\frac{c_-}{c_+}}(-, -) + \sqrt{\frac{c_+}{c_-}}(+, +)$. Define $\varphi : Q_+ \to \tilde{P}_+$ by

$$Q_{+k} = P_{+k} = (\underline{\varepsilon}, \underline{\tilde{\eta}}) \overset{\varphi}{\mapsto} p(\underline{\varepsilon}, \underline{\tilde{\eta}}) \in \tilde{P}_{+k}.$$

Clearly, φ is a $*$-algebra isomorphism. By Proposition 2.3, it is enough to show that φ is equivarent with respect to all tangles having positive colors on their discs and in fact, for the following tangles:

Right-inclusion tangles: For $\underline{\varepsilon}, \underline{\eta} \in I^n$ such that $\mathrm{alt}_+(\underline{\varepsilon}, \underline{\tilde{\eta}}) = e$ (equivalently, $(\underline{\varepsilon}, \underline{\tilde{\eta}})$ has an NC-pairing), we have $\varphi \circ Q_{RI_{+n}}(\underline{\varepsilon}, \underline{\tilde{\eta}}) = p(\underline{\varepsilon}, -, -, \underline{\tilde{\eta}}) + p(\underline{\varepsilon}, +, +, \underline{\tilde{\eta}}) \in \tilde{P}_{+(n+1)}$. Thus, it is enough to show that $p(\underline{\varepsilon}, \underline{\tilde{\eta}}) = p(\underline{\varepsilon}, -, -, \underline{\tilde{\eta}}) + p(\underline{\varepsilon}, +, +, \underline{\tilde{\eta}})$. Let T be the Temperley-Lieb diagram corresponding to an NC-pairing for $(\underline{\varepsilon}, \underline{\tilde{\eta}})$; then, $RI_{+n} \circ T$ gives an NC-pairing for $(\underline{\varepsilon}, \pm, \pm, \underline{\tilde{\eta}})$. Further, we have $c_{(\underline{\varepsilon}, \pm)} = c_{\underline{\varepsilon}} c_{\pm}^{(n+1)}$ and

$$\mathrm{tr}_{M_n}\left(p_{(\underline{\varepsilon}, \pm)}(\tilde{P}_{RI_{+n} \circ T})p_{(\underline{\eta}, \pm)}(\tilde{P}_{(RI_{+n} \circ T)^*})\right) = \mathrm{tr}_{M_{n-1}}\left(p_{\underline{\varepsilon}}(\tilde{P}_T)p_{\underline{\eta}}(\tilde{P}_{T^*})\right)c_{\pm}^{(n+1)}$$

which imply $p(\underline{\varepsilon}, \pm, \pm, \underline{\tilde{\eta}}) = p(\underline{\varepsilon}, \underline{\tilde{\eta}})p_{\pm}^{(n+1)}$. Hence, the required equality holds.

Jones projection tangles: From the action of tangles on $\tilde{P}$, we have

$$
\tilde{P}_{E_{+n}} = \sum_{\underline{\varepsilon}', \underline{\varepsilon}'' \in I^{n+1}} p_{\underline{\varepsilon}'} \tilde{P}_{E_{+n}} p_{\underline{\varepsilon}''} = \sum_{\underline{\varepsilon} \in I^{n-1}, \eta, \nu \in I} p_{(\underline{\varepsilon}, \eta, \eta)} \tilde{P}_{E_{+n}} p_{(\underline{\varepsilon}, \nu, \nu)}
$$

$$
= \sum_{\underline{\varepsilon} \in I^{n-1}, \eta, \nu \in I} \sqrt{\frac{\operatorname{tr}_{M_n}\left(p_{(\underline{\varepsilon}, \eta, \eta)}(\tilde{P}_{E_{+n}}) p_{(\underline{\varepsilon}, \nu, \nu)}(\tilde{P}_{E_{+n}})\right)}{c_{(\underline{\varepsilon}, \eta, \eta)}}} p(\underline{\varepsilon}, \eta, \eta, \nu, \nu, \tilde{\underline{\varepsilon}}).
$$

Now, for $\underline{\varepsilon} \in I^{n-1}$ and $\eta, \nu \in I$, note that $\left(\tilde{P}_{E_{+n}}\right) p_{(\underline{\varepsilon}, \nu, \nu)} \left(\tilde{P}_{E_{+n}}\right) = \frac{c_\nu^{(n)}}{\sqrt{c_- c_+}} p_{\underline{\varepsilon}}\left(\tilde{P}_{E_{+n}}\right)$ and

$$
\operatorname{tr}_{M_n}\left(p_{(\underline{\varepsilon}, \eta, \eta)} p_{\underline{\varepsilon}}(\tilde{P}_{E_{+n}})\right) = \sqrt{c_- c_+} \operatorname{tr}_{M_n}\left((\tilde{P}_{E_{+n}}) p_{(\underline{\varepsilon}, \eta, \eta)}(\tilde{P}_{E_{+n}})\right)
$$

$$
= c_\eta^{(n)} \operatorname{tr}_{M_n}\left(p_{\underline{\varepsilon}}(\tilde{P}_{E_{+n}})\right) = c_\eta^{(n)} \sqrt{c_- c_+} \operatorname{tr}_{M_{n-2}}(p_{\underline{\varepsilon}})
$$

$$
= c_\eta^{(n)} c_{\underline{\varepsilon}} \sqrt{c_- c_+}
$$

Since $c_{(\underline{\varepsilon}, \eta, \eta)} = c_- c_+ c_{\underline{\varepsilon}}$, therefore

$$
\tilde{P}_{E_{+n}} = \sum_{\underline{\varepsilon} \in I^{n-1}, \eta, \nu \in I} \sqrt{\frac{c_\eta^{(n)} c_\nu^{(n)}}{c_- c_+}} p(\underline{\varepsilon}, \eta, \eta, \nu, \nu, \tilde{\underline{\varepsilon}}).
$$

On the other hand, using the weight z of P, $Q_{E_{+n}}$ is equal to

$$
\left[\sum_{\underline{\varepsilon} \in I^{n-1}} \sqrt{\frac{c_{(-)^{n-1}-}}{c_{(-)^{n-1}+}}} (\underline{\varepsilon}, -, -, -, -, \tilde{\underline{\varepsilon}}) + (\underline{\varepsilon}, -, -, +, +, \tilde{\underline{\varepsilon}})\right.
$$

$$
\left. + (\underline{\varepsilon}, +, +, -, -, \tilde{\underline{\varepsilon}}) + \sqrt{\frac{c_{(-)^{n-1}+}}{c_{(-)^{n-1}-}}} (\underline{\varepsilon}, +, +, +, +, \tilde{\underline{\varepsilon}})\right].
$$

Thus, φ preserves the action of E_{+n}.

Left conditional expectation tangle: Fix $\nu, \nu' \in I$ and $\underline{\varepsilon}, \underline{\eta} \in I^{n-1}$ such that $alt_+(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu') = e$. Note that $Q_{E'_{+n}}(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu') = \delta_{\nu=\nu'} \bar{Q}_{E'_{+n}}(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu) = \delta_{\nu=\nu'} \sqrt{\frac{c_{-\nu}}{c_\nu}} \sum_{\gamma \in I} (\gamma, \underline{\varepsilon}, \tilde{\underline{\eta}}, \gamma) \overset{\varphi}{\mapsto} \delta_{\nu=\nu'} \sqrt{\frac{c_{-\nu}}{c_\nu}} \sum_{\gamma \in I} p(\gamma, \underline{\varepsilon}, \tilde{\underline{\eta}}, \gamma)$. On the other hand, $\tilde{P}_{E'_{+n}}(p(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu')) = \tilde{P}_{E'_{+n}}(p_\nu \, p(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu') p_{\nu'}) = \delta_{\nu=\nu'} \tilde{P}_{E'_{+n}}(p(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu))$. Now, we may assume that $alt_+(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu) = e$ which implies $alt_-(\underline{\varepsilon}, \tilde{\underline{\eta}}) = e$ equivalently, $(\underline{\varepsilon}, \tilde{\underline{\eta}})$ has an NC-pairing given by a Temperley-Lieb diagram T (say). Then, $T_1 := \bar{LI}_{-(n-1)} \circ T$ induces an NC-pairing for $(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu)$. So,

we have

$$\tilde{P}_{E'_{+n}}(p(\nu, \underline{\varepsilon}, \tilde{\underline{\eta}}, \nu))$$

$$= \sqrt{\frac{c_{(\nu,\underline{\varepsilon})}}{\mathrm{tr}_{M_{n-1}}\left(p_{(\nu,\underline{\varepsilon})}(\tilde{P}_{T_1})p_{(\nu,\underline{\eta})}(\tilde{P}_{T_1^*})\right)}} \, \tilde{P}_{E'_{+n}}\left(p_{(\nu,\underline{\varepsilon})}(\tilde{P}_{T_1})p_{(\nu,\underline{\eta})}\right)$$

$$= \sqrt{\frac{c_{(\nu,\underline{\varepsilon})}}{\mathrm{tr}_{M_{n-1}}\left(p_{(\nu,\underline{\varepsilon})}(\tilde{P}_{T_1})p_{(\nu,\underline{\eta})}(\tilde{P}_{T_1^*})\right)}} \sqrt{\frac{c_{-\nu}}{c_\nu}} \sum_{\gamma \in I} p_{(\gamma,\underline{\varepsilon})}(\tilde{P}_{T_1})p_{(\gamma,\underline{\eta})}.$$

The last equality is obtained by using the action of tangles on $\tilde{P}$. To show $\varphi \circ Q_{E'_{+n}} = \tilde{P}_{E'_{+n}} \circ \varphi$, it remains to be shown that $\frac{c_{(\nu,\underline{\varepsilon})}}{\mathrm{tr}_{M_{n-1}}(p_{(\nu,\underline{\varepsilon})}(\tilde{P}_{T_1})p_{(\nu,\underline{\eta})}(\tilde{P}_{T_1^*}))}$ is independent of $\nu \in I$. Observe that the denominator

$$\mathrm{tr}_{M_{n-1}}\left(p_{(\nu,\underline{\varepsilon})}(\tilde{P}_{T_1})p_{(\nu,\underline{\eta})}(\tilde{P}_{T_1^*})\right)$$

$$= (c_-c_+)^{\frac{n}{2}} \tilde{P}_{TR^r_{+n}}\left(p_{(\nu,\underline{\varepsilon})}(\tilde{P}_{T_1})p_{(\nu,\underline{\eta})}(\tilde{P}_{T_1^*})\right)$$

$$= (c_-c_+)^{\frac{n}{2}} \sqrt{\frac{c_\nu}{c_{-\nu}}} \sqrt{c_-c_+} \, \tilde{P}_{TR^r_{+n}}\left(p'_{\underline{\varepsilon}}(\tilde{P}_{T_1})p'_{\underline{\eta}}(\tilde{P}_{T_1^*})\right)$$

$$= c_\nu(c_-c_+)^{\frac{n}{2}} \tilde{P}_{TR^r_{+n}}\left(p'_{\underline{\varepsilon}}(\tilde{P}_{T_1})p'_{\underline{\eta}}(\tilde{P}_{T_1^*})\right),$$

where $p'_{(\omega_1,\cdots,\omega_{n-1})} := p^{(2)}_{\omega_1} p^{(3)}_{\omega_2} \cdots p^{(n)}_{\omega_{n-1}} \cdots, \omega_{n-1}) \in I^{n-1}$ and in the second equality, we replace the loop with p_ν in it by a loop with nothing on it, and the numerator $c_{(\nu,\underline{\varepsilon})} = c_\nu c_{-\underline{\varepsilon}}$. Hence, the required fraction becomes independent of $\nu \in I$.

7. Questions

DEFINITION 7.1. A bimodule planar algebra is said to have trivial perturbation class if all its perturbation by positive weights, are spherical.

It will be interesting to find a set of necessary and sufficient conditions for a bimodule planar algebra having trivial perturbation class in terms of its principal graph(s); one can also consider this question in the more specific case of the Bisch-Haagerup planar algebras (see [5], [2] and [4]). Note that all finite depth or irreducible bimodule planar algebras have trivial perturbation class; so, for the Bisch Haagerup planar algebras, this question is relevant ony when the two subgroups have nontrivial intersection and the group generated by them is infinite.

Another interesting problem is to obtain a method of perturbing a bifinite bimodule to a new one whose associated planar algebra is the perturbation of

the one associated to the bimodule which we start with. Note that, by Theorem 5.13, we may find a bifinite bimodule corresponding to a perturbation of the planar algebra associated to a given bifinite bimodule but we don't have any direct relation between these bimodules in terms of the weight. We will give answers to these questions in a forthcoming article.

REFERENCES

1. Bisch, D., *Bimodules, higher relative commutants and the fusion algebra associated to a subfactor*, pp. 13–63 in: Operator algebras and their applications, Fields Inst. Commun. 13, Amer. Math. Soc., Providence, 1997.

2. Bisch, D., Das, P., and Ghosh, S. K., *The planar algebra of group-type subfactors*, J. of Func. Anal. 257 (2009), 20–46.

3. Bisch, D., Das, P., and Ghosh, S. K., *The planar algebra of diagonal subfactors*, pp. 23–47 in: Quanta of maths, Clay Math Proc. 11, Amer. Math. Soc, Providence 2010.

4. Bisch, D., Das, P., and Ghosh, S. K., *The planar algebra of group-type subfactors with cocycle*, in preparation.

5. Bisch, D., and Haagerup, U., *Composition of subfactors: new examples of infinite depth subfactors*, Ann. Sci. École Norm. Sup. 29 (1996), 329–383.

6. Burns, M., *Subfactors, planar algebras, and rotations*, Ph.D. Thesis, Univ. of California, Berkeley 2003, arXiv:1111.1362v1.

7. Evans, D., and Kawahigashi, Y., *Quantum symmetries on operator algebras*, Osford Math. Mon., Oxford Univ. Press, Oxford 1998.

8. Ghosh, S. K., *Planar Algebras: A category theoretic point of view*, J. Algebra 339 (2011), 27–54.

9. Goodman, F., de la Harpe, P., and Jones, V. F. R., *Coxeter graphs and towers of algebras*, Math. Sci. Res. Inst. Pub. 14, Springer, Berlin 1989.

10. Guionnet, A., Jones, V. F. R., and Shlyakhtenko, D., *Random matrices, free probability, planar algebras and subfactors*, pp. 201–238 in: Quanta of maths, Clay Math Proc. 11, Amer. Math. Soc, Providence 2010.

11. Hiai, F., *Minimizing indices of conditional expectations onto a subfactor*, Publ. Res. Inst. Math. Sci. 24 (1988), 673–678.

12. Jones, V. F. R., *Index for subfactors*, Invent. Math. 72 (1983), 1–25.

13. Jones, V. F. R., *Planar algebras*, I, NZ J. Math., to appear, arXiv:math.QA/9909027.

14. Jones, V. F. R., *The planar algebra of a bipartite graph*, pp. 94–117 in: Knots in Hellas'98 (Delphi), World Sci., New York 2000.

15. Jones, V. F. R., *The annular structure of subfactors*, pp. 401–463 in: Essays on geometry and related topics, Mon. Enseign. Math. 38, Geneva 2001.

16. Jones, V. F. R., *Quadratic tangles in planar algebras*, Duke Math. J. 161 (2012), 2257–2295.

17. Jones, V. F. R., and Penneys, D., *The embedding theorem for finite depth subfactor planar algebras*, Quantum Topol. 2 (2011), 302–337.

18. Jones, V. F. R., and Sunder, V. S., *Introduction to subfactors*, London Math. Soc. Lect. Notes Ser. 234, Cambr. Univ. Press, Cambridge 1997.

19. Jones, V. F. R., Shlyakhtenko, D., and Walker, K., *An orthogonal approach to the subfactor of a planar algebra*, Pasific J. Math. 246 (2010), 187–197.

20. Kassel, C., *Quantum groups*, Grad. Texts Math. 155, Springer, Berlin 1995.

21. Kodiyalam, V., and Sunder, V. S., *On Jones' planar algebras*, J. Knot Theory Ramifications 13 (2004), 219–247.

22. Kodiyalam, V., and Sunder, V. S., *From subfactor planar aglebras to subfactors*, Internat. J. Math. 20 (2009), 1207–1231.

23. Leinster, T., *Higher operads, higher categories*, London Math. Soc. Lect. Note Ser. 298, Cambr. Univ. Press, Cambridge 2008.

24. Pimsner, M., and Popa, S., *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. 19 (1986), 57–106.

25. Pimsner, M., and Popa, S., *Iterating the basic construction*, Trans. Amer. Math. Soc. 310 (1988), 127–133.

26. Popa, S., *Classification of subfactors: the reduction to commuting squares*, Invent. Math. 101 (1990), 19–43.

27. Popa, S., *Sur la classification des sous-facteurs d'indice fini du facteur hyperfini*, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), 95–100.

28. Popa, S., *Classification of amenable subfactors of type II*, Acta Math. 172 (1994), 163–255.

29. Popa, S., *An axiomatization of the lattice of higher relative commutants*, Invent. Math. 120 (1995), 427–445.

30. Popa, S., *Universal construction of subfactors*, J. Reine Angew. Math. 543 (2002), 39–81.

31. Sunder, V. S., *II_1-factors, their bimodules and hypergroups*, Trans. Amer. Math. Soc. 330 (1992), 227–256.

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