MATH. SCAND. 114 (2014), 86–109

PARABOLIC STEIN MANIFOLDS

A. AYTUNA and A. SADULLAEV*

Abstract

An open Riemann surface is called parabolic in case every bounded subharmonic function on it reduces to a constant. Several authors introduced seemingly different analogs of this notion for Stein manifolds of arbitrary dimension. In the first part of this note we compile these notions of parabolicity and give some immediate relations among these different definitions. In section 3 we relate some of these notions to the linear topological type of the Fréchet space of analytic functions on the given manifold. In section 4 we look at some examples and show, for example, that the complement of the zero set of a Weierstrass polynomial possesses a continuous plurisubharmonic exhaustion function that is maximal off a compact subset.

1. Introduction

In the theory of Riemann surfaces, simply connected manifolds, which are equal to the complex plane are usually called parabolic and the ones which equal to the unit disk are called hyperbolic. Several authors introduced analogues of these notions for general complex manifolds of arbitrary dimension in different ways; in terms of triviality (parabolic type) and non-triviality (hyperbolic type) of the Kobayashi or Caratheodory metrics, in terms of plurisubharmonic (psh) functions, etc. In some of these considerations existence of rich family of bounded holomorphic functions play a significant role.

On the other hand, attempts to generalize Nevanlinna's value distribution theory to several variables by Stoll, Griffiths, King, et al. produced notions of "parabolicity" in several complex variables defined by requiring the existence of certain special plurisubharmonic functions. The common features of these special plurisubharmonic functions were that they were exhaustive and maximal outside a compact set.

Following Stoll, we will call an n-dimensional complex manifold X, S-parabolic in case there is a plurisubharmonic function ρ on X with the properties:

a) $\{z \in X : \rho(z) \leqslant C\} \subset\subset X, \forall\, C \in \mathsf{R}^+$ (i.e. ρ is exhaustive),

* The first author is partially supported by a grant from Sabanci University. The second author is partially supported by Khorezm Mamun Academy, Grant ΦAΦ1Φ024.

Received 13 February 2011, in final form 23 October 2012.

b) The Monge-Ampère operator $(dd^c\rho)^n$ is zero off a compact K $\subset\subset X$. That is ρ is a maximal plurisubharmonic function outside K.

We note that without the maximality condition b), an exhaustion function $\sigma(z) \in \mathrm{psh}(X) \cap C^{\infty}(X)$ always exist for any Stein manifold X, because such manifolds can be properly embedded in $\mathbb{C}_w^{2n+1}$ and one can take for σ the restriction of $\ln|w|$ to X.

On S-parabolic manifolds any bounded above plurisubharmonic function is constant. In particular, there are no non-constant bounded holomorphic functions on such manifolds. Complex manifolds, on which every bounded above plurisubharmonic function reduces to a constant, a characteristic shared by parabolic open Riemann surfaces and affine-algebraic manifolds, play an important role in the structure theory of Fréchet spaces of analytic functions on Stein manifolds, especially in finding continuous extension operators for analytic functions from complex submanifolds (see [34], [4], [7], [6]). Such spaces will be called "*parabolic*" in this paper.

Special exhaustion functions with certain regularity properties play a key role in the Nevanlinna's value distribution theory of holomorphic maps $f : X \to \mathrm{P}^m$, where $\mathrm{P}^m - m$ dimensional projective manifold (see [19], [25], [29] ,[30]). On the other hand for manifolds which have a special exhaustion function, one can define extremal Green functions as in the classical case (see [14]) and apply pluripotential theory techniques to obtain analogues of some classical results which were proved earlier for $\mathbb{C}^n$ ([11], [41]). In the special case of an affine algebraic manifolds such a program was successfully carried out in [42]

The aim of this paper is to state and analyze different notions of parabolicity, give examples and relate the parabolicity of a Stein manifold X with the linear topological properties of the Fréchet space of global analytic functions on X.

The organization of the paper is as follows: In section 2, we state and compare different definitions of parabolicity. We also bring to attention, a problem in complex pluripotential theory that arise in this context. In the third section we relate the notion of parabolicity of a Stein manifold X with the linear topological type of the Fréchet space $O(X)$, of analytic functions on X. We introduce the notion of *tame isomorphism* of $O(X)$ to the space of entire functions (Definition 3.2) and show (Theorem 3.7) that a Stein manifold of dimension n is S^*-parabolic if and only if $O(X)$ is tamely isomorphic to the space of entire functions in n variables. The final section is devoted to some classes of parabolic manifolds. First we look at complements of the zero sets of entire functions and show that the complement in $\mathbb{C}^n$ of the zero set of a global Weierstrass polynomial (algebroidal function), is S^*-parabolic. Then we generalize a condition of Demailly for parabolicity and use it to show that

Sibony-Wong manifolds (see section 4 for the definition) are parabolic.

Throughout the paper complex manifolds are always assumed to be connected.

2. Different notions of parabolicity

DEFINITION 2.1. A Stein manifold X is called parabolic, in case it does not possess a non-constant bounded above plurisubharmonic function.

Thus, parabolicity of X is equivalent to the following: if $u(z) \in \mathrm{psh}(X)$ and $u(z) \leqslant C$, then $u(z) \equiv$ const. on X. It is convenient to describe parabolicity in term of P-measures of pluripotential theory [20]. We will briefly recall this notion which can be defined for a general Stein manifold X. In the discussion below we will assume without loss a generality that our Stein manifold X is properly imbedded in $\mathbb{C}_w^{2n+1}$, $n = \dim X$, and $\sigma(z)$ is the restriction of $\ln|w|$ to X. Then $\sigma(z) \in \mathrm{psh}(X) \cap C^\infty(X)$, $\{z : \sigma(z) \leqslant C\} \subset\subset X\ \forall C \in R$. We further assume that $0 \notin X$ and $\min \sigma(z) < 0$. We consider σ-balls $B_R = \{z \in X : \sigma(z) < \ln R\}$ and as usual, define the class $\aleph(\overline{B}_1, B_R)$, $R > 1$, of functions $u(z) \in \mathrm{psh}(B_R)$ such, that $u|_{B_R} \leqslant 0$, $u|_{\overline{B}_1} \leqslant -1$. We put

$$\omega(z, \overline{B}_1, B_R) = \sup\{u(z) : u \in \aleph(\overline{B}_1, B_R)\}.$$

The regularization $\omega^*(z, \overline{B}_1, B_R)$ is called the P-measure of $\overline{B}_1$ with respect to B_R, [20].

The P-measure $\omega^*(z, \overline{B}_1, B_R)$ is plurisubharmonic in B_R, is equal to -1 on $\overline{B}_1$ and tends to 0 for $z \to \partial B_R$. Moreover, it is maximal, that is $(dd^c\omega^*)^n = 0$ in $B_R \setminus \overline{B}_1$ and decreasing by R. We put $\omega^*(z, \overline{B}_1) = \lim_{R\to\infty} \omega^*(z, \overline{B}_1, B_R)$.

It follows that

$$\omega^*(z, \overline{B}_1) \in \mathrm{psh}(X), \qquad -1 \leqslant \omega^*(z, \overline{B}_1) \leqslant 0$$

and $(dd^c\omega^*(z, \overline{B}_1))^n = 0$ in $X \setminus \overline{B}_1$.

In the construction of $\omega^*(z, \overline{B}_1)$ we have used the exhaustion function $\sigma(z)$, however it is not difficult to see that $\omega^*(z, \overline{B}_1)$ depends only on X and $\overline{B}_1$; not on the choice of the exhaustion function. Indeed one can define the P-measure for any non pluripolar compact $K \subset X$, by selecting any sequence of domains $K \subset D_j \subset\subset D_{j+1} \subset\subset X$, $j = 1, 2, \ldots$, $X = \bigcup_{j=1}^{\infty} D_j$ and employing the above procedure with B_R's replaced with D_j's. It follows from the definition, that the plurisubharmonic functions $\omega^*(z, K) + 1$ and $\omega^*(z, \overline{B}_1) + 1$ are dominated by a constant multiple of each other. In particular $\omega^*(z, K) \equiv -1$ if and only if $\omega^*(z, \overline{B}_1) \equiv -1$. Hence the later property is an inner property of X. For further properties of P-measures we refer the reader to [20], [22], [37], [38].

Vanishing of $\omega^*(z, K) + 1$ on a parabolic manifold not only imply the triviality of bounded holomorphic functions but also give some information on the growth of unbounded holomorphic functions. In fact on parabolic manifolds, a kind of "Hadamard three domains theorem" with controlled exponents, is true. The precise formulation of this characteristic, that will appear below, is an adaptation of the property (DN) of Vogt [32], which was defined for general Fréchet spaces, to the Fréchet spaces of analytic functions. As usual we will denote by $O(X)$ the Fréchet spaces of analytic functions defined on a complex manifold X with the topology of uniform convergence on its compact subsets. The proposition we will give below is due to Zaharyuta [39] and it has been independently rediscovered by several other authors [4], [35]. We will include a proof of this result for the convenience of the reader.

PROPOSITION 2.2. *The following are equivalent for a Stein manifold X*

a) *X is parabolic;*

b) P-*measures are trivial on X i.e. $\omega^*(z, K) \equiv -1$ for every nonpluripolar compact $K \subseteq X$;*

c) *For every nonpluripolar compact set $K_0 \subset X$ and for every compact set K of X there is another compact set L containing K such that*

$$\|f\|_K \leqslant \|f\|_{K_0}^{\frac{1}{2}} \|f\|_L^{\frac{1}{2}}, \ \forall f \in O(X), \qquad (DN \text{ condition of Vogt})$$

where $\|*\|_H$ denotes the sup norm on H.

PROOF. If X is parabolic, then $\omega^*(z, \overline{B}_1)$ being bounded and plurisubharmonic on X reduces to -1.

Conversely, let $u(z)$ be an arbitrary bounded above psh function on X. Let $u_R = \sup_{B_R} u(z)$, $\infty > R \geqslant 1$. If $u(z) \neq \text{const.}$, then

$$\frac{u(z) - u_R}{u_R - u_1} \in \aleph(\overline{B}_1, B_R) \quad \text{and hence} \quad \frac{u(z) - u_R}{u_R - u_1} \leqslant \omega^*(z, \overline{B}_1, B_R).$$

It follows that

$$(1) \qquad u(z) \leqslant -u_1 \omega^*(z, \overline{B}_1, B_R) + u_R(1 + \omega^*(z, \overline{B}_1, B_R)), \qquad z \in B_R,$$

and as $R \to \infty$ this gives

$$(2) \qquad u(z) \leqslant -u_1 \omega^*(z, \overline{B}_1) + u_\infty(1 + \omega^*(z, \overline{B}_1)), \qquad z \in X.$$

If $\omega^*(z, \overline{B}_1) \equiv -1$, then $u(z) \leq u_1$, $z \in X$, and by maximal principle we have $u(z) = u_1 \equiv \text{const.}$, so that a) and b) are equivalent.

Now we look at the sup norms $\|.\|_{B_m}$ on the sublevel balls $B_m \supset K$. Choose an increasing sequence of norms $\|.\|_k = \|.\|_{m_k}$, $k = 0, 1, \dots, m_0 = 1, \dots,$ that satisfy the condition c) with the dominating norm $\|.\|_0$:

$$\|f\|_k \leqslant \|f\|_0^{\frac{1}{2}} \|f\|_{k+1}^{\frac{1}{2}}, \qquad \forall f \in O(X)$$

Iterating this inequality one gets:

$$\|f\|_1 \leqslant \|f\|_0^{\frac{2^{k-1}-1}{2^{k-1}}} \|f\|_k^{\frac{1}{2^{k-1}}}, \qquad \forall f \in O(X). \tag{3}$$

Denoting the sequence of domains by $D_k = B_{m_k}$ we consider the P-measures $\omega^*(z, \overline{D}_0, D_{k+1})$, $k = 1, 2, \dots$. Since these functions are continuous, by Bremermann's theorem (see [10]) for a fixed k we can find analytic functions $f_1, f_2, \dots, f_m$ on D_{k+1} and positive numbers $a_1, a_2, \dots, a_m$ such that

$$\omega^*(z, \overline{D}_0, D_{k+1}) + 1 - \varepsilon \leqslant \max_{1\leqslant j\leqslant m} (a_j \ln|f_j(z)|) \leqslant \omega^*(z, \overline{D}_0, D_{k+1}) + 1$$

pointwise on $\overline{D}_k$. We note that the compact $\overline{D}_k$ is polynomially convex in $\mathbb{C}^{2n+1} \supset X$, so by Runge's theorem the functions f_j can be uniformly approximated on $\overline{D}_k$ by functions $F \in O(X)$. This in turn by (3) gives us the estimate $\omega^*(z, \overline{D}_0, D_{k+1}) + 1 \leqslant \frac{1}{2^{k-1}} + \varepsilon$, $z \in D_1$. Now playing the same game with D_1 replaced by a given D_j we see that $\omega^*(z, K, D_{k+1})$ converge uniformly to -1 on any compact subset of X, i.e. $\omega^*(z, \overline{D}_0) \equiv -1$. Hence c) ⇒ b).

Conversely, suppose that $\omega^*(z, K) \equiv -1$ for any $K \subset\subset X$. Fix a non-pluripolar compact $K_0 \subset X$ and fix an arbitrary compact set $K \subset X$. Let $B_{k_0} \supset K_0 \cup K$, $k_0 \in \mathbb{N}$. Then, in view of Dini's theorem we can choose k so large that $\omega^*(z, K_0, B_k) \leqslant -1/2$ for $z \in B_{k_0}$. Since $\omega^*(z, K_0, B_k)$ is maximal on $B_k \setminus K_0$, then for arbitrary $f \in O(X)$, $f \neq 0$, the inequality

$$\frac{\ln \frac{|f(z)|}{\|f\|_{K_0}}}{\ln \frac{\|f\|_{B_k}}{\|f\|_{K_0}}} \leqslant \omega^*(z, K_0, B_k) + 1, \qquad z \in B_k,$$

is valid. This in turn implies that

$$\|f\|_K \leqslant \|f\|_{B_{k_0}} \leqslant \|f\|_{K_0}^{1/2} \|f\|_{B_k}^{1/2},$$

for all $f \in O(X)$. Hence a) ⇒ c). This finishes the proof of the proposition.

DEFINITION 2.3. A Stein manifold X is called S-parabolic, if there exit exhaustion function $\rho(z) \in \mathrm{psh}(X)$ that is maximal outside a compact subset of X. If in addition we can choose ρ to be continuous then we will say that X is S^*-parabolic.

In previous papers on parabolic manifolds (see for example [13], [29]) authors usually required the condition of C^∞-smoothness of ρ. Here we only distinguish the cases when the exhaustion function is just psh or continuous psh.

It is not difficult to see that S-parabolic manifolds are parabolic. In fact, since the exhaustion function $\rho(z)$ of the definition of S-parabolicity is maximal off some compact $K \subset\subset X$, the balls $B_r = \{z : \rho(z) < \ln r\}$, $r \geqslant r_0$, contain K for big enough r_0 and hence

$$\omega^*(z, \overline{B}_{r_0}, B_R) = \max\left\{-1, \frac{\rho(z) - R}{R - r_0}\right\}.$$

Consequently,

$$\omega^*(z, \overline{B}_{r_0}) = \lim_{R\to\infty} \omega^*(z, \overline{B}_{r_0}, B_R) \equiv -1, \qquad z \in X.$$

For Stein manifolds of dimension one, the notions of S-parabolicity, S^*-parabolicity, and parabolicity coincide. This is a consequence of the existence of Evans-Selberg potentials (subharmonic exhaustion functions that are harmonic outside a given point) on a parabolic Riemann surfaces [24].

PROBLEM 1. Do the notions of S-parabolicity and S^*-parabolicity coincide for Stein manifolds of arbitrary dimension?

PROBLEM 2. Do the notions of parabolicity and S-parabolicity coincide for Stein manifolds of arbitrary dimension?

3. Spaces of Analytic Functions on Parabolic Manifolds

In this section we will relate the above discussed notions of parabolicity of a Stein manifold X with the linear topological structure of $O(X)$, the Fréchet space of analytic functions on X with the compact open topology. The first result which we will state is due to Aytuna-Krone-Terzioglu and it characterizes parabolicity of a Stein manifold X of dimension n, in terms of the similarity of the linear topological structures of $O(X)$ and $O(\mathbb{C}^n)$. For the proof, we refer the reader to [5].

THEOREM 3.1. *For a Stein manifold X of dimension n the following are equivalent:*

a) *X is parabolic;*

b) *$O(X)$ is isomorphic as Fréchet spaces to $O(\mathbb{C}^n)$.*

The correspondence that sends an entire function f to its Taylor coefficients $(x_m)_{m=0}^{\infty}$ ordered in the usual way, establishes an isomorphism between $O(\mathbb{C}^n)$ and the infinite type power series space

(4)

$$\Lambda_\infty(\alpha_m) := \left\{ x = (x_m)_{m=0}^{\infty} : |x|_k := \sum_{m=0}^{\infty} |x_m| e^{k\alpha_m} < \infty \ \ \forall\, k = 1, 2, \dots . \right\}$$

with $\alpha_m = m^{\frac{1}{n}}$, $m = 0, 1, 2, \dots$.

Recall that a *graded Fréchet space* is a tuple $(X, ^{|*|_s})$, where X is a Fréchet space and $(|*|_s)_{s=1}^{\infty}$ is a fixed system of seminorms defining the topology of X. Whenever we deal with $\Lambda_\infty(\alpha_m)$, for an exponent sequence $\alpha_m \uparrow \infty$ (not necessarily $(m^{1/n})_m$), we will tacitly assume that we are dealing with a graded space and that the grading is given by the norms defined in the above expression. We will need a definition from the structure theory of Fréchet spaces;

Definition 3.2. A continuous linear operator T between two graded Fréchet spaces $(X, ^{|*|_k})$ and $(Y, ^{\|*\|_k})$ is called tame in case:

$$\exists A > 0\ \forall\, k\ \exists C > 0 : \|T(x)\|_k \leqslant C|x|_{k+A}, \qquad \forall\, x \in X.$$

Two graded Fréchet spaces are called tamely isomorphic in case there is a one to one tame linear operator from one onto the other whose inverse is also tame.

The graded space $(O(\mathbb{C}), ^{\|*\|_k})$ where $\|*\|_k$ is the sup norm on the disc with radius e^k, is tamely isomorphic, under the correspondence described above, to the power series space $\Lambda_\infty(m)$ in view of the Cauchy's inequality. This observation motivates our next definition:

Definition 3.3. Let X be a Stein manifold. The space $O(X)$ is said to be tamely isomorphic to an infinite type power series space in case there is an exhaustion of X by connected holomorphically convex compact sets $(K_k)_{k=1}^{\infty}$ with $K_k \subset K_{k+1}^{\circ}$, $k = 1, 2, \dots$, such that the graded space $(O(X), ^{\sup_{K_k} |*|})$ is tamely isomorphic to an infinite type power series space.

The supremum norms are, in some sense, associated with the function theory whereas the power series norms are associated with the structure theory of Fréchet spaces, and tame equivalence gives one, a controlled equivalence between these generating norm systems.

For a graded Fréchet space, linear topological properties that ensure tame equivalence to an infinite type power series space were studied by D. Vogt and his school, in the context of structure theory of nuclear Fréchet spaces [35], [18]. In the proof of the theorem below, we will make use of a specific result of Vogt in this direction. We recall this theorem here for the benefit of the reader.

DEFINITION 3.4. A graded nuclear Fréchet space $(E, ^{|*|_k})$ is said to be a (DN) space in standard form in case, with suitable constants $C_k > 0$,

$$|*|_k^2 \leqslant C_k |*|_{k-1} |*|_{k+1}$$

for all $k = 1, 2, \ldots$.

DEFINITION 3.5. A graded nuclear Fréchet space $(F, ^{|*|_k})$ is said to be a (Ω) space in standard form in case, with suitable constants $D_k > 0$,

$$|*|_k^{*2} \leqslant D_k |*|_{k-1}^* |*|_{k+1}^*$$

for all $k = 1, 2, \ldots$ where $|x^*|_k^* = \sup\{|x^*(y)| : |y|_k \leqslant 1\}$, $k = 1, 2, \ldots$, denotes the dual "norms" on F^*.

THEOREM 3.6 ([35], Theorem 2.3). *Let E be a nuclear (DN) space in standard form, and F an (Ω) space in standard form. Suppose that there exists a tame surjection from F onto E. Then E is tamely isomorphic to an infinite type power series space.*

For two nonnegative real valued functions α and β on a set T we will use the notation $\alpha(t) \prec \beta(t)$ to mean $\exists\, C > 0$ such that $\alpha(t) \leqslant C\beta(t)\ \forall t \in T$.

We can now state the main result of this section:

THEOREM 3.7. *Let X be a Stein manifold. The space of analytic functions on X, $O(X)$, is tamely isomorphic to an infinite type power series space if and only if X is S^*-Parabolic.*

PROOF. $\Rightarrow$: Suppose that $O(X)$ is tamely isomorphic to a power series space $\Lambda_\infty(\alpha_m)$ with $\alpha_m \uparrow \infty$. Fix a tame isomorphism $T : \Lambda_\infty(\alpha_m) \to O(X)$. We also fix an exhaustion $\{K_k\}_{k=0}^\infty$ of X by holomorphically convex compact sets and an integer B' such that for all k

(5) $$\|T(x)\|_k \prec |x|_{k+B'} \text{ and } |x|_k \prec \|T(x)\|_{k+B'}\ \forall x \in \Lambda_\infty(\alpha_m),$$

where $\|*\|_k$ denotes the sup norm on K_k, $k = 0, 1, 2, \ldots$. Let $e_m \doteq T(\varepsilon_m)$ where, as usual, $\varepsilon_m = (0, \ldots, 0, 1, 0, \ldots)$, $m = 1, 2, \ldots$. Set

(6) $$\rho(z) \doteq \limsup_{\xi \to z} \limsup_{m \to \infty} \frac{\log|e_m(\xi)|}{\alpha_m}.$$

Clearly ρ is a plurisubharmonic function on X. If we set $D_\tau \doteq \{z : \rho(z) < \tau\}$ for $\tau \in \mathsf{R}$, we have:

$$K_k \subseteq D_{k+B} \text{ for large } k, \text{ where } B = B' + 1.$$

Now fix an arbitrary $z_0 \in D_\sigma$ and choose, in view of Hartogs lemma, a small $\epsilon > 0$ such that $|e_m(z_0)| \leqslant Ce^{\alpha_m(\sigma-\epsilon)}$ for all m. For any $x = \sum x_m \varepsilon_m \in \Lambda_\infty(\alpha_m)$ we have:

$$|T(x)(z_0)| \leqslant \sum_m |x_m||e_m(z_0)| \prec \sum_m |x_m| e^{(\sigma-\epsilon)\alpha_m} \prec \|T(x)\|_{[[\sigma]]+1+B}$$

Since T is onto and K_m's are holomorphically convex, we have that $z_0 \in K_{[[\sigma]]+1+B}$. Hence $D_\sigma \subseteq K_{[[\sigma]]+1+B}$. Combining this with our previous findings we get

$$\exists\, d > 0 \text{ such that } \overline{D_\sigma} \subseteq D_{\sigma+d} \ \forall \alpha \text{ large} \tag{7}$$

Now fix a nice compact set K, say $K = \overline{D}$ for some domain, with the property that

$$\exists\, \beta > 0 \text{ such that } |x|_\beta \prec \sup_{w \in K} |T(x)(w)| \ \forall x \in \Lambda_\infty(\alpha_m).$$

We wish to show that

$$\Phi(z) \mathrel{\hat{=}} \limsup_{\xi \to z} \{\varphi(\xi) : \varphi \in \mathrm{psh}(X),\ \varphi \mid K \leq 0, \varphi \leqslant \rho + C \text{ for some } C = C(\varphi)\}$$

defines a plurisubharmonic function on X. To this end choose a $\varphi \in \mathrm{psh}(X)$ with $\varphi|K \leqslant 0$ and $\varphi \leqslant \rho + C$ for some $C = C(\varphi) > 0$. By Bremermann's theorem ([10]), we choose a representation

$$\varphi(z) = \limsup_{\xi \to z} \limsup_j \frac{\log|f_j(\xi)|}{c_j}$$

of φ on X for some $f_j \in O(X)$, $j = 1, 2, \ldots$, and positive real numbers $c_j \uparrow \infty$, $j = 1, 2, 3, \ldots$. Using Hartogs lemma in a suitable neighborhood of K we get:

$$\forall \epsilon > 0\, \exists\, j_0 : |f_j(x)| \leqslant e^{\epsilon c_j}, \qquad j \geqslant j_0,\ z \in K$$

In particular if $y_j \mathrel{\hat{=}} T^{-1}(f_j)$ we have:

$$\limsup_j \frac{\log|y_j|_\beta}{c_j} \leqslant 0.$$

Taking into account the relation between φ and ρ and the inclusion $\overline{D_\sigma} \subseteq D_{\sigma+d}$ for large σ, we have, in view of Hartogs lemma:

$$\sup_{w \in D_\sigma} |f_j(w)| \prec e^{(\sigma+d+C)c_j}, \qquad \forall j, \forall \sigma \text{ large}.$$

In particular for large m, we have;

$$|y_j|_m \prec e^{(m+d+C+2B)c_j}, \qquad \forall j.$$

For any non negative number t, we define:

$$h(t) \mathrel{\mathring{=}} \limsup_j \frac{\log |y_j|_t}{c_j}.$$

This function is an increasing convex function on the positive real numbers. Taking into account $h(\beta) \leqq 0$ and $h(m) \leqq m + d + C + 2B$ for large m,it follows, that

$$h(t) \leqslant \left(\frac{N+D}{N-\beta}\right)t - \left(\frac{N+D}{N-\beta}\right)\beta$$

on the interval $[\beta, N]$ for every $N \geqslant \beta$, where $D = d + C + 2B$. Hence $h(t) \leqslant t - \beta$ for $t >> \beta$.

Going back, since

$$\sup_{w \in D_\sigma} |f_j(w)| \prec |y_j|_{\sigma+2+2B}$$

for z with $\rho(z) = \sigma$, we see that,

$$\begin{aligned}\varphi(z) &= \limsup_{\xi \to z} \limsup_n \frac{\log |f_n(\xi)|}{c_n} \leqslant h(\sigma + 2 + 2B + d)\\ &\leqslant \sigma + 2 + 2B + d - \beta = \rho(z) + Q,\end{aligned}$$

where $Q = Q(B, d, \beta) \in \mathsf{R}^+$.

Hence

$$\Phi(z) \leqslant \rho(z) + Q$$

and so $\Phi \in L^\infty_{\text{loc}}(X)$. In particular Φ is a plurisubharmonic function on X and satisfies

$$\exists\, C_1 > 0 \text{ and } C_2 > 0 \text{ such that } \rho(z) - C_1 \leqslant \Phi(z) \leqslant \rho(z) + C_2 \text{ on } X.$$

It follows that Φ is an exhaustion and as a free envelope, is maximal outside a compact set [8].

Observe also that the sublevel sets $\Omega_r \mathrel{\mathring{=}} \{z : \Phi(z) < r\}$ satisfy :

$$\exists\, \kappa_0 > 0 \text{ such that } \overline{\Omega_r} \subseteq \Omega_{r+\kappa_0} \text{ for } r \text{ large enough.}$$

Now fix a decreasing sequence $\{u_j\}$ of continuous plurisubharmonic functions on X converging to Φ. Fix a compact set K and $\epsilon > 0$. Choose an r so large that $\left(\frac{r+\kappa_0-\frac{\epsilon}{2}}{r}-1\right)\max_{\xi\in K}\Phi(\xi)\leqslant\frac{\epsilon}{2}$. There exists an j_0 such that for $j\geqslant j_0$ on Ω_r, $u_j\leqslant r+\kappa_0$ and $u_j|K\leqslant\frac{\epsilon}{2}$. Hence on Ω_r:

$$\frac{u_j-\frac{\epsilon}{2}}{r+\kappa_0-\frac{\epsilon}{2}}\le\omega^*(K,\Omega_r)=\frac{1}{r}\Phi.$$

where ω^* is the corresponding P-measure (see section 2). It follows that on K,

$$0\leqslant u_j-\Phi\leqslant\left(\frac{r+\kappa-\frac{\epsilon}{2}}{r}-1\right)\max_{\xi\in K}\Phi(\xi)+\frac{\epsilon}{2}\le\epsilon\qquad\text{for}\quad j\geqslant j_0.$$

Hence the convergence is uniform on K. So Φ is continuous.

$\Leftarrow$: In the proof of this implication will use the above mentioned theorem of D. Vogt. However, first we wish bring to light a particular Ω-type condition for $O(X)$ provided by a given plurisubharmonic exhaustion function (see also [37]). For this part of the argument, one does not need parabolicity. To stress this point we will summarize our findings separately, in the below Proposition.

Let X be a Stein manifold and $\Phi: X\to[-\infty,\infty)$ a plurisubharmonic function that is an exhaustion. Set $D_t=(x\mid\Phi(x)<t)$ for $t\in\mathsf{R}$. Choose an increasing function ℓ so that for each $t\in\mathsf{R}$, $\overline{D_t}\subset D_{\ell(t)}$. We fix a volume form $d\mu$ on X and using the notation of Lemma 1 [3], we let $d\varepsilon=c\,d\mu$ where c is the strictly positive continuous function that appears in Lemma 1 of [3]. Set;

$$U_t=\left\{f\in O(X):\int_{D_t}|f|^2d\varepsilon\leqslant 1\right\}.$$

Fix positive numbers s_1, s_2, s such that $\ell(0)<s_1\leqslant\ell(s_1)\leqslant s_2\leqslant\ell(s_2)\leqslant s$ and $L\geqslant 0$. Let

$$\Phi_L(z)\stackrel{\circ}{=}\begin{cases}0 & \text{if }\Phi(z)\leqslant 0\\ \dfrac{L\Phi(z)}{s} & \text{otherwise.}\end{cases}$$

Consider an analytic function $f\in U_{s_2}$. Using Lemma 1 of [3], we choose a decomposition of f on $W_+\cap W_-$, $f=f_+-f_-$, with $f_\pm\in O(W_\pm)$, $W_+=(\overline{D_{s_1}})^c$, $W_-=D_{s_2}$, and such that the estimates

$$\int_{W_\pm}|f_\pm|^2e^{-\Phi_L}\,d\varepsilon\leqslant K\int_{W_+\cap W_-}|f|^2e^{-\Phi_L}\,d\mu$$

hold with $K=K(X,s_1,s_2,s,\Phi)>0$. On the other hand, since $f\in U_{s_2}$,

$$\int_{W_+\cap W_-}|f|^2e^{-\Phi_L}\,d\mu\le C\int_{W_+\cap W_-}|f|^2e^{-\Phi_L}\,d\varepsilon\leqslant Ce^{-\frac{Ls_1}{s}}\quad\text{for some }\;C>0.$$

Hence
$$\int_{W_\pm} |f_\pm|^2 e^{-\Phi_L}\, d\varepsilon \leqslant C_1 e^{-\frac{Ls_1}{s}} \qquad \text{for some} \quad C_1 > 0.$$

Now since Φ_L is zero on D_0 we have,

$$\int_{D_0} |f_-|^2\, d\varepsilon = \int_{D_0} |f_-|^2 e^{-\Phi_L}\, d\varepsilon \le \int_{W_-} |f_-|^2 e^{-\Phi_L}\, d\varepsilon \le C_1 e^{-\frac{Ls_1}{s}}$$

and
$$\int_{W_-} |f - f_-|^2\, d\varepsilon \leqslant C_2 e^{\frac{L(s-s_1)}{s}}.$$

Set
$$G = \begin{cases} f_+ & \text{on } W_+ \\ f - f_- & \text{on } W_- \end{cases}$$

Clearly $G \in O(X)$, and,

$$\int_{D_s} |G|^2\, d\varepsilon \leqslant \int_{D_s \cap W_+} |G|^2 e^{-\Phi_L} e^{\Phi_L}\, d\varepsilon + \int_{W_-} |G|^2\, d\varepsilon$$
$$\leqslant C_3\big(e^{\frac{L(s-s_1)}{s}} + e^{\frac{L(s-s_1)}{s}}\big) \leqslant C_4 e^{\frac{L(s-s_1)}{s}}.$$

Moreover

$$\int_{D_0} |G - f|^2\, d\varepsilon = \int_{D_0} |f_-|^2\, d\varepsilon = \int_{D_0} |f_-|^2 e^{-\Phi_L}\, d\varepsilon \leqslant C_1 e^{-\frac{Ls_1}{s}}.$$

Hence we obtain:

$$U_{s_2} \subseteq C e^{-\frac{Ls_1}{s}} U_0 + C e^{\frac{L(s-s_1)}{s}} U_s$$

for some constant $C > 0$ which does not depend upon L.

Set $t \triangleq 1 - \frac{s_1}{s}$, and $r = e^{L(1-t)-\log C}$. Varying the parameter L, a short computation yields

$$\exists\, C > 0 \text{ such that: } U_{s_2} \subseteq \frac{1}{r} U_0 + C r^{\frac{t}{1-t}} U_s \text{ for all } r \in [1, \infty]\,.$$

Since the above inclusion obviously holds for $0 < r \leqslant 1$, and writing the value of t we have:

$$\exists\, D > 0 \text{ such that: } U_{s_2} \subseteq \frac{D}{r} U_0 + \frac{r^{\frac{s}{s_1}}}{r} U_s \text{ for all } r \in (0, \infty).$$

This is an Ω-type condition introduced by Vogt and Wagner [33]. In terms of the "dual norms" this condition can we expressed as (see [33]):

$$\text{(8)}\quad \exists C > 0 \text{ such that } \|x^*\|^*_{s_2} \leqslant C(|||x^*|||^*_0)^{1-\frac{s_1}{s}}(|||x^*|||^*_s)^{\frac{s_1}{s}},\ \forall x^* \in O(X)^*,$$

where $|||x^*|||^*_t \stackrel{\circ}{=} \sup\{|x^*(f)| : f \in O(X), |||f|||_t \leqslant 1\}$, $x^* \in O(X)^*$, $t \in R$ and $|||f|||_t = \left(\int_{D_t} |f|^2\, d\varepsilon\right)^{\frac{1}{2}}$.

We collect our findings, with the above notation, in:

PROPOSITION 3.8. *Let X be a Stein manifold and Φ a plurisubharmonic function on X such that $D_t \stackrel{\circ}{=} \{z : \Phi(z) < t\} \subset\subset X$, $\forall\, t \in \mathsf{R}$. If we have*

$$\text{(9)}\qquad \overline{D_{s_0}} \subseteq D_{s_1} \subseteq \overline{D_{s_1}} \subseteq D_{s_2} \subseteq \overline{D_{s_2}} \subseteq D_s$$

for some indexes $s_0 < s_1 < s_2 < s$, then the Fréchet space $O(X)$, with the norms defined above, satisfies the following Ω-condition:

$$\text{(10)}\qquad \exists\, C > 0 : |||x^*|||^*_{s_2} \leqslant C(|||x^*|||^*_{s_0})^{\frac{s-s_1}{s-s_0}}(|||x^*|||^*_s)^{\frac{s_1-s_0}{s-s_0}},\ \forall\, x^* \in O(X)^*$$

Now we return to the proof of the theorem. Lets fix a continuous proper plurisubharmonic function Φ on X that is maximal outside a compact set. We can arrange things so that Φ is maximal outside a compact subset of D_0, where as usual $D_t = \{x : \Phi(x) < t\}$. Since Φ is continuous, for a given k, by taking $s_0 = k - 1 - \frac{1}{k-1}$, $s_2 = k - \frac{1}{k}$, $s = k + 1 - \frac{1}{k+1}$ and choosing s_1, $s_0 < s_1 < s_2$, so that $\frac{s-s_1}{s-s_0} \leqslant \frac{1}{2}$, the above proposition gives

$$\forall\, k \geqslant 2\ \exists\, C_k > 0 : |||x^*|||^*_{k-\frac{1}{k}} \leq C_k\left(|||x^*|||^*_{k-1-\frac{1}{k-1}}\right)^{\frac{1}{2}}\left(|||x^*|||^*_{k+1-\frac{1}{k+1}}\right)^{\frac{1}{2}},$$

$\forall\, x^* \in O(X)^*$.

Hence $O(X)$, with the grading $|||f|||_k = \left(\int_{D_{k-\frac{1}{k}}} |f|^2\, d\varepsilon\right)^{\frac{1}{2}}$, $k = 1, 2, \ldots$ is an Ω-space in *standard* form. On the other hand the grading $\|f\|_k = \sup_{z\in D_k} |f(z)|$, $k = 0, 1, 2, \ldots$, on $O(X)$, satisfies

$$\|f\|_k^2 \leq \|f\|_{k+1}\|f\|_{k-1}$$

In fact for a non-constant analytic function f on X, and fixed $k \in N$, the plurisubharmonic function

$$\rho(z) = 2\frac{\ln\left(\frac{|f(z)|}{\|f\|_{k+1}}\right)}{\ln\left(\frac{\|f\|_{k+1}}{\|f\|_{k-1}}\right)}$$

is dominated by the maximal function $\Phi - (k+1)$ on the boundary of the region $(z : k-1 < \Phi(z) < k+1)$ and hence is dominated by it on the whole region. Rewriting this domination on the level set $\Phi = k$ yields the desired inequality.

Hence, $O(X)$ with the grading $\|f\|_k = \sup_{D_k} |f|$, $k = 0, 1, 2, \ldots$ is a (DN)-space in standard form.

Moreover for every $k = 1, 2, \ldots$, there is a $K_k > 0$, such that $|||f|||_k \leq K_k\|f\|_k$ and $\|f\|_k \leq K_k|||f|||_{k+2}$. Now all the conditions of Vogt's theorem mentioned above, are satisfied with identity as the required surjection. It follows that $O(X)$ is tamely isomorphic to an infinite type power space. This finishes the proof of the theorem.

The theorem above associates to every special plurisubharmonic continuous exhaustion function Φ on a S^*-parabolic Stein manifold X, an exponent sequence $(\alpha_m)_m$ such that the spaces $(O(X),^{\|*\|_k})$ with grading coming from the sup norms on the level sets of Φ, and $\Lambda_\infty(\alpha_m)$ are tamely isomorphic. It might be of interest to examine the exponent sequences $(\alpha_m)_{m=0}^{\infty}$ obtained in this way and see how they depend upon the special exhaustion function Φ.

To this end let X be a Stein manifold with a continuous plurisubharmonic exhaustion function Φ that is maximal off a compact set that lies in the interior of $K_0 = \{z : \Phi(z) \leq 0\}$. We will choose a hilbertian grading $(\|*\|_k^\wedge)_k$ of $O(X)$ so that the Hilbert spaces $H_k \overset{\circ}{=} \overline{(O(X),^{\|*\|_k^\wedge})}_{k=0}^{\infty}$ satisfy the continuous inclusions;

$$H_k \hookrightarrow O(D_k) \hookrightarrow A(K_0) \hookrightarrow H_0, \qquad \forall\, k = 1, 2, \ldots$$

where $D_k = (z : \Phi(z) < k)$, and $A(K_0)$ is the germs of analytic functions on K_0 with the inductive topology. Moreover we also require that:

a) The tuple (H_0, H_k) is admissible for the pair (K_0, D_k) in the sense of Zaharyuta [39], $\forall\, k \in \mathsf{N}$,
b) The theorem above is valid i.e. there is an infinite type power series space $\Lambda_\infty(\alpha)$ so that $(O(X),^{\|*\|_k^\wedge})_{k=0}^{\infty}$ is tamely isomorphic to $\Lambda_\infty(\alpha)$.

We will only use a special property of admissible pairs, so we will just refer the reader to [40] for the definition, construction and a detailed discussion of this notion. However we should mention that in our case we can take H_0 to be the closure in $L^2(X, dd^c \max(0, \Phi))^n)$, of the space of analytic functions defined near K_0, and H_k to be $O(D_k) \cap L^2(d\varepsilon)$ where $d\varepsilon$ is the measure that appears in the proof of Theorem 2 ([40], [3]) and the existence of an infinite type power series space satisfying the required property for this choice of generating norms follows from the proof of the theorem given above. In what follows, we will denote the corresponding graded space by $(O(X),^{\Phi})$.

Since $O(X)$, for a parabolic Stein manifold X of dimension n, is isomorphic to $\Lambda_\infty(m^{\frac{1}{n}})$, regardless of the special exhaustion function we have:

$$\exists\, C > 0 : \frac{1}{C} \leq \liminf_m \frac{\alpha_m}{m^{\frac{1}{n}}} \leq \limsup_m \frac{\alpha_m}{m^{\frac{1}{n}}} \leq C$$

for all exponent sequences $(\alpha_m)_{m=0}^{\infty}$ such that $O(X)$ and $\Lambda_\infty(\alpha)$ are isomorphic.

To proceed further we need the notion of a Kolmogorov diameter. For a vector space L, let us denote the collection of all subspaces of $Y \subset L$ with $\dim Y \leq m$, by L_m, $m = 1, 2, \ldots$.

DEFINITION 3.9. Let $(\mathsf{X}, {}^{|*|_k})$ be a graded Fréchet space with an increasing sequence of seminorms. Let $U_i = \{x \in \mathsf{X} : |x|_i \leq 1\}$, $i = 1, 2, \ldots$. The m^{th} diameter of U_i with respect to U_j, $i < j$, is defined by

$$d_m(U_i, U_j) \triangleq \inf\{\lambda > 0 : \exists\, Y \in \mathsf{X}_m \text{ such that } U_i \subseteq \lambda U_j + Y\}.$$

Now fix a S^*-parabolic Stein manifold X and suppose that $(O(X), {}^{\Phi})$ and $\Lambda_\infty(\alpha_m)$ are tamely isomorphic under an isomorphism T. In particular there exists an $A > 0$ such that,

$$\forall\, k\, \exists\, C > 0 : \|T(x)\|_k^{\wedge} \leq C|x|_{k+A} \text{ and } C\|T(x)\|_{k+A}^{\wedge} \geqslant |x|_k, \forall\, x \in \Lambda_\infty(\alpha_m).$$

We will denote by U_i and V_i the unit balls corresponding to the i^{th} norms of $(O(X), {}^{\Phi})$ and $\Lambda_\infty(\alpha_m)$ respectively.

Fix a $k >> l$ large and suppose

$$U_k \subseteq \lambda U_l + L,$$

for some $\lambda > 0$ and L some m-dimensional subspace of $O(X)$. Applying T^{-1} to both sides and using the tame continuity estimates we have:

$$\frac{1}{C} V_{k+A} \subseteq T^{-1}(U_k) \subseteq \lambda T^{-1}(U_l) + L' \subseteq \lambda C V_{l-A} + L', \qquad L' \triangleq T^{-1}(L).$$

Hence

$$d_m(V_{k+A}, V_{l-A}) \leq C d_m(U_k, U_l)$$

for all m, where the constant depends only on indices k and l.

Arguing in a similar fashion, we also have

$$d_m(U_{k+A}, U_{l-A}) \leq C d_m(V_k, V_l), \qquad \forall\, m$$

It is a standard fact that $d_m(V_k, V_l) = e^{(l-k)\alpha_m}$ for $k >> l$ ([12]). On the other hand our requirement of admissibility of the norms $(\|*\|_k^{\wedge})_k$ gives, in view of a result of Nivoche-Poletsky-Zaharyuta (Theorem 5 of [40], see also, [16]) the asymptotics

$$\lim_m \frac{-\ln d_m(U_k, U_l)}{m^{\frac{1}{n}}} = \frac{2\pi(n!)^{\frac{1}{n}}}{(C(\overline{D_l}, D_k))^{\frac{1}{n}}}, \qquad \forall\, k >> l$$

where $D_s = (z : \Phi(z) < s)$ is as above, and $C(\overline{D_l}, D_k)$ is the Bedford-Taylor capacity of the condenser $(\overline{D_l}, D_k)$ [8].

Putting all these things together we have:

$$\begin{aligned}\liminf_m \frac{\alpha_m}{m^{\frac{1}{n}}} &\geqslant \lim_m \Bigg[\frac{-\ln d_m(U_k, U_l)}{m^{\frac{1}{n}}}\Bigg(\frac{-\ln C}{(k-l+2A)(-\ln d_m(U_{k+A}, U_{l-A}))} \\ &\qquad + \frac{1}{(k-l+2A)}\Bigg)\Bigg] \\ &= \frac{2\pi(n!)^{\frac{1}{n}}}{(C(\overline{D_l}, D_k))^{\frac{1}{n}}} \cdot \frac{1}{(k-l+A)}.\end{aligned}$$

$$\begin{aligned}\limsup_m \frac{\alpha_m}{m^{\frac{1}{n}}} &\leq \lim_m \Bigg[\frac{-\ln d_m(U_{k+A}, U_{l-A})}{m^{\frac{1}{n}}}\Bigg(\frac{\ln C}{(k-l)(-\ln d_m(U_{k+A}, U_{l-A}))} \\ &\qquad + \frac{1}{(k-l)}\Bigg)\Bigg] \\ &= \frac{2\pi(n!)^{\frac{1}{n}}}{(C(\overline{D_{l-A}}, D_{k+A}))^{\frac{1}{n}}} \cdot \frac{1}{(k-l)}.\end{aligned}$$

On the other hand, since Φ is maximal off a compact set we can use the function

$$\rho(z) = \frac{\Phi - r}{r - s}$$

to compute the capacity of the condenser $(\overline{D_s}, D_r)$ for $r \gg s$ large enough. To be precise, in our case we get [8]:

$$C(\overline{D_s}, D_r) = \frac{1}{(r-s)^n}\int_X (dd^c\Phi)^n.$$

Taking this into account, we obtain:

$$\lim_m \frac{\alpha_m}{m^{\frac{1}{n}}} = 2\pi(n!)^{\frac{1}{n}}\left(\int_X (dd^c\Phi)^n\right)^{-\frac{1}{n}}.$$

We collect our findings in the proposition below. As usual $\|*\|_K$ denote the sup norm on a given compact set K.

PROPOSITION 3.10. *Let X be a S^*-parabolic Stein manifold of dimension n. Fix a plurisubharmonic exhaustion function Φ on X that is maximal outside a compact set. Then the exponent sequence $(\alpha_m)_n$ of the infinite type power series space associated to X by Theorem 3.7 above satisfies:*

$$\lim_m \frac{\alpha_m}{m^{\frac{1}{n}}} = 2\pi (n!)^{\frac{1}{n}} \left(\int_X (dd^c \Phi)^n \right)^{-\frac{1}{n}}$$

Note that one can construct a new plurisubharmonic exhaustion function that is again maximal off a compact set and with a prescribed positive right hand side value in the equation above, by simply multiplying the given exhaustion function with a positive constant. In particular we have:

COROLLARY 3.11. *A Stein manifold X of dimension n is S^*-parabolic if and only if there exists an exhaustion of X by connected, holomorphically convex compact sets $(K_k)_{k=1}^{\infty}$, $K_k \subset (K_{k+1})^{\circ}$, $k = 1, 2, \ldots,$ such that the graded spaces $(O(X), {}^{\|*\|_{K_k}})$ is tamely isomorphic to $(O(\mathbb{C}^n), {}^{\|*\|_{\Delta_k}})$, where Δ_k is the polydisc in $\mathbb{C}^n$ with radius k.*

4. Some classes of parabolic manifolds

An immediate class of parabolic manifolds can be obtained by considering Stein manifolds that admit a proper analytic surjection onto some $\mathbb{C}^n$. Affine algebraic manifolds belong to this class. Moreover such manifolds are S^*-parabolic [30].

In this section we will look at some ways of generating parabolic manifolds and give some nontrivial examples.

4.1. Complements of analytic multifunctions

Let $A \subset \mathbb{C}^n$ be a closed pluripolar set whose complement is pseudoconvex. Such sets are called "analytic multifunctions" by some authors. They are studied extensively by various authors and are extremely important in approximation theory, in the theory of analytic continuation and in the description of polynomial convex hulls (see [1], [9], [15], [17], [23], [27], [28], [36] and others). These sets are removable for the class of bounded plurisubharmonic functions defined on their complements. Hence their complements are parabolic Stein manifolds. We would like to restate Problem 2 given above in this setting since we hope that it will be more tractable.

PROBLEM 3. Let A be as above. Is $X = \mathbb{C}^n \setminus A$, S-parabolic?

In classical case, $n = 1$, every closed polar set $A \subset \mathbb{C}$ is analytic multifunction. As is well-known, if $K \subset\subset \mathbb{C}$ is a closed polar set, then there exist a subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus K$ function $u(z)$, such that $u|_K \equiv -\infty$ and $u(z) - \ln|z| \to 0$ as $z \to \infty$. One can use such functions to construct a special exhaustion function on $\mathbb{C} \setminus A$. To this end fix a $z_0 \notin K \circeq A \cup \{\infty\}$ an arbitrary point. Then there exist $u(z) \in \mathrm{sh}(\overline{\mathbb{C}} \setminus \{z_0\}) \cap \mathrm{har}((\overline{\mathbb{C}} \setminus K) \setminus \{z_0\}) : u|_K \equiv -\infty$ and $u(z) \to +\infty$ as $z \to z_0$. Therefore, $\rho(z) = -u(z)$ is exhaustion for $X = \mathbb{C} \setminus A$, with one singular point z_0.

On the other hand if $A = \{p(z) = 0\} \subset \mathbb{C}^n$ is an algebraic set, then it is easy to see that the function

$$\rho(z) \circeq -\frac{1}{\deg p} \ln|p| + 2\ln|z|$$

is a special exhaustion function for $\mathbb{C}^n \setminus A$ [41].

THEOREM 4.1. *Let* $A = \{('z, z_n) : F('z, z_n) = z_n^k + f_1('z) z_n^{k-1} + \cdots + f_k('z) = 0\}$ *be a Weierstrass polynomial (algebraiodal) set in* $\mathbb{C}^n$, *where* $f_j \in O(\mathbb{C}^{n-1})$ *are entire functions,* $j = 1, 2, \ldots, k$, $k \geqslant 1$. *Then* $X = \mathbb{C}^n \setminus A$ *is* S^**-parabolic.*

PROOF. We put

$$\rho(z) = -\ln|F(z)| + \ln(|'z|^2 + |F(z) - 1|^2). \tag{11}$$

Then $\rho(z) = -\infty$ precisely on the finite set $Q = \{'z = 0, F('0, z_n) = 1\}$. Moreover, ρ is maximal, $(dd^c\rho)^n = 0$ and continuous outside of $A \cup Q$, because $-\ln|F(z)|$ is pluriharmonic and $\ln(|'z|^2 + |F(z) - 1|^2)$ is maximal, since for any holomorphic vector-function $f = (f_1, f_2, \ldots, f_n) : f \neq 0$ the function $\ln\|f\|^2$ is a maximal psh function outside the zero set of f.

We will show, that $\rho(z)$ is exhaustion on $X = \mathbb{C}^n \setminus A$, i.e.

$$\{z : \rho(z) < R\} \subset\subset X \qquad \text{for every} \quad R \in \mathbb{R}. \tag{12}$$

If $F(z) = 0$, then $\rho(z) = +\infty + \ln(|'z|^2 + 1) = +\infty$, so that $\rho|_A = +\infty$. The condition (12) is clear, if all f_j, $j = 0, 1, \ldots, k$, are constant and so we assume that, at least one of them is not constant. Then $M_R = \max_{|'z| \leqslant R}\{|f_1('z)|, \ldots, |f_k('z)|\} \to \infty$. For $|'z| = R \geqslant 1$ and $|z_n| \leqslant M_R^2$ we have

$$\begin{aligned}\rho(z) &= \ln\frac{|'z|^2 + |F(z) - 1|^2}{|F(z)|} \geqslant \ln\frac{|'z|^2 + |F(z) - 1|^2}{1 + |F(z) - 1|}\\ &\geqslant \ln\frac{|'z|^2 + |F(z) - 1|^2}{|'z| + |F(z) - 1|} \geqslant \ln\frac{|'z| + |F(z) - 1|}{2} \geqslant \ln\frac{R}{2}.\end{aligned}$$

On the other hand on $|'z| \leqslant R$ and $|z_n| = M_R^2$ we have:

$$\begin{aligned}\rho(z) &= \ln \frac{|'z|^2 + |F(z) - 1|^2}{|F(z)|} \\ &\geqslant \ln \frac{(M_R^{2k} - M_R M_R^{2k-2} - \ldots - M_R - 1)^2}{M_R^{2k} + M_R M_R^{2k-2} + \ldots + M_R} \\ &= \ln M_R^{2k}(1 + \alpha_k),\end{aligned}$$

where $\alpha_k \to 0$ for $R \to \infty$. It follows that $\rho|_{\partial U_R} \to +\infty$ for $R \to \infty$, where $U_R = \{(|'z| \leqslant R, |z_n| \leqslant M_R^2)\}$.

Let us now consider the level set $D_C = \{z : \rho(z) < C\}$, C-constant. It is an open set and it contains the pole set Q. If R is so big, that $U_R \supset Q$ and $\min\{\ln \frac{R}{2}, \ln M_R^{2k}(1 + \alpha_R)\} \geqslant C$, then $D_C \subset\subset U_R$, since D_C has no any component outside U_R because of maximality of ρ on $X \setminus U_R$. This completes the proof that ρ is an exhaustion function.

COROLLARY 4.2. *The complement,* $\mathbb{C}^n \setminus \Gamma$, *of the graph* $\Gamma = \{('z, z_n) \in \mathbb{C}^n : z_n = f('z)\}$ *of an entire function* f *is* S^**-parabolic.*

4.2. Manifolds, which admit an exhaustion function with small $(dd^c)^n$ *mass*

Demailly [11] considered manifolds X which admit a continuous plurisubharmonic exhaustion function φ, with the property that,

$$\lim_{r\to\infty} \frac{\int_{B_r} (dd^c\varphi)^n}{\ln r} = 0, \tag{13}$$

where $B_r = \{z : \varphi(z) < \ln r\}$.

We note, that S^*-parabolic manifolds satisfy the condition (13). In fact, if $\rho(z)$ is special exhaustion function, then $(dd^c\rho)^n = 0$ off a compact $K \subset\subset X$ so $\int_{B_r} (dd^c\rho)^n = \int_K (dd^c\rho)^n = \text{const.}$, $r \geqslant r_0$. Hence, (13) holds.

If X has a continuous plurisubharmonic exhaustion function satisfying the condition (13), then every bounded above plurisubharmonic function on X is constant [11], so that this kind of manifolds are parabolic. In fact, a more general result is also true.

THEOREM 4.3. *If on a Stein manifold* X *of dimension* n, *there exist a plurisubharmonic* (*not necessary continuous*) *exhaustion function* φ *that satisfies,*

$$\liminf_{r\to\infty} \frac{\int_{B_r} (dd^c\varphi)^n}{[\ln r]^n} = 0, \tag{14}$$

then X *is parabolic.*

PROOF. Let's assume that X satisfies the condition (14), but X is not parabolic. We take a sequence $1 < r_1 < r_2 < \cdots, r_k \to \infty$, such that

$$\lim_{r\to\infty} \frac{\int_{B_{r_k}} (dd^c\varphi)^n}{[\ln r_k]^n} = 0 \tag{15}$$

Without loss of generality we can assume that the ball $B_1 = \{z : \varphi(z) < 0\} \neq \emptyset$. Then according the Proposition 2.2 the P-measure $\omega^*(z, \overline{B}_1, B_{r_k})$ decreases to $\omega^*(z, \overline{B}_1) \neq -1$ as $k \to \infty$. The function $\omega^*(z, \overline{B}_1)$ is maximal, that is $(dd^c\omega^*)^n = 0$ in $X \setminus B_1$ and is equal -1 on $\overline{B}_1$. Hence, by comparison principle of Bedford-Taylor [8] we have:

$$\begin{aligned}\int_{B_{r_k}} [dd^c\omega^*(z, \overline{B}_1, B_{r_k})]^n &= \int_{\overline{B}_1} [dd^c\omega^*(z, \overline{B}_1, B_{r_k})]^n \\ &\geqslant \int_{\overline{B}_1} [dd^c\omega^*(z, \overline{B}_1)]^n = \alpha > 0.\end{aligned}$$

However, if we apply again the comparison principle to $\omega^*(z, \overline{B}_1, B_{r_k})$ and

$$w(z) = \frac{\varphi(z) - \ln r_k}{\ln r_k},$$

then

$$\begin{aligned}\frac{1}{(\ln r_k)^n} \int_{B_{r_k}} [dd^c\varphi(z)]^n &= \int_{B_{r_k}} [dd^c w(z)]^n \\ &\geqslant \int_{B_{r_k}} [dd^c\omega^*(z, \overline{B}_1, B_{r_k})]^n \geqslant \alpha > 0.\end{aligned}$$

This contradiction proves the theorem.

4.3. Sibony-Wong manifolds

We next consider an important class of Stein manifolds (analytic sets) with the Liouville property, which were introduced by Sibony-Wong [26]. To describe these spaces we need to introduce some notation. For an n dimensional closed subvariety X of $\mathbb{C}^N_w$ let us denote, as usual, by σ, the restriction of $\ln|w|$ on X. Denoting the intersection of the r-ball in $\mathbb{C}^N$ with X by $B_r = \{z \in X : \sigma(z) < \ln r\}$ we can describe Sibony-Wong class as those X's, such that

$$\lim_{r\to\infty} \frac{\operatorname{vol}(B_r)}{\ln r} < \infty,$$

where the projective volume, $\mathrm{vol}(B_r)$ is equal to $\frac{H_{2n}(B_r)}{r^{2n}}$, H_{2n}-the Hausdorff measure (R^{2n}-volume) of B_r. Sibony and Wong showed that on such spaces any bounded holomorphic function is constant.

When $n = 1$, a special case of a result by Takegoshi [31] states that if

$$\sup_r \frac{\mathrm{vol}(B_r)}{g(r)} < \infty,$$

where $g : \mathsf{R}^+ \to \mathsf{R}^+$ is a nondecreasing continuous function such that

$$\int_0^\infty \frac{dr}{g(r)} = \infty,$$

then every negative smooth subharmonic function on X reduces to a constant, i.e. X is parabolic.

The proof of this proposition is based on the following estimation:

$$v(r)^2 \leq Cg(r)\frac{d}{dr}(v(r)), \qquad \forall\, v \in \mathrm{sh}(X) \cap C^1(X),$$

where $v(r) = \int_{B_r} dv \wedge d^c v$ and $C > 0$ is a constant. We note that if v is an arbitrary subharmonic function we can approximate it by smooth subharmonic functions $v_j \downarrow v$, we conclude that the above expression is also valid for arbitrary subharmonic functions and hence the proof given in [31] shows that such an X is parabolic. Taking $g(r) = \ln r$, we see that 1-dimensional Sibony-Wong manifolds are parabolic.

For $n > 1$, taking into account that $\mathrm{vol}(B_r) = \int_{B_r} (dd^c\sigma)^n$, by Wirtinger's theorem, we can deduce from Theorem 4.3 above that X is parabolic. Summarizing, we conclude that Sibony-Wong manifolds are parabolic for any $n \in \mathsf{N}$.

In connection with Problem 2 of Section 2 it will be of interest to investigate S^*-parabolicity of Sibony-Wong manifolds. Affine algebraic manifolds are among this class since their projective volume is finite. Moreover they are S^*-parabolic as we have already seen. On the other hand special exhaustion functions for S^*-parabolic Sibony-Wong manifolds other than the algebraic ones can not be asymptotically bigger than $\sigma(z) = \ln |z|$ restricted to X.

THEOREM 4.4. *Let $X \subset \mathsf{C}^N$ be a closed submanifold and $\rho(z)$ a special exhaustion function on it. If*

$$\underline{\lim} \frac{\rho(z)}{\sigma(z)} \geqslant \alpha > 0,$$

then X is an affine-algebraic set in C^N.

PROOF. Taking $C\rho$ instead ρ, if it is necessary, we can assume that, there is some compact $K \subset\subset X$ such, that

$$\frac{\rho(z)}{\sigma(z)} \geqslant 1, \qquad z \in X \setminus K.$$

Let $\sup_K \rho(z) = r_0$. Then $B_r = \{z \in X : \rho(z) < \ln r\}$, $r > r_0$, is not empty and open. Hence, the closure $\overline{B}_r$ is not pluripolar. Therefore, the extremal Green function

$$V_\rho(z, \overline{B}_r) = \sup\{u(z) \in \mathrm{psh}(X) : u|_{B_r} \leqslant 0, u(z) \leqslant C_u + \rho(z)\ \forall z \in X\}$$

is locally bounded on X (see [41]). In the other hand, since $\rho(z) \geqslant \sigma(z)$ outside of compact K, then

$$V(z, \overline{B}_r) \leqslant V_\rho(z, \overline{B}_r), \quad \text{where } V(z, \overline{B}_r) = V_\sigma(w, \overline{B}_r)|_X,$$
$$V_\sigma(w, \overline{B}_r) = \sup\{u(w) \in \mathrm{psh}(\mathbb{C}^N) : u|_{B_r} \leqslant 0, u(w) \leqslant C_u + \ln|w|\}.$$

But the extremal function $V(z, \overline{B}_r)$ is locally bounded on X if and only if X affine-algebraic ([20], [21]). This completes the proof.

REMARK 1. Stoll in [30] introduced and studied analytic sets, for which the solution of the equation (in the notation of the above section),

$$dd^c\omega_R \wedge \Psi = 0, \omega_R|_{\partial B_0} = -1, \omega_R|_{\partial B_R} = 0,$$

has the parabolic property, that $\omega_R \to -1$, for $R \nearrow \infty$, where Ψ is close, positive $(n-1, n-1)$ form. Atakhanov [2] called this kind of sets "parabolic type" and proved that the sets which satisfy

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r)}{\ln r} = 0$$

are of this type. Moreover, he constructed Nevanlinna's equidistribution theory for holomorphic maps $f : X \to P^m$. In particular, on this kind of sets theorems of Picard, Nevanlinna, Valiron on defect hyperplanes are true.

REMARK 2. In the literature there exists quite a number of Liouville-type theorems for specific complex manifolds. However the property that every bounded analytic function reduces to a constant need not imply parabolicity, as is well known to people working in classification theory of open Riemann surfaces. The simple example below illustrates this point.

EXAMPLE. Choose, on complex plane $\mathbb{C}_{z_1}$ a subharmonic function u with the property that $\{z : u(z_1) = -\infty\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$. Let $w(z_1, z_2) = u(z_1) +$

$\ln|z_2|$. Then $w \in \text{psh}(\mathbb{C}^2)$, and the component D of $\{(z_1, z_2) \in \mathbb{C}^2 : w(z_1, z_2) < 0\}$ containing the origin, being pseudoconvex, is a Stein manifold. Any bounded holomorphic function on it is constant by the Liouville's theorem. However, the plurisubharmonic function $w(z_1, z_2) \neq$ const. and is bounded from above i.e. D is not parabolic.

ACKNOWLEDGEMENT. The authors wish to thank the referee for reading the manuscript carefully and for valuable suggestions.

REFERENCES

1. Alexander, H., and Wermer, J., *On the approximation of singularity sets by analytic varieties II*, Michigan Math. J. 32 (1985), 227–235.
2. Atakhanov, K. U., *Defect divisors of holomorphic mappings of parabolic analytic sets*, Siberian Math. J. 28 (1987), 376–381.
3. Aytuna, A., *On Stein manifolds M for which $O(M)$ is isomorphic to $O(\Delta^n)$*, Manuscripta Math. 62 (1988), 297–315.
4. Aytuna, A., *Stein Spaces M for which $O(M)$ is isomorphic to a power series space*, pp. 115–154 in: *Advances in the theory of Fréchet spaces*, Kluwer, Dordrecht 1989.
5. Aytuna, A., Krone, J., and Terzioglu, T., *Complemented infinite type power series subspaces of nuclear Fréchet spaces*, Math. Ann. 283 (1989), 193–202.
6. Aytuna, A, Krone, J., and Terzioglu, T., *Imbedding of Power Series Spaces and Spaces of Analytic Functions*, Manuscripta Math. 67 (1990), 125–142.
7. Aytuna, A., *Algebraicity of certain complex analytic subvarieties of $\mathbb{C}^N$: a functional analysis point of view*, Linear Topol. Spaces Complex Anal. 1 (1994), 1–13.
8. Bedford, E., and Taylor, B. A., *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.
9. Berndtsson, B., and Ransford, T. J., *Analytic multifunctions, the $\bar{\partial}$-equation and a proof of the Corona theorem*, Pacific J. Math. 124 (1986), 57–72.
10. Bremermann, H. J., *On the conjecture of the equivalence of the plurisubharmonic functions and the Hartogs functions*, Math. Ann. 131 (1956), 76–86.
11. Demailly, J .P., *Mesures de Monge – Ampère et caractèrisation gèomètrique des variétés algébriques affines*, Mém. Soc. Math. France 19 (1985), 1–124.
12. Dubinsky, E.,*The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Math. 720, Springer, Berlin 1979.
13. Griffiths, P., and King, J., *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta Math. 130 (1973), 145–220.
14. Klimek, M., *Pluripotential Theory*, London Math. Soc. Mon. 6, Oxford Univ. Press, Oxford 1991.
15. Nishino, T., *Sur les ensembles pseudo concaves*, J. Math. Kyoto Univ. 1 (1962), 225–245.
16. Nivoche, S., *Proof of a conjecture of Zaharyuta concerning a problem of Kolmogorov on the ϵ-entropy*, Invent. Math. 158 (2004), 413–450.
17. Oka, K., *Note sur les familles de fonctions analytiques multiformes etc.*, J. Sci. Hiroshima Univ. Ser. A 4 (1934), 93–98.
18. Poppenberg, M., *Characterization of the subspaces of (s) in the tame category*, Math. Nachr. 150 (1991), 127–141.
19. Sadullaev, A., *Defect divisors in the sense of Valiron*, Mat. Sb. N.S. 108 (1979), 535–547.
20. Sadullaev, A., *Plurisubharmonic measure and capacity on complex manifolds*, Russian Math. Surv. 36 (1981), 61–119.

21. Sadullaev, A., *An estimate for polynomials on analytic set*, Math. USSR-Izv. 46 (1982), 524–534.
22. Sadullaev, A., *Plurisubharmonic functions*, pp. 59–106 in: *Encyclopedia of mathematical sciences* 8, Springer, Berlin 1985.
23. Sadullaev, A., *On analytic multifunctions*, Math. Notes 83 (2008), 652–656.
24. Sario, L., and Nakai, M., *Classification theory of Riemann surfaces*, Grundl. Math. Wiss. 164, Springer, Berlin 1970.
25. Shabat, B.V., *Distribution of the value of holomorphic mappings*, Transl. Math. Monogr. 61, Amer. Math. Soc., Providence 1985.
26. Sibony, N., and Wong, P.-M., *Some remarks on the Casorati-Weierstrass theorem*, Ann. Polon. Math. 39 (1981), 165–174.
27. Slodkowski, Z., *An analytic set-valued selection and its applications to the corona theorem, to polynomial hulls and joint spectra*, Trans. Amer. Math. Soc. 294 (1986), 367–378.
28. Slodkowski, Z., *An open mapping theorem for analytic multifunctions*, Studia Math. 122 (1997), 117–121.
29. Stoll, W., *Value distribution of holomorphic maps into compact complex manifolds*, Lecture Notes Math. 135, Springer, Berlin 1970.
30. Stoll, W.,*Value distribution on parabolic spaces*, Lecture Notes Math. 600, Springer, Berlin 1977.
31. Takegoshi, K, *A Liouville theorem on an analytic space*, J. Math. Soc. Japan 45 (1993), 301–311.
32. Vogt, D., *Charakterisierung der Unterräume von s*, Math. Z. 155 (1977), 109–117.
33. Vogt, D., *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. 67 (1980), 225–240.
34. Vogt, D., *On the functor* $\mathrm{Ext}^1(E, F)$ *for Fréchet spaces*, Studia Math. 85 (1987), 163–197.
35. Vogt, D., *Power series space representations of nuclear Fréchet spaces*, Trans. Amer. Math. Soc. 319 (1990), 191–208.
36. Yamaguchi, H., *Sur une uniformite des surfaces constants dune function entire de deux variables complexes*, J. Math. Kyoto Univ. 13 (1973), 417–433.
37. Zaharyuta, V. P., *Extremal plurisubharmonic functions, Hilbert scales and the isomorphism of spaces of analytic fonctions of several variables, I, II*, Theory of function, Functional Analysis and Their Applications 19 (1974), 133–157, 21 (1974), 65–83.
38. Zaharyuta, V. P., *Separately analytic functions, generalizations of Hartogs' theorem and holomorphic hulls*, Math. USSR-Sb. 101 (1976), 57–76.
39. Zaharyuta, V. P. *Spaces of analytic Functions and Complex Potential theory*, Linear Topol. Spaces Complex Anal. 1 (1994), 74–146.
40. Zaharyuta, V.P., *Kolmogorov prolem on widths asymptotics and pluripotential theory*, pp. 171–196 in: *Functional analysis and complex analysis*, Contemp. Math. 481, Amer. Math. Soc., Providence 2009.
41. Zeriahi, A., *Fonction de Green pluricomplex a pole a l'infini sur un espace de Stein parabolique*, Math. Scand. 69 (1991), 89–126.
42. Zeriahi, A., *Approximation polynomial et extension holomorphe avec croissance sur une variete algebrique*, Ann. Polon. Math. 63 (1996), 35–50.

FENS
SABANCI UNIVERSITY
34956 TUZLA ISTANBUL
TURKEY
E-mail: aytuna@sabanciuniv.edu

MATHEMATICS DEPARTMENT
NATIONAL UNIVERSITY OF
UZBEKISTAN VUZ GORODOK
700174 TASKENT
UZBEKISTAN
E-mail: sadullaev@mail.ru