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TORIC IDEALS OF FINITE GRAPHS AND ADJACENT 2-MINORS

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Abstract

We study the problem when an ideal generated by adjacent 2-minors is the toric ideal of a finite graph.

Let $X = (x_{ij})_{i=1,\dots,m,\,j=1,\dots,n}$ be a matrix of mn indeterminates, and let $A = K[\{x_{ij}\}_{i=1,\dots,m,\,j=1,\dots,n}]$ be the polynomial ring in mn variables over a field K. Given $1 \le a_1 < a_2 \le m$ and $1 \le b_1 < b_2 \le n$, the symbol $[a_1, a_2|b_1, b_2]$ denotes the 2-minor $x_{a_1b_1}x_{a_2b_2} - x_{a_1b_2}x_{a_2b_1}$ of X. In particular $[a_1, a_2|b_1, b_2]$ is a binomial of A. A 2-minor $[a_1, a_2|b_1, b_2]$ of X is *adjacent* ([4]) if $a_2 = a_1 + 1$ and $b_2 = b_1 + 1$. Following [2], we say that a set $\mathscr{M}$ of adjacent 2-minors of X is of *chessboard type* if the following conditions are satisfied:

- if $[a, a+1|b, b+1]$ and $[a, a+1|b', b'+1]$ with $b < b'$ belong to $\mathscr{M}$, then $b + 1 < b'$;
- if $[a, a+1|b, b+1]$ and $[a', a'+1|b, b+1]$ with $a < a'$ belong to $\mathscr{M}$, then $a + 1 < a'$.

Given a set $\mathscr{M}$ of adjacent 2-minors of X of chessboard type, we introduce the finite graph $\Gamma_{\mathscr{M}}$ on the vertex set $\mathscr{M}$, whose edges are $\{[a, a+1|b, b+1], [a', a'+1|b', b'+1]\}$ such that

- $[a, a+1|b, b+1] \neq [a', a'+1|b', b'+1]$,
- $\{a, a+1\} \cap \{a', a'+1\} \neq \emptyset$,
- $\{b, b+1\} \cap \{b', b'+1\} \neq \emptyset$.

For example, if $\mathscr{M} = \{[1, 2|2, 3], [2, 3|3, 4], [3, 4|2, 3], [2, 3|1, 2]\}$, then $\Gamma_{\mathscr{M}}$ is a cycle of length 4. The ideal $I_{\mathscr{M}}$ is generated by $x_{12}x_{23} - x_{13}x_{22}$, $x_{23}x_{34} - x_{24}x_{33}$, $x_{32}x_{43} - x_{33}x_{42}$ and $x_{21}x_{32} - x_{22}x_{31}$. The binomial $x_{32}(x_{13}x_{21}x_{34}x_{42} - x_{12}x_{24}x_{31}x_{43})$ belongs to $I_{\mathscr{M}}$ but neither x_{32} nor $x_{13}x_{21}x_{34}x_{42} - x_{12}x_{24}x_{31}x_{43}$ belongs to $I_{\mathscr{M}}$. Thus $I_{\mathscr{M}}$ is not prime.

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A fundamental fact regarding ideals generated by adjacent 2-minors is

LEMMA 1 ([2]). *Let $\mathcal{M}$ be a set of adjacent 2-minors of X, and let $I_{\mathcal{M}}$ be the ideal of A generated by all 2-minors belonging to $\mathcal{M}$. Then, $I_{\mathcal{M}}$ is a prime ideal if and only if $\mathcal{M}$ is of chessboard type, and $\Gamma_{\mathcal{M}}$ possesses no cycle of length* 4.

A finite graph G is said to be *simple* if G has no loop and no multiple edge. Let G be a finite simple graph on the vertex set $[d] = \{1, \ldots, d\}$, and let $E(G) = \{e_1, \ldots, e_n\}$ be its set of edges. Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ denote the polynomial ring in d variables over K, and let $K[G]$ denote the subring of $K[\mathbf{t}]$ generated by the squarefree quadratic monomials $\mathbf{t}^e = t_i t_j$ with $e = \{i, j\} \in E(G)$. The semigroup ring $K[G]$ is called the *edge ring* of G. Let $K[\mathbf{y}] = K[y_1, \ldots, y_n]$ denote the polynomial ring in n variables over K. The kernel I_G of the surjective homomorphism $\pi : K[\mathbf{y}] \to K[G]$ defined by setting $\pi(y_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, n$ is called the *toric ideal* of G. Clearly, I_G is a prime ideal. It is known that I_G is generated by the binomials corresponding to even closed walks of G. See [7] , [6, Chapter 9] and [5, Lemma 1.1] for details.

EXAMPLE 2. Let G be a complete bipartite graph with the edge set $E(G) = \{\{i, p+j\} \mid 1 \leq i \leq p, 1 \leq j \leq q\}$. Let $X = (x_{ij})_{i=1,\ldots,p,j=1,\ldots,q}$ be a matrix of pq indeterminates and $K[\mathbf{x}] = K[\{x_{ij}\}_{i=1,\ldots,p,j=1,\ldots,q}]$. Then, I_G is the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[G]$ defined by setting $\pi(x_{ij}) = t_i t_{p+j}$ for $1 \leq i \leq p$, $1 \leq j \leq q$. It is known [6, Proposition 5.4] that I_G is generated by the set of all 2-minors of X. Note that each 2-minor $x_{ij}x_{i'j'} - x_{ij'}x_{i'j}$ corresponds to the cycle $\{\{i, p+j\}, \{p+j, i'\}, \{i', p+j'\}, \{p+j', i\}\}$ of G.

In general, a toric ideal is the defining ideal of a homogeneous semigroup ring. We refer the reader to [6] for detailed information on toric ideals. It is known [1] that a binomial ideal I, i.e., an ideal generated by binomials, is a prime ideal if and only if I is a toric ideal. An interesting research problem on toric ideals is to determine when a binomial ideal is the toric ideal of a finite graph.

EXAMPLE 3. The ideal $I = \langle x_1x_2 - x_3x_4, x_1x_2 - x_5x_6, x_1x_2 - x_7x_8\rangle$ is the toric ideal of the semigroup ring $K[t_1t_5, t_2t_3t_4t_5, t_1t_2t_5, t_3t_4t_5, t_2t_3t_5, t_1t_4t_5, t_1t_3t_5, t_2t_4t_5]$. If there exists a graph G such that $I = I_G$, then three quadratic binomials correspond to cycles of length 4. However, this is impossible since these three cycles must have common two edges e_1 and e_2 such that $e_1 \cap e_2 = \emptyset$. Thus, I cannot be the toric ideal of a finite graph. This observation implies that the toric ideal of a finite distributive lattice $\mathscr{L}$ (see [3]) is the toric ideal of a finite graph if and only if $\mathscr{L}$ is planar. In fact, if $\mathscr{L}$ is planar, then it is easy to see that the toric ideal of $\mathscr{L}$ is the toric ideal of a bipartite graph. If $\mathscr{L}$ is not planar, then $\mathscr{L}$ contains a sublattice that is isomorphic to the Boolean lattice B_3 of rank 3. Since the toric ideal of B_3 has three binomials above, the toric ideal of $\mathscr{L}$ cannot be the toric ideal of a finite graph.

Let $\mathscr{M}$ be a set of adjacent 2-minors. Now, we determine when a binomial ideal $I_{\mathscr{M}}$ generated by $\mathscr{M}$ is the toric ideal I_G of a finite graph G. Since I_G is a prime ideal, according to Lemma 1, if there exists a finite graph G with $I_{\mathscr{M}} = I_G$, then $\mathscr{M}$ must be of chessboard type and $\Gamma_{\mathscr{M}}$ possesses no cycle of length 4.

THEOREM 4. *Let $\mathscr{M}$ be a set of adjacent 2-minors. Then, there exists a finite graph G such that $I_{\mathscr{M}} = I_G$ if and only if $\mathscr{M}$ is of chessboard type, $\Gamma_{\mathscr{M}}$ possesses no cycle of length 4, and each connected component of $\Gamma_{\mathscr{M}}$ possesses at most one cycle.*

PROOF. We may assume that $\mathscr{M}$ is of chessboard type and $\Gamma_{\mathscr{M}}$ possesses no cycle of length 4. Let $\mathscr{M} = \mathscr{M}_1 \cup \cdots \cup \mathscr{M}_s$, where $\Gamma_{\mathscr{M}_1}, \ldots, \Gamma_{\mathscr{M}_s}$ is the set of connected components of $\Gamma_{\mathscr{M}}$. If $i \neq j$, then $f \in \mathscr{M}_i$ and $g \in \mathscr{M}_j$ have no common variable. Hence, there exists a finite graph G such that $I_{\mathscr{M}} = I_G$ if and only if for each $1 \leq i \leq s$, there exists a finite graph G_i such that $I_{\mathscr{M}_i} = I_{G_i}$. Thus, we may assume that $\Gamma_{\mathscr{M}}$ is connected. Let p be the number of vertices of $\Gamma_{\mathscr{M}}$, and let q be the number of edges of $\Gamma_{\mathscr{M}}$. Since $\Gamma_{\mathscr{M}}$ is connected, we have $p \leq q + 1$.

Only if. Suppose that there exists a finite graph G with $I_{\mathscr{M}} = I_G$. From [2, Theorem 2.3], the codimension of $I_{\mathscr{M}}$ is equal to p. Let d be the number of vertices of G, and let n be the number of edges of G. Then, we have $d \leq 4p-2q$ and $n = 4p - q$. The height of I_G is given in [7]. If G is bipartite, then the codimension of I_G satisfies $p \geq n-d+1 \geq (4p-q)-(4p-2q)+1 = q+1$. Hence, we have $p = q + 1$ and $\Gamma_{\mathscr{M}}$ is a tree. On the other hand, if G is not bipartite, then the codimension of I_G satisfies $p \geq n - d \geq (4p - q) - (4p - 2q) = q$. Hence, we have $p \in \{q, q + 1\}$ and $\Gamma_{\mathscr{M}}$ has at most one cycle.

If. Suppose that $\Gamma_{\mathscr{M}}$ has at most one cycle. Then, we have $p \in \{q, q + 1\}$.

Case 1. $p = q + 1$, i.e., $\Gamma_{\mathscr{M}}$ is a tree.

Through induction on p, we will show that there exists a connected bipartite graph G such that $I_{\mathscr{M}} = I_G$. If $p = 1$, then $I_{\mathscr{M}} = I_G$ where G is a cycle of length 4. Let $k > 1$, and suppose that the assertion holds for $p = k-1$. Suppose that $\Gamma_{\mathscr{M}}$ has k vertices. Since $\Gamma_{\mathscr{M}}$ is a tree, $\Gamma_{\mathscr{M}}$ has a vertex $v = [a, a+1|b, b+1]$ of degree 1. Let $\mathscr{M}' = \mathscr{M} \setminus \{v\}$. Since $\Gamma_{\mathscr{M}'}$ is a tree, there exists a connected bipartite graph G' such that $I_{\mathscr{M}'} = I_{G'}$ by the hypothesis of induction. From [5, Theorem 1.2], since $I_{G'}$ is generated by quadratic binomials, any cycle of G' of length ≥ 6 has a chord. Let $v' = [a', a' + 1|b', b' + 1]$ denote the vertex of

$\Gamma_{\mathcal{M}}$ that is incident with v. Let $e = \{i, j\}$ be the edge of G' corresponding to the common variable of v and v'. Let $\{1, 2, \dots, d\}$ be the vertex set of G'. We now define the connected bipartite graph G on the vertex set $\{1, 2, \dots, d, d+1, d+2\}$ with the edge set $E(G') \cup \{\{i, d+1\}, \{d+1, d+2\}, \{d+2, j\}\}$. Then, any cycle of G of length ≥ 6 has a chord, and hence, I_G is generated by quadratic binomials. Thus, I_G is generated by the quadratic binomials of $I_{G'}$ together with v corresponding to the cycle $\{\{i, d+1\}, \{d+1, d+2\}, \{d+2, j\}, \{j, i\}\}$. Therefore, $I_{\mathcal{M}} = I_G$.

Case 2. $p = q$, i.e., $\Gamma_{\mathcal{M}}$ has exactly one cycle.

Then, we have $p \geq 8$. Through induction on p, we will show that there exists a graph G such that $I_{\mathcal{M}} = I_G$. If $p = 8$, then $\Gamma_{\mathcal{M}}$ is a cycle of length 8. Then, $I_{\mathcal{M}} = I_G$ where G is the graph shown in Figure 1.

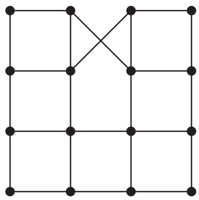

FIGURE 1. Graph for $\mathcal{M}$ such that $\Gamma_{\mathcal{M}}$ is a cycle of length 8.

Let $k > 8$ and suppose that the assertion holds for $p = k - 1$. Suppose that $\Gamma_{\mathcal{M}}$ has k vertices. If $\Gamma_{\mathcal{M}}$ has a vertex $v = [a, a+1|b, b+1]$ of degree 1, then $\Gamma_{\mathcal{M}'}$ where $\mathcal{M}' = \mathcal{M} \setminus \{v\}$ has exactly one cycle, and hence, there exists a graph G' such that $I_{\mathcal{M}'} = I_{G'}$ by the hypothesis of induction. Let $v' = [a', a'+1|b', b'+1]$ denote the vertex of $\Gamma_{\mathcal{M}}$ that is incident with v. Let $e = \{i, j\}$ be the edge of G' corresponding to the common variable of v and v'. Suppose that the vertex set of G' is $\{1, 2, \dots, d\}$. We now define the graph G on the vertex set $\{1, 2, \dots, d, d+1, d+2\}$ with the edge set $E(G') \cup \{\{i, d+1\}, \{d+1, d+2\}, \{d+2, j\}\}$. Since G' satisfies the conditions in [5, Theorem 1.2], it follows that G satisfies the conditions in [5, Theorem 1.2]. Thus, I_G is generated by the quadratic binomials of $I_{G'}$ together with v corresponding to the cycle $\{\{i, d+1\}, \{d+1, d+2\}, \{d+2, j\}, \{j, i\}\}$. Therefore, $I_{\mathcal{M}} = I_G$.

Suppose that $\Gamma_{\mathcal{M}}$ has no vertex of degree 1. Then, $\Gamma_{\mathcal{M}}$ is a cycle of length k. A 2-minor $ad - bc \in \mathcal{M}$ is called *free* if one of the following holds:

- Neither a nor d appears in other 2-minors of $\mathcal{M}$,
- Neither b nor c appears in other 2-minors of $\mathcal{M}$.

From [2, Lemma 1.6], $\mathcal{M}$ has at least two free 2-minors. Let $v = [a, a+1|b, b+1]$ be a free 2-minor of $\mathcal{M}$. We may assume that neither $x_{a,b}$ nor $x_{a+1,b+1}$ appears in other 2-minors of $\mathcal{M}$. Since $\Gamma_{\mathcal{M}}$ is a cycle, $x_{a+1,b}$ appears in exactly two 2-minors of $\mathcal{M}$ and $x_{a,b+1}$ appears in exactly two 2-minors of $\mathcal{M}$. Let $\mathcal{M}' = \mathcal{M} \setminus \{v\}$. Since $\Gamma_{\mathcal{M}'}$ is a tree, there exists a connected bipartite graph G' such that $I_{\mathcal{M}'} = I_{G'}$ by the argument in Case 1. Suppose that the edge $\{1, 3\}$ corresponds to the variable $x_{a+1,b}$ and the edge $\{2, 4\}$ corresponds to the variable $x_{a,b+1}$. We now define the graph G as shown in Figure 2, where vertices 1 and 2 belong to the same part of the bipartite graph G'. Note that G is not bipartite.

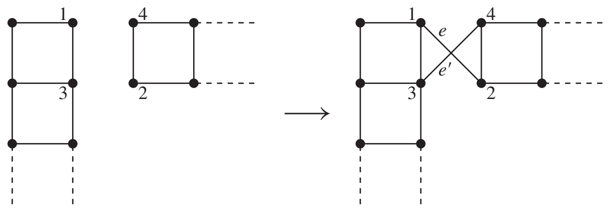

FIGURE 2. New graph G arising from G'.

Let $e = \{1, 2\}$ and $e' = \{3, 4\}$. Since G' is a bipartite graph, it follows that

(a) If either e or e' is an edge of an even cycle C of G, then $\{e, e'\} \subset E(C)$.

(b) If C' is an odd cycle of G, then $\{e, e'\} \cap E(C') \neq \emptyset$.

Let I denote the ideal generated by all quadratic binomials in I_G. Since each quadratic binomial in I_G corresponds to a cycle of G of length 4, it follows that $I_{\mathcal{M}} = I$. Thus, it is sufficient to show that $I_G = I$, i.e., I_G is generated by quadratic binomials. From [5, Theorem 1.2], since G' is bipartite and since $I_{G'}$ is generated by quadratic binomials, all cycles of G' of length ≥ 6 have a chord.

Let C be an even cycle of G of length ≥ 6. If $E(C) \cap \{e, e'\} = \emptyset$, then C has an even-chord since all cycles of the bipartite graph G' of length ≥ 6 have a chord. Suppose that $\{e, e'\} \subset E(C)$ holds. Then, either $\{1, 3\}$ or $\{2, 4\}$ is a chord of C. Moreover, such a chord is an even-chord of C from (b) above.

Let C and C' be odd cycles of G having exactly one common vertex. From (b) above, we may assume that $e \in E(C) \setminus E(C')$ and $e' \in E(C') \setminus E(C)$. If $\{1, 3\}$ does not belong to $E(C) \cup E(C')$, then $\{1, 3\}$ satisfies the condition in [5, Theorem 1.2 (ii)]. If $\{1, 3\}$ belongs to $E(C) \cup E(C')$, then $\{2, 4\} \notin E(C) \cup E(C')$ since C and C' have exactly one common vertex. Hence, $\{2, 4\}$ satisfies the condition in [5, Theorem 1.2 (ii)].

Let C and C' be odd cycles of G having no common vertex. Then, neither $\{1, 3\}$ nor $\{2, 4\}$ belong to $E(C) \cup E(C')$. Hence, $\{1, 3\}$ and $\{2, 4\}$ satisfy the condition in [5, Theorem 1.2 (iii)].

Thus, from [5, Theorem 1.2], I_G is generated by quadratic binomials. Therefore, $I_G = I_{\mathcal{M}}$ as desired.

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