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BEAUVILLE SURFACES WITH ABELIAN BEAUVILLE GROUP

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Abstract

A Beauville surface is a rigid surface of general type arising as a quotient of a product of curves C_1, C_2 of genera $g_1, g_2 \geq 2$ by the free action of a finite group G. In this paper we study those Beauville surfaces for which G is abelian (so that $G \cong \mathsf{Z}_n^2$ with $\gcd(n, 6) = 1$ by a result of Catanese). For each such n we are able to describe all such surfaces, give a formula for the number of their isomorphism classes and identify their possible automorphism groups. This explicit description also allows us to observe that such surfaces are all defined over Q.

1. Introduction and statement of results

A complex surface S is said to be *isogenous to a product* if it is isomorphic to the quotient of a product of curves $C_1 \times C_2$ of genus $g_1 = g(C_1)$, $g_2 = g(C_2) \geq 1$ by the free action of a finite subgroup G of $\mathrm{Aut}(C_1 \times C_2)$, the automorphism group of $C_1 \times C_2$.

If additionally the curves satisfy $g_1, g_2 \geq 2$ we say that S is *isogenous to a higher product*. We will always assume that $g_1 \leq g_2$.

It is not difficult to see that an element of $\mathrm{Aut}(C_1 \times C_2)$ either preserves each curve or interchanges them, the latter being possible only if $C_1 \cong C_2$. Consequently one says that S is of *unmixed type* if no element of G interchanges factors, and of *mixed type* otherwise.

On the other hand if two elements $g, h \in G$ interchange factors it is clear that gh belongs to the subgroup $G^0 < G$ of factor-preserving elements, and therefore $[G : G^0] = 2$.

There is always a minimal realization of S in the sense that G^0 acts faithfully on each factor C_i. This is because if for instance G^0 did not act faithfully on C_1, so that there exists a subgroup $G' \lhd G$ acting trivially on C_1, then we could write

$$S \cong \frac{C_1 \times C_2}{G} = \frac{C_1/G' \times C_2/G'}{G/G'} = \frac{C_1 \times (C_2/G')}{G/G'}.$$

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Hence, from now on we will always assume that the realization $S \cong (C_1 \times C_2)/G$ is minimal.

A Beauville surface is a particular type of surface isogenous to a product. These surfaces were introduced by Catanese in [6] following a construction by Beauville in [5] (see Example 1 below), and they have been subsequently studied by himself, Bauer and Grunewald ([1], [2], [4]), somewhat later by Fuertes, González-Diez and Jaikin ([8], [9]), and more recently by Jones, Penegini, Garion, Larsen, Lubotzky, Guralnick, Malle, Fairbairn, Magaard and Parker ([10], [14], [11], [12], [13], [7]).

A *Beauville surface* is a compact complex surface S isogenous to a higher product, $S \cong (C_1 \times C_2)/G$, where G^0 acts on each of the curves in such a way that each C_i/G^0 is isomorphic to the complex projective line P^1, and the natural projections $C_1 \longrightarrow \mathsf{P}^1$ and $C_2 \longrightarrow \mathsf{P}^1$ ramify over three values.

Catanese proved that both the group G and the curves C_1, C_2 – hence also the genera g_1, g_2 – are invariants of the Beauville surface. Consequently we will say that the surface S has (*Beauville*) *group* G and *covering product* $C_1 \times C_2$.

Here we will consider only unmixed Beauville surfaces. In this case there exist isomorphisms $\varphi_i : G \longrightarrow H_i \leq \mathrm{Aut}(C_i)$ so that the action of an element $g \in G$ on a point $(p_1, p_2) \in C_1 \times C_2$ is given by $g(p_1, p_2) = (\varphi_1(g)(p_1), \varphi_2(g)(p_2))$. From this point of view giving an unmixed Beauville surface amounts to specifying an isomorphism $\phi = \varphi_2 \circ \varphi_1^{-1}$ between groups $H_1 \leq \mathrm{Aut}(C_1)$ and $H_2 \leq \mathrm{Aut}(C_2)$ satisfying the following two conditions

(C1) C_i/H_i is an orbifold of genus zero with three branching values, and

(C2) for any non-identity element $h_1 \in H_1$ which fixes a point on C_1 the element $h_2 = \phi(h_1) \in H_2$ acts freely on C_2 (so that the action $h(p_1, p_2) = (h_1(p_1), h_2(p_2))$ is free on $C_1 \times C_2$).

It turns out that the problem of deciding whether a finite group is an unmixed Beauville group is purely group theoretical [6]. This occurs if and only if G admits two triples of generators (a_1, b_1, c_1) and (a_2, b_2, c_2) such that

(i) $a_1 b_1 c_1 = a_2 b_2 c_2 = 1$;

(ii) $\frac{1}{\mathrm{ord}(a_i)} + \frac{1}{\mathrm{ord}(b_i)} + \frac{1}{\mathrm{ord}(c_i)} < 1, i = 1, 2$;

(iii) $\Sigma(a_1, b_1, c_1) \cap \Sigma(a_2, b_2, c_2) = \{\mathrm{Id}_G\}$, where we define

$$\Sigma(a, b, c) = \left\{ \bigcup_{g \in G} \left(g\langle a \rangle g^{-1} \cup g\langle b \rangle g^{-1} \cup g\langle c \rangle g^{-1} \right) \right\}.$$

This is a very useful characterization since it allows the use of computer programs such as GAP or MAGMA to solve the problem for groups of low order.

The first example of a Beauville surface was given by A. Beauville some twenty years before the term was coined by Catanese. It appears as an exercise at the end of his book on complex surfaces [5] as an example of a complex surface of general type whose two geometrical invariants q and p_g vanish.

EXAMPLE 1 (Beauville). Let us denote by $\mathsf{Z}_n = \mathsf{Z}/n\mathsf{Z}$ the group of integers modulo n and by F_n the Fermat curve of degree n,

$$F_n = \left\{[x : y : z] \in \mathsf{P}^2(\mathsf{C}) : x^n + y^n + z^n = 0\right\}.$$

For $n = 5$ the group $G = \mathsf{Z}_5^2$ acts freely on $F_5 \times F_5$ in the following way: for each $(\alpha, \beta) \in G$ we put

$$(\alpha, \beta)\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \left(\begin{bmatrix} \xi^{\alpha} x_1 \\ \xi^{\beta} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{\alpha+3\beta} x_2 \\ \xi^{2\alpha+4\beta} y_2 \\ z_2 \end{bmatrix}\right)$$

where $\xi = e^{2\pi i/5}$. It is easy to see that the surface $S := (F_5 \times F_5)/\mathsf{Z}_5^2$ is an unmixed Beauville surface.

Obviously the formula

$$(\alpha, \beta)\,([x : y : z]) = [\xi^{\alpha} x : \xi^{\beta} y : z] \qquad (\xi = e^{2\pi i/n})$$

also defines an action of the group Z_n^2 on F_n when $n > 5$. The quotient F_n/Z_n^2 is always an orbifold of genus zero with three branching values, all of them of order n. Therefore any group automorphism $\phi : \mathsf{Z}_n^2 \longrightarrow \mathsf{Z}_n^2$ satisfying condition (C2) above gives rise to a Beauville surface, something that can occur only when $\gcd(n, 6) = 1$ (see [6]). Since any such ϕ is necessarily of the form

$$\phi = \phi_A : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto A \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \text{for some} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathsf{Z}_n),$$

any surface obtained in this way is of the form $S_A^n = (F_n \times F_n)/G_A$, where G_A is the subgroup of $\mathrm{Aut}(F_n \times F_n)$ defined by

$$G_A = \left\{\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in G\right\} \cong G.$$

In this way the action of an element $(\alpha, \beta) \in \mathsf{Z}_n^2$ on $F_n \times F_n$ is given by

$$(\alpha, \beta)\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \left(\begin{bmatrix} \xi^{\alpha} x_1 \\ \xi^{\beta} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{a\alpha+b\beta} x_2 \\ \xi^{c\alpha+d\beta} y_2 \\ z_2 \end{bmatrix}\right).$$

Thus the surface constructed in Example 1 is S^5_A for $A = \left(\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}\right)$.

It is known that, even though in the definition of a Beauville surface both curves C_1 and C_2 are only required to have genus greater than 1, in fact both genera have to be greater than 5 (see [9]). Thus Beauville's example S^5_A above reaches the minimum possible bigenus $(g_1, g_2) = (6, 6)$.

In this paper we prove the following facts:

(1) Each Beauville surface with an abelian group G is isomorphic to one of the form S^n_A (Theorem 1), and is defined over Q (Corollary 1).

(2) The number $\Theta(n)$ of isomorphism classes of Beauville surfaces which have Beauville group Z_n^2 (where $\gcd(n, 6) = 1$) is given by the formula in Theorem 2. One consequence is that asymptotically $\Theta(n) \sim n^4/72$ for prime powers n. Another is that $\Theta(5) = 1$, which means that Beauville's original surface S^5_A mentioned above is the only Beauville surface with group Z_5^2.

(3) The automorphism group of S^n_A is either Z_n^2, or an extension of Z_n^2 by one of the cyclic groups Z_2, Z_3 and Z_6 or by the symmetric group $\mathscr{S}_3$ (Proposition 1).

In [2] Bauer, Catanese and Grunewald have given a lower bound for the asymptotic behaviour of $\Theta(n)$ as $n \to \infty$. More recently, Garion and Penegini in [12] have given upper and lower bounds for $\Theta(n)$ for each n, and in Theorem 2 we extend their results by giving an explicit formula for $\Theta(n)$.

2. Unmixed Beauville structures on the product of Fermat curves

It was proved in [6] that if $S = (C_1 \times C_2)/G$ is a Beauville surface with abelian group G then $G = \mathsf{Z}_n^2$ with $\gcd(n, 6) = 1$, and in [9] that in that case $C_1 = C_2 = F_n$, the Fermat curve of degree n. Combining these two facts one gets the following slightly more precise result:

THEOREM 1. *Any Beauville surface with abelian Beauville group is isomorphic to one of the form* $S^n_A = (F_n \times F_n)/G_A$.

PROOF. In view of the results in [6] and [9] mentioned above it is sufficient to observe that $\mathrm{Aut}(F_n)$ possesses a unique subgroup isomorphic to Z_n^2 satisfying condition (C1) in our definition of Beauville surface, and therefore any Beauville surface with group Z_n^2 is determined by an isomorphism $A : \mathsf{Z}_n^2 \longrightarrow \mathsf{Z}_n^2$ satisfying condition (C2).

This explicit description of Beauville surfaces with abelian group yields the following:

COROLLARY 1. *Each Beauville surface with abelian group is defined over* Q.

PROOF. Since the curves F_n are obviously defined over Q we only have to show that so is the group G_A. That is, we have to show that the obvious action of the absolute Galois group $\mathrm{Gal}(\overline{\mathsf{Q}}/\mathsf{Q})$ on automorphisms of $F_n \times F_n$ leaves the group G_A setwise invariant.

Now, let σ be an arbitrary element of $\mathrm{Gal}(\overline{\mathsf{Q}}/\mathsf{Q})$, and suppose that $\sigma(\xi) = \xi^j$; then, clearly, σ transforms the action of an element (α, β) of G_A into the action of another element of G_A, namely $(j\alpha, j\beta)$.

REMARK 1. In regard to the fields of definition of quotient varieties by abelian groups we draw the reader's attention to the interesting Corollary 1.9 in the article [3] by Bauer and Catanese.

We would like to find which matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathsf{Z}_n)$ define Beauville surfaces S^n_A, that is, we would like to characterize those matrices A such that the group G_A defined above acts freely on $F_n \times F_n$. Note that the Beauville surface S^n_A corresponds, in terms of triples of generators, to a pair of ordered triples satisfying conditions (i) to (iii) where, without loss, we can take the first triple to be the standard triple $((1, 0), (0, 1), (-1, -1))$. Then the second triple is $((a, c), (b, d), (-a - b, -c - d))$, obtained from the first one by the automorphism represented by the matrix $A \in \mathrm{GL}_2(\mathsf{Z}_n)$.

With the previous notation, the elements (α, β) fixing points in the first component are precisely those of the form $(k, 0)$, $(0, k)$ and (k, k). Equivalently the elements (α, β) that fix points in the second curve are those such that $a\alpha + b\beta = 0$, $c\alpha + d\beta = 0$ or $a\alpha + b\beta = c\alpha + d\beta$. We obtain the following:

LEMMA 1. *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathsf{Z}_n)$. *The group* G_A *defined above acts freely on the product* $F_n \times F_n$ *if and only if the following conditions hold:*

$$a, b, c, d, a + b, c + d, a - c, b - d, a + b - c - d \in U(\mathsf{Z}_n) \tag{1}$$

where $U(\mathsf{Z}_n)$ *is the group of units of* Z_n.

PROOF. Let $(\alpha, \beta) \in \mathsf{Z}_n$ be an element fixing some point of the first curve.

If $(\alpha, \beta) = (k, 0)$ the action on the second curve is given by (ak, ck), which fixes no point if and only if $ak \neq 0$, $ck \neq 0$, $ak \neq ck$. This will be true for all k if and only if $a, c, a - c \in U(\mathsf{Z}_n)$.

Arguing the same way with elements of the form $(\alpha, \beta) = (0, k)$ and $(\alpha, \beta) = (k, k)$ one obtains the result.

Note that from this lemma we can deduce the already mentioned fact, due to Catanese, that $\gcd(n, 6)$ must be 1 for Z_n^2 to admit a Beauville structure.

If n is even, the conditions $a, b, a+b \in U(\mathsf{Z}_n)$ cannot hold simultaneously. On the other hand if n is a multiple of 3 then necessarily $a \equiv b \bmod 3$ and $c \equiv d \bmod 3$, but then the matrix A is not invertible.

We will denote by $\mathfrak{F}_n$ the set of matrices in $\mathrm{GL}_2(\mathsf{Z}_n)$ satisfying the conditions in (1), and will write $e^A_{(\alpha,\beta)}$ for the element of G_A corresponding to $(\alpha, \beta) \in \mathsf{Z}_n^2$. In Z_5^2, for instance, there are 24 different matrices satisfying these conditions.

It follows from Theorem 1 that $\mathrm{Aut}(F_n \times F_n)$ acts on the set of groups $\{G_A : A \in \mathfrak{F}_n\}$ by conjugation. Furthermore, two such groups are in the same orbit if and only if the corresponding Beauville surfaces are isomorphic. This is because any isomorphism $\phi : S^n_A \longrightarrow S^n_{A'}$ lifts to an automorphism Φ of the product of curves, yielding a commutative diagram

$$\begin{array}{ccccc} & F_n \times F_n & \xrightarrow{\Phi} & F_n \times F_n & \\ & \downarrow & & \downarrow & \\ S = & \dfrac{F_n \times F_n}{G_A} & \xrightarrow{\phi} & \dfrac{F_n \times F_n}{G_{A'}} & = S' \end{array}$$

such that $\Phi G_A \Phi^{-1} = G_{A'}$ (see [6]).

It is well known that the automorphism group of F_n is $\mathsf{Z}_n^2 \rtimes \mathscr{S}_3$ and so $\mathrm{Aut}(F_n \times F_n) = \langle \mathsf{Z}_n^2 \times \mathsf{Z}_n^2, \mathscr{S}_3 \times \mathscr{S}_3, J\rangle$, where $J(\mu_1, \mu_2) = (\mu_2, \mu_1)$ and the groups $\mathsf{Z}_n^2 \times \mathsf{Z}_n^2$ and $\mathscr{S}_3 \times \mathscr{S}_3$ act separately on each factor, the first one by multiplying the homogeneous coordinates by n-th roots of unity and the second one by permuting them. We now note that every element of $\mathsf{Z}_n^2 \times \mathsf{Z}_n^2$ fixes each element $e^A_{(\alpha,\beta)} \in G_A$ by conjugation, so the action of this subgroup on the set $\{G_A\}$ is trivial. We can therefore restrict our attention to the quotient group $W = \mathrm{Aut}(F_n \times F_n)/(\mathsf{Z}_n^2 \times \mathsf{Z}_n^2)$, which is a semidirect product $(\mathscr{S}_3 \times \mathscr{S}_3) \rtimes \langle J\rangle$ of $\mathscr{S}_3 \times \mathscr{S}_3$ by $\langle J\rangle \cong \mathscr{S}_2$, with the complement $\mathscr{S}_2$ transposing the direct two factors $\mathscr{S}_3$ by conjugation. Thus W is the wreath product $\mathscr{S}_3 \wr \mathscr{S}_2$ of $\mathscr{S}_3$ by $\mathscr{S}_2$.

The group $\mathscr{S}_3$ can be viewed as a subgroup of $\mathrm{GL}_2(\mathsf{Z}_n)$ via the group monomorphism:

$$M : \mathscr{S}_3 \longrightarrow \mathrm{GL}_2(\mathsf{Z}_n)$$

$$\tau \longmapsto M_\tau$$

determined by $M_{\sigma_1} = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ for $\sigma_1 = (1, 3, 2)$ and $M_{\sigma_2} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $\sigma_2 = (1, 2)$.

LEMMA 2. *Let $\Phi = (\tau_1, \tau_2) \in \mathscr{S}_3 \times \mathscr{S}_3$ be a factor-preserving automorphism. Then $\Phi G_A \Phi^{-1} = G_{A'}$, where $A' = M_{\tau_2} A M_{\tau_1}^{-1}$.*

On the other hand, if $\Phi' = \Phi \circ J$ is an automorphism interchanging factors, then $\Phi' G_A \Phi'^{-1} = G_{A'}$, where $A' = M_{\tau_2} A^{-1} M_{\tau_1}^{-1}$.

PROOF. First we note that if $A \in \mathfrak{F}_n$, then $A' = M_{\tau_2} A^{\pm 1} M_{\tau_1}^{-1}$ belongs to $\mathfrak{F}_n$ as well. We will now see how the group $\mathrm{Aut}(F_n \times F_n)$ acts by conjugation on the elements of G_A.

To see how an element $(\tau_1, \tau_2) \in \mathscr{S}_3 \times \mathscr{S}_3$ acts on G_A we will write $(\tau_1, \tau_2) = (\tau_1, \mathrm{Id}) \circ (\mathrm{Id}, \tau_2)$, with $\tau_1, \tau_2 \in \mathscr{S}_3$. If $\Phi = (\mathrm{Id}, \sigma_1)$ then

$$\Phi e^A_{(\alpha,\beta)} \Phi^{-1} \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \left(\begin{bmatrix} \xi^{\alpha} x_1 \\ \xi^{\beta} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{(c-a)\alpha+(d-b)\beta} x_2 \\ \xi^{-a\alpha-b\beta} y_2 \\ z_2 \end{bmatrix} \right)$$

so $\Phi e^A_{(\alpha,\beta)} \Phi^{-1} = e^{A'}_{(\alpha,\beta)}$ with $A' = \left(\begin{smallmatrix} c-a & d-b \\ -a & -b \end{smallmatrix}\right) = M_{\sigma_1} A$. In the same way we have that for $\Phi = (\mathrm{Id}, \sigma_2)$, conjugation yields

$$\Phi e^A_{(\alpha,\beta)} \Phi^{-1} \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \left(\begin{bmatrix} \xi^{\alpha} x_1 \\ \xi^{\beta} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{c\alpha+d\beta} x_2 \\ \xi^{a\alpha+b\beta} y_2 \\ z_2 \end{bmatrix} \right).$$

Thus $\Phi e^A_{(\alpha,\beta)} \Phi^{-1} = e^{A'}_{(\alpha,\beta)}$ again, where $A' = \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right) = M_{\sigma_2} A$ in this case.

Hence conjugation by elements of the form (Id, τ_2) acts by left multiplication of the corresponding matrix on A.

Now taking $\Phi = (\sigma_1, \mathrm{Id})$ and conjugating an element of G_A we have

$$\Phi e^A_{(\alpha,\beta)} \Phi^{-1} \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \left(\begin{bmatrix} \xi^{\beta-\alpha} x_1 \\ \xi^{-\alpha} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{a\alpha+b\beta} x_2 \\ \xi^{c\alpha+d\beta} y_2 \\ z_2 \end{bmatrix} \right).$$

Thus $\Phi e^A_{(\alpha,\beta)} \Phi^{-1} = e^{A'}_{(\beta-\alpha,-\alpha)}$ where $A' = \left(\begin{smallmatrix} b & -a-b \\ d & -c-d \end{smallmatrix}\right) = A M_{\sigma_1}^{-1}$.

Analogously one gets that for $\Phi = (\sigma_2, \mathrm{Id})$

$$\Phi e^A_{(\alpha,\beta)} \Phi^{-1} \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \left(\begin{bmatrix} \xi^{\beta} x_1 \\ \xi^{\alpha} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{a\alpha+b\beta} x_2 \\ \xi^{c\alpha+d\beta} y_2 \\ z_2 \end{bmatrix} \right),$$

and therefore $\Phi e^A_{(\alpha,\beta)} \Phi^{-1} = e^{A'}_{(\beta,\alpha)}$ where $A' = \left(\begin{smallmatrix} b & a \\ d & c \end{smallmatrix}\right) = A M_{\sigma_2}^{-1}$.

Finally, for the automorphism $\Phi = J$ we have

$$\Phi e^A_{(\alpha,\beta)} \Phi^{-1} \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \left(\begin{bmatrix} \xi^{a\alpha+b\beta} x_1 \\ \xi^{c\alpha+d\beta} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{\alpha} x_2 \\ \xi^{\beta} y_2 \\ z_2 \end{bmatrix} \right),$$

which means that $\Phi e^A_{(\alpha,\beta)} \Phi^{-1} = e^{A^{-1}}_{(a\alpha+b\beta,c\alpha+d\beta)}$.

COROLLARY 2. *Two Beauville surfaces S^n_A and S^n_B are isomorphic if and only if $B = M_{\tau_2} A^{\pm 1} M_{\tau_1}$ for some $(\tau_1, \tau_2) \in \mathscr{S}_3 \times \mathscr{S}_3$.*

As noted above, $|\mathfrak{T}_5| = 24$. One can explicitly write down the 24 matrices and check by hand that these matrices lie in a single orbit under the action of $\mathrm{Aut}(F_n \times F_n)$. In this way one obtains the following result:

COROLLARY 3. *Up to isomorphism there is only one Beauville surface with group* Z_5^2*, namely* S_A^5 *where*

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

This result will also follow from the general formula for the number of isomorphism classes of Beauville surfaces with abelian group given in Theorem 2.

REMARK 2. In [1] Bauer and Catanese state that there are two (non isomorphic) Beauville structures on the product $F_5 \times F_5$ originally considered by Beauville. It seems that this discrepancy is due to the fact that they regard two Beauville surfaces as equivalent if there is a factor-preserving isomorphism between them, whereas here we also consider factor-interchanging isomorphisms such as J.

In the next section we will discuss the number of isomorphism classes of Beauville surfaces S_A^n for each n.

3. Isomorphism classes of Beauville surfaces with abelian group

We are interested in finding the number of isomorphism classes of Beauville surfaces with group Z_n^2 for a given n. By Corollary 2 this is equivalent to counting the number of orbits of the group $W = (\mathcal{S}_3 \times \mathcal{S}_3) \rtimes \langle J \rangle = \mathcal{S}_3 \wr \mathcal{S}_2$ on the set $\mathfrak{F}_n$, so by the Cauchy-Frobenius (or Burnside) Lemma we deduce that the number of isomorphism classes of Beauville surfaces is

$$\frac{1}{|W|} \sum_i |\mathcal{C}_i| \, |\operatorname{Fix}(x_i)|,$$

where i indexes the conjugacy classes $\mathcal{C}_i$ of W and x_i is any element of $\mathcal{C}_i$. The conjugacy classes of W are shown in Table 1, where σ_2 and σ_3 are elements of order 2 and 3 in S_3.

First we need to know the cardinality of the set $\mathfrak{F}_n$.

LEMMA 3. *Let n be a natural number. Then*

$$|\mathfrak{F}_n| = n^4 \prod_{p|n} \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)\left(1 - \frac{3}{p}\right)\left(1 - \frac{4}{p}\right)$$

where p ranges over the distinct primes dividing n.

TABLE 1. Conjugacy classes of W.

Conjugacy class	Representative	Order	Number of elements
1	(Id, Id)	1	1
2	(Id, σ_2)	2	6
3	(σ_2, σ_2)	2	9
4	(Id, σ_3)	3	4
5	(σ_3, σ_3)	3	4
6	(σ_2, σ_3)	6	12
7	$(\mathrm{Id}, \mathrm{Id}) \cdot J$	2	6
8	$(\mathrm{Id}, \sigma_2) \cdot J$	4	18
9	$(\sigma_2, \sigma_3\sigma_2) \cdot J$	6	12

PROOF. We will first prove the formula for $n = p$ prime. We want to count the number of matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying condition (1).

Based on the correspondence between matrices $A \in \mathfrak{F}_n$ and pairs of triples $((1,0), (0,1), (-1,-1))$, $((a,c), (b,d), (-a-b, -c-d))$, we deduce that counting matrices in $\mathfrak{F}_n$ is equivalent to counting second triples.

Now, any triple in G projects to an ordered triple of points in the projective line $P = \mathsf{P}^1(\mathsf{Z}_p)$ formed by the 1-dimensional subspaces of the vector space G, so Beauville structures in G correspond to disjoint pairs of triples in P. Conversely, any triple in P is induced by $p-1$ triples in G, all scalar multiples of each other. The standard triple in G induces the triple $(0, \infty, 1) \in P$. Having chosen the standard triple as the first triple in G, there are $|P| - 3 = p - 2$ points of P remaining, so there are $(p-2)(p-3)(p-4)$ choices for the second triple in P, corresponding to $(p-1)(p-2)(p-3)(p-4)$ triples in G.

The case $n = p^e$ follows from the following fact. Having fixed the standard triple as the first one, any second triple given by $a, b, c \in \mathsf{Z}^2_{p^e}$ satisfying (i)–(iii) will give by reduction modulo p a triple $(a \bmod p, b \bmod p, c \bmod p)$. Conversely, for each triple (a, b, c) with $a, b, c \in \mathsf{Z}^2_p$, each of the p^{4e-4} choices for $\mathrm{v} = (h_1, h_2, j_1, j_2)$ with $h_1, h_2, j_1, j_2 = 0, \dots, p^{e-1} - 1$ gives a triple $(a_\mathrm{v}, b_\mathrm{v}, c_\mathrm{v})$ where

$$a_\mathrm{v} = a + (h_1 p, h_2 p)$$
$$c_\mathrm{v} = c + (j_1 p, j_2 p)$$
$$b_\mathrm{v} = -a_\mathrm{v} - c_\mathrm{v}$$

such that $(a_\mathrm{v} \bmod p, b_\mathrm{v} \bmod p, c_\mathrm{v} \bmod p) = (a, b, c)$.

Finally, we can extend the formula to all n by a straightforward application of the Chinese Remainder Theorem.

We note that our formula for $|\mathfrak{F}_n|$ is equivalent to that given by Garion and Penegini in [12, Corollary 3.24] for the function denoted there by N_n.

Lemma 3 immediately gives us information about the asymptotic behaviour of $|\mathfrak{F}_n|$ for large n. If $n = p^e$ for some prime $p \geq 5$ then

$$\frac{|\mathfrak{F}_n|}{n^4} = \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)\left(1 - \frac{3}{p}\right)\left(1 - \frac{4}{p}\right) \geqslant \frac{4!}{5^4},$$

and $|\mathfrak{F}_n|/n^4 \to 1$ as $p \to \infty$. However, if n is divisible by the first k primes $p_i \geq 5$ then since

$$\lim_{k\to\infty} \prod_i \left(1 - \frac{1}{p_i}\right) = 0$$

(see Exercise 9.3 of [15]) we have $|\mathfrak{F}_n|/n^4 \to 0$ as $k \to \infty$.

Since the isomorphism classes of Beauville surfaces with Beauville group Z_n^2 correspond to the orbits of W on $\mathfrak{F}_n$, and $|W| = 72$, there are at least $|\mathfrak{F}_n|/72$ such classes. We will now apply the Cauchy-Frobenius Lemma to find the exact number of isomorphism classes. For this, we need to calculate how many matrices in $\mathfrak{F}_n$ are fixed by each element of the group.

First of all we note that an element of W fixes a matrix in $\mathfrak{F}_n$ if and only if it fixes its components modulo the prime powers $p_i^{e_i}$ in the factorisation of n, so we can again restrict our attention to prime powers.

It is obvious that $x_1 = (\mathrm{Id}, \mathrm{Id})$ fixes every matrix in $\mathfrak{F}_n$, so the number of its fixed points is

$$\Theta_1(n) := |\operatorname{Fix}(x_1)| = |\mathfrak{F}_n|.$$

On the other hand it is easy to see that the conditions on a matrix $A \in \mathrm{GL}_2(\mathsf{Z}_n)$ to be fixed by an element of the conjugacy classes $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$, $\mathscr{C}_6$ or $\mathscr{C}_8$ are incompatible with those defining $\mathfrak{F}_n$.

The action of $x_5 = (\sigma_3, \sigma_3) \in \mathscr{C}_5$ sends a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $(\sigma_3, \sigma_3)A = \left(\begin{smallmatrix} d-b & (a+b)-(c+d) \\ -b & a+b \end{smallmatrix}\right)$. In the case of $n = p^e$ a prime power, the conditions $c = -b$ and $d = a + b$ leave $p^{2e}(1 - 1/p)(1 - 2/p)$ possibilities for the matrix $A = \left(\begin{smallmatrix} a & b \\ -b & a+b \end{smallmatrix}\right)$, from which we have to remove those with a non-unit determinant. If $p \equiv -1 \bmod 3$ then the equation $a^2 + ab + b^2 \equiv 0 \bmod p$ has no solutions, since it is equivalent to $\lambda^2 + \lambda + 1 \equiv 0$ and F_p contains no non-trivial third root of unity. Otherwise, for a fixed a there are two non-valid choices for b. Therefore

$$\Theta_2(p^e) := |\operatorname{Fix}(x_5)| = \begin{cases} p^{2e}\left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right), & \text{if } p \equiv -1 \bmod 3; \\ p^{2e}\left(1 - \frac{1}{p}\right)\left(1 - \frac{4}{p}\right), & \text{if } p \equiv 1 \bmod 3. \end{cases}$$

For $x_7 = (\mathrm{Id}, \mathrm{Id}) \cdot J \in \mathscr{C}_7$, the action on A yields $(\mathrm{Id}, \mathrm{Id})J(A) = A^{-1}$. The equality $A = A^{-1}$ in $\mathfrak{F}_{p^e}$ implies that $a = -d$ and $b(\det(A) + 1) = 0$ and hence $\det(A) = -1$. The general form for a matrix fixed by this element is therefore $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, with $\det(A) = -a^2 - bc = -1$. There are $p^e(1-3/p)$ possibilities for $a \not\equiv 0, \pm 1 \bmod p$, and once a is chosen there are $p^e(1-5/p)$ choices left for $b \not\equiv 0, -a, \frac{1-a^2}{a}, 1-a, -1-a \bmod p$. The final formula gives

$$\Theta_3(p^e) := |\mathrm{Fix}(x_7)| = p^{2e}(1-3/p)(1-5/p)$$

Finally, the element $x_9 = (\sigma_2, \sigma_3\sigma_2) \cdot J \in \mathscr{C}_9$ acts by sending the matrix A to $(\sigma_2, \sigma_3\sigma_2)J(A) = \left(\begin{smallmatrix} -c & c-a \\ d & b-d \end{smallmatrix}\right)/\det(A)$. The conditions imply that $\det(A)$ is different from -1 and $\det(A) = \det(A)^3$, hence $\det(A) = 1$. Writing down the rest of the conditions one gets $a = -c = -d$ and $b = -2a$, so we are looking for elements $a \in \mathsf{Z}_{p^e}$ such that $-3a^2 \equiv 1 \bmod p^e$. Now -3 is a square in Z_{p^e} if and only if it is a quadratic residue modulo p (see e.g. Thm. 7.14 in [15]), and this is the case only when $p \equiv 1 \bmod 3$. To see this note that if $p \equiv \varepsilon \bmod 3$, where $\varepsilon = \pm 1$, by the Law of Quadratic Reciprocity

$$\left(\frac{3}{p}\right) = \begin{cases} \varepsilon, & \text{if } p \equiv 1 \bmod 4; \\ -\varepsilon, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Therefore $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = \varepsilon$. As a consequence

$$\Theta_4(p^e) := |\mathrm{Fix}(x_9)| = \begin{cases} 0, & \text{if } p \equiv -1 \bmod 3; \\ 2, & \text{if } p \equiv 1 \bmod 3. \end{cases}$$

We have thus proved the following:

THEOREM 2. *Let $n = p_1^{e_1} \cdot \ldots \cdot p_s^{e_s}$ be a natural number coprime to 6, where $p_1, \ldots, p_k$ are distinct primes. Then the number of isomorphism classes of Beauville surfaces with Beauville group Z_n^2 is*

$$\Theta(n) = \frac{1}{72}\left(\Theta_1(n) + 4\prod_{i=1}^{s}\Theta_2(p_i^{e_i}) + 6\prod_{i=1}^{s}\Theta_3(p_i^{e_i}) + 12\prod_{i=1}^{s}\Theta_4(p_i^{e_i})\right),$$

where the functions Θ_i are defined as above.

The formulae given above for $\Theta_r(p^e)$ for $r = 2, 3$ and 4 show that the sum on the right-hand side of this equation is dominated by the first term, so that

$$\Theta(n) \sim \frac{1}{72}\Theta_1(n) = \frac{1}{72}|\mathfrak{F}_n| = \frac{n^4}{72}\prod_{p|n}\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\left(1-\frac{3}{p}\right)\left(1-\frac{4}{p}\right)$$

as $n \to \infty$. In particular, we have the following special case:

COROLLARY 4. *For each prime $p \geq 5$ the number $\Theta(p^e)$ of isomorphism classes of Beauville surfaces with Beauville group $\mathsf{Z}_{p^e}^2$ is given by*

$$\frac{1}{72}\big(p^{4e} - 10p^{4e-1} + 35p^{4e-2} - 50p^{4e-3} + 24p^{4e-4} + 10p^{2e} - 60p^{2e-1} + 98p^{2e-2}\big)$$

if $p \equiv -1 \bmod 3$, *and by*

$$\frac{1}{72}\big(p^{4e} - 10p^{4e-1} + 35p^{4e-2} - 50p^{4e-3} + 24p^{4e-4} + 10p^{2e} - 68p^{2e-1} + 106p^{2e-2} + 24\big)$$

if $p \equiv 1 \bmod 3$.

When $e = 1$ this specialises to

$$\Theta(p) = \begin{cases} \dfrac{1}{72}(p^4 - 10p^3 + 45p^2 - 110p + 122), & \text{if } p \equiv -1 \bmod 3, \\ \dfrac{1}{72}(p^4 - 10p^3 + 45p^2 - 118p + 154), & \text{if } p \equiv 1 \bmod 3. \end{cases}$$

REMARK 3. In [2], Bauer, Catanese and Grunewald considered the asymptotic behaviour of the number of Beauville surfaces with Beauville group Z_n^2, where n is coprime to 6. Later, Garion and Penegini [12] considered a wide range of related counting problems; in particular, they obtained bounds similar to those in [2] for Beauville groups $G = \mathsf{Z}_n^2$, specifically that $\Theta(n)$ lies between $|\mathfrak{F}_n|/72$ and $|\mathfrak{F}_n|/6$ (Corollary 3.24). Their results are consistent with ours.

4. The automorphism group of S_A^n

The calculations in the previous section provide some insight into the automorphism group of a Beauville surface with an abelian Beauville group.

The fact that the automorphisms of S_A^n lift to automorphisms of $F_n \times F_n$ shows that all automorphisms of S_A^n are induced by elements of $N(G_A)$, the normaliser of G_A in $\mathrm{Aut}(F_n \times F_n)$ and, in fact, that $\mathrm{Aut}(S_A^n) \cong N(G_A)/G_A$.

Since clearly

$$G_A \leq \mathsf{Z}_n^2 \times \mathsf{Z}_n^2 \leq N(G_A)$$

one sees immediately that each surface S_A^n admits the group $(\mathsf{Z}_n^2 \times \mathsf{Z}_n^2)/G_A$ as a group of automorphisms. Moreover, one can identify the Beauville group Z_n^2 with $(\mathsf{Z}_n^2 \times \mathsf{Z}_n^2)/G_A$ by sending a pair (α, β) to $((0, 0), (\alpha, \beta)) \bmod G_A$, which acts as an automorphism of S_A^n by multiplying the coordinates of the second factor by the corresponding roots of unity. Actually this simply reflects the known fact (see [14]) that a Beauville surface with group G always has the centre $Z(G)$ of G as a group of automorphisms.

This isomorphism $\mathsf{Z}_n^2 \cong (\mathsf{Z}_n^2 \times \mathsf{Z}_n^2)/G_A$ allows a further identification

$$\mathrm{Aut}(S_A^n)/\mathsf{Z}_n^2 = N(G_A)/(\mathsf{Z}_n^2 \times \mathsf{Z}_n^2),$$

where the group on the right is simply the stabiliser in $\mathrm{Aut}(F_n \times F_n)/(\mathsf{Z}_n^2 \times \mathsf{Z}_n^2) = W$ of the matrix A.

In fact we have the following:

PROPOSITION 1. *Let H be the subgroup of W identified as above with the quotient* $\mathrm{Aut}(S_A^n)/\mathsf{Z}_n^2$. *Then* $H \cong \{\mathrm{Id}\}$, Z_2, Z_3, Z_6 *or* $\mathcal{S}_3$.

PROOF. By the comment above, an element of W will induce an automorphism of S_A^n, and therefore belong to H, if and only if it fixes the matrix A. By the proof of Theorem 2, the only elements with fixed points in $\mathfrak{F}_n$ are those in the conjugacy classes $\mathscr{C}_1$, $\mathscr{C}_5$, $\mathscr{C}_7$ and $\mathscr{C}_9$, so $H \subseteq \mathscr{C}_1 \cup \mathscr{C}_5 \cup \mathscr{C}_7 \cup \mathscr{C}_9$.

Let $H^0 = H \cap (\mathcal{S}_3 \times \mathcal{S}_3)$, so that $|H : H^0| \leq 2$ and $H^0 \subseteq \mathscr{C}_1 \cup \mathscr{C}_5$. If $\gamma_1, \gamma_2 \in \mathscr{C}_5$ and $\gamma_2 \neq \gamma_1^{\pm 1}$ then $\gamma_1\gamma_2 \in \mathscr{C}_4$, so $\gamma_1\gamma_2 \notin H$; it follows that $|H^0| = 1$ or 3, and hence $|H| = 1, 2, 3$ or 6. The only groups of these orders are those listed in the Proposition.

Moreover, by the discussion prior to Theorem 2 and the formula for $\Theta_4(p^e)$, if n is divisible by some prime $p \equiv -1 \bmod 3$ then no element of order 6 in W fixes points of $\mathfrak{F}_n$. Similarly, the formula for $\Theta_3(p^e)$, together with the fact that no element of $\mathscr{C}_2 \cup \mathscr{C}_3$ has fixed points, ensures that no element of order 2 can belong to H if 5 divides n. We therefore have the following:

COROLLARY 5. *Let S_A^n be a Beauville surface. If a prime $p \equiv -1 \bmod 3$ divides n, then* $H \cong \{\mathrm{Id}\}$, Z_2, Z_3 *or* $\mathcal{S}_3$. *If 5 divides n then* $H \cong \{\mathrm{Id}\}$ *or* Z_3.

For instance, in Beauville's original example [5], $\mathrm{Aut}(S_A^5)$ is a semidirect product of Z_5^2 by $H \cong \mathsf{Z}_3$.

This study of automorphism groups has been extended to more general Beauville surfaces in [14].

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