MATH. SCAND. 114 (2014), 205–215

CONCRETE REALIZATIONS OF QUOTIENTS OF OPERATOR SPACES

MARC A. RIEFFEL*

Abstract

Let $\mathscr{B}$ be a unital C*-subalgebra of a unital C*-algebra $\mathscr{A}$, so that $\mathscr{A}/\mathscr{B}$ is an abstract operator space. We show how to realize $\mathscr{A}/\mathscr{B}$ as a concrete operator space by means of a completely contractive map from $\mathscr{A}$ into the algebra of operators on a Hilbert space, of the form $A \mapsto [Z, A]$ where Z is a Hermitian unitary operator. We do not use Ruan's theorem concerning concrete realization of abstract operator spaces. Along the way we obtain corresponding results for abstract operator spaces of the form $\mathscr{A}/\mathscr{V}$ where $\mathscr{V}$ is a closed subspace of $\mathscr{A}$, and then for the more special cases in which $\mathscr{V}$ is a $*$-subspace or an operator system.

Introduction

In a recent paper [11] I showed that if $\mathscr{B}$ is a unital C*-subalgebra of a unital C*-algebra $\mathscr{A}$ and if L is the quotient norm on $\mathscr{A}/\mathscr{B}$ pulled back to $\mathscr{A}$, that is,

$$L(A) = \inf\{\|A - B\| \ : B \in \mathscr{B}\}$$

for $A \in \mathscr{A}$, then there is a unital $*$-representation $(\mathscr{H}, \pi)$ of $\mathscr{A}$ and a Hermitian unitary operator U on $\mathscr{H}$ such that

$$L(A) = \|[U, \pi(A)]\|$$

for all $A \in \mathscr{A}$. The consequence of this that most interested me is that it follows that L satisfies the Leibniz inequality

$$L(AC) \le L(A)\|C\| + \|A\|L(C)$$

for all $A, C \in \mathscr{A}$. But another interesting consequence is that the map $A \mapsto [U, \pi(A)]$ gives an isometry of $\mathscr{A}/\mathscr{B}$ into $\mathscr{L}(\mathscr{H})$. Now $\mathscr{A}/\mathscr{B}$ is actually an operator space, in the sense of having a compatible family of norms on all the matrix spaces over it (reviewed below), and this suggests that one should seek a natural construction of a "complete isometry" from $\mathscr{A}/\mathscr{B}$ into the algebra of

* The research reported here was supported in part by National Science Foundation grant DMS-0753228.

Received 24 January 2012.

operators on some Hilbert space (i.e. one respecting the norms on all the matrix spaces). The main purpose of this article is to provide such a construction. In fact, we show that there exists a complete isometry that is again of the form $A \mapsto [U, \pi(A)]$. As a consequence we obtain a "matrix Leibniz seminorm" on $\mathscr{A}$ by taking the norms of the commutators.

Now matrix Leibniz seminorms played a crucial role in my earlier paper [8] relating vector bundles and Gromov-Hausdorff distance, and one of my projects is to generalize the main results of that paper to the setting of non-commutative C*-algebras, so that they can be applied, for example, to the setting of quantizations of coadjoint orbits that I studied in [7], [9], [10]. (Matrix seminorms have already been defined and discussed in this context in [12], [13], [14], but the Leibniz property was not used there.) Actually, for infinite-dimensional C*-algebras, the C*-metrics as defined in [9] are discontinuous and only densely defined. But they are required to be lower semi-continuous with respect to the C*-norm, and in all of the examples that I know of one proves that they are lower semi-continuous by showing that they are the supremum of an infinite family of continuous Leibniz seminorms. This provides ample reason for studying continuous Leibniz seminorms. Thus the results of the present paper provide some interesting information about matrix Leibniz seminorms, and so provide a small step forward in my project.

We will actually develop some of our results in a more general context, namely that in which $\mathscr{V}$ is a closed subspace of a unital C*-algebra $\mathscr{A}$, so that $\mathscr{A}/\mathscr{V}$ is an abstract operator space. We show that in this case there exists a unital $*$-representation $(\mathscr{H}, \pi)$ of $\mathscr{A}$, and projections P and Q on $\mathscr{H}$, such that the linear mapping Ψ from $\mathscr{A}$ to $\mathscr{L}(\mathscr{H})$ defined by

$$\Psi(A) = Q\pi(A)P$$

gives a complete isometry from $\mathscr{A}/\mathscr{V}$ into $\mathscr{L}(\mathscr{H})$. To show this we do not need to use Ruan's construction [2] of complete isometries from abstract operator spaces into operator algebras (essentially because C*-algebras can be considered to be concrete operator spaces, by the Gelfand-Naimark theorem). In fact, our results immediately apply to the situation of a concrete operator space $\mathscr{W}$ and a closed subspace $\mathscr{V}$ of $\mathscr{W}$, so that $\mathscr{W}/\mathscr{V}$ is an abstract operator space, just by considering the unital C*-algebra $\mathscr{A}$ generated by the concrete operator space $\mathscr{W}$. All of this is discussed in Section 1.

Now a C*-subalgebra is in particular a $*$-subspace. For this reason we discuss in Section 2 the situation in which $\mathscr{V}$ is a $*$-subspace of a unital C*-algebra $\mathscr{A}$, so that $\mathscr{A}/\mathscr{V}$ is an abstract operator $*$-space. We show that in this case there exist a unital $*$-representation $(\mathscr{H}, \pi)$ of $\mathscr{A}$, a projection P on $\mathscr{H}$, and a Hermitian unitary operator U on $\mathscr{H}$ that commutes with the representation π, such that when we define the linear $*$-map Ψ from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$ by

$$\Psi(A) = PU\pi(A)P$$

then Ψ gives a completely isometric $*$-map from $\mathscr{A}/\mathscr{V}$ onto a $*$-subspace of $\mathscr{L}(\mathscr{H})$. In Section 3 we then briefly consider the case in which $\mathscr{V}$ is an operator system, that is, $\mathscr{V}$ is a $*$-subspace of $\mathscr{A}$ that contains the identity element of $\mathscr{A}$.

Finally, in Section 4 we discuss the situation in which $\mathscr{V}$ is a unital C*-subalgebra, $\mathscr{B}$, as described above.

1. Quotients of operator spaces

We begin by reviewing here various facts about operator spaces that we need. Let $\mathscr{V}$ be a vector space. For each natural number n we let $M_n(\mathscr{V})$ denote the vector space of $n \times n$ matrices with entries in $\mathscr{V}$. Let $\mathscr{A}$ be a C*-algebra. Then $M_n(\mathscr{A})$ is a $*$-algebra in the evident way, and it has a unique C*-algebra norm. We always view $M_n(\mathscr{A})$ as equipped with this norm. If $\mathscr{V}$ is a subspace of $\mathscr{A}$, then for each natural number n we equip $M_n(\mathscr{V})$ with the restriction to $M_n(\mathscr{V})$ of the norm on $M_n(\mathscr{A})$. The resulting family of norms on all these matrix spaces is called a matrix norm, and when $\mathscr{V}$ is equipped with this family of norms it is called a "concrete operator space". Ruan [2] found axioms that characterize such families of norms. A family of norms that satisfy Ruan's axioms is called an "operator-space matrix norm". A vector space equipped with an operator-space matrix norm (but that is not assumed to be a subspace of a C*-algebra) is called an "abstract operator space". We will not need to use Ruan's axioms, because all of the vector spaces that we consider will either be assumed to be subspaces of C*-algebras, or will eventually be proved to be (at least isomorphic to) such.

If $\mathscr{V}$ and $\mathscr{W}$ are vector spaces and if ϕ is a linear map from $\mathscr{V}$ into $\mathscr{W}$, then by entry-wise application ϕ determines a linear map, ϕ_n, from $M_n(\mathscr{V})$ to $M_n(\mathscr{W})$ for each n. If $\mathscr{V}$ and $\mathscr{W}$ are each equipped with matrix norms, then ϕ is said to be "completely contractive" if the norm of each ϕ_n is no greater than 1, and ϕ is said to be a "complete isometry" if each ϕ_n is an isometry.

If $\mathscr{V}$ is a closed subspace of an operator space $\mathscr{W}$, so that $M_n(\mathscr{V})$ is a subspace of $M_n(\mathscr{W})$ for each n, then for each n we can equip $M_n(\mathscr{W})/M_n(\mathscr{V})$ with the corresponding quotient norm, thus obtaining a "quotient matrix norm" on $\mathscr{W}/\mathscr{V}$. Important perspective for us is given by the fact that $\mathscr{W}/\mathscr{V}$ equipped with this quotient matrix norm is an abstract operator space [2]. But again, in the end we will not actually have used this fact, though we will use this terminology, as we do already in the next proposition.

The main technical step for all of the results of this paper is given by the following proposition, which is closely related to the GNS construction. Here we denote the Banach-space dual of a Banach space X by X'.

PROPOSITION 1.1. *Let $\mathscr{A}$ be a unital C*-algebra, let $\mathscr{V}$ be a closed subspace of $\mathscr{A}$, and equip $\mathscr{A}/\mathscr{V}$ with the corresponding quotient matrix norm (so that $\mathscr{A}/\mathscr{V}$ is an abstract operator space). For a given natural number n let there be given $\psi \in (M_n(\mathscr{A}))'$ with $\psi(M_n(\mathscr{V})) = 0$ and $\|\psi\| = 1$. Then there exist a unital $*$-representation, $(\mathscr{H}, \pi)$, of $\mathscr{A}$, and two projections, P and Q, in $\mathscr{L}(\mathscr{H})$, each of rank no greater than n, such that when we define the completely contractive map $\Psi : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ by*

$$\Psi(A) = Q\pi(A)P$$

for $A \in \mathscr{A}$, then $\Psi(\mathscr{V}) = 0$ and there exist two unit vectors, ξ and η, in $\mathscr{H}^{\oplus n}$ such that

$$\psi(C) = \langle \Psi_n(C)\xi, \eta\rangle \qquad \textit{for all} \quad C \in M_n(\mathscr{A}).$$

PROOF. It is well-known that if $\mathscr{B}$ is a unital C*-algebra and if $\theta \in \mathscr{B}'$ with $\|\theta\| = 1$, then there exist a unital $*$-representation, $(\rho, \mathscr{K})$, of $\mathscr{B}$ and unit vectors ξ^0 and η^0 in $\mathscr{K}$, such that $\theta(B) = \langle \rho(B)\xi^0, \eta^0\rangle$ for all $B \in \mathscr{B}$. See Lemma 3.3 of [11] for a proof of this fact whose main tool is just the Jordan decomposition of a Hermitian linear functional into the difference of two positive linear functionals. Accordingly, we can choose a unital $*$-representation $(\mathscr{K}, \rho)$ of $M_n(\mathscr{A})$, and unit vectors ξ^0 and η^0 in $\mathscr{K}$, such that

$$\psi(C) = \langle \rho(C)\xi^0, \eta^0\rangle \qquad \text{for all} \quad C \in M_n(\mathscr{A}).$$

Let $\{E_{jk}\}$ be the standard matrix-units for $M_n \subseteq M_n(\mathscr{A})$. Then $\rho(E_{11})$ is a projection in $\mathscr{L}(\mathscr{K})$. Set $\mathscr{H} = \rho(E_{11})\mathscr{K}$. Define a unital $*$-representation , π, of $\mathscr{A}$ on $\mathscr{H}$ by $\pi(A) = \rho(A \otimes E_{11})$ where here we view $M_n(\mathscr{A})$ as $A \otimes M_n$. Then it is well-known and easily checked that $(\mathscr{K}, \rho)$ is unitarily equivalent to $(\mathscr{H}^{\oplus n}, \pi_n)$, where by π_n we mean the representation of $M_n(\mathscr{A})$ on $\mathscr{H}^{\oplus n}$ defined by the matrices

$$\pi_n(C) = \{\pi(C_{jk})\}$$

for $C \in M_n(\mathscr{A})$ and $C = \{C_{jk}\}$ with $C_{jk} \in \mathscr{A}$, and where the matrix $\{\pi(C_{jk})\}$ acts on $\mathscr{H}^{\oplus n}$ in the evident way. (This is, for example, essentially proposition 5ii of chapter I of [1].) In particular, there will be unit vectors ξ and η in $\mathscr{H}^{\oplus n}$ such that

$$\psi(C) = \langle \pi_n(C)\xi, \eta\rangle \qquad \text{for all} \quad C \in M_n(\mathscr{A}).$$

Let $\xi = \{\xi_k\}$ and $\eta = \{\eta_j\}$ for $\xi_k, \eta_j \in \mathscr{H}$. (Note that the ξ_k's are generally not orthogonal, and some may be 0, and similarly for the η_j's.) Then for $C = \{C_{jk}\}$ as above, we have

$$\psi(C) = \langle \pi_n(C)\xi, \eta\rangle = \Sigma_{jk}\langle \pi(C_{jk})\xi_k, \eta_j\rangle.$$

Let $D \in \mathscr{V}$, and for fixed p and q with $1 \le p, q \le n$ let $C = D \otimes E_{pq}$, so that $C \in M_n(\mathscr{V})$. Then by assumption on ψ

$$0 = \psi(C) = \langle \pi(D)\xi_q, \eta_p\rangle.$$

Thus for all p and q we have

$$\langle \pi(\mathscr{V})\xi_q, \eta_p\rangle = 0.$$

Let P and Q be the projections onto, respectively, the linear spans of $\{\xi_k\}$ and $\{\eta_j\}$. Thus P and Q are projections on $\mathscr{H}$ of rank at most n. Furthermore, the fact that $\langle \pi(D)\xi_q, \eta_p\rangle = 0$ for all $D \in \mathscr{V}$ and all p and q implies that

$$Q\pi(D)P = 0 \qquad \text{for all} \quad D \in \mathscr{V}.$$

Define the linear mapping Ψ from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$ by

$$\Psi(A) = Q\pi(A)P$$

for all $A \in \mathscr{A}$. Then it is standard and easily checked that Ψ is completely contractive. Of course, $\Psi(\mathscr{V}) = 0$. Furthermore, if we let Ψ_n be the corresponding mapping from $M_n(A)$ into $M_n(\mathscr{L}(\mathscr{H}))$, and if we let P_n and Q_n denote the diagonal $n \times n$ matrices with P, respectively Q, in each diagonal entry, then

$$\Psi_n(C) = Q_n\pi_n(C)P_n$$

for all $C \in M_n(\mathscr{A})$, where π_n is as defined earlier in this proof. Note that $P_n\xi = \xi$ and $Q_n\eta = \eta$. Then as above

$$\psi(C) = \langle \pi_n(C)\xi, \eta\rangle = \langle Q_n\pi_n(C)P_n\xi, \eta\rangle = \langle \Psi_n(C)\xi, \eta\rangle$$

for all $C \in M_n(\mathscr{A})$, as desired.

We remark that if for each non-zero ξ_k we let P_k be the rank-one projection with ξ_k in its range, and if we define Q_j similarly for η_j, then the above proposition can be reformulated in terms of the complete contractions $\Phi_{jk}(A) = Q_j\pi(A)P_k$. But this reformulation seems to be a bit more complicated.

For each natural number n let $(M_n(\mathscr{V}))^\perp$ denote the linear subspace of $(M_n(\mathscr{A}))'$ consisting of the linear functionals that take value 0 on $M_n(\mathscr{V})$. By the Hahn-Banach theorem, for each $C \in M_n(\mathscr{A})$ there is a $\psi \in (M_n(\mathscr{V}))^\perp$ such that $\|\psi\| = 1$ and $\psi(C) = \|C\|_{\mathscr{A}/\mathscr{V}}$, where $\|\cdot\|_{\mathscr{A}/\mathscr{V}}$ denotes the quotient norm on $M_n(\mathscr{A})/M_n(\mathscr{V})$ pulled back to $M_n(\mathscr{A})$. Thus we can choose, in many ways, a subset, $\mathscr{S}_n^{\mathscr{V}}$, of elements of $(M_n(\mathscr{V})^\perp$ of norm 1 such that for every $C \in M_n(\mathscr{A})$ we have

$$\|C\|_{\mathscr{A}/\mathscr{V}} = \sup\{|\psi(C)| : \psi \in \mathscr{S}_n^{\mathscr{V}}\}.$$

For example, $\mathscr{S}_n^{\mathscr{V}}$ could consist of all elements ψ of $(M_n(\mathscr{V}))^\perp$ of norm 1, or of a norm-dense subset thereof, or of the set of extreme points of the unit ball of $(M_n(\mathscr{V}))^\perp$. For each such ψ we obtain from the above proposition a representation $(\mathscr{H}^\psi, \pi^\psi)$ and projections P^ψ and Q^ψ on $\mathscr{H}^\psi$, and the corresponding completely contractive mapping Ψ^ψ from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H}^\psi)$ defined by

$$\Psi^\psi(A) = Q^\psi \pi^\psi(A) P^\psi.$$

Let $\mathscr{H}^{\mathscr{V},n} = \bigoplus\{\mathscr{H}^\psi : \psi \in \mathscr{S}_n^{\mathscr{V}}\}$, the Hilbert space direct sum, and let $\pi^{\mathscr{V},n} = \bigoplus\{\pi^\psi : \psi \in \mathscr{S}_n^{\mathscr{V}}\}$ be the corresponding representation of $\mathscr{A}$ on $\mathscr{H}^{\mathscr{V},n}$. Let $P^{\mathscr{V},n} = \bigoplus\{P^\psi : \psi \in \mathscr{S}_n^{\mathscr{V}}\}$, and define $Q^{\mathscr{V},n}$ similarly. Then define $\Psi^{\mathscr{V},n}$ by

$$\Psi^{\mathscr{V},n}(A) = Q^{\mathscr{V},n}\pi^{\mathscr{V},n}(A)P^{\mathscr{V},n}$$

for all $A \in \mathscr{A}$. Then from the requirements on $\mathscr{S}_n^{\mathscr{V}}$ it is clear that for every $C \in M_n(\mathscr{A})$ we have

$$\|C\|_{\mathscr{A}/\mathscr{V}} = \|\Psi_n^{\mathscr{V},n}(C)\|.$$

Now let $\mathscr{H}^{\mathscr{V}} = \bigoplus\{\mathscr{H}^{\mathscr{V},n} : n \in \mathsf{N}\}$, let $\pi^{\mathscr{V}} = \bigoplus\{\pi^{\mathscr{V},n} : n \in \mathsf{N}\}$, and define projections $P^{\mathscr{V}}$ and $Q^{\mathscr{V}}$ on $\mathscr{H}^{\mathscr{V}}$ similarly. Then from the above considerations we see that we obtain:

THEOREM 1.2. *Let $\mathscr{A}$ be a unital C*-algebra, let $\mathscr{V}$ be a norm-closed subspace of $\mathscr{A}$, and equip $\mathscr{A}/\mathscr{V}$ with the corresponding quotient matrix norm. Then the constructions above provide a unital $*$-representation $(\mathscr{H}^{\mathscr{V}}, \pi^{\mathscr{V}})$ of $\mathscr{A}$, and projections $P^{\mathscr{V}}$ and $Q^{\mathscr{V}}$ on $\mathscr{H}^{\mathscr{V}}$, such that the linear mapping $\Psi^{\mathscr{V}}$ from $\mathscr{A}$ to $\mathscr{L}(\mathscr{H}^{\mathscr{V}})$ defined by $\Psi^{\mathscr{V}}(A) = Q^{\mathscr{V}}\pi^{\mathscr{V}}(A)P^{\mathscr{V}}$ gives a complete isometry from $\mathscr{A}/\mathscr{V}$ into $\mathscr{L}(\mathscr{H}^{\mathscr{V}})$.*

2. Quotients of operator $*$-spaces

My principal aim is to understand quotients of the form $\mathscr{A}/\mathscr{B}$ where $\mathscr{A}$ is a C*-algebra and $\mathscr{B}$ is a C*-subalgebra of $\mathscr{A}$. But both $\mathscr{A}$ and $\mathscr{B}$ are stable under *, and so we will consider first quotients under just that requirement.

DEFINITION 2.1. By a *concrete operator $*$-space* we mean a subspace $\mathcal{W}$ of some C*-algebra $\mathcal{A}$ that is stable under *, that is, if $A \in \mathcal{W}$ then $A^* \in \mathcal{W}$.

If $\mathcal{W}$ is a $*$-stable subspace of some C*-algebra $\mathcal{A}$, then $M_n(\mathcal{W})$ is a $*$-stable subspace of $M_n(\mathcal{A})$ for each natural number n, and the restriction to $M_n(\mathcal{W})$ of the norm on $M_n(\mathcal{A})$ will be a $*$-norm in the sense that $\|C^*\|_n = \|C\|_n$ for all $C \in M_n(\mathcal{W})$.

By a *vector $*$-space* we mean (definition 3.1 of [6]) a vector space $\mathcal{W}$ over $\mathbb{C}$ that is equipped with a $*$-operation (i.e. involution) satisfying the usual properties. Then $M_n(\mathcal{W})$ is also canonically a vector $*$-space where $(C^*)_{jk} = (C_{kj})^*$ for $C = \{C_{jk}\}$ as one would expect.

DEFINITION 2.2. Let $\mathcal{W}$ be a vector $*$-space. By a *matrix $*$-norm* on $\mathcal{W}$ we mean a matrix norm $\{\| \cdot \|_n\}$ on $\mathcal{W}$ such that each $\| \cdot \|_n$ is a $*$-norm. By an *abstract operator $*$-space* we mean a vector $*$-space that is equipped with a matrix $*$-norm that satisfies Ruan's axioms.

Let $\mathcal{W}$ be an operator $*$-space, and let $\mathcal{V}$ be a closed $*$-subspace of $\mathcal{W}$ (that is, $\mathcal{V}$ is stable under the involution on $\mathcal{W}$). Then the involution on $\mathcal{W}$ gives an involution on $\mathcal{W}/\mathcal{V}$ in the evident way, so that $\mathcal{W}/\mathcal{V}$ is a vector $*$-space. Then the quotient norm from each $\| \cdot \|_n$ will be a $*$-norm. In this way $\mathcal{W}/\mathcal{V}$ is an abstract operator $*$-space.

We will now show that if $\mathcal{W}$ is a concrete operator $*$-space then we can use the results of the previous section to obtain a completely isometric $*$-representation of $\mathcal{W}/\mathcal{V}$ as a concrete operator $*$-space. As in the previous section, it suffices to do this for the case in which $\mathcal{W}$ is a unital C*-algebra $\mathcal{A}$. So we now treat that case. Then by Theorem 1.2 there exist a $*$-representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ and projections P and Q in $\mathcal{L}(\mathcal{H})$ such that the linear map $\Psi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ defined by

$$\Psi(A) = Q\pi(A)P$$

gives a complete isometry from $\mathcal{A}/\mathcal{V}$ into $\mathcal{L}(\mathcal{H})$. Define Ψ^* by $\Psi^*(A) = (\Psi(A^*))^*$ as usual. Notice that $\Psi^*(A) = P\pi(A)Q$ for all $A \in \mathcal{A}$, and that $\Psi^*(\mathcal{V}) = 0$. Define $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ by

$$\Phi(A) = \begin{pmatrix} 0 & P\pi(A)Q \\ Q\pi(A)P & 0 \end{pmatrix}.$$

Then it is easily seen that Φ is a $*$-map. Clearly Φ is contractive, and it is a complete isometry since Ψ is. We can rewrite Φ as

$$\Phi(A) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} 0 & \pi(A) \\ \pi(A) & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

and we see that $\left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right)$ is itself a projection. But $\left(\begin{smallmatrix} 0 & \pi(A) \\ \pi(A) & 0 \end{smallmatrix}\right)$ does not quite give a $*$-representation of $\mathscr{A}$. It is thus more attractive to rewrite Φ as

$$\Phi(A) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi(A) & 0 \\ 0 & \pi(A) \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

and to notice that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is a Hermitian unitary on $\mathscr{H} \oplus \mathscr{H}$ that commutes with the representation $\pi \oplus \pi$ of $\mathscr{A}$. This puts Φ into the "commutant representation" form given in Theorems 2.2, 2.9 and 2.10 of [5]. On changing the meaning of the various symbols $\mathscr{H}$, π, P, etc, we thus obtain:

THEOREM 2.3. *Let $\mathscr{A}$ be a unital C*-algebra, let $\mathscr{V}$ be a closed $*$-subspace of $\mathscr{A}$, and equip $\mathscr{A}/\mathscr{V}$ with the corresponding quotient matrix norm (so that $\mathscr{A}/\mathscr{V}$ is an operator $*$-space). Then there exist a unital $*$-representation $(\mathscr{H}, \pi)$ of $\mathscr{A}$, a projection P on $\mathscr{H}$, and a Hermitian unitary operator U on $\mathscr{H}$ that commutes with the representation π, such that the linear $*$-map Ψ from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$ defined by*

$$\Psi(A) = PU\pi(A)P,$$

gives a completely isometric $*$-map from $\mathscr{A}/\mathscr{V}$ onto a $*$-subspace of $\mathscr{L}(\mathscr{H})$.

Notice that we can cut down to the closure of $\pi(\mathscr{A})P\mathscr{H}$, that is, we can assume that $\pi(\mathscr{A})P\mathscr{H}$ is dense in $\mathscr{H}$.

Let E and F be the projections onto the two eigensubspaces of U, so that $U = E - F$ and $E + F = I_{\mathscr{H}}$. Then we can decompose Ψ as

$$\Psi(A) = PE\pi(A)EP - PF\pi(A)FP.$$

The two terms on the right give completely positive maps. Thus this decomposition can be viewed as an analogue for Ψ of the Jordan decomposition of a signed measure. But note that PE is not in general a projection.

3. Quotients of operator systems

In this section we consider quotients of operator systems. As before, it suffices for us to consider $\mathscr{V}$ as a subspace of a C*-algebra $\mathscr{A}$. Thus we assume that $\mathscr{V}$ is an operator system in $\mathscr{A}$, that is, that $\mathscr{V}$ is a closed $*$-subspace that contains the identity element, $1_{\mathscr{A}}$, of $\mathscr{A}$. On applying Theorem 2.3, with the notation used there, we obtain a completely isometric embedding of $\mathscr{A}/\mathscr{V}$ into $\mathscr{L}(\mathscr{H})$ given by a map $\Psi : \mathscr{A} \mapsto \mathscr{L}(\mathscr{H})$ defined by

$$\Psi(A) = PU\pi(A)P.$$

The extra information that we obtain from having $1_{\mathscr{A}} \in \mathscr{V}$ is that

$$0 = \Psi(1_{\mathscr{A}}) = PUP.$$

From this and the fact that U commutes with $\pi(A)$ we see that

$$\Psi(A) = P\pi(A)UP - PUP\pi(A) = P[\pi(A), UP] = PU[\pi(A), P]. \tag{3.1}$$

Let $X = 2P - I$, so that X is a Hermitian unitary. Then it follows that we can express Ψ by

$$\Psi(a) = -(1/2)PU[X, \pi(A)].$$

We can equally well express Ψ by

$$\Psi(A) = PU\pi(A)P - \pi(A)PUP = [PU, \pi(A)]P = [P, \pi(A)]UP.$$

On adding the third term of this equation to that of equation (3.1) we obtain

$$\Psi(A) = P[[P, U]/2, \pi(A)]P.$$

Set $Z = [P, U] = [2P - I, U]/2 = [X, U]/2$. Clearly $Z^* = -Z$ and $\|Z\| \leq 1$. Furthermore, $[U, P^2] = [U, P]P + P[U, P]$ so that $PZ = Z(I - P)$. Thus we obtain:

THEOREM 3.1. *Let $\mathscr{A}$ be a unital C*-algebra, let $\mathscr{V}$ be an operator system in $\mathscr{A}$, and equip $\mathscr{A}/\mathscr{V}$ with the corresponding quotient matrix norm (so that $\mathscr{A}/\mathscr{V}$ is an abstract operator $*$-space). Then there exist a unital $*$-representation $(\mathscr{H}, \pi)$ of $\mathscr{A}$, a projection P on $\mathscr{H}$, and an operator Z on $\mathscr{H}$ satisfying $Z^* = -Z$ and $\|Z\| \leq 1$ and $PZ = Z(I - P)$, such that the linear $*$-map Ψ from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$ defined by*

$$\Psi(A) = (1/2)P[Z, \pi(A)]P$$

gives a completely isometric $*$-map from $\mathscr{A}/\mathscr{V}$ onto a $*$-subspace of $\mathscr{L}(\mathscr{H})$.

We remark that a quite different type of quotient involving operator systems, in which one wants the quotient of an operator system by the kernel of a completely positive map to be an operator system, is studied in [4], [3].

4. Quotients by C*-subalgebras

In this section we assume that $\mathscr{A}$ is a unital C*-algebra and that $\mathscr{B}$ is a unital C*-subalgebra of $\mathscr{A}$ (so $1_{\mathscr{A}} \in \mathscr{B}$). Since $\mathscr{B}$ is, in particular, a $*$-subspace of $\mathscr{A}$, Theorem 2.3 is applicable, and, with the notation used there, we have a completely isometric embedding of $\mathscr{A}/\mathscr{B}$ into $\mathscr{L}(\mathscr{H})$ given by the map $\Psi : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ defined by

$$\Psi(A) = PU\pi(A)P.$$

Now let $\hat{P}$ be the projection onto the closed linear span of $\pi(\mathscr{B})P\mathscr{H}$. Since the range of $\hat{P}$ is $\pi(\mathscr{B})$-invariant, $\hat{P}$ commutes with $\pi(B)$ for all $B \in \mathscr{B}$. Because $1_{\mathscr{A}} \in \mathscr{B}$, the range of $\hat{P}$ contains $P\mathscr{H}$, and so $\hat{P} \geq P$. From the fact that $0 = \Psi(\mathscr{B}) = PU\pi(\mathscr{B})P$ and that $\mathscr{B}$ is an algebra it is easily seen that $PU\pi(\mathscr{B})\hat{P} = 0$. On taking adjoints, we have $\hat{P}U\pi(\mathscr{B})P = 0$, and so in the same way as above we have $\hat{P}U\pi(\mathscr{B})\hat{P} = 0$. Define $\hat{\Psi} : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ by

$$\hat{\Psi}(A) = \hat{P}U\pi(A)\hat{P}.$$

Clearly $\hat{\Psi}$ is completely contractive and $\hat{\Psi}(\mathscr{B}) = 0$. From the fact that $\hat{P} \geq P$ we see that $\|\hat{\Psi}(A)\| \geq \|\Psi(A)\|$ for all $A \in \mathscr{A}$, and it is easily seen that in fact $\|\hat{\Psi}_n(C)\| \geq \|\Psi_n(C)\|$ for all natural numbers n and all $C \in M_n(\mathscr{A})$. Since Ψ gives a complete isometry from $\mathscr{A}/\mathscr{B}$ into $\mathscr{L}(\mathscr{H})$, it follows that $\hat{\Psi}$ does also.

Now let $X = 2\hat{P} - I$. Then X is a Hermitian unitary in $\mathscr{L}(\mathscr{H})$ that commutes with $\pi(B)$ for every $B \in \mathscr{B}$. Notice that because $1_{\mathscr{A}} \in \mathscr{B}$ we have $\hat{P}U\hat{P} = 0$. Then much as in the calculation for equation (3.1) we find that

$$\hat{\Psi}(A) = -(1/2)\hat{P}U[X, \pi(A)].$$

It follows that $\|[X, \pi(A)]\| \geq 2\|\hat{\Psi}(A)\|$ for all $A \in \mathscr{A}$. Define a derivation, Θ, from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$ by

$$\Theta(A) = (1/2)[X, \pi(A)]$$

for all $A \in \mathscr{A}$. Then $\|\Theta(A)\| \geq \|\hat{\Psi}(A)\|$ for all $A \in \mathscr{A}$. Furthermore, Θ is completely contractive. To see this, notice that it is the composition of π with a corner of the completely positive contraction that sends $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ in $M_2(\mathscr{A})$ to

$$\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -X & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & I \end{pmatrix}.$$

A slight modification of the calculations done a few lines above shows easily that $\|\Theta_n(C)\| \geq \|\hat{\Psi}_n(C)\|$ for all natural numbers n and all $C \in M_n(\mathscr{A})$. Notice that $\Theta(B) = 0$ for all $B \in \mathscr{B}$ because $\hat{P}$ commutes with all of the elements of $\pi(\mathscr{B})$. Since $\hat{\Psi}$ gives a complete isometry from $\mathscr{A}/\mathscr{B}$ into $\mathscr{L}(\mathscr{H})$, it follows that Θ does also. Notice that if we replace X by iX then Θ is a $*$-map. We have thus obtained:

THEOREM 4.1. *Let $\mathscr{A}$ be a unital C*-algebra, and let $\mathscr{B}$ be a unital C*-subalgebra of $\mathscr{A}$ (so $1_{\mathscr{A}} \in \mathscr{B}$). Then there exist a unital $*$-representation $(\mathscr{H}, \pi)$ of $\mathscr{A}$, and a Hermitian unitary operator X on $\mathscr{H}$ that commutes with $\pi(B)$ for all $B \in \mathscr{B}$, such that the derivation Θ from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$ defined by*

$$\Theta(A) = (1/2)[iX, \pi(A)]$$

gives a completely isometric $*$-map from $\mathscr{A}/\mathscr{B}$ into $\mathscr{L}(\mathscr{H})$.

This theorem is a strengthening of Corollary 3.4 of [11], and its proof is in part motivated by the proof of Theorem 3.1 of [11].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720-3840
USA
E-mail: rieffel@math.berkeley.edu