

A SHORT RETURN TO SIMPLE AH-ALGEBRAS WITH REAL RANK ZERO

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Abstract

Let A be a unital simple AH-algebra with stable rank one and real rank zero such that $kx = 0$ for all $x \in \ker \rho_A$, the subgroup of infinitesimal elements in $K_0(A)$, and for the same integer $k \geq 1$. We show that A has tracial rank zero and is isomorphic to a unital simple AH-algebra with no dimension growth.

1. Introduction

One of the most successful aspects of operator algebras is the classification of simple separable amenable C^* -algebras, or otherwise known as the Elliott program. The program started with the classification of unital simple AT-algebras with real rank zero up to isomorphisms by their ordered K -theory (with the scale) by G. A. Elliott [4] which was preceded by Elliott's classification of AF-algebras ([3]) some fifteen years earlier. It was followed immediately by a number of earlier results. Then, Elliott and Gong ([5]) made a classification of AH-algebras with slow dimension growth and real rank zero by their scaled ordered K -theory. While the earlier results concentrated in the cases of real rank zero, attention later shifted to the case that C^* -algebras are not assumed to have real rank zero. One of the highlights of the program is the classification (up to isomorphisms) of unital simple AH-algebras with no dimension growth (see [6]). These are C^* -algebras whose real rank may not be zero (in fact these C^* -algebras have real rank one). However, the Elliott invariant this time involves not only the ordered K -groups but also tracial information. Just as one thought a complete classification for simple unital AH-algebras was possible, J. Villadsen provided examples of unital simple AH-algebras whose stable rank may be greater than one and examples of unital simple AH-algebras with stable rank one whose K_0 -group may have perforation ([10] and [11]). It should be noted that unital simple AH-algebras with slow dimension growth have stable rank one and have weakly unperforated K_0 -groups. On the other hand, it was proved in [7] that a unital simple AH-algebra with stable rank one,

real rank zero and with weakly unperforated K_0 -groups has tracial rank zero and therefore it is isomorphic to a unital simple AH-algebra with no dimension growth and with real rank zero. At that point, one might think the next goal would be the classification of unital simple AH-algebras of stable rank one and with weakly unperforated K_0 -groups by their ordered K -groups and tracial information. However, A. Toms provided ([9]) an example of unital simple AH-algebra with stable rank one and with weakly unperforated K_0 -group which is not isomorphic to a unital simple AH-algebra with slow dimension growth and with the same Elliott invariant.

The real rank of these AH-algebras are greater than zero. The purpose of this note is to show that, when a unital simple AH-algebra has real rank zero, then it is more likely classifiable by its Elliott invariant. Toms's first example is an inductive limit of homogeneous C^* -algebras whose spectra are contractive finite CW complexes. As we will show in this short note, this example cannot be made so that it has real rank zero. More precisely, we observe that a unital simple AH-algebra with real rank zero which is an inductive limit of homogeneous C^* -algebras whose spectra are contractive finite CW complexes is in fact a unital simple AH-algebra with tracial rank zero. In particular, it is classifiable and isomorphic to a unital simple AH-algebra with no dimension growth. Using our earlier result in [7], we further show that if A is a unital simple AH-algebra with stable rank one and real rank zero so that the subgroup of infinitesimal elements of $K_0(A)$ is a finite group (or zero), then A is isomorphic to a unital simple AH-algebra with no dimension growth and therefore it is classifiable by the Elliott invariant. We actually prove a slightly more general result.

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2. The cases without torsion

Let A be a unital stably finite C^* -algebra. Denote by $T(A)$ the tracial state space of A . If $\tau \in T(A)$, we will also use τ for the trace $\tau \otimes Tr$ on $M_m(A)$, where Tr is the standard trace on M_m , where $m \geq 1$ is an integer. Let $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$ be the positive homomorphism defined by $\rho_A([p])(\tau) = \tau(p)$ for all projections in $M_m(A)$, $m = 1, 2, \dots$

LEMMA 2.1. *Let $A = \lim_{n \rightarrow \infty} (A_n, \phi_n)$ be a unital C^* -algebra, where A_n is a unital C^* -subalgebra and $\phi_n : A_n \rightarrow A_{n+1}$ is a unital homomorphism. Suppose that $p, q \in A$ are two non-zero projections and there is an integer*

$K \geq 1$ such that

$$K\tau(q) < \tau(p) \quad \text{for all } \tau \in T(A).$$

Then, there exist an integer $n(1) \geq 1$, two projections $p', q' \in A_{n(1)}$ and an integer $m_0 \geq n(1)$ such that $\phi_{n(1),\infty}(p')$ is unitarily equivalent to p , $\phi_{n(1),\infty}(q')$ is unitarily equivalent to q and

$$Kt(\phi_{n(1),m}(q')) < t(\phi_{n(1),m}(p'))$$

for all tracial states t of A_m with $m \geq m_0$.

PROOF. Put $B_n = \phi_{n,\infty}(A_n)$, $n = 1, 2, \dots$. Without loss of generality, we may assume that there are $p', q' \in A_{n(1)}$ for some $n(1)$ such that

$$\phi_{n(1),\infty}(p') = p \quad \text{and} \quad \phi_{n(1),\infty}(q') = q.$$

Suppose that, for some increasing subsequence $\{n(k)\}$,

$$(1) \quad Kt_k(\phi_{n(1),n(k)}(q')) \geq t_k(\phi_{n(1),n(k)}(p'))$$

for some tracial states t_k of $A_{n(k)}$. By a result of Choi and Effros ([2]), there exists a contractive completely positive linear map $L_k : B_k \rightarrow A_k$ such that $\phi_{k,\infty} \circ L_k = \text{id}_{B_k}$, $k = 1, 2, \dots$. Note that, if $m \geq n(1)$,

$$(2) \quad \lim_{k \rightarrow \infty} \|\phi_{m,n(k)}[L_m(\phi_{n(1),\infty}(b)) - \phi_{n(1),m}(b)]\| = 0$$

for all $b \in A_{n(1)}$. In particular, it holds for $m = n(2)$.

Define $s'_k : B_{n(k)} \rightarrow \mathbb{C}$ by $s'_k(b) = t_k \circ L_{n(k)}(b)$ for all $b \in B_{n(k)}$, $k = 1, 2, \dots$. Then s'_k is a state on $B_{n(k)}$. Let s_k be a state of A which extends s'_k . Let τ be a weak limit of $\{t_k\}$. It follows from (2) that τ is a tracial state on $B_{n(1)}$. One would have, by (1),

$$K\tau(q) \geq \tau(p).$$

A contradiction.

DEFINITION 2.2. Recall that a unital C^* -algebra A is said to be AH-algebra if $A = \lim_{n \rightarrow \infty} (A_n, \phi_n)$, where $A_n = P_n M_{r(n)}(C(X_n)) P_n$, X_n is a finite CW complex, $r(n) \geq 1$ is an integer and $P_n \in M_{r(n)}(C(X_n))$ is a projection. We also assume that $\phi_n : A_n \rightarrow A_{n+1}$ is unital. In what follows, we call X_n the spectrum of A_n .

THEOREM 2.3. Let $A = \lim_{n \rightarrow \infty} (A_n, \phi_n)$ be a unital simple AH-algebra such that the spectrum of A_n is a finite disjoint union of contractive spaces and $\phi_n : A_n \rightarrow A_{n+1}$ is a unital homomorphism. Suppose also that A has real rank zero. Then A has tracial rank zero.

PROOF. Let $p, q \in A$ be two projections such that

$$\tau(q) < \tau(p)$$

for all $\tau \in T(A)$. We will show that $q \lesssim p$.

We may assume that $A_n = M_{r(n)}(C(X_n))$, where X_n is a disjoint union of contractive spaces. We may also assume that there are two projections $p', q' \in A_n$ for some $n \geq 1$ such that $\phi_{n,\infty}(p') = p$ and $\phi_{n,\infty}(q') = q$. Moreover, by 2.1, we may assume that

$$t(q') < t(p')$$

for all tracial states t of A_n . Since X_n is a disjoint union of contractive spaces, all projections are trivial. Therefore in A_n , $q' \lesssim p'$. It follows that $q \lesssim p$. This implies that $K_0(A)$ is weakly unperforated.

We will show that projections in A have the cancellation property. Suppose that $p, q \in A$ and p and q are equivalent. As above, there are $p', q' \in A_n$ such that $\phi_{n,\infty}(p') = p$ and $\phi_{n,\infty}(q') = q$. Moreover, we may assume that p' and q' are equivalent. So in A_n , $1 - p'$ and $1 - q'$ have the same rank at each point. Note that, since X_n is a disjoint union of contractive spaces, $1 - p'$ and $1 - q'$ are trivial projections. It follows that $1 - p'$ and $1 - q'$ are equivalent. It follows that $1 - p$ and $1 - q$ are equivalent in A .

Since A is also assumed to have real rank zero, by (the proof of) part (3) of III.2.4 of [1], A has stable rank one. It follows from [7] that A has tracial rank zero.

COROLLARY 2.4. *Let $A = \lim_{n \rightarrow \infty} (A_n, \phi_n)$ be a unital simple AH-algebra such that $\ker \rho_{A_n} = \{0\}$. Suppose that A has real rank zero and stable rank one. Then A has tracial rank zero and is isomorphic to a unital simple AH-algebra with no dimension growth.*

PROOF. The proof is the same as that of 2.3. Note that since $\ker \rho_{A_n} = \{0\}$, two projections p, q in $M_m(A_n)$ with the same rank must give the same element in $K_0(A_n)$.

LEMMA 2.5. *Let A be a unital AH-algebra and $x \in \rho_A(K_0(A))$ such that $x > 0$. Then, there is a projection $e \in M_m(A)$ for some integer $m \geq 1$ such that $\rho_A([e]) = x$.*

PROOF. Note that $M_m(A)$ is also a unital AH-algebra for any integer $m \geq 1$. We may assume that there are two projections $p, q \in A$ such that

$$\rho([p] - [q]) = x.$$

We write $A = \lim_{n \rightarrow \infty} (A_n, \phi_n)$, where $A_n = P_n M_{r(n)}(C(X_n)) P_n$, X_n is a finite CW complex, and $r(n) \geq 1$ is an integer and $P_n \in M_{r(n)}(C(X_n))$ is

a projection. We may assume that there is an integer $n \geq 1$ and projections $p', q' \in A_n$ such that $\phi_{n,\infty}(p') = p$ and $\phi_{n,\infty}(q') = q$. By 2.1, we may also assume that

$$(3) \quad t(q') < t(p')$$

for all tracial states t of A_n . There is an integer $K(n) \geq 1$ and there are trivial projections $p_0, q_0 \in M_{K(n)}(A_n)$ such that p_0 has the same rank at each point of X as that of p' , and q_0 has the same rank at each point of X as that of q' . Therefore

$$(4) \quad t(p_0) = t(p') \quad \text{and} \quad t(q_0) = t(q')$$

for all tracial states t of A_n . It follows that

$$(5) \quad \tau(\phi_{n,\infty}(p_0)) = \tau(p) \quad \text{and} \quad \tau(\phi_{n,\infty}(q_0)) = \tau(q)$$

for all $\tau \in T(A)$. On the other hand, by (3) and (4), since both p_0 and q_0 are trivial, there is a partial isometry $v_0 \in A_n$ such that

$$v_0^* v_0 = q_0 \quad \text{and} \quad v_0 v_0^* \leq p_0.$$

Let $v = \phi_{n,\infty}(v_0)$ and let $e = p_0 - vv^*$. Then $e \in M_m(A)$ is a non-zero projection and, by (5),

$$\rho_A([e]) = x.$$

From the above and combining the result in [7], one has the following corollary:

COROLLARY 2.6. *Let A be a unital simple AH-algebra with real rank zero and stable rank one. Suppose that $\ker \rho_A = \{0\}$. Then A has tracial rank zero.*

Note that, 2.6 is not a generalization of 2.4. In fact a simple AF-algebra may have non-zero infinitesimal elements in its K_0 -group.

We will prove a much more general result that will allow non-zero $\ker \rho_A$.

3. The case that $\rho_A(K_0(A))$ is torsion

LEMMA 3.1. *Let A be a unital simple AH-algebra with stable rank one and with real rank zero. Let $K \geq 1$ be an integer. Suppose that for any $x \in \ker \rho_A$, $Kx = 0$. Suppose that $p, q \in M_m(A)$ are two projections such that $(K + 2)\tau(q) < \tau(p)$ for all $\tau \in T(A)$. Then*

$$[q] \leq [p].$$

PROOF. Since A has real rank zero, by a result of S. Zhang ([12], see also 9.4 of [8]), there exist mutually orthogonal projections $e_1, e_2, \dots, e_{K+1} \in pAp$ such that e_i is equivalent to e_1 for $i = 1, 2, \dots, K$, $e_{K+1} \lesssim e_1$ and $p = \sum_{i=1}^{K+1} e_i$. Write $A = \lim_{n \rightarrow \infty} (A_n, \phi_n)$, where $A_n = P_n M_{r(n)}(C(X)) P_n$, X is a finite CW complex, $r(n) \geq 1$ is an integer and $P_n \in M_{r(n)}(C(X))$ is a projection. Without loss of generality, we may assume that there are projections

$$p', q', e'_1, e'_2, \dots, e'_{K+1} \in M_m(A_n)$$

such that $\phi_{n,\infty}(p') = p$, $\phi_{n,\infty}(q') = q$, $\phi_{n,\infty}(e'_i) = e_i$, $i = 1, 2, \dots, K + 1$. We may also assume that $e'_i \leq p'$, $i = 1, 2, \dots, K + 1$, $e'_1, e'_2, \dots, e'_{K+1}$ are mutually orthogonal, e'_i is equivalent to e'_1 in $M_m(A_n)$, $i = 1, 2, \dots, K$, $e'_{K+1} \lesssim e'_1$ in $M_m(A_n)$ and $\sum_{i=1}^{K+1} e'_i = p'$. Moreover, by 2.1, we may also assume that

$$(K + 2)t(q') < t(p') \leq (K + 1)t(e'_1)$$

for all tracial states t of A_n . Note also, we have

$$(6) \quad (K + 1)t(q') < Kt(e'_1)$$

for all tracial states t of A_n .

There are trivial projections $q'', e''_1 \in M_R(A_n)$ for some $R \geq 1$ such that

$$t(q'') = t(q') \quad \text{and} \quad t(e''_1) = t(e'_1)$$

for all tracial states $t \in T(A_n)$. Let $\bar{q}, \bar{e} \in M_R(A)$ such that $\phi_{n,\infty}(q'') = \bar{q}$ and $\phi_{n,\infty}(e''_1) = \bar{e}$. It follows that

$$\tau(\bar{q}) = \tau(q) \quad \text{and} \quad \tau(\bar{e}) = \tau(e_1)$$

for all $\tau \in T(A)$. Therefore

$$[\bar{q}] - [q] \quad \text{and} \quad [\bar{e}] - [e_1] \quad \text{are in} \quad \ker \rho_A.$$

It follows from the assumption that

$$(7) \quad K[q] = K[\bar{q}] \quad \text{and} \quad K[\bar{e}] = K[e_1].$$

On the other hand, by (6), since both q'' and e''_1 are trivial,

$$(8) \quad q'' \lesssim e''_1.$$

By (7) and (8),

$$(9) \quad K[q] = K[\bar{q}] \leq K[e_1] \leq p.$$

Since A also has stable rank one,

$$(10) \quad q \lesssim e_1 + e_2 + \cdots + e_K \leq p.$$

The following theorem was proved in [7].

THEOREM 3.2. *Let A be a unital simple AH-algebra with stable rank one and real rank zero. Then, for any $\epsilon > 0$, $\sigma > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a non-zero projection $p \in A$ and a finite dimensional C^* -subalgebra $B \subset A$ with $1_B = p$ such that*

$$(11) \quad \|px - xp\| < \epsilon \quad \text{for all } x \in \mathcal{F},$$

$$(12) \quad \text{dist}(pxp, B) < \epsilon \quad \text{for all } x \in \mathcal{F}, \quad \text{and}$$

$$(13) \quad \tau(1 - p) < \sigma \quad \text{for all } \tau \in T(A).$$

COROLLARY 3.3. *Let A be a unital simple AH-algebra with stable rank one and real rank zero. Let $e_1, e_2 \in A$ be two projections such that*

$$(14) \quad \tau(e_1) > \tau(e_2) \quad \text{for all } \tau \in T(A).$$

Then, for any $\epsilon > 0$, $\sigma > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a non-zero projection $p \in A$ and a finite dimensional C^ -subalgebra $B \subset A$ with $1_B = p$ such that*

$$(15) \quad \|px - xp\| < \epsilon \quad \text{for all } x \in \mathcal{F},$$

$$(16) \quad \text{dist}(pxp, B) < \epsilon \quad \text{for all } x \in \mathcal{F}, \quad \text{and}$$

$$(17) \quad \tau(1 - p) < \sigma \quad \text{for all } \tau \in T(A).$$

Moreover, there are projections $e_{j,0} \in (1 - p)A(1 - p)$ and $e_{j,1} \in B$, $j = 1, 2$, such that

$$(18) \quad \|e_{j,0} + e_{j,1} - e_j\| < \min\{1/2, \epsilon\}, \quad j = 1, 2,$$

$$(19) \quad t(e_{1,1}) > t(e_{2,1}) \quad \text{for all tracial states of } B, \quad \text{and}$$

$$(20) \quad \tau(e_{1,0}) > \tau(e_{2,0}) \quad \text{for all } \tau \in T(A).$$

PROOF. Note that A is separable. Let $\epsilon > 0$, $\sigma > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\{x_1, x_2, \dots\}$ be a dense subset of A . Let $\mathcal{F}_n = \mathcal{F} \cup \{e_1, e_2\} \cup \{x_1, x_2, \dots, x_n\}$ and let $\epsilon_n = \epsilon/2^n$ and $\sigma_n = \sigma/2^n$, $n = 1, 2, \dots$. By 3.2, there exists a sequence of projections $p_n \in A$ and finite dimensional C^* -subalgebras $B_n \subset A$ with $1_{B_n} = p_n$ such that

$$(21) \quad \|p_n x - x p_n\| < \epsilon/2^n \quad \text{for all } x \in \mathcal{F}_n,$$

$$(22) \quad \text{dist}(p_n x p_n, B_n) < \epsilon/2^n \quad \text{for all } x \in \mathcal{F}_n \text{ and } \tau(1 - p_n) < \sigma/2^n,$$

$n = 1, 2, \dots$ For all sufficiently large n , there are projections $e^{(j,1,n)} \in B_n$ and $e^{(j,0,n)} \in (1 - p_n)A(1 - p_n)$ such that

$$(23) \quad \|e^{(j,1,n)} + e^{(j,0,n)} - e_j\| < \epsilon/2^{n-2},$$

$j = 1, 2$ and $n = 1, 2, \dots$ We also have

$$(24) \quad \lim_{n \rightarrow \infty} \|p_n e_j p_n - e^{(j,1,n)}\| = 0, \quad j = 1, 2.$$

Suppose that, for a subsequence $\{n_k\}$,

$$(25) \quad t_k(e^{(2,1,n_k)}) \leq t_k(e^{(1,1,n_k)})$$

for some tracial state t_k of B_{n_k} . Let $s'_k : pAp \rightarrow \mathbb{C}$ be a state which extends t_k . Define $s_k : A \rightarrow \mathbb{C}$ by $s_k(a) = s'_k(pap)$ for all $a \in A$. Then s_k is a state on A . Let t be a weak limit of $\{s_k\}$. One checks that t is a tracial state on A . Then (25) and (24) imply that

$$t(e_1) \leq t(e_2)$$

which contradicts with (14).

It follows that, for all sufficiently large n ,

$$(26) \quad t(e^{(2,1,n)}) > t(e^{(1,1,n)}) \quad \text{for all } t \in T(B_n).$$

The lemma follows.

THEOREM 3.4. *Let A be a unital simple AH-algebra with stable rank one and real rank zero. Suppose that there is an integer $K \geq 1$ such that, for any $x \in \ker \rho_A$, $Kx = 0$. Then A has tracial rank zero. Moreover, A is isomorphic to a unital simple AH-algebra with slow dimension growth.*

PROOF. We may assume that A is infinite dimensional. We will show that $K_0(A)$ is weakly unperforated. It then follows from [7] that A has tracial rank zero.

It suffices to show the following: If $p, q \in M_m(A)$ are two non-zero projections for some integer $m \geq 1$ and

$$\tau(p) > \tau(q) \quad \text{for all } \tau \in T(A),$$

then $q \lesssim p$.

To prove this, we note that $M_m(A)$ is also a unital simple AH-algebra with stable rank one, real rank zero and $K_0(M_m(A)) = K_0(A)$, so, to simplify the notation, without loss of generality, we may assume that $p, q \in A$.

Let

$$d_1 = \inf\{\tau(p) - \tau(q) : \tau \in T(A)\}.$$

Since A is simple, $d_1 > 0$. Since A is an infinite dimensional simple C^* -algebra with real rank zero, pAp is also an infinite dimensional simple C^* -algebra with real rank zero. It follows that there is a non-zero projection $e \in pAp$ such that

$$(27) \quad \tau(e) < d_1/2 \quad \text{for all } \tau \in T(A).$$

Put

$$(28) \quad d_2 = \inf\{\tau(e) : \tau \in T(A)\}.$$

Note that $d_2 > 0$. Put $p_0 = p - e$. Then

$$(29) \quad \tau(p_0) > \tau(q) \quad \text{for all } \tau \in T(A).$$

Let

$$\mathcal{F} = \{p, q, e, p_0\}.$$

It follows from 3.2 and 3.3 that, there exists a projection $E \in A$ and a finite dimensional C^* -subalgebra B with $1_B = E$ such that

$$(30) \quad \|Ex - xE\| < \frac{d_2}{64K} \quad \text{for all } x \in \mathcal{F},$$

$$(31) \quad \text{dist}(ExE, B) < \frac{\min\{d_2, 1\}}{64K} \quad \text{for all } x \in \mathcal{F}, \quad \text{and}$$

$$(32) \quad \tau(1 - E) < \frac{d_2}{64K} \quad \text{for all } \tau \in T(A).$$

Moreover, there are projections $p_{0,1}, q_1 \in B$ and $p_{0,0}, q_0 \in (1 - E)A(1 - E)$ such that

$$(33) \quad \|p_{0,1} + p_{0,0} - p_0\| < \frac{1}{16K},$$

$$(34) \quad \|q_1 + q_0 - q\| < \frac{1}{16K}, \quad \text{and}$$

$$(35) \quad t(q_1) < t(p_{0,1})$$

for all tracial state t of B . It follows that, in B ,

$$(36) \quad q_1 \lesssim p_{0,1}.$$

We compute, by (28) and (32) that

$$(37) \quad (K + 2)\tau(q_0) < \tau(e) \quad \text{for all } \tau \in T(A).$$

It follows from 3.1 that

$$(38) \quad q_0 \lesssim e.$$

Combining (36) and (38), we obtain that

$$q_1 + q_0 \lesssim p_{0,1} + e.$$

But, by (33) and (34),

$$[q_1 + q_0] = [q] \quad \text{and} \quad p_{0,1} \leq p_0.$$

Therefore

$$q \lesssim p.$$

COROLLARY 3.5. *Let A be a unital simple AH-algebra with stable rank one and real rank zero. Suppose that $\ker \rho_A$ is finite. Then A has tracial rank zero and $K_0(A)$ is weakly unperforated.*

4. Concluding remarks

REMARK 4.1. Theorem 2.3 shows that that Toms’ example (as in [9]) could not occur under the assumption that A has real rank zero.

REMARK 4.2. More can be said in Theorem 2.3. It is clear that, in Theorem 2.3, it suffices to assume that every A_n has the property that all projections in the matrix algebras of A_n are unitarily equivalent to those constant projections. So it allows A_n to have non-zero K_1 -groups.

Let $A_n = P_n M_{r(n)}(C(X_n)) P_n$, where X_n is a finite CW complex, $r(n) \geq 1$ is an integer and $P_n \in M_{r(n)}(C(X_n))$. Suppose that Y_n is another finite CW complex with covering dimension $hd(X_n)$ which has the same homotopy type of that of X_n . Suppose that

$$\liminf_{n \rightarrow \infty} \left(\sup_{x \in X_n} \frac{hd(X_n)}{\text{Rank } P_n(x)} \right) = 0.$$

Suppose also that $A = \lim_{n \rightarrow \infty} A_n$ is a unital simple C^* -algebra with real rank zero. Then, from the proof of 2.3, one can show that A has tracial rank zero and stable rank one. In other words, if A is homotopically slow dimension growth and is of real rank zero and stable rank one, then A is isomorphic to a unital simple AH-algebra with no dimension growth. One should note that, as in [9], without the assumption of real rank zero, the Cuntz semigroup of A could be very different from those of unital simple AH-algebras with slow dimension growth.

REMARK 4.3. From the proof of 3.4, one sees that Theorem 3.4 holds if the assumption on $\ker \rho_A$ is replaced by the conclusion of 3.1, i.e., there is an integer $K \geq 1$ such that for any pair of projections $p, q \in M_m(A)$ (for any integer $m \geq 1$), $K\tau(p) \leq \tau(q)$ for all $\tau \in T(A)$ implies $p \lesssim q$.

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