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PRIMENESS AND PRIMITIVITY CONDITIONS FOR TWISTED GROUP C^*-ALGEBRAS

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Abstract

For a multiplier (2-cocycle) σ on a discrete group G we give conditions for which the twisted group C^*-algebra associated with the pair (G, σ) is prime or primitive. We also discuss how these conditions behave on direct products and free products of groups.

Introduction

In this paper, G will always denote a discrete group with identity e. The full group C^*-algebra associated with G, $C^*(G)$ is simple only if G is trivial, but other aspects of its ideal structure are of interest. Recall that a C^*-algebra is called *primitive* if it has a faithful irreducible representation and *prime* if nonzero ideals have nonzero intersection. Primeness of a C^*-algebra is in general a weaker property than primitivity. However, according to a result of Dixmier [9], the two notions coincide for separable C^*-algebras.

Furthermore, recall what the *icc property* means for G – that every nontrivial conjugacy class is infinite, and its importance comes to light in the following theorem.

THEOREM A. The following are equivalent:

(i) G has the icc property.

(ii) The group von Neumann algebra $W^*(G)$ is a factor.

(iii) The reduced group C^*-algebra $C_r^*(G)$ is prime.

The equivalence (i) $\Leftrightarrow$ (ii) is a well known result of Murray and von Neumann [19], while (i) $\Leftrightarrow$ (iii) is proved by Murphy [18]. Murphy also shows that the icc property is a necessary condition for primeness of $C^*(G)$. Therefore, for the class of discrete groups, primeness and, in the countable case, primitivity, may be regarded as C^*-algebraic analogs of factors. The theorem gives as a corollary that if G is countable and amenable, then primitivity of $C^*(G)$ is equivalent with the icc property of G. Moreover, since amenability

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of G implies injectivity of $W^*(G)$, this is also equivalent to $W^*(G)$ being the hyperfinite II_1 factor if G is nontrivial, according to Connes [8].

In the present paper, Theorem A will be adapted to a twisted setting where pairs (G, σ) consisting of a group G and a multiplier σ on G are considered. We will show that the reduced twisted group C^*-algebra $C_r^*(G, \sigma)$ is prime if and only if every nontrivial σ-regular conjugacy class of G is infinite, and say that the pair (G, σ) satisfies *condition K* if it possesses this property. It was first introduced by Kleppner [13], who proves that this property is equivalent to the fact that the twisted group von Neumann algebra $W^*(G, \sigma)$ is a factor. The main part of our proof is to show that (G, σ) satisfies condition K if and only if $C_r^*(G, \sigma)$ has trivial center, and this argument is, of course, inspired by the mentioned works of Kleppner and Murphy. As a corollary, we get that primeness of the full twisted group C^*-algebra $C^*(G, \sigma)$ implies condition K on (G, σ). The converse is not true in general, but at least holds if G is amenable, as the full and reduced twisted group C^*-algebras then are isomorphic. Thus, if G is countable and amenable, condition K on (G, σ) is equivalent to primitivity of $C^*(G, \sigma)$ by applying Dixmier's result. This fact is also explained by Packer [22] with a different approach. On the other hand, no examples of nonprimitive, but prime twisted group C^*-algebras are known, so it is not clear whether we need the countability assumption on G.

In the last two sections we will investigate primeness and primitivity of the twisted group C^*-algebras of (G, σ) when $G = G_1 \times G_2$ and when $G = G_1 * G_2$. The free product case turns out to be easier to handle in general, since the corresponding C^*-algebra always decomposes into a free product of the two components. For direct products, however, the multiplier σ on G can have a 'cross-term' which makes a C^*-algebra decomposition into tensor products impossible.

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1. Twisted group C^*-algebras

Let G be a group and $\mathscr{H}$ a nontrivial Hilbert space. The projective unitary group $PU(\mathscr{H})$ is the quotient of the unitary group $U(\mathscr{H})$ by the scalar multiples of the identity, that is,

$$PU(\mathscr{H}) = U(\mathscr{H})/\mathsf{T}1_{\mathscr{H}}.$$

A projective unitary representation G is a homomorphism $G \to PU(\mathscr{H})$. Every lift of a projective representation to a map $U : G \to U(\mathscr{H})$ must satisfy

$$U(a)U(b) = \sigma(a,b)U(ab) \tag{1}$$

for all $a, b \in G$ and some function $\sigma : G \times G \to \mathsf{T}$. From the associativity of U and by requiring that $U(e) = 1_{\mathscr{H}}$, the identities

$$\begin{gathered} \sigma(a,b)\sigma(ab,c) = \sigma(a,bc)\sigma(b,c) \\ \sigma(a,e) = \sigma(e,a) = 1 \end{gathered} \tag{2}$$

must hold for all elements $a, b, c \in G$.

DEFINITION. Any function $\sigma : G \times G \to \mathsf{T}$ satisfying (2) is called a *multiplier on G*, and any map $U : G \to U(\mathscr{H})$ satisfying (1) is called a *σ-projective unitary representation of G on $\mathscr{H}$*.

The lift of a homomorphism $G \to PU(\mathscr{H})$ up to U is not unique, but any other lift is of the form βU for some function $\beta : G \to \mathsf{T}$. Therefore, two multipliers σ and τ are said to be *similar* if

$$\tau(a,b) = \beta(a)\beta(b)\overline{\beta(ab)}\sigma(a,b)$$

for all $a, b \in G$ and some $\beta : G \to \mathsf{T}$. Note that we must have $\beta(e) = 1$ for this to be possible. We say that σ is *trivial* if it is similar to 1 and call σ *normalized* if $\sigma(a, a^{-1}) = 1$ for all $a \in G$.

Moreover, the set of similarity classes of multipliers on G is an abelian group under pointwise multiplication. This group is the *Schur multiplier* of G and will henceforth be denoted by $\mathscr{M}(G)$. Also, we remark that multipliers are often called 2*-cocycles on G with values in* T, and that the Schur multiplier of G coincides with the second cohomology group $H^2(G, \mathsf{T})$.

Let σ be a multiplier on G. We will briefly explain how the operator algebras associated with the pair (G, σ) are constructed and refer to Zeller-Meier [25] for further details. First, the Banach *-algebra $\ell^1(G, \sigma)$ is defined as the set $\ell^1(G)$ together with twisted convolution and involution given by

$$\begin{aligned} (f *_\sigma g)(a) &= \sum_{b \in G} f(b)\sigma(b, b^{-1}a)g(b^{-1}a) \\ f^*(a) &= \overline{\sigma(a, a^{-1})f(a^{-1})} \end{aligned}$$

for elements f, g in $\ell^1(G)$, and is equipped with the usual $\|\cdot\|_1$-norm.

DEFINITION. The full twisted group C^*-algebra $C^*(G, \sigma)$ is the universal enveloping algebra of $\ell^1(G, \sigma)$. Moreover, the canonical injection of G into $C^*(G, \sigma)$ will be denoted by $i_{(G,\sigma)}$ or just i_G if no confusion arise.

For a in G, let δ_a be the function on G defined by

$$\delta_a(b) = \begin{cases} 1 & \text{if } b = a, \\ 0 & \text{if } b \neq a. \end{cases}$$

Then the set $\{\delta_a\}_{a \in G}$ is an orthonormal basis for $\ell^2(G)$ and generates $\ell^1(G, \sigma)$, so that for all a in G, $i_{(G,\sigma)}(a)$ is the image of δ_a in $C^*(G, \sigma)$. The *left regular σ-projective unitary representation λ_σ of G on $B(\ell^2(G))$* is given by

$$(\lambda_\sigma(a)\xi)(b) = (\delta_a *_\sigma \xi)(b) = \sigma(a, a^{-1}b)\xi(a^{-1}b).$$

Note in particular that

$$\lambda_\sigma(a)\delta_b = \delta_a *_\sigma \delta_b = \sigma(a, b)\delta_{ab}$$

for all $a, b \in G$. Moreover, the integrated form of λ_σ on $\ell^1(G, \sigma)$ is defined by

$$\lambda_\sigma(f) = \sum_{a \in G} f(a)\lambda_\sigma(a).$$

DEFINITION. The reduced twisted group C^*-algebra and the twisted group von Neumann algebra of (G, σ), $C_r^*(G, \sigma)$ and $W^*(G, \sigma)$ are, respectively, the C^*-algebra and the von Neumann algebra generated by $\lambda_\sigma(\ell^1(G, \sigma))$, or equivalently by $\lambda_\sigma(G)$.

If τ is similar with σ, then in all three cases, the operator algebras associated with (G, τ) and (G, σ) are isomorphic.

Moreover, there is a natural one-to-one correspondence between the representations of $C^*(G, \sigma)$ and the σ-projective unitary representations of G. In particular, λ_σ extends to a *-homomorphism of $C^*(G, \sigma)$ onto $C_r^*(G, \sigma)$. If G is amenable, then λ_σ is faithful. However, it is not known whether the converse holds unless σ is trivial.

Following the work of Kleppner [13], an element a in G is called *σ-regular* if $\sigma(a, b) = \sigma(b, a)$ whenever b commutes with a, or equivalently if

$$U(a)U(b) = U(b)U(a)$$

for all b commuting with a whenever U is a σ-projective unitary representation of G. If σ and τ are similar multipliers on G, it is easily seen that a in G is σ-regular if and only if it is τ-regular. Furthermore, if a is σ-regular, then cac^{-1} is σ-regular for all c in G, and thus the notion of σ-regularity makes sense for conjugacy classes [13, Lemma 3]. The following theorem may now be deduced from [13, Lemma 4].

THEOREM B. Let σ be a multiplier on G. Then the following are equivalent:

(i) Every nontrivial σ-regular conjugacy class of G is infinite.

(ii) $W^*(G, \sigma)$ is a factor.

DEFINITION. We say that the pair (G, σ) satisfies *condition K* if (i) is satisfied.

If G has the icc property, then (G, σ) always satisfies condition K. If G is abelian, or more generally, if all the conjugacy classes of G are finite, then (G, σ) satisfies condition K only if there are no nontrivial σ-regular elements in G.

EXAMPLE 1.1. For $n \geq 2$, let Z_n denote the cyclic group of order n. Then $\mathscr{M}(\mathsf{Z}_n \times \mathsf{Z}_n) \cong \mathsf{Z}_n$ and its elements may be represented by multipliers σ_k given by

$$\sigma_k((a_1, a_2), (b_1, b_2)) = e^{2\pi i \frac{k}{n} a_2 b_1}$$

for $0 \leq k \leq n-1$. An element $a = (a_1, a_2)$ in $\mathsf{Z}_n \times \mathsf{Z}_n$ is σ_k-regular if and only if both ka_1 and ka_2 belong to $n\mathsf{Z}$. Therefore, $(\mathsf{Z}_n \times \mathsf{Z}_n, \sigma_k)$ satisfies condition K if and only if k and n are relatively prime, in which case we have

$$C^*(\mathsf{Z}_n \times \mathsf{Z}_n, \sigma_k) \cong C_r^*(\mathsf{Z}_n \times \mathsf{Z}_n, \sigma_k) = W^*(\mathsf{Z}_n \times \mathsf{Z}_n, \sigma_k) \cong M_n(\mathsf{C}).$$

EXAMPLE 1.2. It is well known that $\mathscr{M}(\mathsf{Z}^n) \cong \mathsf{T}^{\frac{1}{2}n(n-1)}$ and that the multipliers are, up to similarity, determined by

$$\sigma_\theta(a, b) = e^{2\pi i \sum_{1 \leq i < j \leq n} a_i t_{ij} b_j}$$

for $\theta = (t_{12}, t_{13}, \ldots, t_{n-1,n})$ in $[0, 1)^{\frac{1}{2}n(n-1)}$. Note that the C^*-algebras associated with the pair $(\mathsf{Z}^n, \sigma_\theta)$, $C^*(\mathsf{Z}^n, \sigma_\theta) \cong C_r^*(\mathsf{Z}^n, \sigma_\theta)$, are the noncommutative n-tori when θ is nonzero.

Furthermore, $(\mathsf{Z}^n, \sigma_\theta)$ satisfies condition K if there are no nontrivial σ_θ-regular elements in Z^n, that is, if there for all a in Z^n exists b in Z^n such that

$$\sigma_\theta(a, b)\overline{\sigma_\theta(b, a)} = e^{2\pi i \sum_{1 \leq i < j \leq n} t_{ij}(a_i b_j - b_i a_j)} \neq 1.$$

For $n = 2$ and 3 we can give a good description of this property. Indeed, $(\mathsf{Z}^2, \sigma_\theta)$ satisfies condition K if and only if θ is irrational, and $(\mathsf{Z}^3, \sigma_\theta)$ satisfies condition K if and only if

$$\dim \mathsf{Q}_\theta = 3 \text{ or } 4,$$

where Q_θ denotes the vector space over Q spanned by $\{1, t_{12}, t_{13}, t_{23}\}$.

For $n \geq 4$, the situation is more complicated. In particular, condition K on $(\mathsf{Z}^n, \sigma_\theta)$ does not only depend on the dimension of Q_θ. For example, if $t_{12} = t_{34}$ is some irrational number in $[0, 1)$ and $t_{ij} = 0$ elsewhere, then $\dim \mathsf{Q}_\theta = 2$, and $(\mathsf{Z}^4, \sigma_\theta)$ satisfies condition K. On the other hand, if $t_{12} = t_{23} = t_{34} = 1 - t_{14}$ is some irrational number in $[0, 1)$ and $t_{13} = t_{24} = 0$, then $\dim \mathsf{Q}_\theta = 2$, but it can be easily checked that $(1, 1, 1, 1)$ in Z^4 is σ_θ-regular.

EXAMPLE 1.3. For each natural number $n \geq 2$ let $G(n)$ be the group with presentation

$$G(n) = \langle u_i, v_{jk} : [v_{jk}, v_{lm}] = [u_i, v_{jk}] = e, [u_j, u_k] = v_{jk} \rangle$$

for $1 \leq i \leq n$, $1 \leq j < k \leq n$ and $1 \leq l < m \leq n$. The group $G(n)$ is sometimes called the *free nilpotent group of class* 2 *and rank* n.

In [20], we calculate the multipliers of $G(n)$ and show that

$$\mathcal{M}(G(n)) \cong \mathsf{T}^{\frac{1}{3}(n-1)n(n+1)}.$$

Note that $G(2)$ is isomorphic with the discrete Heisenberg group and this case is already investigated by Packer [21].

To describe our result in the case of $G(3)$, we first remark that $G(3)$ is isomorphic to the group with elements $a = (a_1, a_2, a_3, a_4, a_5, a_6)$, where all entries are integers, and with multiplication defined by

$$a \cdot b = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + a_1 b_2, a_5 + b_5 + a_1 b_3, a_6 + b_6 + a_2 b_3).$$

For every μ in T^8, the element $[\sigma_\mu]$ in $\mathcal{M}(G(3))$ may be represented by

$$\begin{aligned}\sigma_\mu(a, b) = \mu_{13}^{b_6 a_1 + b_3 a_4} & \mu_{22}^{b_5 a_2 + b_3(a_4 - a_1 a_2)} \\ & \cdot \mu_{11}^{b_4 a_1 + \frac{1}{2} b_2 a_1 (a_1 - 1)} \mu_{21}^{a_2(b_4 + a_1 b_2) + \frac{1}{2} a_1 b_2 (b_2 - 1)} \\ & \cdot \mu_{12}^{b_5 a_1 + \frac{1}{2} b_3 a_1 (a_1 - 1)} \mu_{32}^{a_3(b_5 + a_1 b_3) + \frac{1}{2} a_1 b_3 (b_3 - 1)} \\ & \cdot \mu_{23}^{b_6 a_2 + \frac{1}{2} b_3 a_2 (a_2 - 1)} \mu_{33}^{a_3(b_6 + a_2 b_3) + \frac{1}{2} a_2 b_3 (b_3 - 1)}\end{aligned}$$

where $\mu_{ij} \in \mathsf{T}$.

The pair $(G(3), \sigma_\mu)$ satisfies condition K if and only if $G(3)$ has no nontrivial central σ_μ-regular elements, that is, if for all $c = (0, 0, 0, c_1, c_2, c_3)$ in $Z(G(3)) = \mathsf{Z}^3$ there exists a in $G(3)$ such that $\sigma_\mu(a, c)\overline{\sigma_\mu(c, a)} \neq 1$.

Set $\mu_{31} = \mu_{13}\overline{\mu_{22}}$. One can then show that this holds if and only if for each nontrivial c in Z^3 there is some $i = 1, 2$ or 3 such that

$$\prod_{1 \leq j \leq 3} \mu_{ij}^{c_j} \neq 1.$$

2. Primeness and primitivity

Henceforth, we fix a group G and a multiplier σ on G. Consider the right regular $\overline{\sigma}$-projective unitary representation $\rho_{\overline{\sigma}}$ of G on $B(\ell^2(G))$ defined by

$$(\rho_{\overline{\sigma}}(a)\xi)(c) = (\xi *_{\overline{\sigma}} \delta_a^*)(c) = \overline{\sigma(c,a)}\xi(ca).$$

To simplify notation in what follows, we write just $\overline{\rho}$ and λ for $\rho_{\overline{\sigma}}$ and λ_σ. It is straightforward to see that $\lambda(a)$ commutes with $\overline{\rho}(b)$ for all a, b in G, that is, $W^*(G,\sigma)$ is contained in $\overline{\rho}(G)'$. In fact, it is well known that $W^*(G,\sigma) = \overline{\rho}(G)'$. Moreover,

$$(\lambda(a)\overline{\rho}(a)\xi)(c) = \overline{\sigma(a^{-1},c)}\sigma(a^{-1}ca, a^{-1})\xi(a^{-1}ca) \tag{3}$$

for all $a, c \in G$ and all $\xi \in \ell^2(G)$. In particular,

$$\lambda(a)\overline{\rho}(a)\delta_e = \overline{\rho}(a)\lambda(a)\delta_e = \delta_e \tag{4}$$

for all $a \in G$.

REMARK 2.1. The vector δ_e is clearly cyclic for $W^*(G,\sigma)$. It is also separating. Indeed, if $x\delta_e = 0$, then

$$x\delta_a = x\lambda(a)\delta_e = x\overline{\rho}(a)^*\delta_e = \overline{\rho}(a)^*x\delta_e = 0$$

for all $a \in G$. Moreover, the state φ given by $\varphi(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful trace on $W^*(G,\sigma)$. Thus, $W^*(G,\sigma)$ is finite and is therefore a II_1 factor whenever G is infinite and (G,σ) satisfies condition K, according to Theorem B.

LEMMA 2.2. *Let T be an operator in $W^*(G,\sigma)$ and set $f_T = T\delta_e$. Then the following are equivalent:*

(i) *T belongs to the center of $W^*(G,\sigma)$.*

(ii) *$f_T(aca^{-1}) = \sigma(a,c)\overline{\sigma(aca^{-1},a)}f_T(c)$ for all $a, c \in G$.*

Moreover, f_T can be nonzero only on the finite conjugacy classes.

PROOF. The operator T belongs to the center of $W^*(G,\sigma)$ if and only if $T = \lambda(a)T\lambda(a)^*$ for all $a \in G$. Since, by Remark 2.1, δ_e is separating for $W^*(G,\sigma)$, this is equivalent to $f_T = \lambda(a)T\lambda(a)^*\delta_e$ for all $a \in G$. By (4) we have

$$\lambda(a)T\lambda(a)^*\delta_e = \lambda(a)T\overline{\rho}(a)\delta_e = \lambda(a)\overline{\rho}(a)T\delta_e = \lambda(a)\overline{\rho}(a)f_T$$

for all $a \in G$. Thus T belongs to the center if and only if $f_T = \lambda(a)\overline{\rho}(a)f_T$ for all $a \in G$ and the desired equivalence now follows from (3). If a function

f satisfies (ii), then $|f|$ is constant on conjugacy classes. Since f_T belongs to $\ell^2(G)$, it can be nonzero only on the finite conjugacy classes.

REMARK 2.3. Lemma 2.2 is proved in [13, Theorem 1]. However, the proof provided above is shorter. Lemma 2.4 below is proved in [13, Lemma 2] in the case where C is a single point. Also, note that we do not restrict to normalized multipliers as in [13].

LEMMA 2.4. *Let C be a conjugacy class of G. Then following are equivalent:*

(i) *C is σ-regular.*

(ii) *There is a function $f : G \to \mathbb{C}$ satisfying:*

1. *$f(c) \neq 0$ for all $c \in C$.*
2. *$f(aca^{-1}) = \sigma(a, c)\overline{\sigma(aca^{-1}, a)} f(c)$ for all $c \in C$ and all $a \in G$.*

Moreover, f can be chosen in $\ell^2(G)$ if and only if C is finite.

PROOF. (ii) $\Rightarrow$ (i): Suppose c belongs to C and that a commutes with c. Then there is a function $f : G \to \mathbb{C}$ satisfying $0 \neq f(c) = \sigma(a, c)\overline{\sigma(c, a)} f(c)$. Hence $\sigma(a, c) = \sigma(c, a)$, so c is σ-regular.

(i) $\Rightarrow$ (ii): This clearly holds if C is trivial, so suppose C is nontrivial and σ-regular and fix an element c in C. Define a function $f : G \to \mathbb{C}$ by

$$f(x) = \begin{cases} \sigma(a, c)\overline{\sigma(aca^{-1}, a)} & \text{if } x \in C, x = aca^{-1} \text{ for some } a \in G \\ 0 & \text{if } x \notin C \end{cases}$$

First we show that f is well-defined, so assume $aca^{-1} = bcb^{-1}$, and note that

$$\begin{aligned} \sigma(a^{-1}, aca^{-1})\sigma(ca^{-1}, b) &= \sigma(a^{-1}, aca^{-1}b)\sigma(aca^{-1}, b) \\ &= \sigma(a^{-1}, bc)\sigma(bcb^{-1}, b). \end{aligned}$$

As c is σ-regular and commutes with $a^{-1}b$, $\sigma(a^{-1}b, c) = \sigma(c, a^{-1}b)$. Thus

$$\begin{aligned} \sigma(c, a^{-1})\sigma(ca^{-1}, b) &= \sigma(c, a^{-1}b)\sigma(a^{-1}, b) \\ &= \sigma(a^{-1}, b)\sigma(a^{-1}b, c) \\ &= \sigma(a^{-1}, bc)\sigma(b, c). \end{aligned}$$

Together, we get from these equations that

$$\sigma(a^{-1}, aca^{-1})\sigma(b, c) = \sigma(c, a^{-1})\sigma(bcb^{-1}, b). \tag{5}$$

Finally, the two identities

$$\begin{aligned} \sigma(a^{-1}, aca^{-1})\sigma(ca^{-1}, a) &= \sigma(a^{-1}, ac)\sigma(aca^{-1}, a) \\ \sigma(c, a^{-1})\sigma(ca^{-1}, a) = \sigma(a^{-1}, a) &= \sigma(a^{-1}, ac)\sigma(a, c) \end{aligned}$$

give that

$$\sigma(a^{-1}, aca^{-1})\sigma(a, c) = \sigma(c, a^{-1})\sigma(aca^{-1}, a). \tag{6}$$

Combining (5) and (6) we get that

$$\sigma(a, c)\overline{\sigma(aca^{-1}, a)} = \sigma(b, c)\overline{\sigma(bcb^{-1}, b)}.$$

Hence f is well-defined, so $f(aca^{-1}) = f(bcb^{-1})$.

It is easily seen that $|f(x)| = 1$ for all x in C. In fact, if f is any function satisfying (ii), then $|f|$ must be constant and nonzero on C, hence f belongs to $\ell^2(G)$ if and only if C is finite.

In particular, $f(c) = 1$ in our case, so f satisfies part 2 of (ii) for the chosen c in C. It remains to show that f satisfies part 2 of (ii) for all other x in C. Suppose x is an element of C, that is, there exists b in G such that $x = bcb^{-1}$. Note first that

$$f(x) = f(bcb^{-1}) = \sigma(b, c)\overline{\sigma(bcb^{-1}, b)} = \sigma(b, c)\overline{\sigma(x, b)}.$$

Next,

$$\begin{aligned}\sigma(axa^{-1}, a)\sigma(ax, b)\sigma(ab, c) &= \sigma(axa^{-1}, ab)\sigma(a, b)\sigma(ab, c)\\ &= \sigma(axa^{-1}, ab)\sigma(a, bc)\sigma(b, c),\end{aligned}$$

which, since $xb = bc$, gives that

$$\begin{aligned}\sigma(a, x)\overline{\sigma(x, b)} &= \sigma(a, xb)\overline{\sigma(ax, b)} = \sigma(a, bc)\overline{\sigma(ax, b)}\\ &= \sigma(axa^{-1}, a)\sigma(ab, c)\overline{\sigma(axa^{-1}, ab)}\,\overline{\sigma(b, c)}.\end{aligned}$$

Hence

$$\begin{aligned}f(axa^{-1}) &= f(abcb^{-1}a^{-1}) = \sigma(ab, c)\overline{\sigma(abcb^{-1}a^{-1}, ab)}\\ &= \sigma(ab, c)\overline{\sigma(axa^{-1}, ab)} = \sigma(a, x)\overline{\sigma(axa^{-1}, a)}\sigma(b, c)\overline{\sigma(x, b)}\\ &= \sigma(a, x)\overline{\sigma(axa^{-1}, a)}f(x).\end{aligned}$$

Before stating the main theorem, we recall two results which are part of the folklore of operator algebras. The first can be shown as sketched in the proof of [18, Proposition 2.3], while the second is a rather easy consequence of Urysohn's Lemma. Remark that together these two results imply that if A is von Neumann algebra, then A is prime (as a C^*-algebra) if and only if it is a factor.

PROPOSITION 2.5. *If A is a concrete unital C^*-algebra and its bicommutant A'' is a factor, then A is prime.*

PROPOSITION 2.6. *Every prime C^*-algebra has trivial center.*

THEOREM 2.7. *The following conditions are equivalent:*

(i) (G, σ) *satisfies condition K.*

(ii) $W^*(G, \sigma)$ *is a factor.*

(iii) $C_r^*(G, \sigma)$ *is prime.*

(iv) $C_r^*(G, \sigma)$ *has trivial center.*

PROOF. For completeness, we include the few lines required to prove (i) $\Rightarrow$ (ii): Suppose (G, σ) satisfies condition K and let T belong to the center of $W^*(G, \sigma)$. By Lemma 2.2 and Lemma 2.4, f_T can be nonzero only on the finite σ-regular conjugacy classes, hence on $\{e\}$. So $T\delta_e = f_T(e)\delta_e$, thus $T = f_T(e)I$ as δ_e is separating for $W^*(G, \sigma)$ by Remark 2.1.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow from Proposition 2.5 and 2.6.

(iv) $\Rightarrow$ (i): Suppose C is a finite nontrivial σ-regular conjugacy class of G. Let f be a function satisfying part (ii) of Lemma 2.4 and define the operator $T = \sum_{c \in C} f(c)\lambda(c)$. Then T belongs to the center of $C_r^*(G, \sigma)$. Indeed,

$$\begin{aligned}
\lambda(a)T\lambda(a)^* &= \sum_{c \in C} f(c)\lambda(a)\lambda(c)\lambda(a)^* \\
&= \sum_{c \in C} f(c)\sigma(a, c)\overline{\sigma(aca^{-1}, a)}\lambda(aca^{-1}) \\
&= \sum_{b \in aCa^{-1}} f(a^{-1}ba)\sigma(a, a^{-1}ba)\overline{\sigma(b, a)}\lambda(b) \\
&= \sum_{b \in C} f(a^{-1}ba)\overline{\sigma(a^{-1}, b)}\sigma(a^{-1}ba, a^{-1})\lambda(b) \\
&= \sum_{b \in C} f(b)\lambda(b) = T
\end{aligned}$$

for all $a \in G$, where the identity (6) is used to get the fourth equality.

The proof of the following corollary goes along the same lines as the one given in [18, Proposition 2.1] in the untwisted case.

COROLLARY 2.8. *If $C^*(G, \sigma)$ is prime, then (G, σ) satisfies condition K.*

PROOF. Observe that the set $\{\lambda(a)\}_{a \in G}$ is linear independent in $C_r^*(G, \sigma)$, and the universal property of $C^*(G, \sigma)$ ensures that there is a surjective *-homomorphism $C^*(G, \sigma) \to C_r^*(G, \sigma)$ mapping $i_G(a)$ to $\lambda(a)$. Hence, $\{i_G(a)\}_{a \in G}$ is also linear independent and has dense span in $C^*(G, \sigma)$.

Therefore, the result follows by replacing i_G with λ in the proof of Theorem 2.7, and repeating the argument for (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) word by word.

REMARK 2.9. In general, the center of $C^*(G, \sigma)$ is not easily determined.

However, a slightly stronger version of Corollary 2.8 is known in the untwisted case. If $C^*(G)$ has trivial center, then G/H is icc whenever H is a normal subgroup of G satisfying Kazhdan's property T (see e.g. [14]).

COROLLARY 2.10 ([22, Proposition 1.4]). *Assume G is countable and amenable. Then the following conditions are equivalent:*

(i) (G, σ) *satisfies condition K.*

(ii) $C^*(G, \sigma)$ *is primitive.*

PROOF. If (G, σ) satisfies condition K, then $C_r^*(G, \sigma)$ is prime by Theorem 2.7. As G is countable, $C_r^*(G, \sigma)$ is separable and hence primitive by Dixmier's result. Now, the amenability of G implies that $C^*(G, \sigma) \cong C_r^*(G, \sigma)$, so $C^*(G, \sigma)$ is also primitive. Finally, (ii) always implies (i) by Corollary 2.8.

REMARK 2.11. Condition K on (G, σ) does not imply primeness or primitivity of $C^*(G, \sigma)$ in general. To see this, let $G = \mathrm{SL}(3, \mathsf{Z})$ and $\sigma = 1$. Then, G is countable, icc and satisfies Kazhdan's property T. In particular, G is nonamenable. As explained in [4, Proposition 2.5], $C^*(G)$ is not primitive.

On the other hand, I don't know any example of an uncountable and amenable group such that (i) holds, but not (ii).

REMARK 2.12. If G is countable and nilpotent, then condition K on (G, σ) is actually equivalent to simplicity of $C^*(G, \sigma)$ [22, Proposition 1.7]. The same is also true if G is finite.

However, this does not hold for all countable, amenable groups. For example, if G is the group of all finite permutations on a countably infinite set, then G is countable, amenable and icc, so $C^*(G)$ is primitive and nonsimple.

REMARK 2.13. Note that $C_r^*(\mathrm{SL}(3, \mathsf{Z}))$ is known to be simple [5], so Remark 2.11 and 2.12 show that primitivity of a full twisted group C^*-algebra is in general unrelated to simplicity of the corresponding reduced twisted group C^*-algebra.

PROPOSITION 2.14. *The following conditions are equivalent:*

(i) G *is amenable.*

(ii) $C^*(G, \sigma)$ *is nuclear.*

(iii) $C_r^*(G, \sigma)$ *is nuclear.*

(iv) $W^*(G, \sigma)$ *is injective.*

PROOF. This is well known in the untwisted case. The result in the twisted case appeared in a preprint by Bédos and Conti [2], but was left out in the final version. For the convenience of the reader we repeat the argument. First, (i) $\Rightarrow$ (ii) follows from [23, Corollary 3.9]. The implication (ii) $\Rightarrow$ (iii) holds since every quotient of a nuclear C^*-algebra is nuclear. Moreover, the von Neumann algebra generated by a nuclear C^*-algebra is injective, hence (iii) $\Rightarrow$ (iv). Finally, if $W^*(G, \sigma)$ is injective, it has a hypertrace and thus G is amenable by [1, Corollary 1.7], so (iv) $\Rightarrow$ (i).

According to [8], all injective II_1 factors acting on a separable Hilbert space are isomorphic to the hyperfinite II_1 factor. Hence, we get the following corollary.

COROLLARY 2.15. *Assume G is countably infinite. Then the following conditions are equivalent:*

(i) *G is amenable and (G, σ) satisfies condition K.*
(ii) *$C^*(G, \sigma)$ is nuclear and primitive.*
(iii) *$C_r^*(G, \sigma)$ is nuclear and primitive.*
(iv) *$W^*(G, \sigma)$ is the hyperfinite II_1 factor.*

3. Direct products

Let G_1 and G_2 be two groups. A function $f : G_1 \times G_2 \to \mathsf{T}$ is called a *bihomomorphism* if

$$f(a_1b_1, a_2) = f(a_1, a_2)f(b_1, a_2) \quad \text{and} \quad f(a_1, a_2b_2) = f(a_1, a_2)f(a_1, b_2)$$

for all $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$. Let $B(G_1, G_2)$ denote the set of bihomomorphisms $G_1 \times G_2 \to \mathsf{T}$. This is a group under pointwise multiplication and is isomorphic with $\mathrm{Hom}(G_1, \mathrm{Hom}(G_2, \mathsf{T}))$.

It is well known (see e.g. [15]) that the Schur multiplier of $G_1 \times G_2$ decomposes as

$$\mathscr{M}(G_1 \times G_2) \cong \mathscr{M}(G_1) \oplus \mathscr{M}(G_2) \oplus B(G_1, G_2).$$

We will only need to know the following. Let (σ_1, σ_2, f) be a triple where σ_1 and σ_2 are multipliers on G_1 and G_2, respectively, and f belongs to $B(G_1, G_2)$. Then we can define a multiplier σ on $G_1 \times G_2$ by

$$\sigma((a_1, a_2), (b_1, b_2)) = \sigma_1(a_1, b_1)\sigma_2(a_2, b_2)f(b_1, a_2) \tag{7}$$

for $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$, and it can be shown that every multiplier on $G_1 \times G_2$ is similar to such a σ. When σ is a multiplier on $G_1 \times G_2$, we let σ_1 be the multiplier on G_1 defined by

$$\sigma_1(a_1, b_1) = \sigma((a_1, e), (b_1, e))$$

for $a_1, b_1 \in G_1$ and call it the restriction of σ to G_1. Similarly we can define the restriction σ_2 of σ to G_2.

Henceforth, we fix two groups G_1 and G_2, multipliers σ_1 on G_1 and σ_2 on G_2, and a bihomomorphism f in $B(G_1, G_2)$. We set $G = G_1 \times G_2$ and let σ be the multiplier on G defined by (7). Moreover, we write $\sigma = \sigma_1 \times \sigma_2$ if $f = 1$.

It is convenient to record the identity

$$\begin{aligned}(8)\quad \sigma(a,b)\overline{\sigma(b,a)} \cdot f(a_1,b_2)\overline{f(b_1,a_2)}& \\ = \sigma_1(a_1,b_1)&\overline{\sigma_1(b_1,a_1)} \cdot \sigma_2(a_2,b_2)\overline{\sigma_2(b_2,a_2)}\end{aligned}$$

which follows directly from (7). Note also that C is a conjugacy class of G if and only if $C = C_1 \times C_2$ where C_1 and C_2 are conjugacy classes of G_1 and G_2, respectively.

PROPOSITION 3.1. *The following are equivalent:*

(i) $C_r^*(G, \sigma)$ *is prime.*

(ii) *For every finite nontrivial conjugacy class C of G, there exist $a = (a_1, a_2)$ in C and $b = (b_1, b_2)$ in G such that at least one of these conditions hold:*

1. $a_1 b_1 = b_1 a_1$ *and* $f(b_1, a_2) \neq \overline{\sigma_1(a_1, b_1)}\sigma_1(b_1, a_1)$.
2. $a_2 b_2 = b_2 a_2$ *and* $f(a_1, b_2) \neq \sigma_2(a_2, b_2)\overline{\sigma_2(b_2, a_2)}$.

PROOF. Suppose that condition (ii) does not hold. Then there is a finite nontrivial conjugacy class C such that both 1. and 2. fail for all a in C and b in G. Hence, $f(b_1, a_2) = \overline{\sigma_1(a_1, b_1)}\sigma_1(b_1, a_1)$ and $f(a_1, b_2) = \sigma_2(a_2, b_2)\overline{\sigma_2(b_2, a_2)}$ whenever $a = (a_1, a_2)$ is in C, $b = (b_1, b_2)$ in G and b commutes with a. Then C is σ-regular by (8), and therefore (G, σ) does not satisfy condition K, that is, $C_r^*(G, \sigma)$ is not prime by Theorem 2.7. Thus, (i) $\Rightarrow$ (ii).

Conversely, assume that (G, σ) does not satisfy condition K and let $C = C_1 \times C_2$ be a finite nontrivial σ-regular conjugacy class of G. If b_1 in G_1 commutes with a_1 in C_1, then (b_1, e) commutes with (a_1, a_2) for every a_2 in C_2. Hence, the σ-regularity of C and identity (8) give that

$$f(b_1, a_2) = \overline{\sigma_1(a_1, b_1)}\sigma_1(b_1, a_1)$$

whenever a belongs to C and b_1 in G_1 commutes with a_1. Similarly,

$$f(a_1, b_2) = \sigma_2(a_2, b_2)\overline{\sigma_2(b_2, a_2)}$$

whenever b_2 in G_2 commutes with a_2. It follows that for all a in C and b in G, both 1. and 2. fail to hold, hence condition (ii) is not satisfied.

REMARK 3.2. Let G_1 and G_2 be abelian and assume that σ_1 and σ_2 are trivial. Condition (ii) of Proposition 3.1 then says that for all nontrivial (a_1, a_2) in G there exists (b_1, b_2) in G such that $f(a_1, b_2) \neq 1$ or $f(b_1, a_2) \neq 1$. If this holds, σ is called nondegenerate and it was first shown by Slawny [24, Theorem 3.7] that $C^*(G, \sigma) \cong C_r^*(G, \sigma)$ is simple in this case.

LEMMA 3.3. *Let $a = (a_1, a_2)$ be an element in G. If two of the following conditions hold, then all three hold:*

(i) *a is σ-regular.*

(ii) *a_i is σ_i-regular for both $i = 1$ and 2.*

(iii) *$f(a_1, b_2) = f(b_1, a_2)$ whenever $b = (b_1, b_2)$ commutes with a.*

Moreover, (iii) *is equivalent to:*

(iv) *$f(a_1, b_2) = f(b_1, a_2) = 1$ whenever $b = (b_1, b_2)$ commutes with a.*

PROOF. Suppose that (ii) holds and pick any $b = (b_1, b_2)$ in G. Then it follows readily from (8) that (i) holds if and only if (iii) holds.

Next, assume that (iii) holds and let $b = (b_1, b_2)$ commute with a. Then $b' = (b_1, e)$ also commutes with a, so $1 = f(a_1, e) = f(b_1, a_2)$. Similarly, we get $f(a_1, b_2) = 1$ and thus (iv) holds.

Suppose finally that (i) and (iii) hold and pick an element $b = (b_1, b_2)$ in G that commutes with a. As (iv) also holds, we have that $f(b_1, a_2) = 1$. By applying (8) with $b' = (b_1, e)$, we see that a_1 is σ_1-regular. Similarly, $f(a_1, b_2) = 1$ and a_2 is σ_2-regular.

COROLLARY 3.4. *Let $C = C_1 \times C_2$ be a conjugacy class of G. Suppose there is some $a = (a_1, a_2)$ in C such that $f(a_1, b_2) = f(b_1, a_2)$ whenever $b = (b_1, b_2)$ commutes with a. Then the following are equivalent:*

(i) *C is a finite nontrivial σ-regular conjugacy class of G.*

(ii) *C_i is a finite σ_i-regular conjugacy class of G_i for both $i = 1$ and 2 and at least one of C_1 and C_2 is nontrivial.*

COROLLARY 3.5. *Suppose both $C_r^*(G_1, \sigma_1)$ and $C_r^*(G_2, \sigma_2)$ are prime. Let $a = (a_1, a_2)$ be such that $f(a_1, b_2) = f(b_1, a_2)$ whenever $b = (b_1, b_2)$ commutes with a. Then at most one of the following two conditions hold:*

(i) *a is σ-regular.*

(ii) *a belongs to a finite nontrivial conjugacy class of G.*

COROLLARY 3.6. *Suppose $f(a_1, b_2) = f(b_1, a_2)$ whenever $a = (a_1, a_2)$ is σ-regular and $b = (b_1, b_2)$ commutes with a. Then $C_r^*(G, \sigma)$ is prime if both $C_r^*(G_1, \sigma_1)$ and $C_r^*(G_2, \sigma_2)$ are prime.*

REMARK 3.7. In general, primeness of $C_r^*(G,\sigma)$ does not imply primeness of either $C_r^*(G_1,\sigma_1)$ or $C_r^*(G_2,\sigma_2)$. For example, if $G_1 = G_2 = \mathsf{Z}$, then $C^*(G,\sigma)$ can be simple even if both σ_1 and σ_2 are trivial.

Also, $C_r^*(G,\sigma)$ can be nonprime even if both $C^*(G_1,\sigma_1)$ and $C^*(G_2,\sigma_2)$ are simple. To see this, let $G_1 = G_2 = \mathsf{Z}^2$ and consider the case described in the last part of Example 1.2.

PROPOSITION 3.8. *Suppose $f(c_1,c_2) = 1$ whenever c_i belongs to a finite conjugacy class of G_i for either $i = 1$ or 2. Then $C_r^*(G,\sigma)$ is prime if and only if both $C_r^*(G_1,\sigma_1)$ and $C_r^*(G_2,\sigma_2)$ are prime.*

In particular, this holds when $\sigma = \sigma_1 \times \sigma_2$.

PROOF. Suppose $C_r^*(G,\sigma)$ is prime and C_1 is a finite σ_1-regular conjugacy class of G_1. Then $C_1 \times \{e\}$ is σ-regular by Corollary 3.4 so $C_1 = \{e\}$ and hence $C_r^*(G_1,\sigma_1)$ is prime. Similarly we get that $C_r^*(G_2,\sigma_2)$ is prime.

The converse follows directly from Corollary 3.5.

REMARK 3.9. Assume that $\sigma = \sigma_1 \times \sigma_2$. Then $C_r^*(G,\sigma)$ is simple if and only both $C_r^*(G_1,\sigma_1)$ and $C_r^*(G_2,\sigma_2)$ are simple. Indeed, note that the map $\lambda_\sigma(a_1,a_2) \mapsto \lambda_{\sigma_1}(a_1) \otimes \lambda_{\sigma_2}(a_2)$ induces an isomorphism

$$C_r^*(G,\sigma) \cong C_r^*(G_1,\sigma_1) \otimes_{\min} C_r^*(G_2,\sigma_2).$$

The result now follows from the fact that a spatial tensor product of two C^*-algebras is simple if and only if both involved C^*-algebras are simple (see [12, 11.5.5-6]).

The only positive result on primitivity so far in this paper concerns countable, amenable groups. However, Corollary 2.10 relies on Dixmier's result that is not constructive in the sense that it does not give a procedure to construct an explicit faithful irreducible representation.

In some cases, one may construct faithful irreducible representations of $C^*(G,\sigma)$ through an inducing process on representations of $C^*(G_1,\sigma_1)$.

THEOREM 3.10. *Assume that G_2 is amenable. Suppose there exists a faithful irreducible representation π of $C^*(G_1,\sigma_1)$ such that for any given nontrivial a_2 in G_2, there exists a_1 in G_1 such that*

$$f(a_1,a_2)\pi(i_{G_1}(a_1)) \not\simeq \pi(i_{G_1}(a_1)).$$

Then $C^*(G,\sigma)$ is primitive.

PROOF. Recall that there is a twisted action (α,ω) of G_2 on $A = C^*(G_1,\sigma_1)$

satisfying (see e.g. [25])

$$\alpha_{a_2}(i_{G_1}(a_1)) = \overline{f(a_1, a_2)} i_{G_1}(a_1),$$
$$\omega(a_2, b_2) = \sigma_2(a_2, b_2).$$

Hence, there is also a natural action of G_2 on the set $\hat{A}^0$ of equivalence classes of faithful irreducible representations of A given by

$$a_2 \cdot [\psi] = [\psi \circ \alpha_{a_2^{-1}}].$$

For any given nontrivial a_2 in G_2, the assumptions on π gives that

$$\pi \circ \alpha_{a_2^{-1}}(i_{G_1}(a_1)) = f(a_1, a_2)\pi(i_{G_1}(a_1)) \not\simeq \pi(i_{G_1}(a_1))$$

for some a_1 in G_1. Hence

$$a_2 \cdot [\pi] \neq [\pi]$$

for all nontrivial a_2 in G_2. In other words, $[\pi]$ is a free point for this action. The conclusion follows from [4, Theorem 2.1].

EXAMPLE 3.11. Let $G = \mathsf{F}_2 \times \mathsf{Z}$ and let u, v be the generators of F_2. Since $\mathscr{M}(\mathsf{F}_2) = \mathscr{M}(\mathsf{Z}) = \{1\}$, every multiplier on G is, up to similarity, determined by a bihomomorphism $f : \mathsf{F}_2 \times \mathsf{Z} \to \mathsf{T}$. Moreover, f is determined by its values on the generators, that is, by $f(u, 1)$ and $f(v, 1)$. Let σ be the multiplier on G defined by these two numbers, say μ and ν. We remark that

$$C^*(G, \sigma) \cong C^*(\mathsf{F}_2) \rtimes_\alpha \mathsf{Z}$$

where α is determined by $\alpha_k(i_{\mathsf{F}_2}(x)) = \overline{f(x, k)} i_{\mathsf{F}_2}(x)$ for $x \in \mathsf{F}_2$ and $k \in \mathsf{Z}$.

Assume μ is nontorsion and let $A = C^*(\mathsf{F}_2)$ sit inside $B(\mathscr{H})$ for some separable Hilbert space $\mathscr{H}$. Let $U = i_{\mathsf{F}_2}(u)$ and $V = i_{\mathsf{F}_2}(v)$ be the two unitaries in $B(\mathscr{H})$ generating A. Now, proceeding as Choi in [7, Lemma 4], there is an operator D for which $U - D$ is compact and such that the following hold; with respect to a suitable basis on $\mathscr{H}$, D is diagonal with diagonal entries $\{z_i\}_{i=1}^\infty$ satisfying $|z_i| = 1$ for all i, $z_1 = 1$, $z_i \neq z_j$ if $i \neq j$ and $z_i \notin \{\mu^k : k \in \mathsf{Z}\}$ when $i \geq 2$.

Using [7, Lemma 5], we can find a compact perturbation E of V which is a unitary operator having no common nontrivial invariant subspace with D. Then, as explained in [7, Theorem 6], the map $U \mapsto D$, $V \mapsto E$ defines a faithful and irreducible representation π of A on $\mathscr{H}$.

Now we have

$$\pi \circ \alpha_{k^{-1}}(U) = f(u, k)\pi(U) = \mu^k \pi(U) \not\simeq \pi(U)$$

for all k in Z. Indeed, this holds as the point spectrum of $\pi(U) = D$ is different from the point spectrum of $\pi(\alpha_{k^{-1}}(U)) = \mu^k D$ by construction.

A similar argument also holds if ν is nontorsion. Hence, we get from Theorem 3.10 that $C^*(G, \sigma)$ is primitive if either μ or ν is nontorsion.

On the other hand, if (G, σ) satisfies condition K, then at least one of μ and ν must be nontorsion, so this is also a necessary condition for primitivity of $C^*(G, \sigma)$. Indeed, condition (ii) of Proposition 3.1 does not hold if both μ and ν are torsion.

PROPOSITION 3.12. *Assume that $\sigma = \sigma_1 \times \sigma_2$ and that both $C^*(G_1, \sigma_1)$ and $C^*(G_2, \sigma_2)$ are primitive. Then $C^*(G, \sigma)$ is primitive if at least one of G_1 and G_2 is amenable.*

PROOF. Without loss of generality we may assume that G_1 is amenable. Then $C^*(G_1, \sigma_1)$ is nuclear by Proposition 2.14 so the minimal and maximal tensor products of $C^*(G_1, \sigma_1)$ and $C^*(G_2, \sigma_2)$ coincide. According to [11, Section 3], there is a unique isomorphism

$$C^*(G, \sigma) \to C^*(G_1, \sigma_1) \otimes C^*(G_2, \sigma_2)$$

given by $i_G(a_1, a_2) \mapsto i_{G_1}(a_1) \otimes i_{G_2}(a_2)$.

For $i = 1, 2$, let π_i be a faithful irreducible representation of $C^*(G_i, \sigma_i)$ on $\mathscr{H}_i$. Then $\pi = \pi_1 \otimes \pi_2$ is a representation of $C^*(G, \sigma)$ on $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, which is faithful by [17, Theorem 6.5.1] and irreducible by [11, Section 2]. Hence $C^*(G, \sigma)$ is primitive.

REMARK 3.13. Primitivity of $C^*(G, \sigma)$ is in general difficult to decide. For example, let F be a free nonabelian group. Then it is unknown whether $C^*(\mathsf{F} \times \mathsf{F})$ is primitive (see [4, Remark 2.2] for a brief discussion).

4. Free products

In some sense, free products are easier to treat than direct products, since the Schur multiplier decomposes nicely. Indeed, let G_1 and G_2 be two groups. Then we have that (see e.g. [6, page 51])

$$\mathscr{M}(G_1 * G_2) \cong \mathscr{M}(G_1) \oplus \mathscr{M}(G_2). \tag{9}$$

Let σ_1 be a normalized multiplier on G_1 and σ_2 a normalized multiplier on G_2. Following [16, Section 5], we will explain how to obtain a normalized free product multiplier $\sigma_1 * \sigma_2$ on $G_1 * G_2$.

Every nontrivial element x in $G_1 * G_2$ can be uniquely written as a reduced word $x = x_1 x_2 \ldots x_n$, for which the letters with odd index belong to G_i and the letters with even index belong to G_j for $i \neq j$. Define the length function as $l(x) = l(x_1x_2\dots x_n) = n$ and $l(e) = 0$. If $l(x), l(y) \le 1$, we write $x \perp y$ if either $x = e$ or $y = e$ or else if x is in G_i and y is in G_j for $i \neq j$.

Let $s(x)$ and $r(x)$ denote the first and last letter of a nontrivial word x and set $s(e) = r(e) = e$. For a pair of words (x, y), we say that the pair is reduced if $r(x) \neq s(y)^{-1}$.

When (x, y) is not reduced, let w be the longest word such that $r(xw^{-1}) \perp s(w)$ and $r(w^{-1}) \perp s(wy)$. Set $x_w = xw^{-1}$ and $y_w = wy$, so that $x = x_w w$ and $y = w^{-1}y_w$. Let $(x, y)_w = (x_w, y_w)$ be the reduction of (x, y) and note in particular that $x_w y_w = xy$.

If the pair (x, y) is reduced, then we set $(x, y)_w = (x, y)$.

Define now the multiplier τ on $G_1 * G_2$ by

$$\tau(x, y) = \tau((x, y)_w) = \begin{cases} \sigma_1(r(x_w), s(y_w)) & \text{if } r(x_w), s(y_w) \in G_1 \setminus \{e\}, \\ \sigma_2(r(x_w), s(y_w)) & \text{if } r(x_w), s(y_w) \in G_2 \setminus \{e\}, \\ 1 & \text{if } r(x_w) \perp s(y_w), \end{cases}$$

and note that this definition coincides with the one explained in [16, Section 5]. Furthermore, let

$$X = \big\{[a, b] = aba^{-1}b^{-1} : a \in G_1 \setminus \{e\}, b \in G_2 \setminus \{e\}\big\}$$

and recall that the free nonabelian group on X, denoted F_X, may be identified with the normal subgroup of $G_1 * G_2$ generated by X.

Moreover, define a function $\beta : G_1 * G_2 \to \mathsf{T}$ by $\beta(x) = 1$ if $x \notin \mathsf{F}_X$, while for nontrivial $x = q_1^{p_1} \dots q_n^{p_n}$ in F_X, where q_i belongs to X and p_i is an integer, we set

$$\begin{aligned} \beta(x) &= \beta(q_1^{p_1}\dots q_n^{p_n}) \\ &= \begin{cases} \tau(q_1^{p_1}, q_2^{p_2})\tau(q_2^{p_2}, q_3^{p_3}) \dots \tau(q_{n-1}^{p_{n-1}}, q_n^{p_n}) & \text{if } n \ge 2, \\ 1 & \text{if } n = 1. \end{cases} \end{aligned}$$

Now define the multiplier σ on $G_1 * G_2$ by

$$\sigma(x, y) = \beta(x)\beta(y)\overline{\beta(xy)}\tau(x, y).$$

We write $\sigma = \sigma_1 * \sigma_2$ and note that $\sigma \sim \tau$, $\sigma_{|G_i \times G_i} = \sigma_i$ and $\sigma_{|\mathsf{F}_X \times \mathsf{F}_X} = 1$.

On the other hand, if σ is a normalized multiplier on $G_1 * G_2$, we can define the restriction σ_1 on G_1 by

$$\sigma_1(x, y) = \begin{cases} \sigma(x, y) & \text{if } x, y \in G_1 \setminus \{e\}, \\ 1 & \text{if } x \text{ or } y = e. \end{cases}$$

Similarly, we can define the restriction σ_2 of σ to G_2. Next, define the function $\beta : G_1 * G_2 \to \mathsf{T}$ by $\beta(x) = 1$ if $l(x) \leq 1$ and else

$$\beta(x) = \beta(x_1 \dots x_n) = \sigma(x_1, x_2)\sigma(x_1x_2, x_3) \dots \sigma(x_1 \dots x_{n-1}, x_n).$$

Then σ is similar to $\sigma_1 * \sigma_2$ through β.

Remark that every multiplier is similar to a normalized one. Therefore, every multiplier on $G_1 * G_2$ is similar to $\sigma_1 * \sigma_2$ for some normalized multipliers σ_1 on G_1 and σ_2 on G_2.

We are now ready to prove the twisted version of [3, Theorem 1.2].

THEOREM 4.1. *Assume $G = G_1 * G_2$, where G_1 and G_2 are countable and amenable and $(|G_1| - 1)(|G_2| - 1) \geq 2$, and let σ be a multiplier on G. Then $C^*(G, \sigma)$ is primitive.*

PROOF. We may assume that $\sigma = \sigma_1 * \sigma_2$ where σ_1 and σ_2 are normalized multipliers on G_1 and G_2, respectively, and that $\sigma_{|\mathsf{F}_X \times \mathsf{F}_X} = 1$. The proof is only a slight modification of the proof of [3, Theorem 1.2], so we just point out what needs to be adjusted in this proof and use the notation therein. First, recall that there is a twisted action (α, ω) of $(G_1 * G_2)/\mathsf{F}_X \cong G_1 \times G_2$ on $H = \mathsf{F}_X$. Straightforward calculations give that

$$\alpha_{(c,d)}(i_H([a, b])) = \begin{cases} i_H(cd[a, b]d^{-1}c^{-1}) \cdot \sigma_2(d, b) & \text{if } d \neq e \\ i_H(cd[a, b]d^{-1}c^{-1}) \cdot \sigma_1(c, a) & \text{if } d = e \end{cases}$$

for $a, c \in G_1$ and $b, d \in G_2$. Hence the expressions in the equations [3, (2.3), (2.4)] remain unchanged, so it is enough to reconsider [3, Case 3]. More straightforward calculations give that the conditions at the bottom of [3, page 54] must be replaced with:

$k = (s_0, t)$ and $k = (s_l, e_2)$
  if $\lambda(s_0s_l, t)U(s_0s_l, t) \not\simeq \sigma_1(s_l, s_0s_l)U(s_0, t)(\lambda(s_l, t)U(s_l, t))^*$;
$k = (s_0, e_2)$ and $k = (s_l, t)$
  if $\lambda(s_0s_l, t)U(s_0s_l, t) \not\simeq \sigma_1(s_0, s_0s_l)\lambda(s_l, t)U(s_l, t)U(s_0, t)^*$;
$k = (s_0, t)$ and $k = (s_0s_l, e_2)$
  if $\lambda(s_l, t)U(s_l, t) \not\simeq \sigma_1(s_0s_l, s_l)U(s_0, t)(\lambda(s_0s_l, t)U(s_0s_l, t))^*$;
$k = (s_0s_l, t)$ and $k = (s_0, e_2)$
  if $\lambda(s_l, t)U(s_l, t) \not\simeq \sigma_1(s_0, s_l)\lambda(s_0s_l, t)U(s_0s_l, t)U(s_0, t)^*$.

Now it is easily seen that the rest of the proof works with appropriate modifications.

REMARK 4.2. Theorem 4.1 is not surprising. In fact, I am not aware of any pair (G,σ) such that $C^*(G)$ is primitive, but $C^*(G,\sigma)$ is nonprimitive.

REMARK 4.3. Let $G = G_1 * G_2$, let σ be a multiplier on G and assume $\sigma = \sigma_1 * \sigma_2$. Then it is known that (see [16, Section 5])

$$C^*(G,\sigma) = C^*(G_1,\sigma_1) * C^*(G_2,\sigma_2).$$

EXAMPLE 4.4. As explained in Example 1.1 we have that for each natural number n, there exists a multiplier σ_k on $\mathsf{Z}_n \times \mathsf{Z}_n$ such that $C^*(\mathsf{Z}_n \times \mathsf{Z}_n, \sigma_k) \cong M_n(\mathsf{C})$. One immediate consequence of Theorem 4.1 is that

$$M_j(\mathsf{C}) * M_k(\mathsf{C})$$

is primitive for all $j, k \geq 2$. More generally, it has recently been shown [10] that $F_1 * F_2$ is primitive whenever F_1 and F_2 are finite-dimensional C^*-algebras and $(\dim F_1 - 1)(\dim F_2 - 1) \geq 2$.

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