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ON THE CATEGORY OF WEAKLY LASKERIAN COFINITE MODULES

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Abstract

Let R denote a commutative Noetherian (not necessarily local) ring and I be an ideal of R. The main purpose of this note is to show that the category $\mathscr{WL}(R, I)_{cof}$ of I-cofinite weakly Laskerian R-modules forms an Abelian subcategory of the category of all R-modules.

1. Introduction

Let R denote a commutative Noetherian ring and I be an ideal of R. In [5], Hartshorne defined an R-module L to be *I-cofinite*, if $\mathrm{Supp}(L) \subseteq V(I)$ and the R-modules $\mathrm{Ext}^i_R(R/I, L)$ are finitely generated for all i. Then he posed the following question:

> *Is the category $\mathscr{M}(R, I)_{cof}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-homomorphism of I-cofinite modules, are the R-modules* $\ker f$ *and* $\mathrm{coker}\, f$ *I-cofinite?*

Hartshorne in [5] constructed a counterexample to show that this question under the assumption $\dim(R/I) = 2$, has not an affirmative answer in general. On the positive side, Hartshorne proved that if I is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is yes. On the other hand, in [3], Delfino and Marley extended this result to arbitrary complete local rings. Recently, Kawasaki [6] generalized the Delfino and Marley's result for an arbitrary ideal I of dimension one in a local ring R. Finally, more recently, Bahmanpour, Naghipour and Sedghi in [1] completely have removed the local assumption on R.

Recall that, an R-module M is called *weakly Laskerian* if $\mathrm{Ass}(M/N)$ is a finite set for each proper submodule N of M. The class of weakly Laskerian modules introduced in [4], by Divaani-Aazar and Mafi. Recently, Hung Quy [8], introduced the class of FSF modules, modules containing a finitely generated submodule such that the support of the quotient module is finite.

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The main purpose of this paper is to show that for each ideal I of a Noetherian ring R, the category of I-cofinite weakly Laskerian modules forms an Abelian subcategory of the category of all R-modules. In the proof of this result, first we need to prove that over a Noetherian ring R, an R-module M is weakly Laskerian if and only if it is FSF.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. We denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. For any unexplained notation and terminology we refer the reader to [2] and [7].

2. Preliminaries

In this section we bring some technical results, which will be used later. The main goals of this section are Theorem 2.2 and its consequence, Corollary 2.4.

LEMMA 2.1. *Let M be a module over the Noetherian ring R. If $\mathfrak{q} = 0 : x$ is an associated prime ideal of M and $\mathfrak{p}$ is a prime ideal, such that $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{p}x : x = \mathfrak{p}$. Furthermore if $\mathfrak{p}_0 = 0 : x_0$ is another prime ideal associated to M and such that $\mathfrak{p}_0 \nsubseteq \mathfrak{q}$, then $\mathfrak{p}x : x_0 = \mathfrak{p}_0$.*

PROOF. The first statement follows from the easily shown fact that for any $x \in M$ and any ideal $\mathfrak{a}$, if $0 : x \subseteq \mathfrak{a}$, then $\mathfrak{a}x : x = \mathfrak{a}$.

The second statement follows from the stronger assertion that $Rx : x_0 = \mathfrak{p}_0$. To show the nonobvious inclusion $Rx : x_0 \subseteq \mathfrak{p}_0$, let $a \in (Rx : x_0)$, i.e. $ax_0 = bx$ for some $b \in R$. By hypothesis we can take $c \in \mathfrak{p}_0 \backslash \mathfrak{q}$. Then $0 = cax_0 = cbx$, i.e. $cb \in \mathfrak{q}$ and since $c \notin \mathfrak{q}$ we get $b \in \mathfrak{q}$. Hence $bx = 0$ and so $ax_0 = 0$, i.e. $a \in \mathfrak{p}_0$.

Now we are prepared to prove the main result of this section.

THEOREM 2.2. *Let M be a module over the Noetherian ring R and W a set of prime ideals, pairwise incomparable under inclusion. Assume that $W \cap \operatorname{Supp}(M/N)$ is an infinite set for every finitely generated submodule N of M. Then there is a submodule L of M such that $W \cap \operatorname{Ass}(M/L)$ is an infinite set.*

PROOF. We construct inductively an increasing sequence

$$\{0\} = T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$$

of finitely generated submodules of M, distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ from W, (not necessarily distinct) prime ideals $\mathfrak{q}_1, \mathfrak{q}_2, \ldots$ of R, elements $x_1, x_2, \ldots$, such that for $n = 1, 2, \ldots$ we have $\mathfrak{q}_n \subseteq \mathfrak{p}_n$, $\mathfrak{q}_n = T_{n-1} : x_n$ and $\mathfrak{p}_i = T_n : x_i$ when $1 \leq i \leq n$.

The construction is performed as follows:

Choose $\mathfrak{p}_n$ from $W \cap \mathrm{Supp}(M/T_{n-1})$ distinct from $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n-1}$. Choose $\mathfrak{q}_n \in \mathrm{Ass}(M/T_{n-1})$ with $\mathfrak{q}_n \subseteq \mathfrak{p}_n$ and let $x_n \in M$ be taken such that $\mathfrak{q}_n = T_{n-1} : x_n$. Put $T_n = T_{n-1} + \mathfrak{p}_n x_n$. Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are incomparable $\mathfrak{p}_i \nsubseteq \mathfrak{q}_n$ when $1 \leq i \leq n-1$ and by the inductive procedure $\mathfrak{p}_i = T_{n-1} : x_i$ when $1 \leq i \leq n-1$. Therefore the Lemma 2.1 can be applied to conclude that $T_n : x_i = \mathfrak{p}_i$ when $1 \leq i \leq n$.

Let $L = \cup_1^\infty T_n$. Then $L : x_i = \cup_1^\infty T_n : x_i = \mathfrak{p}_i$ for all i. Hence $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ are infinitely many associated prime ideals of M/L in the set W.

The following result, which is an immediately consequence of Theorem 2.2, is needed in the next section. But, first we need the following definition.

DEFINITION 2.3. Let R be a ring an k a nonnegative integer. We define

$$A(R, k) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim(R/\mathfrak{p}) = k\},$$

where $\mathrm{Spec}(R)$ denotes the prime spectrum of R.

COROLLARY 2.4. *Let R be a Noetherian ring, k a nonnegative integer and M be an R-module such that for all finitely generated submodule N of M the intersection* $\mathrm{Supp}(M/N) \cap A(R, k)$ *is not a finite set. Then there exists a submodule L of M such that the intersection* $\mathrm{Ass}(M/L) \cap A(R, k)$ *is an infinite set.*

PROOF. Since $A(R, k)$ is a set of prime ideals, pairwise incomparable under inclusion, the assertion follows from Theorem 2.2.

3. The structure of weakly Laskerian modules over a Noetherian ring

In this section we prove a result concerning the structure of weakly Laskerian modules over a Noetherian ring, which is needed in the section 4.

DEFINITION 3.1 (See [4, Definition 2.1]). An R-module M is said to be *weakly Laskerian*, if $\mathrm{Ass}(M/N)$ is finite, for each proper submodule N of M.

DEFINITION 3.2 (See [8, Definition 2.1]). An R-module M is said to be FSF, if there exists a finitely generated submodule N of M, such that the support of the quotient module M/N is a finite set.

We are now ready to state and prove the main result of this section.

THEOREM 3.3. *Let R be a Noetherian ring and M a nonzero R-module. Then the following statements are equivalent:*

(1) *M is a weakly Laskerian module;*

(2) *M is an* FSF *module.*

PROOF. (1) ⇒ (2) By Corollary 2.4, there exist finitely generated submodules N_0 and N_1 of M such that the intersections

$$\operatorname{Supp}(M/N_0) \cap A(R, 0) \qquad \text{and} \qquad \operatorname{Supp}(M/N_1) \cap A(R, 1),$$

are finite sets. Put $N := N_0 + N_1$. It is enough to prove that

$$\operatorname{Supp}(M/N) \subseteq (\operatorname{Supp}(M/N_0) \cap A(R, 0)) \bigcup (\operatorname{Supp}(M/N_1) \cap A(R, 1)).$$

Suppose the contrary and let $\mathfrak{p}$ be a prime ideal maximal with respect to the property that

$$\mathfrak{p} \in \operatorname{Supp}(M/N) \setminus [(\operatorname{Supp}(M/N_0) \cap A(R, 0)) \bigcup (\operatorname{Supp}(M/N_1) \cap A(R, 1))].$$

In view of

$$\operatorname{Supp}(M/N) \subseteq \operatorname{Supp}(M/N_0) \cap \operatorname{Supp}(M/N_1),$$

we conclude that $\dim(R/\mathfrak{p}) > 1$ and so $V(\mathfrak{p})$ is not finite. For each prime ideal $\mathfrak{Q}$ which properly contains $\mathfrak{p}$, the maximal property of $\mathfrak{p}$, implies that

$$\mathfrak{Q} \in [(\operatorname{Supp}(M/N_0) \cap A(R, 0)) \bigcup (\operatorname{Supp}(M/N_1) \cap A(R, 1))],$$

which contradicts the fact that the union

$$(\operatorname{Supp}(M/N_0) \cap A(R, 0)) \bigcup (\operatorname{Supp}(M/N_1) \cap A(R, 1)),$$

is a finite set.

(2) ⇒ (1) There exists a finitely generated submodule N of M such that $\operatorname{Supp}(M/N)$ is a finite set. Let L be an arbitrary submodule of M. From the short exact sequence

$$0 \to N/N \cap L \to M/L \to M/L + N \to 0,$$

we have

$$\operatorname{Ass}(M/L) \subseteq \operatorname{Ass}(N/N \cap L) \bigcup \operatorname{Ass}(M/N + L).$$

This means that $\operatorname{Ass}(M/L)$ is a finite set and we are done.

The next couple of corollaries show that the property of being weakly Laskerian ascends by tensoring under the completion and descends under a finite integral extension of semi-local rings.

COROLLARY 3.4. *Let R be a Noetherian semi-local ring, $\mathfrak{m}$ the Jacobson radical of R and M be a weakly Laskerian module. Then $M \otimes_R R^*$ is a weakly Laskerian R^*-module, where R^* denotes the $\mathfrak{m}$-adic completion of R.*

PROOF. By Theorem 3.3, there exists a finitely generated submodule N of M such that $\operatorname{Supp}_R(M/N)$ is finite and so $\dim_R(M/N) \leq 1$. Set $J := \bigcap_{\mathfrak{p}\in \operatorname{Supp}(M/N)} \mathfrak{p}$. Then $\dim(R/J) \leq 1$ and $\operatorname{Supp}_R(M/N) = V(J)$. It is easy to see that,

$$\operatorname{Supp}_{R^*}(M/N \otimes_R R^*) \subseteq V(JR^*).$$

Since $\dim R^*/JR^* \leq 1$ and R^* is a semi-local ring then $V(JR^*)$ and, consequently, $\operatorname{Supp}_{R^*}(M/N \otimes_R R^*)$ is a finite set. Now, by Theorem 3.3, we conclude that $M \otimes_R R^*$ is a weakly Laskerian R^*-module.

COROLLARY 3.5. *Let $R \to S$ be a homomorphism, of Noetherian semi-local rings, such that S is a finite R-module. Then every weakly Laskerian S-module is a weakly Laskerian R-module.*

PROOF. Let M be a weakly Laskerian S-module. By Theorem 3.3, there exists a finitely generated submodule of M, N say, such that $\operatorname{Supp}_S(M/N)$ is a finite set. Set $J := \cap_{\mathfrak{p}\in \operatorname{Supp}_S(M/N)}\mathfrak{p}$. Then $\operatorname{Supp}_S(M/N) = V(J)$ and $\dim_S(M/N) = \dim(S/J) \leq 1$. Set $I = J \cap R$. Since S/J is an integral extension of R/I then, by the lying over theorem, one can conclude that $\operatorname{Supp}_R(M/N) \subseteq V(I)$. As $\dim(R/I) \leq 1$ and R is a semi-local ring, then $V(I)$ and so $\operatorname{Supp}_R(M/N)$ is a finite set. Now, the result is evident by Theorem 3.3.

4. Main Results

The purpose of this section is to prove that for an arbitrary ideal I of a Noetherian ring R, the category of I-cofinite weakly Laskerian modules forms an Abelian subcategory of the category of all R-modules.

PROPOSITION 4.1. *Let I be an ideal of a Noetherian ring R and M an R-module such that* $\dim M \leq 1$ *and* $\operatorname{Supp} M \subseteq V(I)$. *Then the following statements are equivalent:*

(i) *M is I-cofinite,*

(ii) *the R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are finitely generated.*

PROOF. See [1, Proposition 2.6].

COROLLARY 4.2. *Let I be an ideal of a Noetherian ring R and M be a weakly Laskerian R-module such that* $\operatorname{Supp} M \subseteq V(I)$. *Then the following statements are equivalent:*

(i) *M is I-cofinite,*

(ii) *the R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are finitely generated.*

PROOF. (i) $\Rightarrow$ (ii) is clear. In order to prove (ii) $\Rightarrow$ (i), by Theorem 3.3 there is a finitely generated submodule N of M such that the R-module M/N has finite support. So $\dim(M/N) \leq 1$ and $\operatorname{Supp} M/N \subseteq V(I)$. Also, the exact sequence

$$(*) \qquad 0 \to N \to M \to M/N \to 0,$$

induces the exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(R/I, N) &\longrightarrow \operatorname{Hom}_R(R/I, M) \\ &\longrightarrow \operatorname{Hom}_R(R/I, M/N) \longrightarrow \operatorname{Ext}^1_R(R/I, N) \longrightarrow \operatorname{Ext}^1_R(R/I, M) \\ &\longrightarrow \operatorname{Ext}^1_R(R/I, M/N) \longrightarrow \operatorname{Ext}^2_R(R/I, N), \end{aligned}$$

which implies that, the R-modules $\operatorname{Hom}_R(R/I, M/N)$ and $\operatorname{Ext}^1_R(R/I, M/N)$ are finitely generated. Therefore, in view of Proposition 4.1, the R-module M/N is I-cofinite. Now it follows from the exact sequence $(*)$ that M is I-cofinite.

Now we are ready to state and prove the main result of this paper.

THEOREM 4.3. *Let I be an ideal of a Noetherian ring R. Let $\mathscr{WL}(R, I)_{cof}$ denote the category of I-cofinite weakly Laskerian R-modules. Then $\mathscr{WL}(R, I)_{cof}$ forms an Abelian subcategory of the category of all R-modules.*

PROOF. Let $M, N \in \mathscr{WL}(R, I)_{cof}$ and let $f : M \longrightarrow N$ be an R-homomorphism. It is enough that to show that the R-modules $\ker f$ and $\operatorname{coker} f$ are I-cofinite.

To this end, the exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \operatorname{im} f \longrightarrow 0,$$

induces an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(R/I, \ker f) &\longrightarrow \operatorname{Hom}_R(R/I, M) \\ &\longrightarrow \operatorname{Hom}_R(R/I, \operatorname{im} f) \longrightarrow \operatorname{Ext}^1_R(R/I, \ker f) \longrightarrow \operatorname{Ext}^1_R(R/I, M), \end{aligned}$$

that implies the R-modules $\operatorname{Hom}_R(R/I, \ker f)$ and $\operatorname{Ext}^1_R(R/I, \ker f)$ are finitely generated. Therefore it follows from Corollary 4.2 that the R-module $\ker f$ is I-cofinite. Now, the assertion follows from the exact sequences

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \operatorname{im} f \longrightarrow 0,$$

and

$$0 \longrightarrow \operatorname{im} f \longrightarrow N \longrightarrow \operatorname{coker} f \longrightarrow 0.$$

Corollary 4.4. *Let R be a Noetherian ring, I be an ideal of R and M be a weakly Laskerian I-cofinite R-module. Then, the R-modules* $\operatorname{Tor}_i^R(N, M)$ *and* $\operatorname{Ext}_R^i(N, M)$ *are I-cofinite and weakly Laskerian, for all finitely generated R-modules N and all integers $i \geq 0$.*

Proof. Since N is finitely generated it follows that N has a free resolution of finitely generated free R-modules. Now the assertion follows using Theorem 4.3 and computing the modules $\operatorname{Tor}_i^R(N, M)$ and $\operatorname{Ext}_R^i(N, M)$, by this free resolution.

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