

SOME SHARP ESTIMATES FOR THE HAAR SYSTEM AND OTHER BASES IN $L^1(0, 1)$

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Abstract

Let $h = (h_k)_{k \geq 0}$ denote the Haar system of functions on $[0, 1]$. It is well known that h forms an unconditional basis of $L^p(0, 1)$ if and only if $1 < p < \infty$, and the purpose of this paper is to study a substitute for this property in the case $p = 1$. Precisely, for any $\lambda > 0$ we identify the best constant $\beta = \beta_h(\lambda) \in [0, 1]$ such that the following holds. If n is an arbitrary nonnegative integer and $a_0, a_1, a_2, \dots, a_n$ are real numbers such that $\|\sum_{k=0}^n a_k h_k\|_1 \leq 1$, then

$$\left| \left\{ x \in [0, 1] : \left| \sum_{k=0}^n \varepsilon_k a_k h_k(x) \right| \geq \lambda \right\} \right| \leq \beta,$$

for any sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ of signs. A related bound for an arbitrary basis of $L^1(0, 1)$ is also established. The proof rests on the construction of the Bellman function corresponding to the problem.

1. Introduction

Our motivation comes from a very natural question about $h = (h_n)_{n \geq 0}$, the Haar system on $[0, 1]$. Recall that this collection of functions is given by (we identify a set with its indicator function):

$$\begin{aligned} h_0 &= [0, 1), & h_1 &= [0, 1/2) - [1/2, 1), \\ h_2 &= [0, 1/4) - [1/4, 1/2), & h_3 &= [1/2, 3/4) - [3/4, 1), \\ h_4 &= [0, 1/8) - [1/8, 1/4), & h_5 &= [1/4, 3/8) - [3/8, 1/2), \\ h_6 &= [1/2, 5/8) - [5/8, 3/4), & h_7 &= [3/4, 7/8) - [7/8, 1) \end{aligned}$$

and so on. A classical result of Schauder [12] states that the Haar system forms a basis of $L^p = L^p(0, 1)$, $1 \leq p < \infty$ (throughout, the underlying measure will be the Lebesgue measure). That is, for every $f \in L^p$ there is a unique sequence $a = (a_n)_{n \geq 0}$ of real numbers satisfying $\|f - \sum_{k=0}^n a_k h_k\|_p \rightarrow 0$. Let $\beta_p(h)$ be the unconditional constant of h , i.e. the least $\beta \in [1, \infty]$ with the

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property that if n is a nonnegative integer and a_0, a_1, \dots, a_n are real numbers such that $\left\| \sum_{k=0}^n a_k h_k \right\|_p \leq 1$, then

$$(1.1) \quad \left\| \sum_{k=0}^n \varepsilon_k a_k h_k \right\|_p \leq \beta$$

for all choices of signs $\varepsilon_k \in \{-1, 1\}$. Using Paley's inequality [10], Marcinkiewicz [3] proved that $\beta_p(h) < \infty$ if and only if $1 < p < \infty$. This fact and its various extensions turned out to be very useful in the study of singular integrals, stochastic integrals, the structure of Banach spaces and in several other areas of mathematics. It follows from the results of Olevskiĭ [8], [9] that the Haar system is extremal in the following sense: if e is another basis of L^p , then

$$(1.2) \quad \beta_p(h) \leq \beta_p(e), \quad 1 < p < \infty.$$

Lindenstrauss and Pełczyński [2] gave a different proof of this fact, using Liapunoff's theorem on the range of a vector measure. The precise value of $\beta_p(h)$ was determined by Burkholder: we have

$$\beta_p(h) = p^* - 1, \quad 1 < p < \infty,$$

where $p^* = \max\{p, p/(p-1)\}$. The original proof of this formula, presented in [1], is quite complicated and technically involved (for the clarification and much more, see the recent paper of Vasyunin and Volberg [14]). The idea rests on the so-called Bellman function method, a powerful tool which has its roots at the optimal control theory. Namely, Burkholder studies the following more general problem: for any $1 < p < \infty$, $F, G \in \mathbb{R}$ and $M \geq |F|$, set

$$(1.3) \quad \mathcal{B}(F, G, M) = \sup \left\{ \left\| G + \sum_{k=1}^n \varepsilon_k a_k h_k \right\|_p \right\},$$

where the supremum is taken over all n , all $a_1, a_2, \dots, a_n \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$ such that $\left\| F + \sum_{k=1}^n a_k h_k \right\|_p \leq M$. The function \mathcal{B} turns out to satisfy a certain second-order partial differential equation, which was successfully solved by Burkholder. Coming back to the original problem, it can be proved that

$$\beta_p(h) = \sup_{M \geq 1} \frac{\mathcal{B}(1, 1, M)}{M} = p^* - 1.$$

We will be interested in finding an appropriate substitute for the above considerations in the limit case $p = 1$. We need to find the right replacement for the

p -th norm appearing in (1.1) and (1.3), and this will be accomplished by the use of a distribution function. To be more precise, suppose that F, G are given real numbers and let $M \geq |F|$. We will determine the least constant $\mathbf{B}(F, G, M)$ with the property that if n is a nonnegative integer and a_1, a_2, \dots, a_n are real numbers such that $\|F + \sum_{k=1}^n a_k h_k\|_1 \leq M$, then

$$\left| \left\{ x \in [0, 1] : \left| G + \sum_{k=1}^n \varepsilon_k a_k h_k(x) \right| \geq 1 \right\} \right| \leq \mathbf{B}(F, G, M).$$

This gives very precise information on the “unconditional” behavior of the Haar series in L^1 . We will also establish related sharp one-sided bounds (obtained earlier by Nazarov et. al. [4] using a slightly different approach) and present some interesting estimates for other types of bases of $L^1(0, 1)$, which can be regarded as weak analogues of Olevskiĭ’s inequality (1.2).

A few words about the proof and the organization of the paper are in order. Our approach rests on the Bellman function method, which is described in the next section. Section 3 contains the study of the one-sided estimate and can be regarded as the preparation for Section 4, where we determine the explicit formula for the above function \mathbf{B} . The final part part of the paper contains some further results concerning weak unconditional constants for arbitrary bases of $L^1(0, 1)$.

2. Bellman function method

We start with the description of the main tool used in the proofs of our results. The technique is well-known and appears in numerous papers in the literature, so we will be brief. For much more detailed exposition, examples and connections we refer the interested reader to the papers [5], [6], [14], [13] and [15].

Let $V : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ be a fixed function and put

$$\mathcal{D} = \{(F, G, M) \in \mathbf{R} \times \mathbf{R} \times [0, \infty) : |F| \leq M\}.$$

For any $(F, G, M) \in \mathcal{D}$, introduce the class $\mathcal{C}(F, G, M)$ which consists of all pairs (f, g) of functions on $[0, 1]$, which are of the form

$$f = F + \sum_{k=1}^n a_k h_k, \quad g = G + \sum_{k=1}^n \varepsilon_k a_k h_k$$

for some n , some $a_1, a_2, \dots, a_n \in \mathbf{R}$ and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$, and such that $\|f\|_1 \leq M$. We define the *Bellman function* $\mathbf{B} : \mathcal{D} \rightarrow \mathbf{R} \cup \{\infty\}$ by

$$(2.1) \quad \mathbf{B}(F, G, M) = \sup \left\{ \int_0^1 V(f(x), g(x)) dx : (f, g) \in \mathcal{C}(F, G, M) \right\}.$$

Observe that the problem described in the previous section can be rewritten in the above form, with $V(x, y) = 1_{\{|y| \geq 1\}}$.

The fundamental property of the function \mathbf{B} is described in the statement below.

THEOREM 2.1. *The function \mathbf{B} is the smallest function on \mathcal{D} for which the two following conditions hold:*

- (a) (*Majorization*) We have $\mathbf{B}(F, G, M) \geq V(F, G)$ for all $(F, G, M) \in \mathcal{D}$.
- (b) (*Diagonal concavity*) For any $(F_-, G_-, M_-), (F_+, G_+, M_+) \in \mathcal{D}$ such that $|F_+ - F_-| = |G_+ - G_-|$, we have

$$(2.2) \quad \mathbf{B}\left(\frac{F_- + F_+}{2}, \frac{G_- + G_+}{2}, \frac{M_- + M_+}{2}\right) \geq \frac{1}{2} \mathbf{B}(F_-, G_-, M_-) + \frac{1}{2} \mathbf{B}(F_+, G_+, M_+).$$

PROOF. Let us start with showing that \mathbf{B} satisfies (a) and (b). The first condition follows immediately from the observation that the functions $f \equiv F$, $g \equiv G$ belong to $\mathcal{C}(F, G, M)$. To prove the second property, pick $(f_-, g_-) \in \mathcal{C}(F_-, G_-, M_-)$ and $(f_+, g_+) \in \mathcal{C}(F_+, G_+, M_+)$ and splice them together into one pair, given by

$$(f(x), g(x)) = \begin{cases} (f_-(2x), g_-(2x)) & \text{if } x < 1/2, \\ (f_+(2x-1), g_+(2x-1)) & \text{if } x \geq 1/2. \end{cases}$$

From the structure of the Haar system, we see that there is a finite N such that

$$f = \frac{F_- + F_+}{2} + \sum_{k=1}^N a_k h_k, \quad g = \frac{G_- + G_+}{2} + \sum_{k=1}^N b_k h_k.$$

The assumption $|F_+ - F_-| = |G_+ - G_-|$ implies that $a_1 = \pm b_1$. Furthermore, for any $n \geq 2$ we have $a_n = \pm b_n$, since, by the structure of the Haar system, a_n, b_n are the corresponding coefficients of the functions f_- and g_- , or the functions f_+ and g_+ (depending on whether the support of h_n is contained in the left or in the right half of $[0, 1)$). Finally, by the triangle inequality, we have

$$\|f\|_1 \leq \frac{1}{2} \|f_-\|_1 + \frac{1}{2} \|f_+\|_1 \leq \frac{M_- + M_+}{2},$$

which gives $(f, g) \in \mathcal{C}((F_- + F_+)/2, (G_- + G_+)/2, (M_- + M_+)/2)$. In consequence,

$$\begin{aligned} & \mathbf{B}\left(\frac{F_- + F_+}{2}, \frac{G_- + G_+}{2}, \frac{M_- + M_+}{2}\right) \\ & \geq \int_0^1 V(f(x), g(x)) \, dx \\ & = \frac{1}{2} \int_0^1 V(f_-(x), g_-(x)) \, dx + \frac{1}{2} \int_0^1 V(f_+(x), g_+(x)) \, dx. \end{aligned}$$

Since the pairs (f_-, g_-) , (f_+, g_+) were arbitrary elements of $\mathcal{C}(F_-, G_-, M_-)$ and $\mathcal{C}(F_+, G_+, M_+)$, respectively, the condition (b) follows.

Next, suppose that $B : \mathcal{D} \rightarrow \mathbf{R}$ is any function satisfying the properties (a) and (b). Pick $(F, G, M) \in \mathcal{D}$ and a pair $(f, g) \in \mathcal{C}(F, G, M)$. There is a nonnegative integer N and appropriate coefficients a_k and ε_k such that

$$f = F + \sum_{k=1}^N a_k h_k \quad \text{and} \quad g = G + \sum_{k=1}^N \varepsilon_k a_k h_k.$$

For any $n \geq 0$, let $f_n = F + \sum_{k=1}^n a_k h_k$, $g_n = G + \sum_{k=1}^n \varepsilon_k a_k h_k$ and M_n be, respectively, the projections of f , g and $|f|$ on the space spanned by h_0, h_1, \dots, h_n . Note that $|f_n| \leq M_n$ almost everywhere, which can be showed, for example, by the use of a backward induction. The key step lies in proving that for all $n \geq 0$,

$$\int_0^1 B(f_{n+1}(x), g_{n+1}(x), M_{n+1}(x)) \, dx \leq \int_0^1 B(f_n(x), g_n(x), M_n(x)) \, dx.$$

To do this, let I denote the support of h_{n+1} . The functions $B(f_n, g_n, M_n)$ and $B(f_{n+1}, g_{n+1}, M_{n+1})$ coincide on $[0, 1] \setminus I$, so it suffices to show that

$$\int_I B(f_{n+1}(x), g_{n+1}(x), M_{n+1}(x)) \, dx \leq \int_I B(f_n(x), g_n(x), M_n(x)) \, dx.$$

However, f_n , g_n and M_n are constant on I ; denote the corresponding three values by x , y and z , respectively. Then the triple $(f_{n+1}, g_{n+1}, M_{n+1})$ equals $(x + a_{n+1}, y + \varepsilon_{n+1} a_{n+1}, z + b_{n+1})$ on the left half of I and $(x - a_{n+1}, y - \varepsilon_{n+1} a_{n+1}, z - b_{n+1})$ on the right half of this interval (here b_{n+1} is the appropriate coefficient of $|f|$). Consequently, the above estimate can be transformed into

the equivalent bound

$$\begin{aligned} & \frac{1}{2}B(x + a_{n+1}, y + \varepsilon_{n+1}a_{n+1}, z + b_{n+1}) \\ & + \frac{1}{2}B(x - a_{n+1}, y - \varepsilon_{n+1}a_{n+1}, z - b_{n+1}) \leq B(x, y, z), \end{aligned}$$

which follows immediately from (b). Thus, by (a),

$$\begin{aligned} \int_0^1 V(f(x), g(x)) dx & \leq \int_0^1 B(f(x), g(x), |f(x)|) dx \\ & = \int_0^1 B(f_N(x), g_N(x), M_N(x)) dx \\ & \leq \int_0^1 B(f_0(x), g_0(x), M_0(x)) dx \\ & = B(F, G, \|f\|_1). \end{aligned}$$

However, we have $\|f\|_1 \leq M$ and the class $\mathcal{C}(F, G, M)$ grows when we increase the third parameter. Therefore,

$$\int_0^1 V(f(x), g(x)) dx \leq B(F, G, M)$$

and taking the supremum over all (f, g) yields the desired bound $B \leq B$. This proves the claim.

Before we proceed, let us make here several observations. Let us first take a look at the diagonal concavity of B , i.e., the condition (b) above. Obviously, it is equivalent to the following statement:

(b') For any $(F, G, M) \in \mathcal{D}$, any $\varepsilon \in \{-1, 1\}$ and $m \in \mathbb{R}$, the function

$$\xi : t \mapsto B(F + t, G + \varepsilon t, M + mt)$$

is mid-point concave on the interval $\{t : (F + t, G + \varepsilon t, M + mt) \in \mathcal{D}\}$.

In all the situations we are interested in, the function V is nonnegative and hence bounded from below. Thus, by (a), the function B also has this property and its mid-point concavity implies that it is merely concave.

A natural question is: given V , how to find the corresponding function B ? Let us now present some intuitive observations which may be helpful during the search. We would also like to point out here that similar argumentation appears, for example, in the analysis of optimal stopping problems [11]. See

also [13] for more detailed discussion and examples. The “state space” \mathcal{D} can be split into two sets:

$$\begin{aligned} D_1 &= \{(F, G, M) : B(F, G, M) = V(F, G)\}, \\ D_2 &= \{(F, G, M) : B(F, G, M) > V(F, G)\} \end{aligned}$$

(in the theory of the optimal stopping, these are the so-called the stopping and the continuation region, respectively). Since B is *the least* diagonally concave majorant of V , it seems plausible to assume the following. For each $(F, G, M) \in D_2$ there is a direction along which B is locally linear (otherwise, roughly speaking, it would be possible to make B smaller). More precisely, for such (F, G, M) , there are $\varepsilon \in \{-1, 1\}$ and $m \in \mathbb{R}$ such that $t \mapsto B(F + t, G + \varepsilon t, M + mt)$ is linear for t lying in some neighborhood of 0. In other words, the whole set D_2 can be “foliated” into line segments of appropriate slope along which the function B is linear. If B is twice differentiable on D_2 , this yields the following second-order differential equation which should be satisfied by B : for each $(F, G, M) \in D_2$,

$$\det \begin{bmatrix} B_{FF} + 2B_{FG} + B_{GG} & B_{FM} + B_{GM} \\ B_{FM} + B_{GM} & B_{MM} \end{bmatrix} (F, G, M) = 0$$

or

$$\det \begin{bmatrix} B_{FF} - 2B_{FG} + B_{GG} & B_{FM} - B_{GM} \\ B_{FM} - B_{GM} & B_{MM} \end{bmatrix} (F, G, M) = 0.$$

Sometimes this system of differential equations can be explicitly solved: see e.g. [1], [14], [13], and this brings the candidate for the Bellman function. Then one proves rigorously that the function has all the desired properties.

Our approach will be slightly different and will not rest on solving the above system of differential equations. We will guess the right formula for B by indicating the appropriate foliation of the set D_2 .

3. One-sided bound

This section is devoted to the analysis of the function

$$B^o(F, G, M) = \sup\{|\{x \in [0, 1] : g(x) \geq 1\}|\} : (f, g) \in \mathcal{C}(F, G, M)\}.$$

We will use the technique described in the preceding section, with the choice $V(F, G) = 1_{\{G \geq 1\}}$. The calculations will be rather easy and we will gain some information which will be needed in the study of the two-sided case. We would like to stress here that the result is not new: it has already been established by Nazarov, Reznikov, Vasyunin and Volberg in an unpublished paper [4], with the use of similar methods.

3.1. An explicit formula for \mathbf{B}^o

Let $B : \mathcal{D} \rightarrow \mathbf{R}$ be given by

$$B(F, G, M) = \begin{cases} 1 & \text{if } G + M \geq 1, \\ 1 - \frac{(1-G-M)^2}{(1-G)^2 - F^2} & \text{if } G + M < 1. \end{cases}$$

THEOREM 3.1. *We have $\mathbf{B}^o \leq B$.*

PROOF. By Theorem 2.1, it suffices to verify that the function B satisfies the conditions (a) and (b'). The majorization

$$B(F, G, M) \geq 1_{\{G \geq 1\}}$$

is straightforward. Indeed, the estimate is obvious for $G + M \geq 1$, while for remaining (F, G, M) , we observe that

$$\frac{(1-G-M)^2}{(1-G)^2 - F^2} \leq \frac{(1-G-M)^2}{(1-G)^2 - M^2} = \frac{1-G-M}{1-G+M} \leq 1$$

and hence $B(F, G, M) \geq 0 = 1_{\{G \geq 1\}}$. To check the property (b'), fix $(F, G, M) \in \mathcal{D}$ with $G + M < 1$, let $\varepsilon \in \{-1, 1\}$ and $m \in \mathbf{R}$. Define $\xi = \xi_{F, G, M, \varepsilon, m}$ by

$$\xi(t) = B(F + t, G + \varepsilon t, M + mt).$$

for t such that $(F + t, G + \varepsilon t, M + mt) \in \mathcal{D}$. It is easy to check that this function is of class C^1 , and we must prove that it is concave. Fix t belonging to the domain of ξ and let $\tilde{F} = F + t$, $\tilde{G} = G + \varepsilon t$ and $\tilde{M} = M + mt$. If $\tilde{G} + \tilde{M} > 1$, then $\xi''(t) = 0$; if $\tilde{G} + \tilde{M} < 1$, then $|\tilde{F}| \leq \tilde{M} < 1 - \tilde{G}$ and a straightforward computation gives

$$\xi''(t) = -\frac{2}{(\tilde{G} - 1)^2 - \tilde{F}^2} \left(m + \varepsilon - \frac{(\tilde{M} + \tilde{G} - 1)(2\tilde{G}\varepsilon - 2\varepsilon - 2\tilde{F})}{(\tilde{G} - 1)^2 - \tilde{F}^2} \right)^2 \leq 0.$$

This yields the desired concavity, since ξ is smooth.

THEOREM 3.2. *We have $\mathbf{B}^o \geq B$.*

PROOF. The function \mathbf{B}^o is the least function on \mathcal{D} which satisfies (a) and (b'). Observe that $\mathbf{B}^o(F, G, M) = \mathbf{B}^o(-F, G, M)$ for all F, G, M , since otherwise the formula $(F, G, M) \mapsto \min\{\mathbf{B}^o(F, G, M), \mathbf{B}^o(-F, G, M)\}$ would define a function satisfying (a) and (b'), but smaller than \mathbf{B}^o . In consequence, it suffices to prove the inequality $\mathbf{B}^o(F, G, M) \geq B(F, G, M)$ for positive F only. For the sake of clarity, we split the reasoning into several steps.

Step 1. If $G \geq 1$, then $\mathbf{B}^o(F, G, M) \geq V(F, G) = 1 = B(F, G, M)$.

Step 2. Now suppose that $G < 1$, but $F + G \geq 1$. Below, we will frequently use the following argument: we will write the point (F, G, M) as a convex combination of appropriate two points (at which we have already proved the majorization), and then apply the diagonal concavity (2.2), thus obtaining the desired lower bound for $\mathbf{B}^o(F, G, M)$. Here, for any $p \in (0, 1)$, we have

$$\begin{aligned} \mathbf{B}^o(F, G, M) &\geq p \mathbf{B}^o(0, F + G, M - F) \\ &\quad + (1 - p) \mathbf{B}^o\left(\frac{F}{1 - p}, G - \frac{p}{1 - p}F, M + \frac{p}{1 - p}F\right) \\ &\geq p \mathbf{B}^o(0, F + G, M - F) \geq p, \end{aligned}$$

where the latter passage is due to Step 1 considered above. Since p was arbitrary, we obtain that $\mathbf{B}^o(F, G, M) \geq 1 = B(F, G, M)$ provided $F + G \geq 1$.

Step 3. Suppose that $F + G < 1$ and $F = M$. Then, by the diagonal concavity, we may write

$$\begin{aligned} \mathbf{B}^o(F, G, M) &\geq \frac{2F}{F - G + 1} \mathbf{B}^o\left(\frac{F - G + 1}{2}, \frac{-F + G + 1}{2}, \frac{F - G + 1}{2}\right) \\ &\quad + \frac{1 - F - G}{F - G + 1} \mathbf{B}^o(0, G - F, 0) \\ &\geq \frac{2F}{F - G + 1} = B(F, G, M), \end{aligned}$$

where in the last estimate we have used Step 2 and the fact that \mathbf{B}^o is nonnegative.

Step 4. Finally, let $F + G < 1$ and $F < M$. Fix $p \in (0, 1)$ and put

$$F_+ = \frac{F}{1 - p} + \frac{p}{1 - p} \frac{1 - F - G}{2}, \quad M_+ = \frac{M}{1 - p} - \frac{p}{1 - p} \frac{1 - F - G}{2}.$$

We have

$$(3.1) \quad M_+ - F_+ = \frac{M - F - p(1 - F - G)}{1 - p}.$$

Therefore, if $M + G \geq 1$, then the latter numerator is nonnegative for all p , and the diagonal concavity of \mathbf{B}^o gives

$$\begin{aligned} \mathbf{B}^o(F, G, M) &\geq p \mathbf{B}^o\left(\frac{F + G - 1}{2}, \frac{F + G + 1}{2}, \frac{1 - F - G}{2}\right) \\ (3.2) \quad &\quad + (1 - p) \mathbf{B}^o(F_+, G - (F_+ - F), M_+) \\ &\geq p, \end{aligned}$$

in view of Step 2. Letting $p \rightarrow 1$ gives $\mathbf{B}^\circ(F, G, M) \geq 1 = B(F, G, M)$. On the other hand, if $M + G < 1$, then the expression in (3.1) vanishes for $p = (M - F)/(1 - F - G) \in (0, 1)$ and hence, repeating the first inequality from (3.2) and using Steps 2 and 3, we get

$$\begin{aligned} \mathbf{B}^\circ(F, G, M) &\geq p + (1 - p) \frac{2F_+}{F_+ - (G - (F_+ - F)) + 1} \\ &= \frac{M - F}{1 - F - G} + \frac{1 - G - M}{1 - F - G} \frac{M + F}{1 + F - G} \\ &= B(F, G, M). \end{aligned}$$

This completes the proof of the desired estimate.

3.2. On the search of the Bellman function

Here we sketch some steps which led us to the discovery of the function B above. First, it is more convenient to work with

$$\mathbf{B}(F, G, M) = \sup\{|\{x \in [0, 1] : g(x) \geq 0\}| : (f, g) \in \mathcal{C}(F, G, M)\},$$

which is related to \mathbf{B}° via the identity $\mathbf{B}^\circ(F, G, M) = \mathbf{B}(F, G - 1, M)$ for all $(F, G, M) \in \mathcal{D}$. Consequently, by Theorem 2.1, we see that \mathbf{B} is diagonally concave and satisfies the majorization

$$(3.3) \quad \mathbf{B}(F, G, M) \geq 1_{\{G \geq 0\}}.$$

Furthermore, directly from its definition, we see that \mathbf{B} enjoys the homogeneity-type property

$$(3.4) \quad \mathbf{B}(\pm\alpha F, \alpha G, \alpha M) = \mathbf{B}(F, G, M), \quad \alpha > 0.$$

This follows immediately from the observation that

$$|\{x \in [0, 1] : g(x) \geq 0\}| = |\{x \in [0, 1] : \alpha g(x) \geq 0\}|$$

combined with the equivalence $(f, g) \in \mathcal{C}(F, G, M)$ if and only if $(\pm\alpha f, \alpha g) \in \mathcal{C}(\alpha F, \alpha G, \alpha M)$. In particular, this gives that the function $x \mapsto \mathbf{B}(x, -x, x)$ is constant on $(0, \infty)$. On the other hand, this function is concave on \mathbf{R} , in view of the diagonal concavity of \mathbf{B} . In consequence, we get

$$(3.5) \quad \mathbf{B}(1/2, -1/2, 1/2) \geq \mathbf{B}(0, 0, 0) = 1$$

(the latter equality follows from (3.3) and the obvious bound $\mathbf{B} \leq 1$). The next step in the analysis is to introduce the function

$$b(x, y) = \mathbf{B}\left(\frac{x+1}{2}, \frac{x-1}{2}, y\right),$$

given on $D = \{(x, y) \in \mathbb{R}^2 : y \geq |\frac{x+1}{2}|\}$. Using (3.4), we see that for $F \neq \pm G$,

$$\begin{aligned} b\left(\frac{F+G}{F-G}, \frac{M}{F-G}\right) &= \mathbf{B}(F, G, M) = \mathbf{B}(-F, G, M) \\ &= b\left(\frac{F-G}{F+G}, -\frac{M}{F+G}\right), \end{aligned}$$

from which we infer that b satisfies

$$(3.6) \quad b(x, y) = b\left(\frac{1}{x}, -\frac{y}{x}\right).$$

Furthermore, since \mathbf{B} is diagonally concave, we have that b is a concave function, and the majorization (3.3) implies that $b(x, y) \geq 1_{\{x \geq 1\}} \geq 0$. The condition (3.5) implies that $b(0, 1/2) \geq 1$; hence, using the concavity of b along the halflines starting from $(0, 1/2)$ and contained in D , we infer that $b(x, y) \geq 1$ (and hence $b(x, y) = 1$) provided $y \geq -x/2 + 1/2$. Thus, all we need is to identify the explicit formula for b on the set

$$\Omega = \{(x, y) \in D : y \leq -x/2 + 1/2\}.$$

It is easy to show that $b(-1, 0) = \mathbf{B}(0, -1, 0) = 0$: indeed, $\mathcal{C}(0, -1, 0)$ contains only the constant pair. The line segment which joins $(-1, 0)$ and $(0, 1/2)$ is a part of the boundary of Ω , so it seems plausible to guess that b is linear along this segment: $b(2y - 1, y) = 2y$ for $y \in [0, 1/2]$.

Next, we assume that b is of class C^1 in the interior of Ω . By (3.6), we may restrict our search to the triangle $\Omega \cap \{(x, y) : x \geq -1\}$. Let us try to identify the foliation \mathcal{F} of b restricted to this set (i.e., split the triangle into the union of maximal segments along which b is linear). We already know that the segment with the endpoints $(0, 1/2)$ and $(-1, 1)$, as well as the boundary segment with endpoints $(-1, 0)$, $(0, 1/2)$, belong to the foliation. Now pick a segment $I \in \mathcal{F}$ which contains the point $(-1, y)$ for a given $y \in (0, 1)$. If I intersects one of the two boundary segments (call it J), at a point different from $(0, 1/2)$, then b must be linear in the triangle spanned by I and J (i.e., the convex hull of $I \cup J$). In particular, this implies that b must be linear along the segment which joins $(-1, y)$ with $(0, 1/2)$. Consequently, we see that the

only foliation is possible, namely, the fan of segments from the vertex $(0, 1/2)$. This implies

$$b(-1, y) - 1 = -b_x(-1, y) + b_y(-1, y) \left(y - \frac{1}{2} \right).$$

On the other hand, differentiating (3.6) with respect to x at the point $(-1, y)$, $y \in (0, 1)$, yields

$$2b_x(-1, y) = yb_y(-1, y).$$

If we combine the two latter identities, we obtain the following differential equation. If $\varphi(y) = b(-1, y)$, $y \in [0, 1]$, we have

$$\varphi(y) - 1 = \varphi'(y) \cdot \frac{y - 1}{2}.$$

Therefore, $\varphi(y) = K(y - 1)^2 + 1$ for some parameter K . Moreover, we already know that $\varphi(0) = \mathbf{B}(0, -1, 0) = 0$; this yields $K = -1$ and hence

$$b(x, y) = (1 + x)b\left(0, \frac{1}{2}\right) - xb\left(-1, \frac{1 + x - 2y}{2x}\right) = 1 - \frac{\left(\frac{x-1}{2} + y\right)^2}{x}$$

for $(x, y) \in \Omega$, $x \in [-1, 0]$. By (3.6), the same formula is valid on the whole Ω . This gives us the candidate

$$B(F, G, M) = \mathbf{B}(F, G - 1, M) = b\left(\frac{F + G - 1}{F - G + 1}, \frac{M}{F - G + 1}\right)$$

studied in the previous subsection.

4. Two-sided bound

We turn to the proof of the main result of this paper. We will provide the explicit formula for the function

$$B(F, G, M) = \sup\{|\{x \in [0, 1] : |g(x)| \geq 1\}| : (f, g) \in \mathcal{C}(F, G, M)\}.$$

This will be accomplished by the technique described in Section 2, with $V(F, G) = 1_{\{|G| \geq 1\}}$.

4.1. An explicit formula for B

Introduce the following subsets of \mathcal{D} :

$$\begin{aligned} \mathcal{D}_1 &= \{(F, G, M) : |F| + |G| \geq 1\} \\ &\quad \cup \{(F, G, M) : |F| + |G| < 1, M \geq \frac{1}{2}(F^2 - G^2 + 1)\}, \\ \mathcal{D}_2 &= \{(F, G, M) : |F| + |G| < 1, M < F^2 - G^2 + |G|\}, \\ \mathcal{D}_3 &= \{(F, G, M) : |F| + |G| < 1, F^2 - G^2 + |G| \leq M < \frac{1}{2}(F^2 - G^2 + 1)\}. \end{aligned}$$

Note that if $|F| + |G| < 1$, then $F^2 - G^2 + |G| < \frac{1}{2}(F^2 - G^2 + 1)$; thus the subsets are pairwise disjoint. Let $B : \mathcal{D} \rightarrow \mathbf{R}$ be given by

$$(4.1) \quad B(F, G, M) = \begin{cases} 1 & \text{on } \mathcal{D}_1, \\ 1 - \frac{(1-|G|-M)^2}{(1-|G|)^2 - F^2} & \text{on } \mathcal{D}_2, \\ 2M - F^2 + G^2 & \text{on } \mathcal{D}_3. \end{cases}$$

THEOREM 4.1. *We have $B \leq B$.*

PROOF. As previously, we verify that the function B satisfies the conditions (a) and (b'). The first of them is very easy: if $|G| \geq 1$, then $|F| + |G| \geq 1$ and $B(F, G, M) = V(F, G)$; for $|G| < 1$ it is not difficult to see that B takes nonnegative values only. To check (b'), fix $(F, G, M) \in \mathcal{D}$, $\varepsilon \in \{-1, 1\}$, $m \in \mathbf{R}$ and consider the function

$$\xi(t) = B(F + t, G + \varepsilon t, M + mt),$$

given on the interval $\{t : (F + t, G + \varepsilon t, M + mt) \in \mathcal{D}\}$. The domain of this function can be split into a finite family (I_k) of intervals which have the property that on each I_k , ξ coincides with ξ_1 , ξ_2 or ξ_3 . Here $\xi_1(t) \equiv 1$,

$$\xi_2(t) = 1 - \frac{(1 - |G + \varepsilon t| - M - mt)^2}{(1 - |G + \varepsilon t|)^2 - (F + t)^2}$$

and

$$\xi_3(t) = 2M + 2mt - (F + t)^2 + (G + \varepsilon t)^2.$$

It is not difficult to check that the function ξ is continuous and that ξ_1 and ξ_3 are concave on \mathbf{R} . Furthermore, if $\xi = \xi_2$ on I_k , then by the definition of \mathcal{D}_2 we infer that $G + t\varepsilon$ is bounded away from 0; this implies that $\xi_2|_{I_k}$ is concave (see the one-sided case, this function has already appeared there, with $|G + t\varepsilon|$ replaced by $G + t\varepsilon$). This implies that ξ is concave on each of the intervals I_k . Furthermore, B is C^1 -smooth on the boundary between \mathcal{D}_2 and \mathcal{D}_3 (on the surface $M = F^2 - G^2 + |G|$) and any point which belongs to

$\partial D_1 \cap \partial D_2$, automatically lies in ∂D_3 . Therefore, to get the concavity of ξ on the whole domain, it suffices to check only the jumps of its first derivative on the boundary between \mathcal{D}_1 and \mathcal{D}_3 (formally, we need to look at the one-sided derivatives of ξ at those t , for which $(F + t, G + \varepsilon t, M + mt) \in \partial \mathcal{D}_1 \cap \partial \mathcal{D}_3$). However, the derivatives behave appropriately, since B equals 1 on \mathcal{D}_1 and $B < 1$ on \mathcal{D}_3 . This completes the proof.

THEOREM 4.2. *We have $B \geq B$.*

PROOF. Arguing as in the setting of the one-sided estimate, it suffices to show the desired bound for nonnegative F and G only. Of course, the function B majorizes the Bellman function B^o corresponding to the one-sided estimate. Consequently, the desired inequality holds for $G + M \geq 1$ and for $(F, G, M) \in \mathcal{D}_2$ (if the second possibility occurs, we obtain equality or the trivial bound $B \leq 1$). Now suppose that $G + M < 1$ and $M \geq \frac{1}{2}(F^2 - G^2 + 1)$, so that $B(F, G, M) = 1$. Then $M > F^2 - G^2 + G$ (see the sentence below the definitions of $\mathcal{D}_1 - \mathcal{D}_3$) and hence $F < G$: indeed, otherwise we would have $2M - F^2 + G^2 = M + (M - F^2 + G^2) < M + G < 1$. Obviously, we have

$$B(F, G, M) \geq B\left(F, G, \frac{F^2 - G^2 + 1}{2}\right)$$

and we can express the point on the right as the following convex combination:

$$\begin{aligned} & \left(F, G, \frac{F^2 - G^2 + 1}{2}\right) \\ &= \frac{1 - F + G}{2} \cdot (F_-, G_-, M_-) + \frac{1 + F - G}{2} \cdot (F_+, G_+, M_+), \end{aligned}$$

where

$$F_- = F - \frac{1 + F - G}{2}, \quad G_- = G + \frac{1 + F - G}{2}, \quad M_- = |F_-| = -F_-$$

and

$$F_+ = F + \frac{1 - F + G}{2}, \quad G_+ = G - \frac{1 - F + G}{2}, \quad M_+ = F_+.$$

Since $|F_+ - F_-| = |G_+ - G_-|$, (2.2) gives

$$B(F, G, M) \geq \frac{1 - F + G}{2} B(F_-, G_-, M_-) + \frac{1 + F - G}{2} B(F_+, G_+, M_+).$$

But $M_- + |G_-| = M_+ + |G_+| = 1$, so $B(F_{\pm}, G_{\pm}, M_{\pm}) \geq 1$, by the above reasoning. This yields the desired bound $B(F, G, M) \geq 1 = B(F, G, M)$.

Finally, suppose that $M + G < 1$ and $(F, G, M) \in \mathcal{D}_3$, and consider the maximal line segment of the form

$$I = \{(F + s, G - s, M + s) : s \in (t_-, t_+)\},$$

contained in \mathcal{D}_3 . It is not difficult to derive that

$$t_+ = \frac{F^2 - G^2 + 1 - 2M}{2(1 - F - G)}, \quad t_- = -\frac{M - (F^2 - G^2 + G)}{2(1 - F - G)}.$$

The endpoint of I , corresponding to $s = t_-$, lies in $\partial\mathcal{D}_2$; the other endpoint belongs to $\partial\mathcal{D}_1$. We have already verified the majorization on $\mathcal{D}_1 \cup \mathcal{D}_2$, so

$$\begin{aligned} \mathbf{B}(F, G, M) &\geq \frac{-t_-}{t_+ - t_-} \mathbf{B}(F + t_+, G - t_+, M + t_+) \\ &\quad + \frac{t_+}{t_+ - t_-} \mathbf{B}(F + t_-, G - t_-, M + t_-) \\ &\geq \frac{-t_-}{t_+ - t_-} (2(M + t_+) - (F + t_+)^2 + (G - t_+)^2) \\ &\quad + \frac{t_+}{t_+ - t_-} (2(M + t_-) - (F + t_-)^2 + (G - t_-)^2) \\ &= 2M - F^2 + G^2. \end{aligned}$$

This completes the proof.

4.2. On the search of the Bellman function

Again, we write down the definition of \mathbf{B} :

$$\mathbf{B}(F, G, M) = \sup\{|\{x \in [0, 1] : |g(x)| \geq 1\}| : (f, g) \in \mathcal{C}(F, G, M)\}.$$

In comparison to the one-sided case, the situation is more difficult since the function \mathbf{B} does not seem to have any homogeneity-type property. Nevertheless, it majorizes the Bellman function corresponding to the one-sided estimate, which gives

$$(4.2) \quad \mathbf{B}(F, G, M) \geq \begin{cases} 1 & \text{if } |G| + M \geq 1, \\ 1 - \frac{(1 - |G| - M)^2}{(1 - |G|)^2 - F^2} & \text{if } |G| + M < 1. \end{cases}$$

This, in particular, yields

$$(4.3) \quad \mathbf{B}(F, G, M) = 1 \quad \text{provided} \quad |G| + M \geq 1.$$

Next, we proceed as follows. Fix $a \in (0, 1)$ and consider the function

$$b(x, y) = \mathbf{B}\left(\frac{x+a}{2}, \frac{x-a}{2}, y\right),$$

given on the set $\{(x, y) \in \mathbb{R}^2 : y \geq |\frac{x+a}{2}|\}$. This function is concave and, by (4.3), we have $b(x, y) = 1$ for $y \geq 1 - |\frac{x-a}{2}|$. Thus all we need is to determine the formula for b on the parallelogram $\mathcal{P} = \{(x, y) : |\frac{x+a}{2}| \leq y < 1 - |\frac{x-a}{2}|\}$ (see Figure 1).

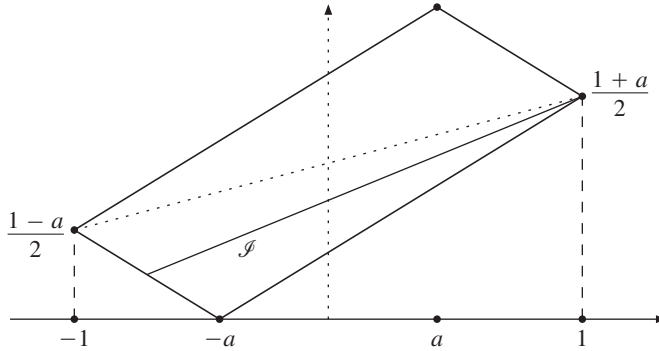


FIGURE 1. The parallelogram \mathcal{P}

Directly from the concavity of b , we obtain that $b(x, y) = 1$ if (x, y) lies on or above the dotted diagonal of \mathcal{P} – precisely, the line segment with endpoints $(-1, \frac{1-a}{2})$ and $(1, \frac{1+a}{2})$ – due to the fact that b equals 1 when evaluated at the sides of \mathcal{P} lying above this segment. For (x, y) lying below the diagonal we have, by (4.2),

$$b(x, y) \geq \zeta(x, y) = 1 - \frac{(1 - |\frac{x-a}{2}| - y)^2}{(1-a)(1+x)}.$$

Let us search for the least concave majorant of ζ . Some experiments lead to the following idea. Take an interval \mathcal{I} with endpoints $(1, \frac{1+a}{2})$ and $(t, -\frac{t+a}{2})$, where $t \in (-1, -a]$ (see Figure 1). It is easy to check that ζ is not concave along this interval and that the least concave majorant of $\zeta|_{\mathcal{I}}$ is given by

$$b_0(x, y) = \begin{cases} \zeta(x, y) & \text{if } (x, y) \in \mathcal{I}, y < \frac{a}{2} - (\frac{1}{2} - a)x, \\ 2y - ax & \text{if } (x, y) \in \mathcal{I}, y \geq \frac{a}{2} - (\frac{1}{2} - a)x. \end{cases}$$

Assuming $b = b_0$ for all (x, y) below the diagonal, we obtain the candidate for the Bellman function, given by (4.1).

5. A weak unconditional constant for an arbitrary basis of $L^1(0, 1)$

The estimates obtained in the previous sections can be used to obtain some interesting bounds for an arbitrary basis of $L^1(0, 1)$. For any sequence $e = (e_0, e_1, e_2, \dots)$ in $L^1(0, 1)$ and $\lambda > 0$ we define the weak unconditional constant $\beta_e(\lambda)$ as the least number β with the following property. If n is a nonnegative integer and a_0, a_1, \dots, a_n are real numbers such that $\|\sum_{k=0}^n a_k e_k\|_1 \leq 1$, then

$$(5.1) \quad \left| \left\{ x \in [0, 1] : \left| \sum_{k=0}^n \varepsilon_k a_k e_k(x) \right| \geq \lambda \right\} \right| \leq \beta$$

for all choices of signs $\varepsilon_k \in \{-1, 1\}$. If we plug λa_k in the place of a_k above, $k = 0, 1, \dots, n$, we see that the results of Section 3 imply that

$$\beta_h(\lambda) = \min \left\{ \frac{2}{\lambda}, 1 \right\}$$

(see also [1]). The main theorem of this section gives a related estimate for a different choice of a basis of L^1 , which should be compared to (1.2).

THEOREM 5.1. *If e is a basis of $L^1(0, 1)$, then $\beta_e(\lambda) \geq \beta_h(\lambda)$ for all $\lambda > 0$.*

In the proof of this statement we will need the following auxiliary fact. Roughly speaking, it says that any finite subsequence of Haar functions can be approximated using pairwise disjoint blocks of elements of e .

LEMMA 5.2. *Let $e = (e_n)_{n \geq 0}$ be an arbitrary basis of $L^1(0, 1)$. Suppose that $(h_k)_{k=0}^N$ is a finite collection of Haar functions. Then for any $\delta > 0$ there is an increasing sequence $(n_k)_{k=0}^{N+1}$ of integers, a sequence $(b_n)_{n=0}^{n_{N+1}-1}$ of real numbers and two sequences $(f_n)_{n=0}^{n_{N+1}-1}$, $(r_n)_{n=0}^{n_{N+1}-1}$ of real-valued functions on $(0, 1)$ such that the following holds:*

(i) *we have the decomposition*

$$\sum_{n=n_k}^{n_{k+1}-1} b_n e_n = f_k + r_k, \quad k = 0, 1, 2, \dots, N,$$

(ii) *we have $\|r_k\|_1 \leq \delta$ for $k = 0, 1, 2, \dots, N$,*

(iii) *there is a measure-preserving transformation $T : [0, 1] \rightarrow [0, 1]$ such that $f_k(Tx) = h_k(x)$ for all $x \in (0, 1)$ and $k = 0, 1, 2, \dots, N$.*

This result can be obtained by a slight modification of the construction presented in Olevskiĭ [7]; see also Theorem 3 in [8] and references therein. We omit the details.

PROOF OF THEOREM 5.1. Pick arbitrary $\lambda, \kappa > 0$ and $\gamma \in (0, 1)$. There is a nonnegative integer N , a sequence a_0, a_1, \dots, a_N of real numbers and a sequence $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N$ of signs such that

$$(5.2) \quad \left\| \sum_{k=0}^N a_k h_k \right\|_1 \leq 1$$

and

$$(5.3) \quad \left\{ x \in [0, 1] : \left| \sum_{k=0}^N \varepsilon_k a_k h_k(x) \right| \geq \frac{\lambda + 1 - \gamma}{\gamma} \right\} \\ \geq \beta_h \left(\frac{\lambda + 1 - \gamma}{\gamma} \right) - \kappa.$$

Now we apply Lemma 5.2 to the finite family $(h_k)_{k=0}^N$ of Haar functions and a fixed $\delta > 0$. As the result we obtain the corresponding sequence $(n_k)_{k=0}^N$, the coefficients $(b_n)_{n \geq 0}$ and the appropriate functions $(f_k)_{k \geq 0}$ and $(r_k)_{k \geq 0}$. Putting $\tilde{a}_k = \gamma a_k$ for $k = 0, 1, \dots, N$, we obtain, by Lemma 5.2,

$$(5.4) \quad \left\| \sum_{k=0}^N \tilde{a}_k \sum_{n=n_k}^{n_{k+1}-1} b_n e_n \right\|_1 \leq \left\| \sum_{k=0}^N \tilde{a}_k f_k \right\|_1 + \left\| \sum_{k=0}^N \tilde{a}_k r_k \right\|_1 \\ \leq \left\| \sum_{k=0}^N \tilde{a}_k h_k \right\|_1 + \delta \sum_{k=0}^N |a_k| = \gamma \left\| \sum_{k=0}^N a_k h_k \right\|_1 + \delta \sum_{k=0}^N |a_k| \leq 1,$$

provided δ is sufficiently small (it suffices to take $\delta < (1 - \gamma)(\sum_{k=0}^N |a_k|)^{-1}$: see (5.2)). In consequence, we get

$$(5.5) \quad \left\{ x \in [0, 1] : \left| \sum_{k=0}^N \varepsilon_k \tilde{a}_k \sum_{n=n_k}^{n_{k+1}-1} b_n e_n(x) \right| \geq \lambda \right\} \geq I - II,$$

where

$$I = \left\{ x \in [0, 1] : \left| \sum_{k=0}^N \varepsilon_k \tilde{a}_k f_k(x) \right| \geq \lambda + 1 - \gamma \right\} \\ = \left\{ x \in [0, 1] : \left| \sum_{k=0}^N \varepsilon_k a_k h_k(x) \right| \geq \frac{\lambda + 1 - \gamma}{\gamma} \right\} \\ \geq \beta_h \left(\frac{\lambda + 1 - \gamma}{\gamma} \right) - \kappa,$$

by virtue of (5.3), and

$$\begin{aligned} II = \left\| \left\{ x \in [0, 1] : \left| \sum_{k=0}^N \varepsilon_k \tilde{a}_k r_k(x) \right| \geq 1 - \gamma \right\} \right\| &\leq (1 - \gamma)^{-1} \left\| \sum_{k=0}^N \varepsilon_k \tilde{a}_k r_k \right\|_1 \\ &\leq \frac{\gamma \delta}{1 - \gamma} \sum_{k=0}^N |a_k|, \end{aligned}$$

by Chebyshev's inequality. Thus, combining (5.4) and (5.5), we see that

$$\beta_e(\lambda) \geq \beta_h \left(\frac{\lambda + 1 - \gamma}{\gamma} \right) - \kappa - \frac{\gamma \delta}{1 - \gamma} \sum_{k=0}^N |a_k|.$$

Therefore, letting $\delta \rightarrow 0$ and then $\gamma \rightarrow 1, \kappa \rightarrow 0$, we obtain $\beta_e(\lambda) \geq \beta_h(\lambda)$, since the function β_h is continuous. This completes the proof.

REMARK 5.3. It is easy to see that when $\lambda > 2$, then there is a basis e for which we have the strict inequality $\beta_e(\lambda) > \beta_h(\lambda)$. In fact, it is not difficult to construct a basis e for which $\beta_e \equiv 1$. For example, let h be the Haar system. Consider the basis e such that for any $n \geq 0$,

$$e_{2^n} = h_0 - 2^{-n-1}(h_0 + h_1 + 2h_2 + 4h_4 + \cdots + 2^n h_{2^n})$$

is the indicator function of the set $[2^{-n-1}, 1)$, and $e_k = h_k$ for remaining k . Suppose that λ is a given positive number and let n be an integer satisfying $2^{n+3} \geq \lambda$. Then $\| -2^{n+2}e_{2^n} + 2^{n+2}e_{2^{n+1}} \|_1 = 1$ and for any $x \in [2^{-n-1}, 1)$ we have the inequality $2^{n+2}e_{2^n}(x) + 2^{n+2}e_{2^{n+1}}(x) = 2^{n+3} \geq \lambda$. Letting $n \rightarrow \infty$ yields $\beta_e(\lambda) = 1$, directly from the definition of the weak unconditional constant. Thus, the function β_e is identically 1.

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