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NORM OF THE BERGMAN PROJECTION

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Abstract

This paper deals with the the norm of the weighted Bergman projection operator $P_\alpha : L^\infty(B) \to \mathscr{B}$ where $\alpha > -1$ and $\mathscr{B}$ is the Bloch space of the unit ball B of the C^n. We consider two Bloch norms, the standard Bloch norm and invariant norm w.r.t. automorphisms of the unit ball. Our work contains as a special case the main result of the recent paper [6].

1. Introduction and preliminaries

Introduce first the notation which will be used in this paper. We follow the Rudin monograph [7]. Throughout the whole paper $n \geq 1$ is an integer. Let $\langle \cdot, \cdot \rangle$ stands for the inner product in the complex n-dimensional space C^n given by

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n,$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are coordinate representation of $z, w \in \mathsf{C}^n$ in the standard base $\{e_1, \ldots, e_n\}$ of C^n. The inner product induces the Euclidean norm

$$|z| = \langle z, z \rangle^{1/2}.$$

Denote by B the unit ball $\{z \in \mathsf{C}^n : |z| < 1\}$; let $S = \partial B$ be its boundary.

We let v be the volume measure in C^n, normalized so that $v(B) = 1$. We will also consider a class of weighted volume measures on B. For $\alpha > -1$ we define a finite measure v_α on B by

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha \, dv(z),$$

where c_α is a normalizing constant so that $v_\alpha(B) = 1$. Using polar coordinates, one can easily calculate that

$$c_\alpha = \binom{n+\alpha}{n}. \tag{1}$$

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It is well known that the bi-holomorphic mappings of B onto itself have the following form

$$\varphi_a(w) = \frac{a - \frac{\langle w,a\rangle}{|a|^2}a - (1-|a|^2)^{1/2}\big(w - \frac{\langle w,a\rangle}{|a|^2}a\big)}{1-\langle w,a\rangle} \qquad \text{for} \quad a \in B,$$

up to unitary transformations; for $a = 0$, we set $\varphi_a = -\mathrm{Id}_B$. In the case $n = 1$ this is simply the equality $\varphi_a(w) = (a-w)/(1-\overline{a}w)$. Traditionally, these mappings are also called bi-holomorphic automorphisms. By $\mathrm{Aut}(B) = \{U \circ \varphi_a : a \in B, U \in \mathcal{U}\}$, where $\mathcal{U}$ is the group of all unitary transformations of the space $\mathbf{C}^n$, is denoted the group of all bi-holomorphic automorphisms of the unit ball. One often calls $\mathrm{Aut}(B)$ the group of Möbius transformations of B.

Observe that $\varphi_a(0) = a$. Since φ_a is involutive, i.e., $\varphi_a \circ \varphi_a = \mathrm{Id}_B$, we also have $\varphi_a(a) = 0$.

The real Jacobian of φ_a is given by the expression

$$(J_{\mathbf{R}}\varphi_a)(w) = \left(\frac{1-|a|^2}{|1-\langle w,a\rangle|^2}\right)^{n+1} \quad \text{on} \quad B.$$

Two identities

$$1 - |\varphi_a(w)|^2 = \frac{(1-|a|^2)(1-|w|^2)}{|1-\langle w,a\rangle|^2} \tag{2}$$

and

$$(1-\langle w,a\rangle)(1-\langle \varphi_a(w),a\rangle) = 1-|a|^2 \tag{3}$$

for $a, w \in B$, will also be useful.

By using (2) we obtain the next relation

$$\begin{aligned} dv_\alpha(\varphi_a(w)) &= (1-|\varphi_a(w)|^2)^\alpha (J_{\mathbf{R}}\varphi_a)(w)dv(w) \\ &= \left(\frac{(1-|w|^2)(1-|a|^2)}{|1-\langle w,a\rangle|^2}\right)^\alpha \left(\frac{1-|a|^2}{|1-\langle w,a\rangle|^2}\right)^{n+1} dv(w) \\ &= \left(\frac{(1-|a|^2)}{|1-\langle w,a\rangle|^2}\right)^{n+1+\alpha} dv_\alpha(w). \end{aligned}$$

For a holomorphic function $f(z) = f(z_1, \dots, z_n)$ with $\nabla f(z)$ we denote the complex gradient

$$\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \dots, \frac{\partial f(z)}{\partial z_n}\right).$$

The Bloch space $\mathscr{B}$ contains all functions f holomorphic in B for which the semi-norm

$$\|f\|_\beta = \sup_{z\in B}(1-|z|^2)\,|\nabla f(z)|$$

is finite. One can obtain a true norm by adding $|f(0)|$, more precisely in the following way

$$\|f\|_{\mathscr{B}} = |f(0)| + \|f\|_\beta, \qquad f \in \mathscr{B}.$$

It is well known that $\mathscr{B}$ is a Banach space with the above norm. The standard reference for Bloch space of the unit disc is [1]. For the high dimensions case we refer to [8], [9], [11].

When $1 \le p < \infty$, let L^p stands for the Lebesgue space of all measurable functions in B which modulus with the exponent p is integrable in the unit ball; for $p = \infty$ let it be the space of all essentially bounded measurable functions in the unit ball. Denote by $\|\cdot\|_p$ the norm on L^p (for all $1 \le p \le \infty$). For $\alpha > -1$ the Bergman projection operator P_α is defined by

$$P_\alpha g(z) = \int_B K_\alpha(z,w)g(w)\,dv_\alpha(w), \qquad g \in L^p,$$

where

$$K_\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}, \qquad z, w \in B$$

is the weighted Bergman kernel.

Bergman type projections are central operators when dealing with questions related to analytic function spaces. One wants to prove that Bergman projections are bounded and the exact operator norm of the operator is often difficult to obtain. By the F. Forelli-W. Rudin theorem [3], the operator $P_\alpha : L^p \to L^p \cap H(B)$ is bounded if and only if $\alpha > 1/p-1$; here $1 \le p < \infty$ and $H(B)$ is the space of holomorphic functions in the unit ball. In the same paper they obtain the norm of P_α for $p = 1$ and $p = 2$. M. Mateljević and M. Pavlović [5] extended this result when $0 < p < 1$. On the other hand, for $n = 1$, the Bergman projection $P_\alpha : L^\infty \to \mathscr{B}$ is bounded and onto; see [10]. For $n > 1$ the operator $P_\alpha : L^\infty \to \mathscr{B}$ is surjective what can be seen from [11, Theorem 3.4] in the Zhu book.

The β-norm and $\mathscr{B}$-norm of the Bergman projection $P_\alpha : L^\infty \to \mathscr{B}$ are

$$\|P_\alpha\|_\beta = \sup_{\|g\|_\infty \le 1} \|P_\alpha g\|_\beta, \qquad \|P_\alpha\|_{\mathscr{B}} = \sup_{\|g\|_\infty \le 1} \|P_\alpha g\|_{\mathscr{B}},$$

respectively.

There are several equivalent ways to introduce the Bloch space in the unit ball of $\mathbb{C}^n$. The preceding one is natural and straightforward but the norm defined in that way is not invariant with respect to the group $\mathrm{Aut}(B)$. The following Bloch norm has this property. For $f \in H(B)$, we define the invariant gradient $|\tilde{\nabla} f(z)|$, where

$$\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

and where φ_z is an automorphisms of the unit ball such that $\varphi_z(0) = z$. This norm is invariant w.r.t. automorphisms of the unit ball. Namely,

$$|\tilde{\nabla}(f \circ \varphi)| = |(\tilde{\nabla} f) \circ \varphi|$$

for all $\varphi \in \mathrm{Aut}(B)$. Then the Bloch space $\mathscr{B}$ contains all holomorphic functions f in the ball B for which

$$\|f\|_{\tilde{\beta}} := \sup_{z \in B} |\tilde{\nabla} f(z)| < \infty$$

(cf. [11, Theorem 3.4] or [8]).

For $n = 1$ we have

$$|\tilde{\nabla} f(z)| = (1 - |z|^2)|\nabla f(z)|,$$

but for $n > 1$ this is no longer true. Notice that $\|\cdot\|_{\tilde{\beta}}$ is also a semi-norm. One can obtain a norm in the following way

$$\|f\|_{\tilde{\mathscr{B}}} = |f(0)| + \|f\|_{\tilde{\beta}}, \qquad f \in \mathscr{B}.$$

The $\tilde{\beta}$-norm and $\tilde{\mathscr{B}}$-norm of the Bergman projection is

$$\|P_\alpha\|_{\tilde{\beta}} = \sup_{\|g\| \le 1} \|P_\alpha g\|_{\tilde{\beta}}, \qquad \|P_\alpha\|_{\tilde{\mathscr{B}}} = \sup_{\|g\| \le 1} \|P_\alpha g\|_{\tilde{\mathscr{B}}}.$$

From the proof of [11, Theorem 3.4] we find out that

$$\|P_\alpha g\|_{\tilde{\beta}} \le C \|g\|_\infty,$$

where C is a positive constant. The later implies that P_α is a bounded operator since

$$\|P_\alpha\|_{\tilde{\mathscr{B}}} \le 1 + \|P_\alpha\|_{\tilde{\beta}}.$$

Before stating the main results let us prove the following simple lemma.

LEMMA 1.1. *For every $\alpha > -1$ we have*

$$\|P_\alpha\|_{\mathscr{B}} \le 1 + \|P_\alpha\|_{\beta} \tag{4}$$

and

$$\|P_\alpha\|_{\tilde{\mathscr{B}}} \leq 1 + \|P_\alpha\|_{\tilde{\beta}}. \tag{5}$$

PROOF. Since

$$|P_\alpha g(0)| = \left|\int_B g(w)\, dv_\alpha(w)\right| \leq \|g\|_\infty$$

it follows that

$$\|P_\alpha g\|_{\mathscr{B}} = |P_\alpha g(0)| + \|P_\alpha g\|_\beta \leq \|g\|_\infty + \|P_\alpha\|_\beta \|g\|_\infty.$$

This implies (4). The relation (5) can be proved similarly.

In this paper we find the exact norm of P_α w.r.t. β-Bloch (semi-)norm. It is the content of our Theorem 1.2 which generalizes the result from the recent paper [6] in two directions. We also estimate the $\tilde{\beta}$-Bloch (semi-)norm in Theorem 1.3.

For simplicity in computation which follows, it is convenient to introduce

$$\theta = n + 1 + \alpha.$$

Let

$$C_\alpha = \frac{\Gamma(\theta+1)}{\Gamma^2((\theta+1)/2)},$$

where Γ is Euler's Gamma function. In this paper we prove the following two theorems and present their proofs in the following two sections.

THEOREM 1.2. *For the β-(semi-)norm of the Bergman projection P_α we have*

$$\|P_\alpha\|_\beta = C_\alpha.$$

In order to formulate the next theorem, assume that $n > 1$ and define the following function on the real line:

$$\ell(t) = \theta \int_B \frac{|(1-w_1)\cos t + w_2 \sin t|}{|w_1 - 1|^\theta}\, dv_\alpha(w). \tag{6}$$

THEOREM 1.3. *For $\alpha > -1$ we have*

$$\ell(\pi/2) = \frac{\pi}{2}\ell(0) = \frac{\pi}{2} C_\alpha. \tag{7}$$

For the $\tilde{\beta}$-(semi-)norm of the Bergman projection P_α we have

$$\|P_\alpha\|_{\tilde{\beta}} = \tilde{C}_\alpha := \max_{0\leq t\leq \pi/2} \ell(t) \tag{8}$$

and

$$\frac{\pi}{2}C_\alpha \le \|P_\alpha\|_{\tilde\beta} \le \frac{\sqrt{\pi^2+4}}{2}C_\alpha. \tag{9}$$

REMARK 1.4. For $\alpha = 0$ we put $P = P_0$ and we have

$$\|P\|_\beta = \frac{(n+1)!}{\Gamma^2\left(1+\frac{n}{2}\right)}.$$

Moreover, for $n = 1$ we obtain

$$\|P\|_\beta = \frac{8}{\pi},$$

which presents the main result in [6]. As an immediate corollary of Theorem 1.2, Theorem 1.3 and Lemma 1.1 we have the following norm estimates of the Bergman projection

$$C_\alpha \le \|P_\alpha\|_{\mathscr{B}} \le 1 + C_\alpha$$

and

$$\frac{\pi}{2}C_\alpha \le \|P_\alpha\|_{\tilde{\mathscr{B}}} \le 1 + \frac{\sqrt{\pi^2+4}}{2}C_\alpha.$$

CONJECTURE 1.5. *In connection with Theorem 1.3, we conjecture that*

$$\tilde C_\alpha = \frac{\pi}{2}C_\alpha.$$

The next corollary is an immediate consequence of the boundedness of P_α.

COROLLARY 1.6. *If f is holomorphic in B and if $\Re f \in L^\infty$, then $f \in \mathscr{B}$. Moreover, there exist C such that*

$$\|f\|_\beta, \|f\|_{\tilde\beta} \le C\|\Re f\|_\infty.$$

PROOF. Note that for $g \in H^\infty(B)$,

$$P_\alpha g = g, \qquad P_\alpha \overline{g} = \overline{g(0)}.$$

Assume that f is holomorphic and moreover that $f(0) = 0$. Let $u = \Re f$ and $f_r(z) = f(rz)$ for $0 < r < 1$. We have

$$f_r = P_\alpha f_r = P_\alpha(f_r + \overline{f_r}) = 2P_\alpha u_r$$

and follows

$$\|f_r\|_\beta = \|2P_\alpha u_r\|_\beta = 2\|P_\alpha u_r\|_\beta \le 2C_\alpha \|u_r\|_\infty.$$

Letting $r \to 1$ and $\alpha \to -1$, we obtain

$$\|f\|_\beta \le 2C_{-1}\|\Re f\|_\infty,$$

where

$$C_{-1} = \lim_{\alpha\to -1} C_\alpha = n!\Gamma^{-2}\left(\frac{n+1}{2}\right).$$

Thus if f is holomorphic in B and if $\Re f \in L^\infty$, then $f \in \mathscr{B}$. If we remove the assumption $f(0) = 0$, we obtain

$$\|f\|_\beta = \|f - f(0)\|_\beta \le 2C_{-1}\|\Re(f - f(0))\|_\infty \le C\|\Re f\|_\infty,$$

where we set $C = 4C_{-1}$.

2. Proof of Theorem 1.2

What we have to find is

$$\|P_\alpha\|_\beta = \sup\Big\{(1 - |z|^2)\,|\nabla_z(P_\alpha g)(z)| : |z| < 1,\ \|g\|_\infty \le 1\Big\}.$$

A straightforward calculation yields

$$\nabla_z K_\alpha(z, w) = \frac{\theta\overline{w}}{(1 - \langle z, w\rangle)^{\theta+1}}, \qquad z, w \in B, \tag{10}$$

and this implies the formula

$$\nabla_z(P_\alpha g)(z) = \int_B \nabla_z K_\alpha(z, w)g(w)\,dv_\alpha(w), \qquad z \in B.$$

For a fixed $z \in B$ and for $\|g\|_\infty \le 1$ we have the following estimates

$$\begin{aligned}
|\nabla(P_\alpha g)(z)| &= \sup_{\zeta\in S} |\langle \nabla P_\alpha g(z), \zeta\rangle| \\
&= \sup_{\zeta\in S}\left|\int_B \langle \nabla_z K_\alpha(z, w)g(w), \zeta\rangle\,dv_\alpha(w)\right| \\
&\le \sup_{\zeta\in S}\int_B |\langle \nabla_z K_\alpha(z, w)g(w), \zeta\rangle|\,dv_\alpha(w)
\end{aligned}$$

$$= \sup_{\zeta\in S}\int_B \left|\left\langle \frac{\theta\overline{w}}{(1-\langle z,w\rangle)^{\theta+1}},\zeta\right\rangle\right| |g(w)|\,dv_\alpha(w)$$

$$\leq \sup_{\zeta\in S}\int_B \frac{\theta|\langle\overline{w},\zeta\rangle|}{|1-\langle z,w\rangle|^{\theta+1}}\,dv_\alpha(w).$$

Denote

$$F_\zeta(z)=\theta\int_B \frac{(1-|z|^2)|\langle w,\overline{\zeta}\rangle|}{|1-\langle z,w\rangle|^{\theta+1}}\,dv_\alpha(w).$$

The statement of the Theorem 1.2 will follow directly from the following two equalities

$$\|P_\alpha\|_\beta = \sup\{F_\zeta(z) : z\in B, \zeta\in S\} = C_\alpha,$$

which will be proved through the following lemmas.

LEMMA 2.1. *For every $\alpha > -1$ we have*

$$\sup\{F_\zeta(z) : z\in B, \zeta\in S\} \leq C_\alpha.$$

LEMMA 2.2. *For every $\alpha > -1$ there exists a sequence of functions $\{g_m \in L^\infty : \|g_m\|_\infty = 1\}$ and a sequence of vectors $\{z_m\in B\}$ such that*

$$\lim(1-|z_m|^2)|\nabla(P_\alpha g_m)(z_m)| = C_\alpha.$$

In order to give proofs of the previous lemmas we need [7, Proposition 1.4.10] and some its corollaries collected in the following proposition.

PROPOSITION 2.3. a) *For $z\in B$, c real, $t>-1$ define*

$$J_{c,t}(z)=\int_B \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}\,dv(w).$$

When $c<0$, then $J_{c,t}$ is bounded in B. Moreover,

$$J_{c,t}(z)=\frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma(\lambda_1)^2}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+\lambda_1)|z|^{2k}}{\Gamma(k+1)\Gamma(n+1+t+k)}, \tag{11}$$

where $\lambda_1=(n+1+t+c)/2$.

b) *Further we can write $J_{c,t}$ in the closed form as*

$$J_{c,t}(z)=\frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma(n+1+t)}\,{}_2F_1(\lambda_1,\lambda_1,n+1+t,|z|^2), \tag{12}$$

where ${}_2F_1$ is the Gauss hypergeometric function. In particular

$$J_{c,t}\,(z/|z|) = \frac{\Gamma(n+1)\Gamma(t+1)\Gamma(-c)}{\Gamma^2((n+1+t-c)/2)}, \qquad z \neq 0. \tag{13}$$

PROOF. The first part of this proposition coincides with the first part of [7, Proposition 1.4.10] together with its proof. In order to prove the part b) we recall the classical definition of the Gauss hypergeometric function:

$$F(a,b,c,z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{14}$$

where $(d)_k = d(d+1)\dots(d+k-1)$ is the Pochhammer symbol. The series converges at least for complex $z \in \mathsf{U} := \{z : |z| < 1\} \subseteq \mathsf{C}$ and for $z \in \mathsf{T} := \{z : |z| = 1\}$, if $c > a+b$. For $\Re(c) > \Re(b) > 0$ we have the following well-known formula

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\, dt. \tag{15}$$

In particular the Gauss theorem states that

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c) > \Re(a+b). \tag{16}$$

In order to derive (12) from (11), we use the formula $\Gamma(x+1) = x\Gamma(x)$ and obtain

$$\Gamma(k+\lambda_1) = (\lambda_1)_k \Gamma(\lambda_1) \tag{17}$$

and

$$\Gamma(n+1+t+k) = (n+1+t)_k \Gamma(n+1+t). \tag{18}$$

From (14), (17) and (18), by taking $a = b = \lambda_1$ and $c = n+1+t$, we derive (12). The formula (13) follows from (16) and observing that $c > a+b = n+1+t+c$.

Also we need the Vitali theorem, and include its formulation (cf. [4, Theorem 26.C]).

THEOREM 2.4 (Vitali). *Let X be a measure space with finite measure μ, and let $h_m : X \mapsto \mathsf{C}$ be a sequence of functions that is uniformly integrable, i.e., such that for every $\varepsilon > 0$ there exists $\delta > 0$, independent of k satisfying*

$$\mu(E) < \delta \Longrightarrow \int_E |h_m|\, d\mu < \varepsilon. \tag{$\dagger$}$$

Now: if $\lim h_m(x) = h(x)$ *a.e., then*

$$\lim \int_X h_m \, d\mu = \int_X h \, d\mu. \tag{‡}$$

In particular, if

$$\sup \int_X |h_m|^p \, d\mu < \infty \qquad \textit{for some} \quad p > 1,$$

then (†) *and* (‡) *hold.*

PROOF OF LEMMA 2.1. For fixed $z \in B$ let us make the change of variables $w = \varphi_z(\omega)$, $\omega \in B$ in the integral which represent $F_\zeta(z)$. In previous section we obtained the next relation for pull-back measure

$$dv_\alpha(\varphi_z(\omega)) = \frac{\left(1 - |z|^2\right)^\theta}{|1 - \langle z, \omega\rangle|^{2\theta}} dv_\alpha(\omega).$$

By using this result and relation (3) in the forth equality, we find

$$\begin{aligned}
\theta^{-1} F_\zeta(z) &= \int_B \frac{(1-|z|^2)|\langle w, \overline{\zeta}\rangle|}{|1 - \langle z, w\rangle|^{\theta+1}} \, dv_\alpha(w) \\
&= \int_B \frac{(1-|z|^2)|\langle \varphi_z(\omega), \overline{\zeta}\rangle|}{|1 - \langle z, \varphi_z(\omega)\rangle|^{\theta+1}} \frac{(1-|z|^2)^\theta}{|1 - \langle z, \omega\rangle|^{2\theta}} \, dv_\alpha(\omega) \\
&= \int_B \frac{(1-|z|^2)^{\theta+1}|\langle \varphi_z(\omega), \overline{\zeta}\rangle|}{|1 - \langle z, \varphi_z(\omega)\rangle|^{\theta+1}|1 - \langle z, \omega\rangle|^{2\theta}} \, dv_\alpha(\omega) \\
&= \int_B \frac{(|1 - \langle z, \omega\rangle|^{\theta+1}|1 - \langle z, \varphi_z(\omega)\rangle|)^{\theta+1}|\langle \varphi_z(\omega), \overline{\zeta}\rangle|}{|1 - \langle z, \varphi_z(\omega)\rangle|^{\theta+1}|1 - \langle z, \omega\rangle|^{2\theta}} \, dv_\alpha(\omega) \\
&= \int_B \frac{|\langle \varphi_z(\omega), \overline{\zeta}\rangle|}{|1 - \langle z, \omega\rangle|^{\theta-1}} \, dv_\alpha(\omega).
\end{aligned}$$

Therefore

$$\theta^{-1} F_\zeta(z) = \int_B \frac{|\langle \varphi_z(\omega), \overline{\zeta}\rangle|}{|1 - \langle z, \omega\rangle|^{\theta-1}} \, dv_\alpha(\omega). \tag{19}$$

From the last representation of $F_\zeta(z)$, since

$$|\langle \varphi_z(\omega), \overline{\zeta}\rangle| \le 1, \qquad z \in B, \zeta \in S,$$

we infer

$$\theta^{-1} F_\zeta(z) = \int_B \frac{|\langle \varphi_z(\omega), \overline{\zeta}\rangle|}{|1-\langle z,\omega\rangle|^{\theta-1}}\, dv_\alpha(\omega)$$

$$\leq c_\alpha \int_B \frac{(1-|\omega|^2)^\alpha\, dv(\omega)}{|1-\langle z,\omega\rangle|^{\theta-1}} = c_\alpha J_{c,t}(z),$$

where we set $t=\alpha$, $c=-1$;

$$c_\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)}$$

as in (1). Then $\lambda_1 = (n+\alpha)/2$ (the parameter from Proposition 2.3).

For $z \in B$, $z \neq 0$ we have

$$J_{c,t}(z) \leq J_{c,t}(z/|z|) = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma^2((n+1+\alpha+1)/2)}.$$

Thus

$$\theta^{-1} F_\zeta(z) \leq c_\alpha \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma^2((n+1+\alpha+1)/2)} = \theta^{-1} C_\alpha \tag{20}$$

what is the statement of this lemma.

Proof of Lemma 2.2. Take $\zeta = e_1$ and $z_m = \frac{m}{m+1}\zeta$. Define

$$g_m(w) = \frac{w_1}{|w_1|} \frac{|1-\langle z_m, w\rangle|^{\theta+1}}{(1-\langle w, z_m\rangle)^{\theta+1}}, \qquad w \in B,\ w_1 \neq 0.$$

Then $g_m \in L^\infty$ and $\|g_m\|_\infty = 1$. Further from (19) and (10) we obtain

$$\begin{aligned}(1-|z_m|^2)|\nabla(P_\alpha g_m)(z_m)| &\geq (1-|z_m|^2)|\langle \nabla(P_\alpha g_m)(z_m), \zeta\rangle| \\ &= (1-|z_m|^2)\left|\int_B \langle \nabla_z K_\alpha(z,w) g_m(w), \zeta\rangle\, dv_\alpha(w)\right| \\ &= \theta \int_B \frac{(1-|z_m|^2)|w_1|}{|1-\langle z_m, w\rangle|^{\theta+1}}\, dv_\alpha(w) \\ &= \theta \int_B \frac{|\langle \varphi_{z_m}(\omega), \overline{\zeta}\rangle|}{|1-\langle z_m,\omega\rangle|^{\theta-1}}\, dv_\alpha(\omega) =: G_m.\end{aligned}$$

For $p = \frac{\theta-1/2}{\theta-1}$ (note that $p>1$), according to Proposition 2.3 (take $c=-1/2$ and $t=\alpha$)

$$\sup \int_B \left(\frac{|\langle \varphi_{z_m}(\omega), \overline{\zeta}\rangle|}{|1-\langle z_m,\omega\rangle|^{\theta-1}}\right)^p dv_\alpha(\omega) < \infty$$

(notice also that $|\langle\varphi_{z_m}(\omega),\overline{\zeta}\rangle|\le 1$). Therefore by the Vitali theorem

$$\begin{aligned}\lim G_m &= \theta \lim \int_B \frac{|\langle\varphi_{z_m}(\omega),\overline{\zeta}\rangle|}{|1-\langle z_m,\omega\rangle|^{\theta-1}}\,dv_\alpha(\omega)\\ &= \theta \int_B \lim \frac{|\langle\varphi_{z_m}(\omega),\overline{\zeta}\rangle|}{|1-\langle z_m,\omega\rangle|^{\theta-1}}\,dv_\alpha(\omega).\end{aligned}$$

For fixed $\omega\in B$ we have

$$\lim \frac{|\langle\varphi_{z_m}(\omega),\overline{\zeta}\rangle|}{|1-\langle z_m,\omega\rangle|^{\theta-1}} = \frac{|\langle\zeta,\overline{\zeta}\rangle|}{|1-\langle\zeta,\omega\rangle|^{\theta-1}} = \frac{1}{|1-\langle\zeta,\omega\rangle|^{\theta-1}}.$$

Therefore, by using again Proposition 2.3, we obtain

$$\begin{aligned}\lim(1-|z_m|^2)|\nabla(P_\alpha g)(z_m)| &= \theta c_\alpha J_{-1,\alpha}(e_1)\\ &= \theta\binom{n+\alpha}{n}\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma^2((n+1+\alpha+1)/2)}\\ &= \frac{\Gamma(\theta+1)}{\Gamma^2((\theta+1)/2)} = C_\alpha,\end{aligned}$$

what finishes the proof of this lemma.

3. Proof of Theorem 1.3

We have to find and estimate the following extremum

$$\tilde{C}_\alpha = \|P_\alpha\|_{\tilde{\beta}} = \sup\big\{|\tilde{\nabla}_z(P_\alpha g)(z)| : |z|<1,\ \|g\|_\infty\le 1\big\}.$$

We first prove (8). It follows from the following two lemmas.

LEMMA 3.1. *For $\alpha>-1$ and ℓ defined in* (6) *we have*

$$\tilde{C}_\alpha \le \max_{0\le t\le\pi/2}\ell(t).$$

PROOF. Let $f=P_\alpha g$. We begin as in the proof of [11, Theorem 3.4]. We have

$$\begin{aligned}(f\circ\varphi_a)(z) = (P_\alpha g\circ\varphi_a)(z) &= \int_{\mathsf{B}} K_\alpha(\varphi_a(z),w)g(w)\,dv_\alpha(w)\\ &= \int_{\mathsf{B}} K_\alpha(\varphi_a(z),\varphi_a(w))g(\varphi_a(w))\,dv_\alpha(\varphi_a(w))\end{aligned}$$

Since

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)},$$

we have

$$f \circ \varphi_a(z) = \frac{(1 - \langle z, a \rangle)^\theta}{(1 - |a|^2)^\theta} \int_B \frac{(1 - \langle a, w \rangle)^\theta}{(1 - \langle z, w \rangle)^\theta} g \circ \varphi_a(w)\, dv_\alpha(\varphi_a(w)).$$

Differentiating in z at 0 by using the product rule we have

$$\tilde{\nabla} f(a) = \theta \int_B \frac{(\overline{w} - \overline{a})(1 - \langle a, w \rangle)^\theta}{(1 - |a|^2)^\theta} g \circ \varphi_a(w)\, dv_\alpha(\varphi_a(w)),$$

where

$$dv_\alpha(\varphi_a(w)) = \left(\frac{(1 - |a|^2)}{|1 - \langle w, a \rangle|^2} \right)^\theta dv_\alpha(w).$$

Thus

$$\tilde{\nabla} f(a) = \theta \int_B \frac{(\overline{w} - \overline{a})(1 - \langle a, w \rangle)^\theta}{|1 - \langle w, a \rangle|^{2\theta}} g \circ \varphi_a(w)\, dv_\alpha(w), \tag{21}$$

and consequently for

$$\theta' = \binom{n + \alpha}{n} \theta, \tag{22}$$

$$\begin{aligned} |\tilde{\nabla} f(a)| &= \theta \sup_{\zeta \in S} \left| \int_B \langle \frac{(\overline{w} - \overline{a})(1 - \langle a, w \rangle)^\theta}{|1 - \langle w, a \rangle|^{2\theta}} g \circ \varphi_a(w), \zeta \rangle\, dv_\alpha(w) \right| \\ &\le \theta' \sup_{\zeta \in S} \int_B \left| \left\langle \frac{(\overline{w} - \overline{a})}{|1 - \langle w, a \rangle|^\theta}, \zeta \right\rangle \right| |g \circ \varphi_a(w)| (1 - |w|^2)^\alpha\, dv(w) \\ &\le \theta' \|g\|_\infty \sup_{\zeta \in S} \int_B |\langle \overline{w} - \overline{a}, \zeta \rangle| \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, a \rangle|^\theta}\, dv(w). \end{aligned}$$

Let

$$L(a) = \sup_{\zeta \in S} \int_B |\langle \overline{w} - \overline{a}, \zeta \rangle| \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, a \rangle|^\theta}\, dv(w)$$

and define

$$L = \sup_{a \in \overline{B}} L(a).$$

Then

$$L = \sup_{a \in \overline{B}} \sup_{\zeta \in S} \int_B |S_{\zeta, w}(a)|\, dv_\alpha(w),$$

where

$$S_{\zeta,w}(a) = \frac{\langle w - a, \zeta\rangle}{(1 - \langle w, a\rangle)^\theta}.$$

Observe that $S_{\zeta,w}(a)$ is a subharmonic function in a. It follows that $a \to L(a)$ is subharmonic and its maximum is achieved on the boundary of the unit ball. Therefore there exist $a_0, \zeta_0 \in S$ such that

$$L = \int_B |\langle w - a_0, \zeta_0\rangle| \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, a_0\rangle|^\theta}\, dv(w).$$

Let U be an unitary transformation of $\mathbf{C}^n$ onto itself such that $Ua_0 = e_1$ and $U\zeta_0 = \cos t e_1 + \sin t e_2$ for some $t \in [0, \pi]$ (Here $t = \arg(a_0, \zeta_0)$). Take the substitution $w = U\omega$. Then we obtain

$$\begin{aligned}
L &= \int_B |\langle U\omega - Ue_1, \zeta_0\rangle| \frac{(1 - |U\omega|^2)^\alpha}{|1 - \langle U\omega, a_0\rangle|^\theta}\, dv(U\omega) \\
&= \int_B |\langle \omega - e_1, U\zeta_0\rangle| \frac{(1 - |\omega|^2)^\alpha}{|1 - \langle \omega, Ua_0\rangle|^\theta}\, dv(\omega) \\
&= \int_B |\langle \omega - e_1, \cos t e_1 + \sin t e_2\rangle| \frac{(1 - |\omega|^2)^\alpha}{|1 - \langle \omega, e_1\rangle|^\theta}\, dv(\omega) \\
&= \int_B \frac{|(1 - w_1)\cos t + w_2 \sin t|}{|w_1 - 1|^\theta}\, dv_\alpha(w).
\end{aligned}$$

LEMMA 3.2. *Let ℓ be defined as in* (6). *Then*

$$\tilde{C}_\alpha \geq \ell(\pi/2).$$

PROOF. Let $\zeta = e_2$, $a = \epsilon_m e_1$, where $\epsilon_m = \frac{m}{m+1}$. Then

$$|\langle \tilde{\nabla} f(a), \zeta\rangle| = \theta' \left| \int_B \frac{\overline{w}_2 (1 - \epsilon_m w_1)^\theta (1 - |w|^2)^\alpha}{|1 - \epsilon_m w_1|^{2\theta}} g \circ \varphi_a(w)\, dv(w) \right|.$$

Define g_m such that

$$\overline{w}_2 (1 - \epsilon_m w_1)^\theta g_k \circ \varphi_a(w) = |\overline{w}_2 (1 - \epsilon_m w_1)^\theta|$$

and let $f_m = P_\alpha g_m$. We obtain

$$|\langle \tilde{\nabla} f_m(a), \zeta\rangle| = \theta' \int_B |w_2| \frac{(1 - |w|^2)^\alpha}{|1 - \epsilon_m w_1|^\theta}\, dv(w).$$

Thus

$$\tilde{C}_\alpha \geq \sup_{m,\zeta,a} |\langle \tilde{\nabla} f_m(a), \zeta \rangle| \geq \ell(\pi/2).$$

In order to prove (7) and (9) we need the following lemma (which is an extension of a corresponding result of L. Bungart, G. Folland and Ch. Fefferman, cf. [7, Proposition 1.4.9]).

LEMMA 3.3. *For a multi-index $\eta = (\eta_1, \ldots, \eta_n)$ we have*

$$\int_S |\zeta^\eta| \, d\sigma(\zeta) = \frac{(n-1)! \prod_{j=1}^n \Gamma\left(1 + \frac{\eta_j}{2}\right)}{\Gamma\left(n + \frac{|\eta|}{2}\right)} \tag{23}$$

and

$$\int_B |z^\eta| \, dv_\alpha(z) = \frac{\Gamma(\theta)}{\Gamma\left(\theta + \frac{|\eta|}{2}\right)} \prod_{j=1}^n \Gamma\left(1 + \frac{\eta_j}{2}\right); \tag{24}$$

here, $w^\eta = \prod_{j=1}^n w_j^{\eta_j}$, $|\eta| = \sum_{j=1}^n \eta_j$; σ is the area normalized measure on S.

PROOF. We have to adapt the proof of corresponding result in Rudin's book, where it is proved the same statement for $\eta = 2\chi$, where χ is a multi-index. We only sketch the proof.

Denote

$$I = \int_{\mathbf{C}^n} |z^\eta| \exp(-|z|^2) \, dV(z),$$

where dV is the ordinary Lebesgue measure on $\mathbf{C}^n$. By Fubini's theorem

$$I = \prod_{j=1}^n \int_{\mathbf{C}} |\lambda|^{\eta_j} \exp(-|\lambda|^2) \, dV(\lambda).$$

One can easily derive

$$I = \pi^n \prod_{j=1}^n \Gamma(1 + \eta_j/2).$$

On the other hand, by applying the polar coordinates transformation in I, we obtain (ω_{2n} is the volume measure of unit ball)

$$I = 2n\omega_{2n} \int_0^\infty r^{|\eta|+2n-1} \exp(-r^2) \, dr \int_S |\zeta^\eta| \, d\sigma(\zeta).$$

From the last two expression for I it follows the first result of this lemma.

Let us prove now (24). In

$$\int_B f(x)\,dv_\alpha(x) = 2n\binom{n+\alpha}{n}\int_0^1 r^{2n-1}(1-r^2)^\alpha\,dr\int_S f(r\zeta)\,d\sigma(\zeta),$$

take $f(z) = |z|^\eta$. Since

$$\int_S f(r\zeta)\,d\sigma(\zeta) = r^{|\eta|}\frac{(n-1)!\prod_{j=1}^n \Gamma\big(1+\frac{\eta_j}{2}\big)}{\Gamma\big(n+\frac{|\eta|}{2}\big)}$$

and

$$2n\binom{n+\alpha}{n}\int_0^1 r^{2n+|\eta|-1}(1-r^2)^\alpha\,dr = \frac{\Gamma(1+\alpha+n)\Gamma\big(n+\frac{|\eta|}{2}\big)}{\Gamma(n)\Gamma\big(1+\alpha+n+\frac{|\eta|}{2}\big)},$$

it follows the relation.

The relation (7) and the left-hand inequality in (9) follows from the following lemma (in view of (7)).

LEMMA 3.4. *Let $\ell(t)$ be defined as in* (6)*. Then $\ell(0) = C_\alpha$ and $\ell(\pi/2) = \frac{\pi}{2}C_\alpha$.*

PROOF. The relation $\ell(0) = C_\alpha$ follows at once.

Prove the second relation. Observe first that for $l \neq k$

$$\int_B w_1^l\overline{w}_1^k|w_2|\,dv(w) = 0.$$

By choosing $\eta(k) = (2k, 1, 0, \ldots, 0)$ we obtain

$$\begin{aligned} J &= \int_B \frac{|w_2|}{|1-w_1|^\theta}\,dv_\alpha(w) = \int_B \frac{|w_2|}{|(1-w_1)^{\theta/2}|^2}\,dv_\alpha(w) \\ &= \sum_{k=0}^\infty \binom{-\theta/2}{k}^2\int_B |w_1|^{2k}|w_2|\,dv_\alpha(w), \end{aligned}$$

$w = (w_1, \ldots, w_n)$. From (24) we find that

$$\int_B |w_1|^{2k}|w_2|\,dv_\alpha(w) = \int_B |z^{\eta(k)}|\,dv_\alpha(z) = \frac{\Gamma(\theta)\Gamma\big(\frac{3}{2}\big)\Gamma(1+k)}{\Gamma\big(\theta+\frac{|\eta|}{2}\big)}.$$

Therefore

$$J = \Gamma\big(\tfrac{3}{2}\big)\Gamma(\theta)\sum_{k=0}^{\infty}\binom{-\theta/2}{k}^2 \frac{k!}{\Gamma\big(\theta+k+\frac{1}{2}\big)} = \frac{\Gamma\big(\frac{3}{2}\big)\Gamma(\theta)}{\Gamma\big(\theta+\frac{1}{2}\big)}\sum_{k=0}^{\infty}\frac{\big(\big(\frac{\theta}{2}\big)_k\big)^2}{\big(\theta+\frac{1}{2}\big)_k k!}$$

$$= \frac{\Gamma\big(\frac{3}{2}\big)\Gamma(\theta)}{\Gamma\big(\theta+\frac{1}{2}\big)}\, {}_2F_1\left(\frac{\theta}{2},\frac{\theta}{2};\theta+\frac{1}{2};1\right) = \frac{\pi\Gamma(\theta)}{2\Gamma^2\big(\frac{\theta}{2}+\frac{1}{2}\big)}.$$

The last equality is derived with help of Gauss theorem, i.e., of the relation (16). Hence

$$\ell(\pi/2) = \theta\int_B \frac{|w_2|}{|1-w_1|^{n+1}}\,dv_\alpha(w)$$

$$= \frac{\pi\theta\Gamma(\theta)}{2\Gamma^2\big(\frac{\theta}{2}+\frac{1}{2}\big)} = \frac{\pi}{2}\frac{\Gamma(\theta+1)}{\Gamma^2\big(\frac{\theta}{2}+\frac{1}{2}\big)} \quad \left(> \frac{\Gamma(\theta+1)}{\Gamma^2\big(\frac{\theta}{2}+\frac{1}{2}\big)} = \|P\|_\beta\right).$$

To finish the proof of Theorem 1.3 we need to prove the right inequality in (9). It follows from this simple observation

$$\tilde{C}_\alpha \le |\sin t|\ell(0) + |\cos t|\ell(\pi/2) \le \sqrt{\ell(0)^2 + \ell(\pi/2)^2}.$$

REMARK 3.5. If $g \in C(\overline{B})$ and $f = P_\alpha[g]$, then it follows from (21) and Vitali's theorem that there exist a mapping $\Phi : S \to \mathbf{C}^n$ such that for $\zeta \in S$

$$\lim_{a\to\zeta}\tilde{\nabla} f(a) = g(\zeta)\Phi(\zeta).$$

But if g is a polynomial, then we know that $\lim_{a\to t}\tilde{\nabla} f = 0$, implying that

$$\Phi(\zeta) = \int_B \frac{(\overline{w}-\overline{\zeta})(1-\langle\zeta,w\rangle)^\theta}{|1-\langle w,\zeta\rangle|^{2\theta}}\,dv_\alpha(w) = 0 \qquad \text{for} \quad \zeta\in S.$$

If by $\mathscr{B}_0$ we denote the little Bloch space, i.e. the space of holomorphic mappings f defined on the unit ball such that

$$\lim_{|z|\to 1}|\tilde{\nabla} f(z)| = 0,$$

and consider the Bergman projection

$$P_\alpha : C(\overline{B}) \to \mathscr{B},$$

then by the previous consideration we obtain

$$P_\alpha(C(\overline{B})) \subset \mathscr{B}_0 \subset \mathscr{B}.$$

It follows that
$$\|P_\alpha : C(\overline{B}) \to \mathscr{B}_0\| \leq \tilde{C}_\alpha$$

w.r.t. invariant $\tilde{\beta}$ Bloch semi-norm. Moreover, since the extremal sequence (see the proof of Lemma 3.2) is consisted of continuous functions g_k, we obtain that
$$\|P_\alpha : C(\overline{B}) \to \mathscr{B}_0\| = \tilde{C}_\alpha.$$

The same can be repeated for the standard β Bloch semi-norm. We refer to [2, Table 2] for the image under Bergman projection P of some spaces different from L^∞ and $C(\overline{B})$ (for $n = 1$).

REFERENCES

1. Anderson, J. M., Clunie, J., and Pomerenke, C., *On Bloch functions and normal functions*, J. Reine Angew. Math. 270 (1974), 12–37.
2. Axler, S., and Zhu, K., *Boundary behavior of derivatives of analytic functions*, Michigan Math. J. 39 (1992), 129–143.
3. Forelli, F., and Rudin, W., *Projections on spaces of holomorphic functions in balls*, Indiana Univ. Math. J. 24 (1974), 593–602.
4. Halmos, P. R., *Measure Theory*, Van Nostrand, New York 1950.
5. Mateljević, M., and Pavlović, M., *An extension of the Forelli-Rudin projection theorem*, Proc. Edinburgh Math. Soc. 36 (1993), 375–389.
6. Perälä, A., *On the optimal constant for the Bergman projection onto the Bloch space*, Ann. Acad. Sci. Fen. Math. 37 (2012), 245–249.
7. Rudin, W., *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Grundl. Math. Wiss. 241, Springer, Berlin 1980.
8. Timoney, R. M., *Bloch functions in several complex variables I*, Bull. Lond. Math. Soc. 12 (1980), 241–267.
9. Timoney, R. M., *Bloch functions in several complex variables II*, J. Reine Angew. Math. 319 (1980), 1–22.
10. Zhu, K., *Operator Theory in Function Spaces*, Mon. Textb. Pure Appl. Math. 139, Marcel Dekker, New York 1990.
11. Zhu, K., *Spaces of Holomorphic Functions in the Unit Ball*, Grad. Texts Math. 226, Springer, Berlin 2005.

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