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STRONG n-GENERATORS AND THE RANK OF SOME NOETHERIAN ONE-DIMENSIONAL INTEGRAL DOMAINS

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1. Introduction

Throughout this paper any ring is a commutative Noetherian ring with identity. If A is a ring, I and J are ideals of A such that $I \supset J$, and S is a multiplicative set of A then let $\mu(I)$, $\mu(I/J)$, and $\mu(S^{-1}I)$ respectively be the minimal number of generators of I, I/J, and $S^{-1}I$ respectively as an A, A/J, or $S^{-1}A$-module respectively.

Definition. Let A be a ring and n a positive integer. An ideal I of A is *n-generated* if $\mu(I) \leq n$, and if $\mu(I) \leq n$ for each ideal I of A then I.S. Cohen [6] says that A is of *finite rank n*. S.T. Chapman and N.H. Vaughan [5] call a non-zero element a in A a *strong n-generator* if a can be chosen as the first of n generators of each ideal in which it is contained, and if moreover a is contained in some ideal I such that $\mu(I) = n$ then we introduce the concept *proper* strong n-generator for a. An ideal I of A is *strongly n-generated* if I is n-generated and $\mu(I/(a)) < n$ for each non-zero element a of I.

Notation. If A is an integral domain then let $\overline{A}$ be the integral closure of A, and if A is of finite rank then let $\mu_*(A) = \max\{\mu(I) : I \text{ is an ideal of } A\}$. Let R be a one-dimensional integral domain, let F be the field of fractions of R, and let C be the conductor of R in $\overline{R}$, i.e. $C = (R : \overline{R})_R$. If I is a non-zero ideal of R then $\mathrm{Ass}(I)$ is the set of maximal ideals that contain I.

We have that $\overline{R}$ is an integrally closed one-dimensional domain, and hence $\overline{R}$ is a Dedekind domain. It is well-known that a Dedekind domain is strongly two-generated, and we also know that $\mathrm{Ass}(I)$ is finite for each non-zero ideal I of R since R is Noetherian.

Let C be non-zero which is the same as $\overline{R}$ being a fractional ideal of R, and hence $\overline{R}$ is a finitely generated R-module. By I.S. Cohen [6], Theorem 10 and Corollary 3, R is of finite rank, and if $\mu_*(R_M) \leq n-1$ for each maximal

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ideal M of R then $\mu_*(R) \leq n$ for some integer n. By H. Bass [4], Proposition 1.4, $\mu_*(R) \leq \max(2,k)$ if $\mu_*(R_M) \leq k$ for each maximal ideal M of R. A specification of these results is that if R is not integrally closed then $\mu_*(R) = \mu(C) = \mu(C_M) = \mu_*(R_M)$ for some $M \in \mathrm{Ass}(C)$ (Theorem 15). In Proposition 3 is proved that an ideal I of R is strongly $\mu(I)$-generated if and only if $\mu(I) = 2$, and $\mu(I/I^2) = 1$. Let $M \in \mathrm{Ass}(C)$. Then there is a unique biggest M-primary ideal I such that $\mu(I) = \mu_*(R_M)$ (Proposition 16), and hence if M is such that $\mu_*(R_M) = \mu_*(R)$ then $\mu(I) = \mu_*(R)$. In [12], Theorem 21 we have already proved that the set of strong $\mu_*(R)$-generators of R is $R \setminus \bigcup_{i\in S}(I_iM_i)$ where $\{M_i\}_{i\in S}$ is the set of maximal ideals of R such that $\mu_*(R_{M_i}) = \mu_*(R)$, and I_i is the unique biggest M_i-primary ideal of R such that $\mu(I_i) = \mu_*(R)$. Let P be the set of *proper* strong $\mu_*(R)$-generators of R. If $\mu_*(R) = 2$, and $\{M_i\}_{i\in T}$ is the set of maximal ideals of R such that $M_i \notin \mathrm{Ass}(C)$, and $M_i\overline{R} \neq r\overline{R}$ for any $r \in R$ then $P = \bigcup_{M\in Ass(C)}(M \setminus M^2) \bigcup_{i\in T} M_i$ (Theorem 11). If $\mu_*(R) > 2$ then the set $\{M_i\}_{i\in S}$ of maximal ideals of R such that $\mu_*(R_{M_i}) = \mu_*(R)$ is finite and contained in $\mathrm{Ass}(C)$. If $S = \{1,\dots,k\}$ then $P = \bigcup_{i=1}^{k}(I_i \setminus I_iM_i)$, where I_i is the unique biggest M_i-primary ideal of R such that $\mu(I_i) = \mu_*(R)$, $i = 1,\dots,k$ (Theorem 18). In Section 4 we treat rings strictly between K and $K[X]$ where K is a field. In Proposition 19 we prove that any such ring has non-zero conductor in its integral closure. If R is a ring such that $K[X^k] \subseteq R \subseteq K[X]$ where k is a positive integer, and $R \nsubseteq K[X^l]$ for any $l > 1$ then the field of fractions of R is $K(X)$ (Lemma 20), the integral closure of R is $K[X]$, and $\mu_*(R) \leq k$ (Proposition 21). For the special case $R = K[X^k, aX^l]$ with a in $K[X^k] \setminus K$ and some $k, l \in \mathsf{N}$ such that $(k,l) = 1$ we show that $\mu_*(R) = k$, and for each maximal ideal M such that $\mu_*(R_M) = k$, M^{k-1} is the biggest M-primary ideal I such that $\mu(I) = k$ (Proposition 23). Let $n_1,\dots,n_l$ be positive integers such that $(n_1,\dots,n_l) = 1$, and let N be the least integer such that for any $i \geq 0$, $N + i$ belongs to the numerical semigroup generated by $\{n_1,\dots,n_l\}$. If the numerical semigroup ring $K[X^{n_1},\dots,X^{n_l}]$ is contained in R, and R is contained in $K[X]$ then $C = X^N K[X]$, and if $M = XK[X] \cap R$ then $\mu_*(R) = \min v(M)$ where $v : K(X) \longrightarrow \mathsf{Z}$ is the valuation of $K(X)$ with $K[X]_{(X)}$ as valuation ring (Proposition 24).

2. Preliminaries

Let A be a domain, let I be a non-zero ideal of A, and let $S_I = A \setminus \bigcup_{i\in S} M_i$ where $\{M_i\}_{i\in S}$ is the set of maximal ideals of A which contain I.

PROPOSITION 1. *Let A be a one-dimensional domain, and let I be an ideal of A such that $\mu(I) > 2$. Then $\mu(I_M) = \mu(S_I^{-1}I) = \mu(I)$ for some $M \in \mathrm{Ass}(I)$.*

PROOF. We have $\mu(I_M) \leq \mu(I)$ for each maximal ideal M of A. By [8], Satz 1, $\mu(I) \leq \max\{\mu(I_p) + \dim A/p : p \in \text{Spec } A\} \leq \max\{\mu(I_M), 2 : M \in \text{Ass}(I)\} \leq \mu(I)$ whence $\mu(I) = \mu(S_I^{-1}I) = \mu(I_M)$ for some $M \in \text{Ass}(I)$.

LEMMA 2. *Let A be an integral domain with non-zero conductor C in the integral closure $\overline{A}$ of A. If s is comaximal to C, $a \in \overline{A}$, and $sa \in A$ then $a \in A$.*

PROOF. We have $sb = 1 + r$ for some $r \in C$, and some $b \in A$. Then $asb = a + ra \in A$, and hence $a \in A$.

3. One-dimensional integral domains

Let R be a one-dimensional integral domain with field of fractions F, and let the conductor C of R in $\overline{R}$ be non-zero. If M is a maximal ideal of R then $\overline{R_M} = (R \setminus M)^{-1}\overline{R}$ by [2], Proposition 5.12.

PROPOSITION 3. *Let I be an ideal of R. Then the following are equivalent:*
1) *I is strongly $\mu(I)$-generated.*
2) *$\mu(I) = 2$, and $\mu(S_I^{-1}I) = 1$.*
3) *$\mu(I) = 2$, and $\mu(I/I^2) = 1$.*
4) *$\mu(I) = 2$, and $\mu(I_M) = 1$ for each maximal ideal M of R.*

PROOF. 1) $\Rightarrow$ 2) Let I be an ideal of R which is strongly $\mu(I)$-generated. By [1], Theorem 8, $\mu(S_I^{-1}I) < \mu(I)$, and hence $\mu(I) = 2$ by Proposition 1, and $\mu(S_I^{-1}I) = 1$.

2) $\Leftrightarrow$ 3) By [1], Proposition 1, $\mu(I/I^2) = \mu(S_I^{-1}I)$.

2) $\Rightarrow$ 4) is obvious.

4) $\Rightarrow$ 1) By [12], Proposition 19, I is strongly two-generated.

LEMMA 4. *Let M be a maximal ideal of R such that $M \notin \text{Ass}(C)$. Then there is a unique maximal ideal N of $\overline{R}$ such that $N \cap R = M$. Moreover $M\overline{R} = N$, and $\overline{R_M} = R_M$ is a discrete valuation ring* (DVR).

PROOF. By [14], Ch. 5, §5, Lemma, $R_M = \overline{R_M}$. By [2], Proposition 9.2, R_M is a DVR, and by [2], Proposition 3.11, $MR_M = N\overline{R_M}$ for some unique maximal ideal N of $\overline{R}$. Then $N \cap R = M$, and by [2], Proposition 1.14, $M\overline{R}$ is N-primary. Since $\overline{R}$ is a Dedekind domain $M\overline{R} = N^l$ for some $l \in \mathsf{N}$. But $N\overline{R_M} = MR_M = M\overline{R_M} = N^l\overline{R_M}$, and hence $l = 1$.

COROLLARY 5. *Let I be an ideal of R such that $\mu(I) > 2$. Then $\mu(I_M) = \mu(I)$ for some $M \in Ass(C)$.*

PROOF. The result follows from Proposition 1 and the previous lemma.

If $R \neq \overline{R}$, $\mu(I) = 2$, and I is not comaximal to C then we see in the following example that $\mu(S_I^{-1}I)$ is not in general equal to $\mu(I)$, and $\mu(I_M)$ can be less than $\mu(I)$ for each $M \in \mathrm{Ass}(C)$ whence I is strongly $\mu(I)$-generated by Proposition 3.

EXAMPLE. Let $R = K[X^2, X^3(1+X^2)]$, let $M_1 = (X^2, X^3(1+X^2))$, $M_2 = (1+X^2, X^3(1+X^2))$, and let $I = (X^3(1+X^2), (1+X^2)^2)$. Then $\mu(I) = 2$, $C = X^2(1+X^2)(1,X)$ (see Proposition 22), $\mathrm{Ass}(C) = \{M_1, M_2\}$, and I is M_2-primary. Hence $I_{M_1} = R_{M_1}$, and $S_I^{-1}I = I_{M_2}$. Since $(1+X^2)^2 = (X^3(1+X^2))^2/X^6$ we have $\mu(I_{M_2}) = 1$.

E.D. Davis and A.V. Geramita have proved in [7], Theorem 1 that $\mu(M) = \mu(M_M)$ if R_M is not regular, i.e. if R_M is not a DVR.

LEMMA 6. *Suppose $R \neq \overline{R}$. If $M \in \mathrm{Ass}(C)$ then $\mu(M) = \mu(M_M) \geq 2$, and M is not strongly $\mu(M)$-generated. Moreover $\mu_*(R) = 2$ if and only if $\mu_*(R_M) = 2$ for each $M \in \mathrm{Ass}(C)$.*

PROOF. Suppose $M \in \mathrm{Ass}(C)$. If M is principal then M_M is principal, and by [2], Proposition 9.2, R_M is integrally closed. By [14], Ch. 5, §5, Lemma, $C \not\subseteq M$ which is a contradiction. Thus $\mu(M) \geq \mu(M_M) \geq 2$. By Proposition 1, $\mu(M) = \mu(M_M)$, and by Proposition 3, M is not strongly $\mu(M)$-generated. The last statement is true since $2 \leq \mu_*(R_M) \leq \mu_*(R)$.

LEMMA 7. *Let I be an ideal of R which is comaximal to C. Then I is a product of maximal ideals.*

PROOF. By [2], Proposition 9.1, $I = \prod_{i=1}^{l} q_i$ where q_i is M_i-primary for $i = 1, \ldots, l$, and $M_i \neq M_j$ if $i \neq j$. By Lemma 4, R_{M_i} is a DVR, and by [2], Proposition 9.2, $q_{iM_i} = M_{iM_i}^{k_i}$ for some $k_i \in \mathsf{N}$, $i = 1, \ldots, l$. By [2], Propositions 3.11 and 4.8, $q_i = M_i^{k_i}$, $i = 1, \ldots, l$.

LEMMA 8. *Let I be an ideal of R which is comaximal to C. Then $I = I\overline{R} \cap R$.*

PROOF. By Lemma 7, $I\overline{R} \cap R = (\prod_{i=1}^{l} M_i^{k_i})\overline{R} \cap R = (\prod_{i=1}^{l} (M_i\overline{R})^{k_i}) \cap R$. By Lemma 4, $I\overline{R} \cap R = \bigcap_{i=1}^{l} (M_i\overline{R})^{k_i} \cap R \subseteq \bigcap_{i=1}^{l} ((M_i R_{M_i})^{k_i} \cap R)$ and by [2], Proposition 3.11 and 4.8, $I\overline{R} \cap R \subseteq \bigcap_{i=1}^{l} M_i^{k_i} = I$. Hence $I\overline{R} \cap R = I$.

PROPOSITION 9. *Let I be an ideal of R which is comaximal to C. Then I is strongly two-generated.*

PROOF. If $M \in \mathrm{Ass}(C)$ then $I_M = R_M$, and if $M \notin \mathrm{Ass}(C)$ then by Lemma 4, R_M is a DVR. Hence I_M is principal for each maximal ideal M of R, and by [12], Proposition 19, I is strongly two-generated.

LEMMA 10. *Let I be an ideal of R which is comaximal to C. Then I is principal if and only if there is $r \in R$ such that $I\overline{R} = r\overline{R}$.*

Proof. $\Rightarrow$) Suppose I is principal. Then $I = (r)$ for some $r \in R$. Hence $I\overline{R} = r\overline{R}$.

$\Leftarrow$) Assume there is $r \in R$ such that $I\overline{R} = r\overline{R}$. By Proposition 9, $I = (s, t)$ for some $s, t \in R$, and $(s, t)\overline{R} = r\overline{R}$. Hence $s = ur$ and $t = vr$ for some $u, v \in \overline{R}$. Since $r\overline{R} \nsubseteq M\overline{R}$ for any $M \in \mathrm{Ass}(C)$ we have that r is comaximal to C, and by Lemma 2, $u, v \in R$, and thus (u, v) is comaximal to C. By Lemma 8, $(u, v) = (u, v)\overline{R} \cap R = R$, and thus I is principal.

Theorem 11. *Let R be a one-dimensional domain with non-zero conductor C in the integral closure $\overline{R}$ of R. Let $\mu_*(R) = 2$, and let $\{M_i\}_{i\in S}$ be the maximal ideals of R such that $M_i \notin \mathrm{Ass}(C)$, and $M_i\overline{R} \neq r\overline{R}$ for any $r \in R$ and $i \in S$. Then the set of proper strong two-generators of R is $\bigcup_{M\in\mathrm{Ass}(C)}(M \setminus M^2) \bigcup_{i\in S} M_i$.*

Proof. Let I be an ideal of R such that $\mu(I) = 2$. If I is not comaximal to C then $I \subseteq M$ for some $M \in \mathrm{Ass}(C)$, $R \neq \overline{R}$ and $\mu(M) = \mu(M_M) = 2$ by Lemma 6. If I is comaximal to C then I is a product of maximal ideals by Lemma 7, and hence there is a non-principal maximal ideal M which contains I and is comaximal to C. By Lemma 10, $M \in \{M_i\}_{i\in S}$. By [12], Theorem 21 the set of proper strong two-generators of R is $\bigcup_{M\in\mathrm{Ass}(C)}(M \setminus M^2) \bigcup_{i\in S} M_i$.

Definition. Let (A, M) be a one-dimensional local Noetherian domain, let I be an ideal of A, and let $\ell(B)$ be the length of a finitely generated A-module B. Then C. Gottlieb [9] calls I a *maximally generated* ideal if $\mu(I) = \min\{\ell(A/(a)) : a \in M\}$ or equivalently $MI = aI$ for some $a \in M$, and then $\mu(I) = \mu_*(A)$. An element a of M is *superficial* if there exists a positive integer c such that $(M^l : a) \cap M^c = M^{l-1}$ for any sufficiently large integer l cf. [14], p. 285.

Remark 12. If (A, M) is a one-dimensional local Noetherian domain then A is Cohen-Macaulay. Suppose a is a superficial element of M. By definition $a^{-1}M^l \cap M^c = M^{l-1}$, and thus $aM^{l-1} = a(a^{-1}M^l \cap M^c) = M^l \cap aM^c$. Since a is M-primary, $M^{c+i} \subseteq aM^c$ for some positive integers c, i, and hence $aM^k = M^{k+1}$ for each integer k which is large enough. Then M^k is a maximally generated ideal, and $\min\{\ell(A/(b)) :\ b \in M\} = \ell(A/(a)) = \mu(M^k) = \mu_*(A)$. By [10], Theorems 14.13 and 17.11, $e(A) = e(M, A) = e((a), A) = \ell(A/(a)) = \mu(M^k)$ where $e(A)$ is the multiplicity of A. If A has infinite residue field then A has a superficial element by [14], p. 287.

Lemma 13. *Let (A, M) be a one-dimensional local domain with a superficial element s. Let $\overline{A}$ be the integral closure of A, and let C be the non-zero conductor of A in $\overline{A}$. Then C is a maximally generated ideal of A.*

Proof. By Remark 12, $sM^k = M^{k+1}$ for some $k \in \mathbb{N}$, and hence

$s\overline{A}(M\overline{A})^k = sM^k\overline{A} = M^{k+1}\overline{A} = (M\overline{A})^{k+1}$. Because of unique factorization of an ideal in $\overline{A}$ as a product of prime ideals we have $s\overline{A} = M\overline{A}$. Since C is an ideal of $\overline{A}$ we have $MC = MC\overline{A} = sC\overline{A} = sC$, and C is maximally generated by definition.

NOTATION. If M is a maximal ideal of R then let $R_M(u) = R[u]_{M[u]}$.

REMARK 14. Since $R_M(u)$ is CM and has infinite residue field, $R_M(u)$ has a superficial element and a maximally generated ideal by Remark 12. If I is an ideal of R then $\mu(IR_M(u)) = \ell_{R_M(u)}(IR_M(u)/IMR_M(u)) = \ell_{R_M}(I_M/I_M M_M) = \mu(I_M)$, and since any ideal of $R_M(u)$ is an extended ideal of R_M, $\mu_*(R_M(u)) = \mu_*(R_M)$. By [10], Theorem 12.4, $\overline{R_M}[u]$ is integrally closed, and hence $\overline{R_M}(u)$ is also. The integral closure of $R_M[u]$ in $F[u]$ is $\overline{R_M}[u]$ by [2], Ch. 5, Exercise 9, and thus $\overline{R_M[u]} = \overline{R_M}[u]$, and by [2], Proposition 5.12, $\overline{R_M(u)} = (R[u] \setminus MR[u])^{-1}\overline{R_M}[u]$.

THEOREM 15. *Let R be a one-dimensional domain with non-zero conductor C in the integral closure $\overline{R}$ of R, and let $R \neq \overline{R}$. Then $\mu_*(R) = \mu(C) = \mu(C_M) = \mu_*(R_M)$ for some $M \in$ Ass(C). If moreover $\overline{R}$ is a principal ideal domain* (PID) *then $\mu_*(R) = \mu(\overline{R})$ where $\overline{R}$ is considered as a fractional ideal of R.*

PROOF. Let $M \in$ Ass(C). By [14], Ch. 5, §5, Lemma, C_M is the conductor of R_M in $\overline{R_M}$. Obviously $CR_M[u]$ is contained in the conductor of $R_M[u]$ in $\overline{R_M}[u]$. If $b \notin CR_M[u]$ then there is $a \in \overline{R_M}$ such that $ab \notin R_M[u]$. Hence $CR_M[u]$ is the conductor of $R_M[u]$ in $\overline{R_M[u]}$, and by [14], Ch. 5, §5, Lemma, the conductor of $R_M(u)$ in $\overline{R_M(u)}$ is $CR_M(u)$. By Remark 14 there is a superficial element s in $R_M(u)$, and by Lemma 13, $CR_M(u)$ is a maximally generated ideal of $R_M(u)$. By Remark 14, $\mu(C) \geq \mu(C_M) = \mu(CR_M(u)) = \mu_*(R_M(u)) = \mu_*(R_M)$. Let I be an ideal of R such that $\mu(I) = \mu_*(R) > 2$. By Corollary 5, $\mu(I) = \mu(I_{M'})$ for some $M' \in$ Ass(C), and hence $\mu_*(R) = \mu_*(R_{M'})$ for some $M' \in$Ass(C). If $\mu_*(R) = 2$ then $\mu_*(R_{M'}) = 2$ for any $M' \in$Ass(C) by Lemma 6. Let M be such that $\mu_*(R_M) = \mu_*(R)$. Then $\mu(C) = \mu(C_M) = \mu_*(R_M) = \mu_*(R)$. If $\overline{R}$ is a PID then $C = c\overline{R}$ for some $c \in R$, and hence $\mu(C) = \mu(\overline{R})$ if $\overline{R}$ is considered as a fractional ideal of R.

PROPOSITION 16. *Let $R \neq \overline{R}$ and $M \in$ Ass(C). Then there is a unique biggest M-primary ideal I of R such that $\mu(I) = \mu_*(R_M)$.*

PROOF. Let $\mu_*(R) > 2$, and let I_M and J_M be maximal among ideals of R_M such that $\mu(I_M) = \mu(J_M) = \mu_*(R_M)$. By Remark 14, $\mu(IR_M(u)) = \mu(JR_M(u)) = \mu_*(R_M(u))$, and $IR_M(u)$ and $JR_M(u)$ are maximally generated ideals of $R_M(u)$. By [9], Lemma 4, $\mu(IR_M(u) + JR_M(u)) = \mu_*(R_M(u))$, and by Remark 14, $\mu(I_M + J_M) = \mu_*(R_M)$ whence $I_M = J_M$, and I_M is the unique biggest ideal of R_M such that $\mu(I_M) = \mu_*(R_M)$. Let I be the ideal of R such that $I = I_M \cap R$. Then I is M-primary, and $S_I^{-1} I = I_M$. By Lemma 6, $\mu(I) \geq \mu(I_M) = \mu_*(R_M) \geq 2$, and by Proposition 1, $\mu(I) = \mu(I_M) = \mu_*(R_M)$. Let q be an M-primary ideal of R such that $\mu(q) = \mu_*(R_M)$. If $\mu_*(R_M) = 2$ then $\mu(M_M) = 2$ by Lemma 6, and $q \subseteq M = I$. If $\mu_*(R_M) > 2$ then $\mu(q_M) = \mu(q)$ by Proposition 1, and hence $q_M \subseteq I_M$, whence $q \subseteq I$. If $\mu_*(R) = 2$ then $\mu(M) = \mu_*(R_M)$ by Lemma 6.

Remark 17. Let $R \neq \overline{R}$ and $M \in \mathrm{Ass}(C)$. If $(\overline{R_M}, m)$ is local, and $\overline{R_M}/m \simeq R_M/M_M$ then let I be the biggest maximally generated ideal of $R_M(u)$, s a superficial element of $R_M(u)$, and J a fractional ideal of $R_M(u)$ which fulfils $M_M(u) = (s) + sJ$ and such that $J \subseteq \overline{R_M(u)}$. Then by [3], Corollary 4, $I = (\ldots((C : J)_{R_M(u)} : J)_{R_M(u)} \ldots : J)_{R_M(u)}$, and $I' = I \cap R$ is the biggest M-primary ideal of R such that $\mu(I') = \mu_*(R_M)$. If there is a superficial element in R_M, and $(\overline{R_M}, m)$ is as above then the biggest maximally generated ideal of R_M can be determined by [3], Corollary 4.

Theorem 18. *Let R be a one-dimensional domain with non-zero conductor C in the integral closure $\overline{R}$ of R. Let $\mu_*(R) > 2$, let $\{M_i\}_{i=1}^k$ be the set of maximal ideals of R such that $\mu_*(R_{M_i}) = \mu_*(R)$. Then there is a unique biggest M_i-primary ideal I_i of R such that $\mu(I_i) = \mu_*(R)$, $i = 1, \ldots, k$, $\{M_i\}_{i=1}^k \subseteq \mathrm{Ass}(C)$, and the set of proper strong $\mu_*(R)$-generators of R is $\bigcup_{i=1}^k (I_i \setminus I_i M_i)$.*

Proof. Since $\mu_*(R) > 2$ we have $R \neq \overline{R}$. By Theorem 15, R has finite rank. By Lemma 4, $M_i \in \mathrm{Ass}(C)$, $i = 1, \ldots, k$. By Proposition 16 there is a unique biggest M_i-primary ideal I_i of R such that $\mu(I_i) = \mu_*(R_{M_i}) = \mu_*(R)$, $i = 1, \ldots, k$. Let I be an ideal of R such that $\mu(I) = \mu_*(R)$. By Corollary 5, $\mu(I_{M_i}) = \mu_*(R)$, and $I_{M_i} \cap R$ is M_i-primary for some $i \in \{1, \ldots, k\}$. Since $\mu(I_{M_i} \cap R) \geq \mu((I_{M_i} \cap R)_{M_i}) = \mu(I_{M_i}) = \mu_*(R)$ we have $I_i \supseteq I_{M_i} \cap R \supseteq I$. By [12], Theorem 21, the set of proper strong $\mu_*(R)$-generators is $\bigcup_{i=1}^k (I_i \setminus I_i M_i)$.

4. Integral domains contained in $K[X]$

Let K be a field, and let R be a subring of $K[X]$ which strictly contains K. Then $K[X]$ is integral over R, and hence R is a one-dimensional domain. By the next proposition R has non-zero conductor in its integral closure.

Notation. Let $v : K(X) \longrightarrow \mathsf{Z}$ be the valuation of $K(X)$ with $K[X]_{(X)}$ as the valuation ring.

From Lemma 20 on, the special case when $X^k \in R$ for some positive integer k is treated, and in Proposition 24, $X^l \in R$ for each sufficiently large integer l, i.e. R contains the semigroup ring $K[X^{n_1}, \ldots, X^{n_t}]$ for some positive integers $n_1, \ldots, n_t$ such that $(n_1, \ldots, n_t) = 1$. In that case we show that $\mu_*(R) = \min v(M)$ where $M = XK[X] \cap R$, and the conductor of R in $K[X]$ is equal to $X^N K[X]$ where N is the least integer such that $X^{N+i} \in R$, $i = 0, 1, \ldots$ The notation of Section 3 are used throughout this section.

PROPOSITION 19. *Let K be a field, let R be a subring of $K[X]$ which strictly contains K, and let F be the field of fractions of R. Then the conductor C of R in its integral closure $\overline{R}$ is non-zero, and $\overline{R} = F \cap K[X]$.*

PROOF. Since X is integral over R, $R[X] = K[X]$ is a finitely generated R-module, and hence $K[X]$ is a Noetherian R-module. By [2], Proposition 7.8, R is Noetherian. Since $K[X]$ is integrally closed, and $F \subseteq K(X)$ we have that $\overline{R} \subseteq K[X]$. Therefore $\overline{R} = F \cap K[X]$ is a finitely generated R-module, and hence $C \neq 0$.

LEMMA 20. *Let K be a field, and let k be a positive integer. If R is such that $K[X^k] \subseteq R \subseteq K[X]$, and $R \nsubseteq K[X^l]$ for any $l > 1$ then $F = K(X)$.*

PROOF. We have that $K(X)$ contains F, and F strictly contains $K(X^k)$. The minimum polynomial of X over $K(X^k)$ is $t^k - X^k$. By [13], Theorem 3.2 the minimum polynomial m of X over F divides $t^k - X^k$ in F. By [13], Theorem 4.2 and Proposition 4.3 the degree of m is less than k. Let r be the degree of m, and let G be a splitting field of m. Then the zeros of m in G are $g_1 X, \ldots, g_r X$ for some non-zero $g_1, \ldots, g_r \in G$, and hence the constant term of m is $(-1)^r \prod_{i=1}^r g_i X^r$ where $\prod_{i=1}^r g_i$ is non-zero. Then $K(X^r) \subseteq F$, and by induction $K(X) \subseteq F$.

PROPOSITION 21. *Let K be any field, let k be a positive integer, and let R be a ring such that $K[X^k] \subseteq R \subseteq K[X]$, $X^l \notin R$ if $l < k$, and $R \nsubseteq K[X^l]$ for any $l > 1$.*

1) *Then the integral closure of R is $K[X]$, and $\mu_*(R) = \mu(K[X]) \leq k$ where $K[X]$ is considered as a fractional ideal of R.*

2) *If $\sum_{i=1}^{k-1} a_i X^i \in R$ for some $a_i \in K[X^k]$, $i = 1, \ldots, k-1$, and $a_j \in K \setminus \{0\}$ for some $j \in \{1, \ldots, k-1\}$ then $\mu_*(R) < k$.*

PROOF. 1) By Proposition 19 and Lemma 20, $K[X]$ is the integral closure of R. As $K[X] = (1, X, \ldots, X^{k-1})R$ we have that $\mu(K[X]) \leq k$. By Theorem 15, $\mu_*(R) = \mu(K[X])$.

2) We have $K[X] = (1, X, \ldots, X^{k-1})R$, and by assumption $X^j \in (1, X, \ldots, X^{j-1}, X^{j+1}, \ldots, X^{k-1})R$. Hence $\mu(K[X]) < k$, and by 1) $\mu_*(R) < k$.

PROPOSITION 22. *Let $R = K[X^k, aX^l]$ for some positive integers k, l and some $a \in K[X^k] \setminus X^k K[X^k]$ such that $k > 1$, $(k, l) = 1$ and $X^l \notin R$ if $l < k$. Then*

1) $C = X^{(k-1)(l-1)}a^{k-1}K[X]$, *and*

2) C_M *is a maximally generated ideal of* R_M *for each* $M \in \mathrm{Ass}(C)$.

Proof. 1) Let $f = X^{(k-1)(l-1)}a^{k-1}$. By Proposition 21, $\overline{R} = K[X]$. We will show that $fK[X] \subseteq R$. Since $K[X] = (1, X, \ldots, X^{k-1})R$ it is enough to show that $fX^i \in R$ if $i \in \{0, \ldots, k-1\}$. Let $i \in \{0, \ldots, k-1\}$. As $(k, l) = 1$ there is $\mu,\ \nu \in \mathbf{Z}$ such that $fX^i = X^{(k-1)(l-1)+i}a^{k-1} = X^{k\nu+l\mu}a^{k-1}$. Since $\mu = tk + j$ for some $j \in \{0, \ldots, k-1\}$, and $t \in \mathbf{Z}$ we have $k\nu + l\mu = k(\nu + lt) + lj$, and therefore we can assume that $0 \le \mu \le k-1$. Then $(k-1)(l-1) + i = k\nu + l\mu$, and $(k-1-\mu)l + i + 1 = k\nu + k$. Thus $1 \le k\nu + k$, and hence $0 \le \nu$ whence $fX^i \in R$, and $fK[X] \subseteq R$. If $c \in R$ is such that $C = cK[X]$ then c divides f in $K[X]$. We have $i = k-1$ if and only if $(k-1-\mu)l = k\nu$, i.e. if and only if $\nu = 0$ and $\mu = k-1$. Hence a^{k-1} divides c. If $(k-1) \times (l-1) - 1 = k\nu + l\mu$ then $(k-1-\mu)l = k(\nu+1)$ which is impossible. Hence $fX^{-1} \notin C$, and thus $c = f$.

2) If $M = XK[X] \cap R$ then $M_M = (X^k, X^l)$, and $C_M = X^{(k-1)(l-1)} \cdot (1, X, \ldots, X^{\min\{k,l\}})$. Thus $X^{\min\{k,l\}}C_M = M_M C_M$. If $M \in \mathrm{Ass}(C)$, and $a \in M$ then $M = (b, aX^l)$ for some $b \in K[X^k]$ such that b divides a in $K[X^k]$, and $C_M = a^{k-1}(1, X, \ldots, X^{k-1})$. Hence $bC_M = M_M C_M$, and C_M is maximally generated for each $M \in \mathrm{Ass}(C)$.

Proposition 23. *Let* $R = K[X^k, aX^l]$ *for some positive integers* k, l *and some* $a \in K[X^k] \setminus X^k K[X^k]$ *such that* $k > 1$, $(k,l) = 1$, *and* $X^l \notin R$ *if* $l < k$. *Let* $\mathrm{Ass}(C) = \{M_i\}_{i=1}^t$, *and let* $M_1 = XK[X] \cap R$.

1) *Then* $\mu(C_{M_1}) = \min\{k, l\}$, *and if* $a \notin K$ *then* $\mu(C_{M_i}) = \mu_*(R) = k$, *for* $i = 2, \ldots, t$.

2) *Let* I_i *be the unique biggest* M_i*-primary ideal of* R *such that* $\mu(I_i) = \mu_*(R_{M_i})$, $i = 1, \ldots, t$. *Then* $I_i = M_i^{\mu(C_{M_i})-1}$, $i = 1, \ldots, t$.

Proof. 1) Suppose $a \notin K$, let $i \in \{2, \ldots, t\}$ and let $S = R \setminus M_i$, where $M_i = (b, aX^l)$ for some $b \in K[X^k]$ such that b divides a in $K[X^k]$. By Proposition 22, $C_{M_i} = a^{k-1}(1, X, \ldots, X^{k-1})$, and if $\mu(C_{M_i}) < k$ then $a^{k-1}X^j = a^{k-1}\sum_{\nu=1,\nu\neq j}^{k-1} \frac{r_\nu}{s_\nu} X^\nu$ for some $j \in \{1, \ldots, k-1\}$, $r_\nu \in R$ and $s_\nu \in S$, such that $r_\nu = 1$ or b divides r_ν, $\nu = 1, \ldots, j-1, j+1, \ldots, k-1$. Let $s = \prod_{\nu=1,\nu j}^{k-1} s_\nu$. Then $s(X^j - \sum_{\nu, r_\nu = 1} \frac{X^\nu}{s_\nu}) = s(\sum_{\nu, r_\nu \neq 1} \frac{r_\nu}{s_\nu} X^\nu)$, and b divides the right side but not the left one which is impossible. Thus $\mu(C_{M_i}) = k$, and by Proposition 21, $\mu_*(R) = k$. Since $M_{1M_1} = (X^k, X^l)$, and $C_{M_1} = X^{(k-1)(l-1)}(1, X, \ldots, X^{\min\{k,l\}})$ we have $\mu(C_{M_1}) = \min\{k, l\}$.

2) We have that the embedding dimension emdim $R_{M_i} = \ell(M_{iM_i}/M_{iM_i}^2) = 2$, $i = 1, \ldots, t$, and by [3], Theorem 1, I_{iM_i} is the conductor of R_{M_i} in its blowing up ring at M_{iM_i}. By [11], Proposition 1.8, $I_{iM_i} = M_{iM_i}^{e(R_{M_i})-1}$, and by [2], Proposition 4.8, $I_i = M_i^{e(R_{M_i})-1}, i = 1, \ldots, t$ where $e(R_{M_i})$ is the multiplicity of

R_{M_i}. By Proposition 22, C_{M_i} is maximally generated, and by definition and Remark 12, $e(R_{M_i}) = \mu(C_{M_i})$, $i = 1, \dots, t$.

By Proposition 23 and Theorems 11 and 18 the set of proper strong $\mu_*(R)$-generators of R is completely determined when $R = K[X^k, aX^l]$.

Let B be a primitive numerical semigroup generated by some positive integers $n_1, \dots, n_l$. Then $(n_1, \dots, n_l) = 1$, and there is a least integer $g(B)$ such that $g(B) + i \in B$ for $i = 1, 2, \dots$ The integer $g(B)$ is called the Frobenius number of B.

PROPOSITION 24. *Let $n_1, \dots, n_t$ be positive integers such that $(n_1, \dots, n_t) = 1$, and let $K[X; B] = K[X^{n_1}, \dots, X^{n_t}]$. Let R be a ring such that $K[X; B] \subseteq R \subseteq K[X]$ where B is maximal such that $K[X; B] \subseteq R$. Let $N = g(B) + 1$, and $M = XK[X] \cap R$. Then*

1) $C = X^N K[X]$,
2) $\mu_*(R) = \min v(M)$, *and if* $\min v(M) = 1$ *then* $R = K[X]$.

PROOF. 1) By Proposition 21, $\overline{R} = K[X]$. As $X^N K[X] \subseteq C$, and $X^{N-1} \notin R$ we have $C = X^N K[X]$.

2) Let $\min v(M) = l$, and let f be a polynomial in M such that $v(f) = l$. Let $m \in \mathsf{N}$ be minimal such that $X^m K[X] \subseteq (X^N, \dots, X^{N+l-1})R$. If $m > N$ then let $i \in \{0, \dots, l-1\}$ and $j \geq 0$ be such that $N + jl + i = m - 1$. Hence $X^{N+i} f^j$ and $X^{N+i}(f^j - aX^{jl})$ belong to $(X^N, \dots, X^{N+l-1})$ for some $a \in K$ such that $f^j - aX^{jl} = 0$ or $v(f^j - aX^{jl}) > jl$. Thus $X^{N+i+jl} = X^{m-1} \in (X^N, \dots, X^{N+l-1})$ which is a contradiction. Hence $m = N$, and $\mu(C) \leq l$. Since $C_M/MC_M = (\overline{X}^N, \overline{X}^{N+1}, \dots, \overline{X}^{N+l-1})$ where $\overline{X}$ is the image of X in R_M/MR_M, and $\overline{X}^N, \overline{X}^{N+1}, \dots, \overline{X}^{N+l-1}$ are linearly independent in the R_M/MR_M-vector space we have $\mu(C) = \mu(C_M) = l$. By Theorem 15, $\mu_*(R) = l$. If $\min v(M) = 1$ then R is a PID, and hence integrally closed. Thus $R = K[X]$.

REMARK 25. By Proposition 24, C is primary for $M = XK[X] \cap R$, and by Theorem 15, $\mu_*(R_M) = \mu_*(R)$. We have $K[X]_M / XK[X]_M \simeq R_M/M_M$, and $K[X]_M$ is local. By Remarks 12 and 17 the biggest M-primary ideal I such that $\mu(I) = \mu_*(R)$ can be determined, and by Theorems 11 and 18 the set of proper strong $\mu_*(R)$-generators in R is determined. If moreover K is infinite then R_M/M_M is infinite, and by Remark 12 there is a superficial element in R_M.

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