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FRACTIONAL INTEGRATION ALONG HOMOGENEOUS CURVES IN R^3

SILVIA SECCO

1. Introduction

Given the curve

$$\Gamma(t) = (t^\alpha, t^\beta, t^\gamma), \qquad t > 0 \tag{1}$$

in R^3, with α, β, γ positive exponents, we consider the "fractional integration operator of order σ along Γ"

$$T^\sigma f(x) = \int_0^{+\infty} f(x - \Gamma(t)) t^{\sigma-1}\, dt, \qquad x \in \mathsf{R}^3 \tag{2}$$

defined for $\Re e\sigma > 0$.

We are interested in studying the boundedness properties of T^σ as an operator from $L^p(\mathsf{R}^3)$ to $L^q(\mathsf{R}^3)$. In order to have positive results, we shall suppose that α, β, γ are distinct. Therefore we may assume $\alpha < \beta < \gamma$. The homogeneity of the operator T^σ with respect to the dilations

$$\delta \cdot (x_1, x_2, x_3) = (\delta^\alpha x_1, \delta^\beta x_2, \delta^\gamma x_3) \tag{3}$$

implies the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\Re e\sigma}{Q}$$

where $Q = \alpha + \beta + \gamma$ is the homogeneous dimension of R^3 with respect to the dilations (3). If $\sigma \in \mathsf{R}$, another necessary condition on p and q is due to the fact that T^σ dominates the operator T_0 which is defined by limiting the integration in (2) to a bounded interval of R which doesn't contain the origin. According to what has been proved in [3] and later improved in [4], T_0 is bounded from $L^p(\mathsf{R}^3)$ to $L^q(\mathsf{R}^3)$ when $(\frac{1}{p}, \frac{1}{q})$ belongs to the closed trapezoid with vertices $A = (0,0)$, $B = (1,1)$, $C = (\frac{2}{3}, \frac{1}{2})$, $D = (\frac{1}{2}, \frac{1}{3})$. As we will prove, this condition is also necessary when $\sigma \in \mathsf{C}$.

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In the special case of $\sigma = \frac{Q}{6}$, Drury has proved in [2] that the operator T^σ is bounded from $L^{\frac{3}{2}}(\mathsf{R}^3)$ to $L^2(\mathsf{R}^3)$ for $(\alpha, \beta, \gamma) = (1, 2, k)$ and $k \geq 4$; subsequently Pan has extended the same estimate to $3 \leq k < 4$, [5]. More recently Pan has shown in [6] that this result holds also for $\beta + \gamma \geq 5\alpha$.

In this paper we will prove the following Theorem.

Theorem 1. *For $\Re e\sigma > 0$, T^σ is bounded from $L^p(\mathsf{R}^3)$ to $L^q(\mathsf{R}^3)$ if and only if*

(i) $(\frac{1}{p}, \frac{1}{q}) \in ABCD$,

(ii) $\frac{1}{p} - \frac{1}{q} = \frac{\Re e\sigma}{Q}$,

where $ABCD$ is the closed trapezoid with vertices

$$A = (0, 0), \; B = (1, 1), \; C = (\tfrac{2}{3}, \tfrac{1}{2}), \; D = (\tfrac{1}{2}, \tfrac{1}{3}).$$

We consider now the arc-length ds on the arc Γ_0 obtained by taking $t \in [0, 1]$ and the convolution operator

$$f(x) \longmapsto \int_{\Gamma_0} f(x - \Gamma(t))\, ds, \qquad x \in \mathsf{R}^3. \tag{4}$$

In order to express ds it is convenient to renormalize the exponents of the curve Γ so to obtain $\alpha = 1$. This can be done without loss of generality, by changing variable $t^\alpha = u$. The operator in (4) becomes essentially

$$Tf(x) = \int_0^1 f(x - (t, t^\beta, t^\gamma))\, dt, \qquad x \in \mathsf{R}^3. \tag{5}$$

As a consequence of Theorem 1 we have the following result for the operator T.

Corollary 2. *Let T be the operator defined in* (5). *Then*

(i) *if $\beta + \gamma \geq 5$ the typeset for T is the closed trapezoid with vertices $A = (0, 0)$, $B = (1, 1)$, $C' = (\frac{Q-2}{Q}, \frac{Q-3}{Q})$, $D' = (\frac{3}{Q}, \frac{2}{Q})$;*

(ii) *if $\beta + \gamma < 5$ the typeset for T is the whole trapezoid with vertices $A = (0, 0)$, $B = (1, 1)$, $C = (\frac{2}{3}, \frac{1}{2})$, $D = (\frac{1}{2}, \frac{1}{3})$.*

This paper is organized as follows. In the next section we present an estimate on the decay of certain oscillatory integrals that will be used in the proof of Theorem 1. In the third section we prove Theorem 1 with the method used by Christ in [1] which consists in introducing a Littlewood-Paley decomposition adapted to the homogeneity of the curve (1).

2. Notation and preliminary estimates

DEFINITION 3. Given a function f on R^3 and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathsf{R}^3$, we put

$$D_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}f(x_1, x_2, x_3) = f(\varepsilon_1 x_1, \varepsilon_2 x_2, \varepsilon_3 x_3), \qquad (x_1, x_2, x_3) \in \mathsf{R}^3.$$

DEFINITION 4. Given a vector-valued function $f = \{f_j, \} \in L^p(l^r)$ where j varies over Z, we define

$$\|f\|_{L^p(l^r)} = \left\| \left(\sum_{j\in\mathsf{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(\mathsf{R}^3)}$$

and we denote by $\|\cdot\|_{p,q,r}$ the norm of an operator from $L^p(l^r)$ to $L^q(l^r)$.

Now let $I \subset \mathsf{R}$ be a closed interval and let $\mathscr{C}$ be a curve in R^3 defined by

$$\mathscr{C} : t \in I \longmapsto (\psi_1(t), \psi_2(t), \psi_3(t)) = \psi(t) \in \mathsf{R}^3,$$

where ψ_1, ψ_2, ψ_3 are smooth real-valued functions and let ω be a smooth function with compact support in I. For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathsf{R}^3$ we define the oscillatory integral

$$I(\xi) = \int_I e^{-i\xi\cdot\psi(t)}\omega(t)\, dt.$$

We have the following result:

LEMMA 5. *Suppose that for every $t \in I$, the vectors $\psi'(t), \psi''(t), \psi'''(t)$ span R^3 and that $\psi_1'(t) \neq 0, \psi_2'(t) \neq 0, \psi_3'(t) \neq 0$ for all $t \in I$. Then*

$$I(\xi) = O(|\xi|^{-\frac{1}{3}}) \qquad \text{as } |\xi| \to \infty.$$

Moreover there exists a constant $C > 1$ such that

$$I(\xi) = O(|\xi|^{-N}) \qquad \text{as } |\xi| \to \infty$$

for all $N \geq 0$, in the regions of the space ξ_1, ξ_2, ξ_3 where

$$\begin{aligned} &|\xi_1| > C(|\xi_2| + |\xi_3|) \\ &|\xi_2| > C(|\xi_1| + |\xi_3|) \\ &|\xi_3| > C(|\xi_1| + |\xi_2|). \end{aligned} \tag{6}$$

The proof of Lemma 5 is an application of van der Corput's lemma and therefore we omit the details.

3. Proof of Theorem 1

For $\Re e\sigma > 0$, let $K^\sigma(x_1, x_2, x_3)$ be the kernel on R^3 defined by

$$\langle K^\sigma, f\rangle = \int_0^{+\infty} f(t^\alpha, t^\beta, t^\gamma)t^{\sigma-1}\,dt$$

so that

$$T^\sigma f(x_1, x_2, x_3) = f * K^\sigma(x_1, x_2, x_3).$$

We prove first the necessity of conditions (i) and (ii). As we have already observed, condition (ii) is necessary for $\sigma \in \mathsf{C}$ and condition (i) is necessary for $\sigma \in \mathsf{R}$. Therefore it remains only to prove the necessity of condition (i) for $\sigma \in \mathsf{C}$. Given $0 < \epsilon < 1$, let φ be a smooth positive real-valued function in R^3 compactly supported in a ball of radius ϵ centered at the point $x_0 = (1, 1, 1)$ and identically one in a ball of radius $\frac{\epsilon}{2}$ centered at x_0. We may write

$$\begin{aligned}\varphi(x) &= \int e^{-i\xi\cdot x}(\mathscr{F}^{-1}\varphi)(\xi)\,d\xi\\ &= \int \widehat{\delta_\xi}(\mathscr{F}^{-1}\varphi)(\xi)\,d\xi.\end{aligned}$$

Then if we define

$$K^\sigma\varphi = \int \widehat{\delta_\xi}K^\sigma(\mathscr{F}^{-1}\varphi)(\xi)\,d\xi,$$

we have

$$\begin{aligned}\|K^\sigma\varphi\|_{pq} &\le \int \|\widehat{\delta_\xi}K^\sigma\|_{pq}|\mathscr{F}^{-1}\varphi(\xi)|\,d\xi\\ &\le \|K^\sigma\|_{pq}\|\mathscr{F}^{-1}\varphi\|_1\\ &\le C\|K^\sigma\|_{pq}.\end{aligned}\tag{7}$$

Since

$$\Re e(t^{\sigma-1}) = t^{\Re e\sigma-1}\cos(\Im \mathrm{m}\sigma\,\mathrm{log t}),$$

then if ϵ has been chosen sufficiently small and $f \in L^p(\mathsf{R}^3)$ is a positive real-valued function, we obtain

$$\begin{aligned}\|K^{\Re e\sigma}\varphi * f\|_q &\le C\|K^\sigma\varphi * f\|_q\\ &\le C\|K^\sigma\varphi\|_{pq}\|f\|_p.\end{aligned}\tag{8}$$

By combining (7) and (8) we get

$$\|K^{\Re e\sigma}\varphi\|_{pq} \leq C\|K^{\sigma}\|_{pq}.$$

The necessity of condition (i) then follows from the fact that the kernel $K^{\Re e\sigma}\varphi$ defines a convolution operator which is bounded only on the closed trapezoid $ABCD$.

To prove the sufficiency of the conditions (i) and (ii) we first observe that since

$$\|T^{\sigma}\|_{pq} \leq \|T^{\Re e\sigma}\|_{pq},$$

we may assume $\sigma \in \mathsf{R}$ and $\sigma > 0$.

Then we make a dyadic decomposition of the kernel K^{σ}. Let $\omega(t)$ be a smooth function on R^{+}, supported on $\{\, t : \frac{1}{2} < t < 4 \,\}$ and such that $\sum_{j\in\mathsf{Z}} \omega(2^{j}t) = 1$ for $t > 0$. We define the kernels

$$\langle K_j^{\sigma}, f\,\rangle = \int f(t^{\alpha}, t^{\beta}, t^{\gamma})\omega(2^{j}t)t^{\sigma-1}\,dt. \tag{9}$$

Every kernel K_j^{σ} may be obtained from the kernel K_0^{σ} by dilation and with a multiplicative factor in fact, by changing variable in (9), we have

$$\langle K_j^{\sigma}, f\,\rangle = 2^{-\sigma j}\langle K_0^{\sigma}, D_{(2^{-\alpha j}, 2^{-\beta j}, 2^{-\gamma j})} f\,\rangle.$$

Then we define the operators

$$T_j^{\sigma} f(x_1, x_2, x_3) = \int f(x_1 - t^{\alpha}, x_2 - t^{\beta}, x_3 - t^{\gamma})\omega(2^{j}t)t^{\sigma-1}\,dt$$

so that $T^{\sigma} = \sum_{j\in\mathsf{Z}} T_j^{\sigma}$.

The Fourier multiplier corresponding to the operator T_j^{σ} is

$$\widehat{K_j^{\sigma}}(\xi_1, \xi_2, \xi_3) = 2^{-\sigma j}\widehat{K_0^{\sigma}}(2^{-\alpha j}\xi_1, 2^{-\beta j}\xi_2, 2^{-\gamma j}\xi_3)$$

and by Lemma 5, we know that the multiplier

$$\widehat{K_0^{\sigma}}(\xi_1, \xi_2, \xi_3) = \int_{\frac{1}{2}}^{4} e^{-i(\xi_1,\xi_2,\xi_3)\cdot(t^{\alpha},t^{\beta},t^{\gamma})}\omega(t)t^{\sigma-1}\,dt$$

decays slowly in the complementary set of the region of the space ξ_1, ξ_2, ξ_3 defined by (6). We say that this set contains the singular directions for $\widehat{K_0^{\sigma}}$ and we decompose it into four subsets:

$$\Gamma_0^1 = \{ (\xi_1, \xi_2, \xi_3) \in \mathsf{R}^3 : a|\xi_3| < |\xi_2| < b|\xi_3|,\ |\xi_1| < \delta|\xi_3| \}$$

$$\Gamma_0^2 = \{ (\xi_1, \xi_2, \xi_3) \in \mathsf{R}^3 : a|\xi_3| < |\xi_1| < b|\xi_3|,\ |\xi_2| < \delta|\xi_3| \}$$

$$\Gamma_0^3 = \{ (\xi_1, \xi_2, \xi_3) \in \mathsf{R}^3 : a|\xi_2| < |\xi_1| < b|\xi_2|,\ |\xi_3| < \delta|\xi_2| \}$$

$$\Gamma_0^4 = \{ (\xi_1, \xi_2, \xi_3) \in \mathsf{R}^3 : a|\xi_3| < |\xi_1| < b|\xi_3|,\ a|\xi_3| < |\xi_2| < b|\xi_3| \}$$

where a, b, δ are positive constants and δ is sufficiently small.

Since the region $\cup_{i=1}^4 \Gamma_0^i$ contains the singular directions for $\widehat{K_0^\sigma}$, then the singular directions for $\widehat{K_j^\sigma}$ are contained in the union of the cones $\Gamma_j^i, i = 1, \ldots, 4$, where

$$\Gamma_j^i = \{(\xi_1, \xi_2, \xi_3) \in \mathsf{R}^3 : (2^{-\alpha j}\xi_1, 2^{-\beta j}\xi_2, 2^{-\gamma j}\xi_3) \in \Gamma_0^i \}.$$

For fixed $i = 1, \ldots, 4$, as j varies in Z, the sets Γ_j^i are essentially disjoint.

Now, for $i = 1, \ldots, 4$, we introduce a C^∞ partition of unity $\{\eta_j^i\}$ in R^3 minus the coordinate planes with η_j^i homogeneous of degree zero (with respect to the Euclidean dilation group), supported in a set like Γ_j^i, but a little bit larger and identically one in the set Γ_j^i. Moreover let η_j^i be such that $\eta_j^i(\xi_1, \xi_2, \xi_3) = \eta_0^i(2^{-\alpha j}\xi_1, 2^{-\beta j}\xi_2, 2^{-\gamma j}\xi_3)$.

Let $Q_j^i, i = 1, \ldots, 4$, be the operator with η_j^i as Fourier multiplier and let ϕ be a smooth function on R^3, compactly supported and identically one in a neighborhood of the origin. Finally let P_j be the operator with $\phi(2^{-\alpha j}\xi_1, 2^{-\beta j}\xi_2, 2^{-\gamma j}\xi_3)$ as Fourier multiplier. Following Christ [1], for fixed σ we write

$$\begin{aligned}
(10)\qquad T^\sigma &= \sum_{j\in\mathsf{Z}} T_j^\sigma \\
&= \sum_{j\in\mathsf{Z}} T_j^\sigma P_j + \sum_{j\in\mathsf{Z}} T_j^\sigma (I - P_j) \\
&= \sum_{j\in\mathsf{Z}} T_j^\sigma P_j + \sum_{j\in\mathsf{Z}} T_j^\sigma (I - P_j) Q_j^1 + \sum_{j\in\mathsf{Z}} T_j^\sigma (I - P_j)(I - Q_j^1) \\
&\ldots \\
&= \sum_{j\in\mathsf{Z}} T_j^\sigma P_j + \sum_{j\in\mathsf{Z}} T_j^\sigma (I - P_j) Q_j^1 + \sum_{j\in\mathsf{Z}} T_j^\sigma (I - P_j)(I - Q_j^1) Q_j^2 \\
&\quad + \sum_{j\in\mathsf{Z}} T_j^\sigma (I - P_j) \left(\prod_{i=1}^{2} (I - Q_j^i) \right) Q_j^3
\end{aligned}$$

$$+\sum_{j\in\mathbb{Z}} T_j^\sigma(I-P_j)\left(\prod_{i=1}^{3}(I-Q_j^i)\right)Q_j^4$$

$$+\sum_{j\in\mathbb{Z}} T_j^\sigma(I-P_j)\prod_{i=1}^{4}(I-Q_j^i)$$

where I is the identity operator.

We consider first the operator $\sum_{j\in\mathbb{Z}} T_j^\sigma P_j$. Each term $T_j^\sigma P_j$ has a smooth, compactly supported Fourier multiplier given by

$$2^{-j\sigma}\widehat{K_0^\sigma}(2^{-\alpha j}\xi_1, 2^{-\beta j}\xi_2, 2^{-\gamma j}\xi_3)\phi(2^{-\alpha j}\xi_1, 2^{-\beta j}\xi_2, 2^{-\gamma j}\xi_3)$$

therefore it is a convolution operator with a Schwartz kernel. Let $H_j(x_1, x_2, x_3)$ be such a kernel. By homogeneity we have

$$H_j(x_1, x_2, x_3) = 2^{(Q-\sigma)j}H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)$$

where H_0 is the convolution kernel corresponding to the operator $T_0^\sigma P_0$. Let $\rho(x_1, x_2, x_3)$ be a homogeneous norm with respect to the dilations (3), we prove that the sum over all j of the absolute values of the convolution kernels of $T_j^\sigma P_j$ is bounded by a constant times the kernel $[\rho(x_1, x_2, x_3)]^{-Q+\sigma}$. We consider the series

$$\begin{aligned}(11)\quad \sum_{j\in\mathbb{Z}}|H_j(x_1, x_2, x_3)| &= \sum_{j\in\mathbb{Z}} 2^{(Q-\sigma)j}|H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)|\\ &= \sum_{2^j\rho(x_1,x_2,x_3)\le 1} 2^{(Q-\sigma)j}|H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)|\\ &\quad + \sum_{2^j\rho(x_1,x_2,x_3)>1} 2^{(Q-\sigma)j}|H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)|.\end{aligned}$$

For $N \in \mathbb{N}$ we denote by $\|H_0\|_{(N)}$ the norm of H_0 in $\mathcal{S}(\mathbb{R}^3)$ given by

$$\|H_0\|_{(N)} = \sum_{|\nu|\le N}\sup_{x\in\mathbb{R}^3}(1+|x|)^N|\partial^\nu H_0(x)|$$

where ν is a multi–index.

Observe that from hypotheses (i) and (ii) we get $Q-\sigma > 0$. In the first sum of (11) we put

$$|H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)| \le \|H_0\|_{(0)}$$

so that

$$\sum_{2^j\rho(x_1,x_2,x_3)\le 1} 2^{(Q-\sigma)j}|H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)| \le C[\rho(x_1,x_2,x_3)]^{-Q+\sigma}. \tag{12}$$

Since $\rho(x_1,x_2,x_3) \le C|(x_1,x_2,x_3)|^{\frac{1}{\alpha}}$ when $|(x_1,x_2,x_3)| \ge 1$, in the second sum of (11) we put

$$\begin{aligned}|H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)| &\le \frac{\|H_0\|_{(M)}}{1 + |(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)|^M}\\ &\le \frac{C}{[2^j\rho(x_1,x_2,x_3)]^{M\alpha}}\end{aligned}$$

for a positive integer M such that $M\alpha > Q - \sigma$. In this way we have

$$\sum_{2^j\rho(x_1,x_2,x_3)\ge 1} 2^{Q-\sigma}|H_0(2^{\alpha j}x_1, 2^{\beta j}x_2, 2^{\gamma j}x_3)| \le C[\rho(x_1,x_2,x_3)]^{-Q+\sigma}. \tag{13}$$

By combining (12) and (13) we get

$$\sum_{j\in \mathsf{Z}} |H_j(x_1,x_2,x_3)| \le C[\rho(x_1,x_2,x_3)]^{-Q+\sigma}.$$

Therefore by applying the Hardy-Littlewood-Sobolev's Theorem, it follows that the operator $\sum_{j\in\mathsf{Z}} T_j^\sigma P_j$ is bounded from L^p to L^q whenever $\frac{1}{p} - \frac{1}{q} = \frac{\sigma}{Q}$. We may repeat the previous arguments also for the operator

$$\sum_{j\in\mathsf{Z}} T_j^\sigma (I - P_j) \prod_{i=1}^{4} (I - Q_j^i)$$

because every term $T_j^\sigma(I - P_j)\prod_{i=1}^4 (I - Q_j^i)$ has a Fourier multiplier which belongs to the Schwartz class. This is because the multiplier of the operator T_j^σ decays rapidly with all its derivatives outside the regions $\Gamma_j^i, i = 1, \dots, 4$.

So the operator $\sum_{j\in\mathsf{Z}} T_j^\sigma(I - P_j)\prod_{i=1}^4 (I - Q_j^i)$ is bounded from L^p to L^q when $\frac{1}{p} - \frac{1}{q} = \frac{\sigma}{Q}$. Therefore it remains only to study the central terms of (10). We treat explicitly only the operator $\sum_{j\in\mathsf{Z}} T_j^\sigma(I - P_j)Q_j^1$, since the other ones are analogous.

Let $\{\widetilde{\eta}_j^1\}$ another C^∞ partition of unity in R^3 minus the coordinate planes with $\widetilde{\eta}_j^1$ homogeneous of degree zero (with respect to the Euclidean dilation group), supported in the set Γ_j^1 ulteriorly enlarged and identically one on the support of η_j^1. Moreover let $\widetilde{\eta}_j^1$ be such that $\widetilde{\eta}_j^1(\xi_1,\xi_2,\xi_3) = \widetilde{\eta}_0^1(2^{-\alpha j}\xi_1, 2^{-\beta j}\xi_2, 2^{-\gamma j}\xi_3)$. Let $\widetilde{Q}_j^1$ be the operator with $\widetilde{\eta}_j^1$ as Fourier multiplier, then

$$Q_j^1 \circ \widetilde{Q}_j^1 = Q_j^1 \tag{14}$$

for all $j \in \mathsf{Z}$.

We prove that

$$\left\| \sum_{j\in\mathbb{Z}} T_j^{\sigma}(I-P_j)Q_j^1 f \right\|_q \leq C\|\{T_j^{\sigma}(I-P_j)Q_j^1 f\}\|_{L^q(l^2)} \tag{15}$$

for any $f \in \mathcal{S}(\mathbb{R}^3)$.

Since (14) holds we have

$$\left\| \sum_{j\in\mathbb{Z}} T_j^{\sigma}(I-P_j)Q_j^1 f \right\|_q = \left\| \sum_{j\in\mathbb{Z}} \widetilde{Q}_j^1 T_j^{\sigma}(I-P_j)Q_j^1 f \right\|_q .$$

Moreover the linear operator $\sum_{j\in\mathbb{Z}} \epsilon_j \widetilde{Q}_j^1$ is bounded on $L^q(\mathbb{R}^3)$, uniformly in all choices of $\epsilon_j = \pm 1$. This is because the multiplier corresponding to the operator $\sum_{j\in\mathbb{Z}} \epsilon_j \widetilde{Q}_j^1$ is a Marcinkiewicz multiplier. Therefore by applying the Littlewood-Paley inequality, we get (15). Since P_j is a convolution operator with a Schwartz kernel and the vector-valued Hardy-Littlewood maximal function is bounded on $L^q(l^2)$ for $1 < q < +\infty$, we get from (15) the following estimates

$$\begin{aligned}
\left\| \sum_{j\in\mathbb{Z}} T_j^{\sigma}(I-P_j)Q_j^1 f \right\|_q &\leq C\|\{T_j^{\sigma}(I-P_j)Q_j^1 f\}\|_{L^q(l^2)} \\
&\leq C[\|\{T_j^{\sigma}Q_j^1 f\}\|_{L^q(l^2)} + \|\{P_j T_j^{\sigma}Q_j^1 f\}\|_{L^q(l^2)}] \\
&\leq C[\|\{T_j^{\sigma}Q_j^1 f\}\|_{L^q(l^2)} + C_1\|\{M T_j^{\sigma}Q_j^1 f\}\|_{L^q(l^2)}] \\
&\leq C\|\{T_j^{\sigma}Q_j^1 f\}\|_{L^q(l^2)} \\
&= C\|\{T_j^{\sigma}f_j\}\|_{L^q(l^2)} \\
&\leq \|\{T_j^{\sigma}\}\|_{p,q,2}\|\{f_j\}\|_{L^p(l^2)}
\end{aligned} \tag{16}$$

where $f_j = Q_j^1 f$.

Since also the linear operator $\sum_{j\in\mathbb{Z}} \epsilon_j Q_j^1$ is bounded on $L^p(\mathbb{R}^3)$, uniformly in all choices of $\epsilon_j = \pm 1$, because its Fourier multiplier is a Marcinkiewicz multiplier, by the Littlewood-Paley theory we get

$$\|\{Q_j^1 f\}\|_{L^p(l^2)} \leq C\|f\|_p. \tag{17}$$

By combining (16) and (17) we have

$$\left\| \sum_{j\in\mathbb{Z}} T_j^{\sigma}(I-P_j)Q_j^1 f \right\|_q \leq C\|\{T_j^{\sigma}\}\|_{p,q,2}\|f\|_p. \tag{18}$$

By repeating the previous considerations, we get an analogous estimate also for the remaining operators in (10).

Now we consider exponents $p \leq 2$ because the other ones may be treated by duality. For $0 < \theta < 1$, we write $\frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{\infty}$. Given the vector-valued operator

$$\begin{aligned} \{T_j^\sigma\} : L^p(l^2) &\to L^q(l^2) \\ \{g_j\} &\mapsto \{T_j^\sigma g_j\}, \end{aligned}$$

by Holder's inequality we have that

$$\|\{T_j^\sigma\}\|_{p,q,2} \leq \|\{T_j^\sigma\}\|_{p,q,p}^{\theta} \, \|\{T_j^\sigma\}\|_{p,q,\infty}^{1-\theta}. \tag{19}$$

Therefore, by the previous estimates on the operators in (10) and by (19), we have proved that

$$\|T^\sigma\|_{pq} \leq C[1 + \|\{T_j^\sigma\}\|_{p,q,p}^{\theta} \, \|\{T_j^\sigma\}\|_{p,q,\infty}^{1-\theta}]. \tag{20}$$

Since T_j^σ is a convolution operator with a positive kernel we get

$$\|\{T_j^\sigma\}\|_{p,q,\infty} \leq \|T^\sigma\|_{pq}. \tag{21}$$

Furthemore, by Minkowski's integral inequality we get, for $x \in \mathsf{R}^3$

$$\begin{aligned} \|\{T_j^\sigma g_j\}\|_{L^q(l^p)} &= \left(\int \left(\sum_{j \in \mathsf{Z}} |T_j^\sigma g_j(x)|^p \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \\ &= \left(\int \left(\sum_{j \in \mathsf{Z}} |T_j^\sigma g_j(x)|^p \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}.\frac{1}{p}} \\ &\leq \left(\sum_{j \in \mathsf{Z}} \left(\int |T_j^\sigma g_j(x)|^q \, dx \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\ &\leq \sup_{j \in \mathsf{Z}} \|T_j^\sigma\|_{pq} \left(\sum_{j \in \mathsf{Z}} \|g_j\|_p^p \right)^{\frac{1}{p}} \\ &= \sup_{j \in \mathsf{Z}} \|T_j^\sigma\|_{pq} \|\{g_j\}\|_{L^p(l^p)} \end{aligned}$$

that is

$$\|\{T_j^\sigma\}\|_{p,q,p} \leq \sup_{j \in \mathsf{Z}} \|T_j^\sigma\|_{pq}. \tag{22}$$

Then we shall prove that $T_j^\sigma, j \in \mathsf{Z}$, are uniformly bounded from $L^p(\mathsf{R}^3)$ to $L^q(\mathsf{R}^3)$.

By homogeneity we have

$$T_j^\sigma(x_1,x_2,x_3) = 2^{-j\sigma} D_{(2^{\alpha j},2^{\beta j},2^{\gamma j})}\big(T_0^\sigma D_{(2^{-\alpha j},2^{-\beta j},2^{-\gamma j})} f\big)(x_1,x_2,x_3)$$

and since $\frac{1}{p}-\frac{1}{q}=\frac{\sigma}{Q}$ we get

$$\begin{aligned}
\|T_j^\sigma f\|_q &= 2^{-j\sigma}\|D_{(2^{\alpha j},2^{\beta j},2^{\gamma j})}\big(T_0^\sigma D_{(2^{-\alpha j},2^{-\beta j},2^{-\gamma j})} f\big)\|_q \\
&= 2^{-j(\sigma+\frac{Q}{q})}\|T_0^\sigma D_{(2^{-\alpha j},2^{-\beta j},2^{-\gamma j})} f\|_p \\
&\le 2^{-j(\sigma+\frac{Q}{q})}\|T_0^\sigma\|_{pq}\|D_{(2^{-\alpha j},2^{-\beta j},2^{-\gamma j})} f\|_p \\
&= 2^{-j(\sigma+\frac{Q}{q}-\frac{Q}{p})}\|T_0^\sigma\|_{pq}\|f\|_p \\
&= \|T_0^\sigma\|_{pq}\|f\|_p.
\end{aligned} \tag{23}$$

But

$$T_0^\sigma f(x_1,x_2,x_3) = \int_{\frac{1}{2}}^{4} f(x_1-t^\alpha, x_2-t^\beta, x_3-t^\gamma)\omega(t)t^{\sigma-1}\,dt$$

is a convolution operator with a finite measure supported on the curve $\Gamma(t)$ for $t\in[\frac{1}{2},4]$. Therefore by Pan's theorem [4], T_0^σ is bounded on the whole trapezoid $ABCD$ and so $T_j^\sigma, j\in\mathsf{Z}$, are uniformly bounded from $L^p(\mathsf{R}^3)$ to $L^q(\mathsf{R}^3)$ when $\frac{1}{p}-\frac{1}{q}=\frac{\sigma}{Q}$.

By combining (19) with (20), (21), (22), we get

$$\|T^\sigma\|_{pq} \le C[1+\|T^\sigma\|_{pq}^{1-\theta}]. \tag{24}$$

Now, by replacing on both sides of (24) the operator T^σ by the operator

$$T_N^\sigma = \sum_{|j|\le N} T_j^\sigma \tag{25}$$

for a finite N, we may repeat the previous arguments and prove that

$$\|T_N^\sigma\|_{pq} \le C \tag{26}$$

where C is a positive constant independent of N. By taking the limit of (25) and (26) as $N\to+\infty$ we conclude the proof.

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DIPARTIMENTO DI MATEMATICA
POLITECNICO DI TORINO
CORSO DUCA DEGLI ABRUZZI 24
10129 TORINO
ITALY