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CODIMENSION TWO DETERMINANTAL VARIETIES WITH ISOLATED SINGULARITIES

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Abstract

We study codimension two determinantal varieties with isolated singularities. These singularities admit a unique smoothing, thus we can define their Milnor number as the middle Betti number of their generic fiber. For surfaces in C^4, we obtain a Lê-Greuel formula expressing the Milnor number of the surface in terms of the second polar multiplicity and the Milnor number of a generic section. We also relate the Milnor number with Ebeling and Gusein-Zade index of the 1-form given by the differential of a generic linear projection defined on the surface. To illustrate the results, in the last section we compute the Milnor number of some normal forms from Frühbis-Krüger and Neumer [7] list of simple determinantal surface singularities.

1. Introduction

The goal of this paper is to study codimension two determinantal varieties X with an isolated singularity. These conditions imply that $\dim(X) \leq 4$. In the cases of surfaces in C^4 and 3-dimensional varieties in C^5, these singularities admit a unique smoothing, hence the topological type of their Milnor fiber is well defined.

Let X be a codimension two determinantal variety with isolated singularity and X_t its generic fiber. We define the Milnor number of X as the middle Betti number of X_t. The conditions that X has isolated singularity and also be smoothable implies that $\dim(X) = 2,\ 3$. Since these are normal singularities, it follows from a result of Greuel and Steenbrink ([11], p. 540) that $b_1(X_t) = 0$, where b_1 is the first Betti number. They also prove in [11] that for every complex analytic space with isolated singularity one has $\pi_i(X_t) = 0$, for $i \leq \dim(X) - \operatorname{codim}(X)$, where $\pi_i(X_t)$ is the i-th homotopy group of X_t. Thus, it follows that the generic fiber of a determinantal variety X_t with isolated singularity is connected. When $\dim(X) = 3$, it also follows that X_t is 1-connected.

For determinantal surfaces X in C^4, we use these results and Morse theory to obtain a Lê-Greuel formula expressing the Milnor number $\mu(X)$, in terms

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of the second polar multiplicity $m_2(X)$ and the Milnor number of a generic section of X. This formula holds for 3-dimensional determinantal varieties in $\mathbb{C}^5$, under the additional hypothesis that $b_2(X_t) = 0$.

We do not know an algebraic formula to compute $\mu(X_t)$. Our approach in this paper, in order to calculate this invariant is to further investigate its geometric interpretation. For this we relate $m_2(X)$, and consequently $\mu(X)$, to the Ebeling and Gusein-Zade index of the 1-form dp, where p is a generic linear projection defined on X. We show in the last section how to use the results to compute the Milnor number of some normal forms from Frühbis-Krüger and Neumer [7] list of simple determinantal surface singularities.

For recent related results on determinantal varieties with isolated singularities see Nuño-Ballesteros, Oréfice and Tomazella [16], Damon and Pike [3].

2. Basic Definitions

Let $\mathrm{Mat}_{(n,p)}(\mathbb{C})$ be the set of all $n \times p$ matrices with complex entries, $\Delta_t \subset \mathrm{Mat}_{(n,p)}(\mathbb{C})$ the subset formed by matrices that have rank less than t, with $1 \leq t \leq \min(n, p)$. It is possible to show that Δ_t is an irreducible singular algebraic variety of codimension $(n-t+1)(p-t+1)$ (see [1]). Moreover the singular set of Δ_t is exactly Δ_{t-1}. The set Δ_t is called *generic determinantal variety*.

Definition 2.1. Let $M = (m_{ij}(x))$ be an $n \times p$ matrix whose entries are complex analytic functions on $U \subset \mathbb{C}^r$, $0 \in U$ and f the function defined by the $t \times t$ minors of M. We say that X is a determinantal variety of codimension $(n-t+1)(p-t+1)$ if X is defined by the equation $f = 0$.

We can look to a matrix $M = (m_{ij}(x))$ as a map $M : \mathbb{C}^r \longrightarrow \mathrm{Mat}_{(n,p)}(\mathbb{C})$, with $M(0) = 0$. Then, the determinantal variety in $\mathbb{C}^r$ is the set $X = M^{-1}(\Delta_t)$, with $1 \leq t \leq \min\{n, p\}$. The singular set of X is given by $M^{-1}(\Delta_{t-1})$. We denote $X_{\mathrm{reg}} = M^{-1}(\Delta_t \backslash \Delta_{t-1})$, the regular part of X. Notice that X has an isolated singularity at the origin if and only if $r \leq (n-t+2)(p-t+2)$.

Let $\mathcal{O}_r$ be the ring of germs of analytic functions on $\mathbb{C}^r$. We denote by $\mathrm{Mat}_{(n,p)}(\mathcal{O}_r)$ the set of all matrices $n \times p$ with entries in $\mathcal{O}_r$. This set can be identified with $\mathcal{O}_r^{np}$, where $\mathcal{O}_r^{np}$ is a free module of rank np.

We concentrate our attention in this paper to codimension 2 determinantal singularities and their deformations. It is a consequence of Auslander-Buchsbaum formula and the Hilbert-Burch's Theorem that any deformation of a Cohen-Macaulay variety of codimension 2 can be given as a perturbation of the presentation matrix (see [6], p. 3994). Therefore we can study these varieties and their deformations using their representation matrices. We can express the normal module and the space of the first order deformations in terms of matrices, hence we can treat the base of the semi-universal deformation using matrix representation.

The singularity theory of $(n+1) \times n$ matrices has been studied in [6] and [17].

3. The Generic Fiber

Let $X_0 \subset \mathbb{C}^r$ be the germ of an analytic d-dimensional variety, on some open set of $\mathbb{C}^r$ with isolated singularity at the origin. A smoothing of X_0 is a flat deformation with the property that its generic fiber is smooth. More precisely:

DEFINITION 3.1. We say that a germ of analytic variety $(X_0, 0)$ with isolated singularity of complex dimension $d \geq 1$ has a smoothing, if there exist an open ball $B_\epsilon(0) \subset \mathbb{C}^r$ centered at the origin, a closed subspace $X \subset B_\epsilon(0) \times D$, where $D \subset \mathbb{C}$ is an open disc with center at zero and a proper analytic map $F : X \longrightarrow D$, with the restriction to X of the projection $p : B_\epsilon(0) \times D \longrightarrow D$ such that

(a) F is flat;
(b) $(F^{-1}(0), 0)$ is isomorphic to $(X_0, 0)$;
(c) $F^{-1}(t)$ is non singular for $t \neq 0$.

It follows from the above definition that X has isolated singularity at the origin and is a normal variety if X_0 is normal at zero. Moreover,

$$F \mid_{F^{-1}(D-\{0\})} : F^{-1}(D - \{0\}) \longrightarrow D - \{0\}$$

is a fiber bundle whose fibers $X_t = F^{-1}(t)$ are non singular.

The topology of the generic fiber of a reduced curve has been intensively studied (see [2]). For instance, the following result holds:

THEOREM 3.2 ([2], p. 258). *Let $f : Y \longrightarrow D$ be a good representative of a flat family $f : (Y, 0) \longrightarrow (D, O)$ of reduced curves. Then, for all $t \in D$ the fiber Y_t is connected.*

For d-dimensional analytic spaces the following result is due to Greuel and Steenbrink.

THEOREM 3.3 ([11], p. 17). *Let $(X, 0)$ be a complex analytic space, d-dimensional, with isolated singularity and X_t the Milnor fiber of a smoothing of $(X, 0)$. Then, $\Pi_i(X_t) = 0$ for $i \leq \dim X - \operatorname{codim} X$.*

It follows from the previous theorem that if $(X, 0)$ is Cohen-Macaulay of codimension 2 with isolated singularity, then its Milnor fiber is $(\dim X - 2)$-connected.

As a consequence of Sard's Theorem, it follows that complete intersections are smoothable; moreover the base of their semiuniversal deformations is smooth whence the existence and uniqueness of the smoothing hold for them. For determinantal singularities, the existence and uniqueness of the smoothing do not occur in general (see [11]). But the following result was proved by Wahl:

THEOREM 3.4 ([20], p. 241). *Let $(X, 0)$ be a determinantal variety with isolated singularity at the origin defined by $t \times t$ minors of an $n \times p$ matrix M, whose entries are in $\mathcal{O}_r$, $2 \leq t \leq n \leq p$. If $\dim(X) < n + p - 2t + 3$, then X has a smoothing.*

In particular, it follows from this result that if $(X, 0)$ is Cohen-Macaulay with codimension less than or equal to 2 and $\dim(X, 0) \leq 3$, then $(X, 0)$ admits a smoothing. We also observe that for Cohen-Macaulay singularities of codimension less than or equal to 2, there is no obstruction for lifting second-order deformations, the basis of the semi-universal deformation is smooth ([6]).

The following result was proved by Greuel and Steenbrink in [11].

THEOREM 3.5 (See [11], p. 540). *Let X_t be the Milnor fiber of a smoothing of a normal singularity, then $b_1(X_t) = 0$.*

4. Morse Theory and the Topology of Varieties with Isolated Singularity

Let $(X, 0) \subset (\mathbb{C}^r, 0)$ be a d-dimensional variety with isolated singularity at the origin. Suppose that X has a smoothing, that is, there exists a flat family $\Pi : \mathfrak{X} \longrightarrow D \subset \mathbb{C}$, restriction of the projection $\Phi : B_\epsilon(0) \times D \longrightarrow D$, such that $X_t = \Pi^{-1}(t)$ is smooth for all $t \neq 0$ and $X_0 = X$.

The variety $\mathfrak{X}$ also has isolated singularity at the origin. Let p be a complex analytic function defined in X with isolated singularity at the origin. Let

$$\begin{aligned} \widetilde{p} : \mathfrak{X} \subset \mathbb{C}^r \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (x, t) &\longrightarrow \widetilde{p}(x, t), \end{aligned}$$

such that $\widetilde{p}(x, 0) = p(x)$ and for all $t \neq 0$, $\widetilde{p}(\cdot, t) = p_t$ is a Morse function in X_t.

Thus we have the following diagram

$$\begin{array}{ccc} X_t \subset & \mathfrak{X} & \subset \mathbb{C}^r \times \mathbb{C} \\ \Big\downarrow{\scriptstyle p_t} & & \Big\downarrow{\scriptstyle (\Pi, p)} \\ \mathbb{C} \times \{t\} & & \mathbb{C} \times \mathbb{C} \end{array} \tag{1}$$

Notice that the number of critical points of p_t is finite. In fact, x is a critical point of p_t if and only if x is a critical point of the function $\mathrm{Re}(p_t) : X_t \longrightarrow \mathsf{R}$. Since the real part of p_t is an analytic function on X_t, the number of critical points of $\mathrm{Re}(p_t)$ and, hence of p_t, is finite.

PROPOSITION 4.1. *Let X be a d-dimensional variety with isolated singularity at the origin admitting a smoothing and $p_t : X_t \longrightarrow \mathsf{C}$, $p_t = \widetilde{p}(\cdot, t)$ as above. Then,*

(a) *If $t \neq 0$*

$$X_t \simeq p_t^{-1}(0) \mathbin{\dot{\cup}} \{\textit{cells of dimension } d\}, \tag{2}$$

where $\dot{\cup}$ indicates the gluing of the spaces and $\simeq$ indicates that the spaces have the same homotopy type.

(b)

$$\chi(X_t) = \chi(p_t^{-1}(0)) + (-1)^d n_\sigma, \tag{3}$$

where n_σ is the number of critical points of p_t and $\chi(X_t)$ denotes the Euler characteristic of X_t.

PROOF. Let $x_1, \ldots, x_\nu$ be the critical points of p_t and $y_i = p_t(x_i)$, $1 \leq i \leq \nu$, their critical values. Suppose that 0 is a regular value of p_t, for all $t \neq 0$. We denote by E_i the line segments connecting the points y_i to 0, $E_i \cap E_j = \{0\}$ for $i \neq j$ and $E = \cup E_i$. Take $\eta > 0$ small enough such that $y_i \in D_\eta(0)$ for all $1 \leq i \leq \nu$.

The set $D_\eta(0)$ is a regular neighborhood of E that retracts to E.

We can realize this retraction through a smooth vector field that can be lifted into the stratified space X_t. Integrating this vector field, the space $p_t^{-1}(D_\eta)$ retracts by deformation on $p_t^{-1}(E)$.

Then,

$$X_t = p_t^{-1}(D_\eta) \simeq p_t^{-1}(E) = \bigcup_i (p_t^{-1}(E_i)) = p_t^{-1}(0) \cup \overline{\left(\bigcup_i p_t^{-1}(E_i - \{0\}) \right)}.$$

Observe first that x_i is a critical point of the restriction of p_t to $p_t^{-1}(E_i - \{0\})$ if and only if x_i is a critical point of the restriction of the real part of p_t to $p_t^{-1}(E_i - \{0\})$. Therefore, it follows from the classical Morse theory (see [15]) that

(4)

$$X_t = p_t^{-1}(0) \cup \overline{\left(\bigcup_i p_t^{-1}(E_i - \{0\}) \right)} \simeq p_t^{-1}(0) \mathbin{\dot{\cup}} \{\text{cells of dimension } d\}.$$

As the Euler characteristic is homotopy invariant, using the decomposition (2) we have

$$\chi(X_t) = \chi(p_t^{-1}(0)) + (-1)^d n_\sigma, \tag{5}$$

where n_σ is the number of critical points of p_t.

A consequence of the decomposition (2) is that only $\overline{p_t^{-1}(E_i - \{0\})}$ contributes to the free part of $H(X_t, \mathsf{Z})$. Hence, $b(X_t)$ is less than or equal to the number of critical points of p_t.

REMARK 4.2. This result also appears in [12].

Formula (5) above can also be expressed replacing n_σ by $m_d(X)$, the d-th polar multiplicity of X. We refer to [19] for the definition and properties of polar varieties. Here, we follow Gaffney in [8] to define the d-th polar multiplicity by the following construction: Let $\mathfrak{X} \subset \mathsf{C}^r \times \mathsf{C}^s$ be a complex analytic variety of complex dimension $d+s$ and $\Pi : \mathfrak{X} \longrightarrow \mathsf{C}^s$ an analytic function such that $\Pi^{-1}(0) = X$. Let $\widetilde{p} : \mathfrak{X} \subset \mathsf{C}^r \times \mathsf{C}^s, 0 \to \mathsf{C}^s, 0$ be such that $\widetilde{p}|_X$ has isolated singularity at the origin. Then, we can define $m_d(X, \widetilde{p}, \Pi) = m_0(P_d(\Pi, \widetilde{p}))$, where $P_d(\Pi, \widetilde{p})$ is the polar variety of $\mathfrak{X}$ with respect to $(\Pi, \widetilde{p})$. In general, $m_d(X, \widetilde{p}, \Pi)$ depends on the choices of $\mathfrak{X}$ and $\widetilde{p}$, but when $\mathfrak{X}$ is a versal deformation of X or in the case that X has a unique smoothing, m_d depends only on X and $\widetilde{p}$. Furthermore, if $\widetilde{p}$ is a generic linear projection, m_d is an invariant of the analytic variety X, which we denote by $m_d(X)$.

When $s = 1$ and $\widetilde{p}$ is a generic linear projection, we recover the conditions in diagram (1) and we can relate n_σ and $m_d(X)$. In fact, the following result is a direct consequence of the definitions of these two invariants.

PROPOSITION 4.3. *Under the conditions of Proposition 4.1, $n_\sigma = m_d(X)$.*

5. Determinantal Varieties

In this section, we restrict our attention to Cohen-Macaulay singularities of codimension 2 with isolated singularity at the origin. These include determinantal surfaces in C^4 and 3- dimensional determinantal varieties in C^5. These varieties admit a unique smoothing (see [17]), thus the following definition makes sense:

DEFINITION 5.1. Let $(X, 0) \subset (\mathsf{C}^r, 0)$ be the germ of a codimension 2 determinantal variety with isolated singularity at the origin, $\dim(X) = 2, 3$.The Milnor number of X, denoted $\mu(X)$, is defined by $\mu(X) = b_d(X_t)$, where X_t is the generic fiber of X and $b_d(X_t)$ is the d-th Betti number of X_t, $d = \dim(X)$.

Let $p : X \longrightarrow \mathbb{C}$ be a complex analytic function with isolated singularity at the origin. Then, $Y = X \cap p^{-1}(0)$ is a variety of dimension $d-1$, with isolated singularity at 0.

In particular, when $p : \mathbb{C}^r \longrightarrow \mathbb{C}$ is a linear function, it follows from the presentation matrix that Y is also a Cohen-Macaulay determinantal variety of dimension $d-1$ in $\mathbb{C}^{r-1}$.

In the following theorem, we obtain a formula of type Lê-Greuel for germs of Cohen-Macaulay determinantal surfaces of codimension 2 with isolated singularity at the origin. This formula was first obtained for linear sections of hypersurfaces by Lê Dung Trang in [13], and later extended to complete intersections with isolated singularities by Giusti and Henry in [10] (see also [14], where Massey obtains the formula in a more general setting).

THEOREM 5.2. *Let $(X, 0) \subset (\mathbb{C}^4, 0)$ be the germ of a determinantal surface with isolated singularity at the origin. Then,*

$$m_2(X) = \mu(p^{-1}(0) \cap X) + \mu(X),$$

where $m_2(X)$ is the second polar multiplicity of X.

PROOF. From (2), we have

$$\chi(X_t) = \chi(p_t^{-1}(0)) + (-1)^2 n_\sigma, \tag{6}$$

where n_σ is the number of critical points of $p_t : X_t \longrightarrow \mathbb{C}$. From Proposition 4.3 it follows that $n_\sigma = m_2(X)$.

Moreover, $\chi(X_t) = b_0(X_t) - b_1(X_t) + b_2(X_t)$. Then, using (6) we get

$$b_0(X_t) - b_1(X_t) + b_2(X_t) = \chi(p_t^{-1}(0)) + m_2(X). \tag{7}$$

We know that $p_t^{-1}(0) \subset X_t$ is the generic fiber of the determinantal curve $p^{-1}(0) \subset X$. Therefore, $p_t^{-1}(0)$ has the homotopy type of a bouquet of spheres of real dimension 1. Let $C = p^{-1}(0) \cap X$. Then, $\chi(C) = 1 - \mu(C, 0)$, where $\mu(C, 0)$ is the Milnor number of the curve ([2]).

Since determinantal varieties are normal varieties, it follows from Proposition 3.5 that $b_1(X_t) = 0$. Moreover X_t is connected. Therefore,

$$1 + b_2(X_t) = 1 - \mu(p^{-1}(0) \cap X) + m_2(X).$$

Hence, $m_2(X) = \mu(p^{-1}(0) \cap X) + \mu(X)$.

When $\dim(X) = 3$, we obtain an expression which reduces to the Lê-Greuel formula when $b_2(X_t) = 0$.

PROPOSITION 5.3. *Let* $(X, 0) \subset (\mathbb{C}^5, 0)$ *be the germ of a determinantal variety of codimension* 2 *with isolated singularity at the origin. Then,*

$$m_3(X) = \mu(p^{-1}(0) \cap X) + \mu(X) + b_2(X_t),$$

where $m_3(X)$ *is the polar multiplicity of* X.

6. Index of 1-Forms on Determinantal Varieties

In this section we relate the formulas of the previous section with Ebeling and Gusein-Zade index formulas in ([4]). They define indices of 1-forms on determinantal varieties having an essential isolated singularity, EIDS. These singularities can be represented by a matrix $M = (m_{ij}(x))$, $x \in \mathbb{C}^r$, which is transverse, away from the origin to the rank stratification of $\mathrm{Mat}_{(n,p)}(\mathbb{C})$, (see [4] for more details).

In particular, codimension two determinantal varieties with isolated singularities are EIDS, and the results in [4] apply to this class of singularities.

Let X be a germ of a codimension two determinantal variety with isolated singularity and ω the germ of a 1-form on $\mathbb{C}^r$ whose restriction to $(X, 0)$ has an isolated singular point at the origin.

Ebeling and Gusein-Zade definition of the Poincaré-Hopf index of ω reduces in our case to the following:

DEFINITION 6.1. The Poincaré-Hopf index (PH-index), $\mathrm{ind}_{\mathrm{PH}}\,\omega$, is the sum of the indices of the zeros of a generic pertubation $\widetilde{\omega}$ of the 1-form ω on $\widetilde{X}$, a smoothing of X.

PROPOSITION 6.2 ([4], p. 117). *The PH-index* $\mathrm{ind}_{\mathrm{PH}}\,\omega$ *of the* 1*-form* ω *on the EIDS* $(X, 0)$ *is equal to the number of non-degenerate singular points of a generic deformation* $\widetilde{\omega}$ *of the* 1*-form* ω *on* $\widetilde{X}_{\mathrm{reg}}$, *the regular part of* $\widetilde{X}$.

For determinantal varieties with isolated singularity, the relation between the PH-index and the radial index (see [5] for the definition of the radial index), is given by

$$\mathrm{ind}_{\mathrm{PH}}(\omega; X, 0) = \mathrm{ind}_{\mathrm{rad}}(\omega; X, 0) + (-1)^{\dim(X)}\overline{\chi}(X, 0),$$

where $\overline{\chi}(X, 0) = \chi(X, 0) - 1$. The PH-index is closely related to the d-th polar multiplicity as we can see in the following proposition.

PROPOSITION 6.3. *Let* $(X, 0) \subset (\mathbb{C}^4, 0)$ *be a determinantal surface, with isolated singularity at the origin. Then,* $\mathrm{ind}_{\mathrm{PH}}(\omega; X, 0) = m_2(X)$, *where* $\omega = dp$, *here* p *is a generic projection and* $m_2(X)$ *is the second polar multiplicity of* X.

PROOF. Let $p : (X, 0) \longrightarrow \mathbb{C}$ be a generic linear projection and $\epsilon > 0$ small enough such that the restriction of p to $X \cap B_\epsilon(0)$ has isolated critical point at the origin. Then, by Theorem 3 of [5], $\mathrm{ind}_{\mathrm{rad}}(dp; X, 0) = -\overline{\chi}(p^{-1}(t))$, $t \neq 0$, where $\overline{\chi}(X) = \chi(X) - 1$. Then, $\mathrm{ind}_{\mathrm{PH}}(dp; X, 0) = -\overline{\chi}(p^{-1}(t)) + (-1)^2\overline{\chi}(X, 0)$. Therefore,

$$\mathrm{ind}_{\mathrm{PH}}(dp; X, 0) = -(\chi(p^{-1}(t)) - 1) + (\chi(X, 0) - 1) = \mu(C) + \mu(X),$$

where $C = p^{-1}(0) \cap X$.

REMARK 6.4.

(a) This result is useful in calculations of $\mu(X)$, since in many cases one can use geometric methods to calculate $\mathrm{ind}_{\mathrm{PH}}(\omega; X, 0)$. This procedure will be useful in the calculations in the next section.

(b) An important problem not addressed in this work is the determination of an algebraic formula for the polar multiplicity as in Lê-Greuel's formula for ICIS. See [9], for an algebraic approach characterizing the d-th polar multiplicity of d-dimensional singular spaces.

7. Examples

In this section, we compute the Milnor number $\mu(X)$ for some normal forms of the simple determinantal surfaces X in $\mathbb{C}^4$ classified by Frühbis-Krüger and Neumer [7].

To calculate the Milnor number, we use the formula $m_2(X) = \mu(C)+\mu(X)$ from Theorem 5.2, where $\mu(C)$ is the Milnor number of the curve $C = X \cap p^{-1}(0)$, where $p : X \to \mathbb{C}$ is a generic linear projection.

Using Proposition 6.3, $m_2(X) = \mathrm{ind}_{\mathrm{PH}}(\omega, X, 0)$. Moreover, if X is simple and p is a generic linear projection, $C = X \cap p^{-1}(0)$ is a simple determinantal curve and its Milnor number can be calculated or we can directly use the table of simple curves in [6], p. 4008–4009.

To find $m_2(X)$, or equivalently, $\mathrm{ind}_{\mathrm{PH}}(\omega, X, 0)$, we can follow one of the following procedures:

(a) To use the algorithm proposed by Ebeling and Gusein-Zade in [4];

(b) To calculate the number of non degenerate singular points of the linear form ω defined on a smoothing of X.

(c) To obtain a perturbation X_t of X with singular points points $p_1, \ldots, p_l$ and we use the fact that $\mu(X) = \sum_{i=1}^{l} \mu(X_t, p_i)$.

We illustrate each one of these procedures in the examples below. Calculations are invariant by the group of contact equivalences acting in the space of matrices (see [7] and [18]). More details on the calculations can be found in [17].

EXAMPLE 1 (see [4], p. 123). Let $\begin{pmatrix} z & y+w & x \\ w & x & y \end{pmatrix}$ be the normal form in [4], which is contact equivalent to $M = \begin{pmatrix} x & y & z \\ w & x & y \end{pmatrix}$. To apply Ebeling and Gusein-Zade method let $p : \mathbb{C}^4 \longrightarrow \mathbb{C}$, $p(x, y, z, w) = w$ and $\omega = dp$. We consider the space curve $(C, 0) = X \cap p^{-1}(0)$ represented by the matrix $N = \begin{pmatrix} x & y & z \\ 0 & x & y \end{pmatrix}$. The family $\begin{pmatrix} x & y+b & z+c \\ a & x & y \end{pmatrix}$, is the versal unfolding of $(C, 0)$, whose discriminant is $a(b^2 - c^2) = 0$ (see [6]).

We obtain M from the versal deformation of N taking $a = w$, $b = c = 0$ and, moreover, a smoothing M_λ to M is obtained taking $a = b = w$, $c = \lambda \neq 0$. For each fixed λ, M_λ intersects the discriminant in 3 distinct points where the function $p(x, y, z, w) = w$ has non-degenerate critical points. Using 5.2, we obtain $\mu(X) + \mu(C) = 3$. Now $\mu(C) = 2$, then $\mu(X) = 1$.

EXAMPLE 2. Let $\begin{pmatrix} z & w+x & y^k \\ w & y & x \end{pmatrix}$. This normal form is contact equivalent to the second normal form in table $2a$ in [7]. Let $p : X \to \mathbb{C}$ be defined by $p(x, y, z, w) = w$, $\omega = dp$ and $(C, 0)$ the determinantal curve given by $\begin{pmatrix} z & x & y^k \\ 0 & y & x \end{pmatrix}$, $k \geq 1$, whose versal unfolding is given by

$$\begin{pmatrix} z & x+b & y^k + \sum_{i=0}^{k-1} c_i y^i \\ a & y & x \end{pmatrix}.$$

A smoothing of X is obtained by taking $c_0 = \lambda \neq 0$, $a = b = w$ and $c_i = 0$ for $i \neq 0$. Let M_λ the matrix obtained in this way.

For each λ fixed, let $f_\lambda : \mathbb{C}^4 \longrightarrow \mathbb{C}^3$ be the map determined by the maximal minors of M_λ, Jf_λ the Jacobian matrix of f_λ, and $[Jf_\lambda, \omega]$ the 4×4 matrix whose first three rows are the rows of Jf_λ and the last row is given by the coefficients of the form ω.

To determine the non-degenerate critical points of ω in M_λ, we find the solutions of the system whose equations $f_\lambda = 0$, and the 3×3 minors of the matrix $[Jf_\lambda, \omega]$. There are $2k$ solutions of the form $\left(y, (ky^{k+1})^{\frac{1}{2}}, 2ky^k, -2(ky^{k+1})^{\frac{1}{2}}\right)$ and $\lambda = (k+1)y^k$. We can verify that these solutions are non- degenerate singular points. Then, $\mathrm{ind}_{\mathrm{PH}}(\omega) = 2k$ and $\mu(X) = k - 1$.

EXAMPLE 3. Let $M = \begin{pmatrix} z & y & x \\ x & w & yz + y^k w \end{pmatrix}$. We now use procedure (c) and induction on k. We first consider the case $k = 1$.

Let $p : \mathbb{C}^4 \longrightarrow \mathbb{C}$ be given by $p(x, y, z, w) = y - z$ and $\omega = dp$. In this case, the determinantal curve given by $C = X \cap p^{-1}(0)$ is defined by the matrix $\begin{pmatrix} z & z & x \\ x & w & z^2+zw \end{pmatrix}$. By change of coordinates and applying Mather's lemma [21], we can reduce this normal form to $\begin{pmatrix} z & 0 & x \\ x & w & z^2 \end{pmatrix}$. This, in turn, is contact equivalent to the first normal form in [6], with $\mu(C)=3$. A smoothing of X is given by

$$\begin{pmatrix} z & y & x \\ x+\lambda & w & tz^2+yz+yw \end{pmatrix},$$

with $t \in \mathbb{C}$, and $t \neq 0$. We proceed as in the previous example to find that $\mathrm{ind}_{\mathrm{PH}}(X;\omega,0)=8$ and $\mu(X)=5$.

We now suppose that for $k-1$, the Milnor number of M is $2k+1$. To show that for k, $\mu(X)=2k+3$, we consider the following 1-parameter deformation of M

$$M_t = \begin{pmatrix} z & y & x \\ x & w & yz+y^k w+ty^{k-1}w \end{pmatrix},$$

$t \in \mathbb{C}$, $t \neq 0$. The variety X_t defined by the maximal minors of M_t is singular at the origin and at the points $(0,-t,0,\pm\sqrt{-t^3})$. Then,

$$\mu(X) = \mu(X_t,0)+\mu(X_t,u_1)+\mu(X_t,u_2),$$

where 0 is the origin in $\mathbb{C}^4$, $u_i=(0,-t,0,(-1)^i\sqrt{-t^3})$, $i=1,2$.

We can see that the germ of X_t at $x=y=z=0$ and $t\neq 0$, is equivalent to the normal form $\begin{pmatrix} z & y & x \\ x & w & yz+wy^{k-1} \end{pmatrix}$. Then, by the induction hypothesis, $\mu(X_t,0)=2k+1$.

Now calculating the 1-jet of X_t at the points $u_i=(0,-t,0,(-1)^i\sqrt{-t^3})$, we find that $\mu(X,u_i)=1$, $i=1,2$. Therefore, $\mu(X)=2k+3$.

REMARK 7.1. In [17], p. 82, it was conjectured that the formula $\tau(X)=\mu(X)+1$, holds for weighted homogeneous determinantal surfaces in $\mathbb{C}^4$ with isolated singularities, where $\tau(X)$ is the Tjurina number of X. This formula holds in the above examples, and it follows from the calculations in [17] and [3] that they hold more generally for all simple singularities of determinantal surfaces from Frühbis-Krüger and Neumer's list.

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