MATH. SCAND. 115 (2014), 173–188

SOME SUFFICIENT CONDITIONS FOR HEEGAARD GENERA TO BE ADDITIVE UNDER ANNULUS SUM

FENGLING LI*, FENGCHUN LEI† and GUOQIU YANG‡

Abstract

Let M_i be a compact orientable 3-manifold, and A_i an incompressible annulus on a component F_i of ∂M_i, $i = 1, 2$. Suppose A_1 is separating on F_1 and A_2 is non-separating on F_2. Let M be the annulus sum of M_1 and M_2 along A_1 and A_2. In the present paper we show that if M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with Heegaard distance $d(S_i) \geq 2g(M_i) + 5$ for $i = 1, 2$, then $g(M) = g(M_1) + g(M_2)$. Moreover, when $g(F_2) \geq 2$, the minimal Heegaard splitting of M is unique up to isotopy.

1. Introduction

Let M_i be a compact connected orientable bordered 3-manifold, and A_i an incompressible annulus on ∂M_i, $i = 1, 2$. Let $h : A_1 \to A_2$ be a homeomorphism. The manifold M obtained by gluing M_1 and M_2 along A_1 and A_2 via h is called an *annulus sum* of M_1 and M_2 along A_1 and A_2, and is denoted by $M_1 \cup_h M_2$ or $M_1 \cup_{A_1=A_2} M_2$.

Let $V_i \cup_{S_i} W_i$ be a Heegaard splitting of M_i, $i = 1, 2$, and $M = M_1 \cup_{A_1=A_2} M_2$. M has a natural Heegaard splitting $V \cup_S W$ induced from $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ with genus $g(S) = g(S_1) + g(S_2)$ (refer to Schultens [15]). Let $g(M)$ be the minimal genus among all Heegaard surfaces of M. Then we always have $g(M) \leq g(M_1) + g(M_2)$.

Some sufficient conditions for tunnel number of knots not to go down under connected sum are given in [17]. When both A_1 and A_2 are non-separating on the corresponding boundary component, there are some sufficient conditions for the equality $g(M) = g(M_1) + g(M_2)$ to hold, see [2], [8].

In the present paper, we consider the case that A_1 is separating on F_1 (so necessarily $g(F_1) > 1$) and A_2 is non-separating on F_2. The following are the main results of the paper.

* Supported in part by two grants of NSFC (No. 11101058 and No. 11329101).
† Supported in part by a key grant of NSFC (No. 10931005) and a grant of NSFC (No. 11329101).
‡ Supported in part by two grants of NSFC (No. 11001065 and No. 11071106).
Received 2 July 2012, in final form 24 June 2013.

THEOREM 1.1. *Let M be an irreducible 3-manifold, and A a properly embedded essential annulus separating M into M_1 and M_2. Suppose $A_i = A$ lies in an incompressible boundary component F_i of M_i for $i = 1, 2$, and A_1 is separating on F_1 and A_2 is non-separating on F_2. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $d(S_i) \geq 2g(M_i)+5$ for $i = 1, 2$. Then $g(M) = g(M_1)+g(M_2)$.*

THEOREM 1.2. *Let M be an irreducible 3-manifold, and A a properly embedded essential annulus separating M into M_1 and M_2. Suppose $A_i = A$ lies in an incompressible boundary component F_i of M_i for $i = 1, 2$, and A_1 is separating on F_1 and A_2 is non-separating on F_2. Suppose M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $d(S_i) \geq 2g(M_i) + 5$ for $i = 1, 2$, then*

(1) *If $g(F_2) \geq 2$, then the minimal Heegaard splitting of M is unique up to isotopy;*

(2) *If $g(F_2) = 1$, then there are at most two minimal Heegaard splittings of M up to isotopy.*

Note that papers [8] and [17] only consider the case that both A_1 and A_2 are non-separating. In [2], the bound of the Heegaard distance for the additivity to hold is "$d(S_i) \geq 2(g(M_1) + g(M_2))$, $i = 1, 2$", in other words, the bound of $d(S_1)$ ($d(S_2)$, resp.) of M_1 (M_2, resp.) not only relates to the genus of M_1 (M_2, resp.), but also relates to the genus of the other manifold M_2 (M_1, resp.). While in our results, the bound is "$d(S_i) \geq 2g(M_i) + 5$, $i = 1, 2$", that is, $d(S_1)$ only relates to $g(M_1)$ and $d(S_2)$ only relates to $g(M_2)$. And with the weaker assumptions, we obtain the stronger conclusion that the minimal Heegaard splitting of M is in some sense unique. Hence we remark that the situations here are quite different from those in [2], [8] and [17], and the arguments there are not applicable to the main cases here.

The paper is organized as follows. In Section 2, we review some preliminaries which will be used later. In Section 3, we give the proof of the main results.

2. Preliminaries

In this section, we will review some fundamental facts on surfaces in 3-manifolds. Definitions and terms which have not been defined are all standard, refer to, for example, [5].

A *Heegaard splitting* of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V and W are compression bodies and $S = V \cap W = \partial_+ V = \partial_+ W$ is the Heegaard surface. $V \cup_S W$ is said to be *weakly reducible* if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $V \cup_S W$ is *strongly irreducible* (see [1]).

Suppose $M = V \cup_S W$ is a Heegaard splitting. Let $\mathscr{C}_V(S)$ ($\mathscr{C}_W(S)$, resp.), denote the set of all simple closed curves on S which bound essential disks in V (W, resp.). Define $d(S) = \min\{d(\alpha, \beta) \mid \alpha \in \mathscr{C}_V(S), \beta \in \mathscr{C}_W(S)\}$, where $d(\alpha, \beta)$ is measured in the curve complex $\mathscr{C}(S)$ of S (see [4]).

Let F be a surface properly embedded in M. We say that F is *∂-parallel* if F cuts off a 3-manifold homeomorphic to $F \times [0, 1]$ from M. A properly embedded surface is *essential* if it is incompressible and not ∂-parallel.

Let P be a properly embedded separating surface in a 3-manifold M which cuts M into two 3-manifolds M_1 and M_2. Then P is *bicompressible* if P has compressing disks in both M_1 and M_2. P is *strongly irreducible* if it is bicompressible and each compressing disk in M_1 meets each compressing disk in M_2.

Now let P be a closed bicompressible surface in an irreducible 3-manifold M. By maximally compressing P in both sides of P and removing any resulting 2-sphere components, we obtain two surfaces that we denote by P_+ and P_-. Let H_1^P denote the closure of the region that lies between P and P_+ and similarly define H_2^P to denote the closure of the region that lies between P and P_-. Then H_1^P and H_2^P are compression bodies. If P is strongly irreducible in M, then the Heegaard splitting $H_1^P \cup_P H_2^P$ is strongly irreducible. Two strongly irreducible surfaces P and Q are said to be *well-separated* in M if $H_1^P \cup_P H_2^P$ may be isotoped to be disjoint from $H_1^Q \cup_Q H_2^Q$.

Scharlemann and Thompson ([13]) showed that any irreducible and ∂-irreducible Heegaard splitting $M = V \cup_S W$ has an *untelescoping* as

$$V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \dots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m),$$

such that each $V_i \cup_{S_i} W_i$ is a strongly irreducible Heegaard splitting with $F_i = \partial_- W_i \cap \partial_- V_{i+1}$, $1 \le i \le m-1$, $\partial_- V_1 = \partial_- V$, $\partial_- W_m = \partial_- W$, and for each i, each component of F_i is a closed incompressible surface of positive genus, and only one component of $M_i = V_i \cup_{S_i} W_i$ is not a product. It is easy to see that when $m \ge 2$, $g(S) \ge g(S_i) + 1 \ge g(F_i) + 2$ for each i. From $V_1 \cup_{S_1} W_1, \dots, V_m \cup_{S_m} W_m$, we can get a Heegaard splitting of M by a process called amalgamation (see [16]).

The following are some basic facts and results on Heegaard splittings.

LEMMA 2.1 ([11]). *Suppose $(Q, \partial Q) \subset (M, \partial M)$ is an essential surface and Q' is the result of ∂-compressing Q in M. Then Q' is essential.*

LEMMA 2.2 ([15], [9]). *Let F be an incompressible surface (not a 2-sphere, a 2-disk or a projective plane) properly embedded in $M = V \cup_S W$. If $V \cup_S W$ is strongly irreducible, then F can be isotoped so that $S \cap F$ consists of loops that are essential in both F and S.*

The following Lemma is a well known fact (see [15]).

LEMMA 2.3. *An incompressible surface F in a compression body V with $\partial F \subset \partial_+ V$ cuts V into compression bodies.*

LEMMA 2.4 ([9]). *Let V be a non-trivial compression body and $\mathscr{A}$ be a disjoint union of essential annuli properly embedded in V. Then there is an essential disk D in V with $D \cap \mathscr{A} = \emptyset$.*

LEMMA 2.5 ([3], [11]). *Let $V \cup_S W$ be a Heegaard splitting of M and F be a properly embedded incompressible surface (maybe not connected) in M. Then either any component of F is ∂-parallel in M or $d(S) \leq 2 - \chi(F)$.*

LEMMA 2.6 ([14]). *Let P and Q be strongly irreducible connected closed separating surfaces in a 3-manifold M. Then either*

(1) *P and Q are well-separated,*

(2) *P and Q are isotopic, or*

(3) *$d(P) \leq 2g(Q)$.*

LEMMA 2.7 ([6]). *Let $M = V \cup_S W$ be a Heegaard splitting with $d(S) > 2g(M)$. Then $V \cup_S W$ is the unique minimal Heegaard splitting of M up to isotopy.*

LEMMA 2.8 ([12]). *Let V be a non-trivial compression body and $\mathscr{A}$ be a disjoint union of incompressible annuli properly embedded in V. If U is a component of $V \backslash \mathscr{A}$ with $U \cap \partial_- V \neq \emptyset$, then $\chi(U \cap \partial_- V) \geq \chi(U \cap \partial_+ V)$.*

LEMMA 2.9 ([8], [17]). *Let N be a compact orientable 3-manifold which is not a compression body, and $F = \partial N$. Suppose Q is a properly embedded connected separating surface in N with ∂Q essential in F, and Q cuts N into two compression bodies N_1 and N_2 with $Q = \partial_+ N_1 \cap \partial_+ N_2$ and $F \cap N_2$ a collection of annuli. If Q is compressible in both N_1 and N_2, and Q can be compressed to Q^* in N_1 such that any component of Q^* is ∂-parallel in N, then N has a Heegaard splitting $V \cup_S W$ with $g(S) = 1 - \frac{1}{2}\chi(Q)$ and $d(S) \leq 2$.*

3. The proof of the main results

In $M = M_1 \cup_h M_2$, let $A = A_2 = h(A_1)$ and F_i be the incompressible boundary component of M_i in which A_i lies, $i = 1, 2$. We denote the two components of $F_1 - int A$ by F_1^1 and F_1^2, and let $F_3 = F_1^1 \cup (F_2 - int A) \cup F_1^2$, then F_3 is a boundary component of M. Let $I = [0, 1]$ and $F_i \times I$ be a regular neighborhood of F_i in M_i with $F_i = F_i \times \{0\}$. We denote the surface $F_i \times \{1\}$ by F^i, it's clear that F^i is incompressible in M_i. Let $M^i = M_i - F_i \times [0, 1)$ for $i = 1, 2$, and $M^0 = F_1 \times I \cup_A F_2 \times I$. Then $M = M^1 \cup_{F^1} M^0 \cup_{F^2} M^2$. Note that M^0 contains three boundary components F^1, F^2 and F_3. By [2] Lemma 2.3, M^0 contains only two essential surfaces up to isotopy, F_1^* and F_2^*, say, where $F_1^* = X_1 \cup X_2 \cup X_3$ such that X_1 and X_3 are isotopic to F_1^1, and X_2 is a copy of $F_2 - int A$. And $F_2^* = Y_1 \cup Y_2 \cup Y_3$ such that Y_1 and Y_3 are isotopic to F_1^2, and Y_2 is a copy of $F_2 - int A$.

LEMMA 3.1. *Let M^1, M^0, F^1 and F^2 be as above. If M_1 has a Heegaard splitting $V_1 \cup_{S_1} W_1$ with $d(S_1) \geq 2g(M_1) + 5$, then any minimal Heegaard splitting of $M^1 \cup_{F^1} M^0$ is the amalgamation of minimal Heegaard splittings of M^1 and M^0, and $g(M^1 \cup_{F^1} M^0) = g(M_1) + g(F_2)$.*

PROOF. Now suppose that $V \cup_S W$ is a minimal Heegaard splitting of $M^1 \cup_{F^1} M^0$ with $F_3 \subset \partial_- V$. Since $M^1 \cup_{F^1} M^0 = M_1 \cup_A (F_2 \times I)$, by a result of Schultens [15], $g(S) \leq g(M_1) + g(F_2)$.

First we show that M_1 is not a compression body. Otherwise, M_1 is either a (closed surface)$\times I$ or ∂-reducible. In each case, M_1 cannot have a Heegaard splitting $V_1 \cup_{S_1} W_1$ with distance $d(S_1) \geq 2g(M_1) + 5$.

We will show that $V \cup_S W$ is weakly reducible. So suppose for contradiction that $V \cup_S W$ is strongly irreducible. First note that $S \cap A \neq \emptyset$ since F^1 is essential in M and there is no closed essential surface in a compression body. By Lemma 2.2, we may assume that each component of $S \cap A$ is essential in both S and A, and $|S \cap A|$ is minimal. Since A is an essential annulus in M, by Lemma 2.4, $S \backslash A$ has compressing disks in both V and W. If there are distinct components of $S \backslash A$, say P_1 and P_2, such that P_1 has a compressing disk D_V in V and P_2 has a compressing disk D_W in W, but $P_1 \neq P_2$, then we have $\partial D_V \cap \partial D_W = \emptyset$, contradicting the strong irreducibility of $V \cup_S W$. Hence we assume that all components of $S \backslash A$ are incompressible in $M \backslash A$ except exactly one bicompressible component.

CLAIM 1. $\chi(S \cap M_1) \leq -2g(M_1)$.

PROOF. Now if $S \cap M_1$ is incompressible, then it is essential in M_1. Otherwise, any component of $S \cap M_1$ is ∂-parallel in M_1, which means that M_1 is a compression body, a contradiction. By Lemma 2.5, $2 - \chi(S \cap M_1) \geq d(S_1) \geq 2g(M_1) + 5$, thus $\chi(S \cap M_1) \leq -3 - 2g(M_1)$.

Now suppose $S \cap M_1$ is bicompressible. We denote the bicompressible component of $S \cap M_1$ by P. In fact, P is strongly irreducible in M_1. We claim that $\chi(P) \leq -2$. If not, P is either a disk, an annulus, a pair of pants, or a once punctured torus. In each case we conclude that a component of ∂P bounds a disk in M_1, therefore A is compressible in M_1, a contradiction. If there exists an incompressible component Q of $S \cap M_1$ which is essential in M_1, by Lemma 2.5, $2 - \chi(Q) \geq d(S_1) \geq 2g(M_1) + 5$, and then $\chi(S \cap M_1) \leq \chi(Q) + \chi(P) \leq -5 - 2g(M_1)$. Hence in the following we may assume that the incompressible components of $S \cap M_1$ are all ∂-parallel in M_1. Let P^V be the surface obtained by maximally compressing P in V and removing all possible 2-sphere components. Since P is strongly irreducible, we can see that, by the No Nesting Lemma ([10]) and [10, Lemma 5.5], P^V is incompressible in M_1. Now if P^V is essential in M_1, by Lemma 2.5, $2 - \chi(P^V) \geq d(S_1) \geq 2g(M_1) + 5$, and then $\chi(S \cap M_1) \leq \chi(P) \leq \chi(P^V) - 2 \leq -5 - 2g(M_1)$. Therefore we may assume that each component of P^V is ∂-parallel in M_1.

Since A is an essential annulus in M and by Lemma 2.3, each component of $V \cap M_1$ and $W \cap M_1$ is a compression body. Let U_1 be the component of $V \cap M_1$ containing P and U_2 be the component of $W \cap M_1$ containing P. Since any component of $S \cap M_1$ other than P is ∂-parallel, $U_1 \cup_P U_2 \cong M_1$ and $\partial_+ U_1 \cap \partial_+ U_2 = P$. Since M_1 is not a compression body and A is an annulus, by Lemma 2.9, there exists a Heegaard surface S^1 of M_1 with $d(S^1) \leq 2$ and $g(S^1) \leq 1 - \frac{1}{2}\chi(P)$. Since $d(S^1) \leq 2$, by Lemma 2.7, S^1 is not isotopic to the unique minimal Heegaard surface S_1 of M_1, we have $g(S^1) \geq g(M_1) + 1$. Hence $\chi(S \cap M_1) \leq \chi(P) \leq 2 - 2g(S^1) \leq -2g(M_1)$.

This completes the proof of Claim 1.

Claim 2. $\chi(S \cap (F_2 \times I)) \leq \chi(F_2)$.

Proof. Since A is an essential annulus in M, if we denote the component of $V \cap (F_2 \times I)$ or $W \cap (F_2 \times I)$ which contains F^2 by U, by Lemma 2.8, we have $\chi(S \cap (F_2 \times I)) \leq \chi(U \cap (S \cap (F_2 \times I))) \leq \chi(U \cap F^2) = \chi(F_2)$.

This completes the proof of Claim 2.

Now by the proof of Claim 1, we know that if $S \cap M_1$ is incompressible in M_1 then it is essential in M_1, and therefore by Lemma 2.5, $\chi(S \cap M_1) \leq -3 - 2g(M_1)$. Then $2g(S) = 2 - \chi(S \cap M_1) - \chi(S \cap (F_2 \times I)) \geq 2g(M_1) + 2g(F_2) + 3$, a contradiction. Hence $S \cap M_1$ is bicompressible in M_1 and $\chi(S \cap M_1) \leq -2g(M_1)$. Note that S is a strongly irreducible Heegaard surface of M, so $S \cap (F_2 \times I)$ is incompressible in $F_2 \times I$. By Lemma 2.1, $S \cap (F_2 \times I)$ is ∂-parallel in $F_2 \times I$ since any incompressible and ∂-incompressible surface in a trivial compression body is just a spanning annulus.

If $\chi(S \cap (F_2 \times I)) < \chi(F_2)$, then $2g(S) = 2 - \chi(S \cap M_1) - \chi(S \cap (F_2 \times I)) > 2g(M_1) + 2g(F_2)$, a contradiction. Thus $\chi(S \cap (F_2 \times I)) = \chi(F_2)$. Now if $\chi(F_2) = 0$, i.e., $g(F_2) = 1$, then $g(S) \leq g(M_1) + 1$. Since S is a Heegaard surface of $M_1 \cup_A F_2 \times I$ while S_1 is a Heegaard surface of M_1, S and S_1 are not well-separated, furthermore, S is not isotopic to S_1. Then by Lemma 2.6, $d(S_1) \leq 2g(S) \leq 2g(M_1) + 2 < 2g(M_1) + 5$, a contradiction. So $g(F_2) > 1$ and $S \cap (F_2 \times I)$ has one and only one component. Since A_2 is non-separating, we have that $|S \cap A| = 2$.

We take an essential arc α in $S \cap (F_2 \times I)$ such that α is adjacent to the two components of $S \cap A$. Let $N^* = N((S \cap A) \cup \alpha, S \cap (F_2 \times I))$ be the regular neighborhood of $(S \cap A) \cup \alpha$ in $S \cap (F_2 \times I)$), and denote the component of ∂N^* which is not a component of $S \cap A$ by β^*, from the construction of regular neighborhood we may assume that $\beta^* \subset F_2 \times \left\{\frac{1}{2}\right\}$ in $F_2 \times I$. Set $A^* = \beta \times I$ with $\beta \subset F_2 \backslash A_2$ and $\beta^* = \beta \times \left\{\frac{1}{2}\right\}$. Denote the component of $(F_2 \backslash A_2) \backslash \beta$ which doesn't contain ∂A_2 by F^*. By substituting $F^* \times I$ with (a disk)$\times I$, we get a 3-manifold M' with a Heegaard surface S' induced from the Heegaard surface S of $M^1 \cup_{F^1} M^0$. Since S is strongly irreducible, S' is strongly irreducible with $g(S') \leq g(M_1) + 1$. Then by Lemma 2.6, $d(S_1) \leq 2g(S') \leq 2g(M_1) + 2 < 2g(M_1) + 5$, a contradiction.

Hence $V \cup_S W$ is weakly reducible. Then $V \cup_S W$ has an untelescoping as $V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} \dots \cup_{H_{n-1}} (V_n' \cup_{S_n'} W_n')$, where $n \geq 2$, and each component of H_i, $1 \leq i \leq n-1$, is a closed essential surface in $M^1 \cup_{F^1} M^0$. Let $\mathcal{H} = \bigcup_{i=1}^{n-1} H_i$. We may assume that any component of $\mathcal{H} \cap A$ is essential in both A and $\mathcal{H}$, and $|\mathcal{H} \cap A|$ is minimal. If $\mathcal{H} \cap A \neq \emptyset$, let H be a component of $\mathcal{H}$ with $H \cap A \neq \emptyset$. If $H \cap M_1$ is essential in M_1 and $H \cap (F_2 \times I)$ is ∂-parallel in $F_2 \times I$, by Lemma 2.5, $2 - \chi(H \cap M_1) \geq d(S_1) \geq 2g(M_1) + 5$, $\chi(H \cap (F_2 \times I)) \leq \chi(F_2)$, and then $2g(S) \geq 2g(H) + 4 = 6 - \chi(H \cap M_1) - \chi(H \cap (F_2 \times I)) \geq 2g(M_1) + 2g(F_2) + 4$, a contradiction.

Hence if $H \cap A \neq \emptyset$, $H \cap M_1$ is ∂-parallel in M_1 and $H \cap (F_2 \times I)$ is ∂-parallel in $F_2 \times I$. Then H is an essential closed surface in M^0, hence H is isotopic to either F_1^* or F_2^*. We may assume that H is isotopic to F_1^*.

If there is no component of $\mathcal{H}$ in M_1, we denote the Heegaard splitting in the untelescoping between F_1^* and F_3 by $N_1 = V_1' \cup_{S_1'} W_1'$. Since A is essential in M, it is essential in N_1. Note that $N_1 \cap M_1 \cong M_1$ and $N_1 \cap M_2 \cong F_2 \times I$. Since $V_1' \cup_{S_1'} W_1'$ is strongly irreducible, by Claim 1 and Claim 2 applied to this splitting we have $\chi(S_1' \cap M_1) \leq -2g(M_1)$, $\chi(S_1' \cap (F_2 \times I)) \leq \chi(F_2)$. Then $2g(S) \geq 4 - \chi(S_1') \geq 2g(M_1) + 2g(F_2) + 2$, a contradiction.

Hence there is some component of $\mathcal{H}$ in M_1. We denote the outermost one by F^*, and suppose F^* is essential in M_1. We denote the Heegaard splitting in the untelescoping between F^* and F_3 by $N_1 = V_1' \cup_{S_1'} W_1'$.

Claim 3. $\chi(S_1' \cap M_1) \leq -3 - 2g(M_1)$.

Proof. Since F^* is essential in M_1, by Lemma 2.5 $2 - \chi(F^*) \geq d(S_1) \geq 2g(M_1) + 5$. If we denote the component of $V_1' \cap M_1$ or $W_1' \cap M_1$ which contains the essential component F^* by U, since A is an essential annulus in M, by Lemma 2.8, we have $\chi(S_1' \cap M_1) \leq \chi(U \cap (S_1' \cap M_1)) \leq \chi(U \cap F^*) \leq \chi(F^*) \leq -3 - 2g(M_1)$.

This completes the proof of Claim 3.

Now by Claim 2, we have $\chi(S_1' \cap (F_2 \times I)) \leq \chi(F_2)$. Then $2g(S) \geq 4 - \chi(S_1') \geq 2g(M_1) + 2g(F_2) + 5$, a contradiction.

Hence $S_1' \cap A = \emptyset$, and since F^* is the outermost component of $\mathscr{H}$ in M_1, we know that some component of $\mathscr{H}$ must be parallel to F^1 in M_1.

Then we get a generalized Heegaard splitting as: $V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} (V_2' \cup_{S_2'} W_2')$, and H_1 is isotopic to F^1. We may further assume that $V_1' \cup_{S_1'} W_1'$ is a Heegaard splitting of M^1, and $V_2' \cup_{S_2'} W_2'$ is a Heegaard splitting of M^0. Since A is separating on F_1 and non-separating on F_2, M^0 contains only three boundary components F^1, F^2 and F_3. Note that $g(F_3) = g(F_1) + g(F_2) - 1$, hence $g(S_2') \geq g(M^0) \geq g(F_1) + g(F_2)$. Then we have $g(S) = g(S_1') + g(S_2') - g(H_1) \geq g(M_1) + g(F_2)$, and equality holds if and only if $g(S_1') = g(M_1)$, $g(S_2') = g(F_1) + g(F_2)$. Combining this with Schultens' result we see that $g(S) = g(M_1) + g(F_2)$, and therefore the previous inequality is an equality, implying that $g(S_1') = g(M_1)$ and $g(S_2') = g(F_1) + g(F_2)$. Hence $V \cup_S W$ is the amalgamation of minimal Heegaard splittings of M^1 and M^0.

LEMMA 3.2. *Let M^2, M^0 and F^2 be as above. If M_2 has a Heegaard splitting $V_2 \cup_{S_2} W_2$ with $d(S_2) \geq 2g(M_2) + 5$, then any minimal Heegaard splitting of $M^2 \cup_{F^2} M^0$ is the amalgamation of minimal Heegaard splittings of M^2 and M^0, and $g(M^2 \cup_{F^2} M^0) = g(M_2) + g(F_1)$.*

PROOF. The proof is analogous to the proof of Lemma 3.1. The only difference is that A is separating in F_1 but non-separating in F_2.

Now suppose that $V \cup_S W$ is a minimal Heegaard splitting of $M^2 \cup_{F^2} M^0$ with $F_3 \subset \partial_- V$. Since $M^2 \cup_{F^2} M^0 = (F_1 \times I) \cup_A M_2$, we have $g(S) \leq g(M_2) + g(F_1)$. Following the lines of the proof of Lemma 3.1, M_2 is not a compression body.

We will show that $V \cup_S W$ is weakly reducible. So suppose for contradiction that $V \cup_S W$ is strongly irreducible. First note that $S \cap A \neq \emptyset$, since F^2 is essential in M. By Lemma 2.2, we may assume that each component of $S \cap A$ is essential in both S and A, and $|S \cap A|$ is minimal.

By proofs similar to those of Claim 1 and 2 we have the following two Claims.

CLAIM 4. $\chi(S \cap M_2) \leq -2g(M_2)$.

CLAIM 5. $\chi(S \cap (F_1 \times I)) \leq \chi(F_1)$.

By arguments similar to those in the proof of Lemma 3.1, we know that $S \cap M_2$ is bicompressible in M_2 while $S \cap (F_1 \times I)$ is ∂-parallel in $F_1 \times I$. If all components of $S \cap (F_1 \times I)$ are parallel to the same component of F_1^1 and F_1^2, say F_1^1, then in V, a component of $A \cap V$ cuts off a trivial compression body $F^1 \times I$ from V, but this is impossible since the component of $A \cap V$ is a spanning annulus in V. Hence at least one component of $S \cap (F_1 \times I)$ is parallel to F_1^1 and at least one component of $S \cap (F_1 \times I)$ is parallel to F_1^2.

Now if $\chi(S \cap (F_1 \times I)) < \chi(F_1)$, then $2g(S) = 2 - \chi(S \cap M_2) - \chi(S \cap (F_1 \times I)) > 2g(M_2) + 2g(F_1)$, a contradiction. Thus $\chi(S \cap (F_1 \times I)) = \chi(F_1)$. Since A is separating in F_1 and $|\partial A| = 2$, $\chi(F_1^1) \leq -1$ and $\chi(F_1^2) \leq -1$. Hence if $\chi(S \cap (F_1 \times I)) = \chi(F_1)$, $S \cap (F_1 \times I)$ has only two components with one parallel to F_1^1 and the other parallel to F_1^2. This means that $|S \cap A| = 2$. Suppose that the boundary components of A are simple closed curves α_1 and α_2. Now if $g(F_1^1) = r > 1$, without loss of generality, we may assume that $\partial F_1^1 = \alpha_1$. Take a separating simple closed curve γ in F_1^1 such that $F_1^1 \backslash \gamma = T_{1,2} \cup \Sigma_{r-1,1}$, where $T_{1,2}$ is a twice punctured torus with boundary α_1 and γ, and $\Sigma_{r-1,1}$ is a once punctured genus $r-1$ surface with boundary γ. In $M^2 \cup_{F^2} M^0$, cut $F_1^1 \times I$ along $\gamma \times I$, we get two 3-manifolds $T_{1,2} \times I$ and $\Sigma_{r-1,1} \times I$. Get rid of $\Sigma_{r-1,1} \times I$ and keep $T_{1,2} \times I$, we add a 2-handle along γ to get $\widetilde{M}_1 = (T_{1,2} \times I) \cup_\gamma 2$-handle. And if $g(F_1^1) = 1$, we can take $\widetilde{M}_1 = F_1^1 \times I$. For F_1^2, we do the same operation to get $\widetilde{M}_2$. Then we get a 3-manifold $M' = \left(\widetilde{M}_1 \cup_{\alpha_1 \times I} (A \times I) \cup_{\alpha_2 \times I} \widetilde{M}_2\right) \bigcup_A M_2$ (Actually, $\widetilde{M}_1 \cup_{\alpha_1 \times I} (A \times I) \cup_{\alpha_2 \times I} \widetilde{M}_2$ is a product of a genus 2 closed surface with I) with a Heegaard surface S' induced from S. Since S is strongly irreducible, S' is strongly irreducible with $g(S') \leq g(M_2) + 2$. Then by Lemma 2.6, $d(S_2) \leq 2g(S') \leq 2g(M_2) + 4 < 2g(M_2) + 5$, a contradiction.

Hence $V \cup_S W$ is weakly reducible. Then $V \cup_S W$ has an untelescoping as $V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} \ldots \cup_{H_{n-1}} (V_n' \cup_{S_n'} W_n')$, where $n \geq 2$, and each component of H_i, $1 \leq i \leq n-1$, is a closed essential surface in $M^2 \cup_{F^2} M^0$. Let $\mathscr{H} = \bigcup_{i=1}^{n-1} H_i$. We may assume that any component of $\mathscr{H} \cap A$ is essential in both A and $\mathscr{H}$, and $|\mathscr{H} \cap A|$ is minimal. Let $\mathscr{H}' = \bigcup H_i$ where $H_i \in \mathscr{H}$ and $H_i \cap M_2$ is essential in M_2. If some component H' of $\mathscr{H}'$ and F^1 cobound a Heegaard splitting in the untelescoping, we denote the Heegaard splitting between H' and F^1 by $N_j = V_j' \cup_{S_j'} W_j'$. Since $H' \cap M_2$ is essential in M_2, by Claim 3 in the proof of Lemma 3.1, we have $\chi(S_j' \cap M_2) \leq -3 - 2g(M_2)$, and by Claim 5, $\chi(S_j' \cap (F_1 \times I)) \leq \chi(F_1)$. Then we have $2g(S) \geq 4 - \chi(S_j') \geq 2g(M_2) + 2g(F_1) + 5$, a contradiction.

Hence the outermost component with $H \cap A \neq \emptyset$ must be ∂-parallel in M_2. We may assume that H is isotopic to F_1^*. Let $N_1 = V_1' \cup_{S_1'} W_1'$ be the Heegaard splitting bounded by F^1, F_1^* and F_3 in the untelescoping. Then $g(N_1) \geq \min\{g(F_1) + g(F_1^*), g(F_1) + g(F_3), g(F_1^*) + g(F_3)\}$. Note that $g(F_3) = g(F_1) + g(F_2) - 1$ and $g(F_1^*) = g(F_2) + 2g(F_1^1) - 1$, hence $g(S_1') \geq g(N_1) \geq g(F_1) + g(F_2)$.

If there is no other component of $\mathscr{H}$, we denote the Heegaard splitting in the untelescoping bounded by F_1^* by $N_2 = V_2' \cup_{S_2'} W_2'$. A is essential in M, and hence in N_2. Note that $N_2 \cap M_1 \cong F_1^1 \times I$ and $N_2 \cap M_2 \cong M_2$. By Claim 4, we have $\chi(S_2' \cap M_2) \leq -2g(M_2)$. Since S_2' is separating in N_2, $|S_2' \cap A|$ is even while $|\partial F_1^1| = 1$. This means that $S_2' \cap (F_1 \times I)$ has at least two components. Then by Claim 5, $\chi(S_2' \cap (F_1 \times I)) \leq 2\chi(F_1^1)$. Then we have $g(S) = g(S_1') + g(S_2') - g(F_1^*) \geq g(M_2) + g(F_1) + 1$, a contradiction.

Hence there is some other component F^* of $\mathcal{H}$. If $F^* \cap M_2$ is essential in M_2, we denote the Heegaard splitting in the untelescoping between F_1^* and F^* by $N_2 = V_2' \cup_{S_2'} W_2'$. Then by Claim 3 in the proof of Lemma 3.1, we have $\chi(S_2' \cap M_2) \leq -3 - 2g(M_2)$. By Claim 5, we have $\chi(S_2' \cap (F_1 \times I)) \leq 2\chi(F_1^1)$. Then $g(S) = g(S_1') + g(S_2') - g(F_1^*) \geq g(M_2) + g(F_1) + 2$, a contradiction.

Hence one component of $\mathcal{H}$ must be parallel to F^2 in M_2. Then by the same arguments as in the last paragraph of the proof of Lemma 3.1, only replacing M_1 by M_2, we have that $V \cup_S W$ is the amalgamation of minimal Heegaard splittings of M^2 and M^0, and $g(S) = g(M_2) + g(F_1)$.

Now we come to the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Let $V \cup_S W$ be a minimal Heegaard splitting of M. Then $g(S) \leq g(M_1) + g(M_2)$.

If $V \cup_S W$ is strongly irreducible, then by Lemma 2.2 we may assume $S \cap A$ is a collection of essential simple closed curves on both S and A, and $|S \cap A|$ is minimal. By Claim 1 in the proof of Lemma 3.1, $\chi(S \cap M_1) \leq -2g(M_1)$, and by Claim 4 in the proof of Lemma 3.2, we have that $\chi(S \cap M_2) \leq -2g(M_2)$. Then $2g(S) = 2 - \chi(S \cap M_1) - \chi(S \cap M_2) \geq 2g(M_1) + 2g(M_2) + 2$, a contradiction.

Hence $V \cup_S W$ is weakly reducible. Thus $V \cup_S W$ has an untelescoping as

$$V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} \ldots \cup_{H_{m-1}} (V_m' \cup_{S_m'} W_m'),$$

where $m \geq 2$, and for each i, each component of H_i is a closed essential surface in M. Let $\mathcal{F} = \bigcup_{i=1}^{m-1} H_i$.

CLAIM 6. *For any $i \in \{2, \ldots, m-1\}$, there are no two components H_{i-1}, H_i of $\mathcal{F}$ so that $H_{i-1} \cap M_1$ is essential in M_1 and $H_i \cap M_2$ is essential in M_2 whether $H_{i-1} \cap M_1$ and $H_i \cap M_2$ have boundary or not.*

PROOF. Suppose there exist two components of $\mathcal{F}$ so that $H_{i-1} \cap M_1$ is essential in M_1 and $H_i \cap M_2$ is essential in M_2. Suppose $V_i' \cup_{S_i'} W_i'$ is the Heegaard splitting in the untelescoping between them. Then by Claim 3 in the proof of Lemma 3.1, we have $\chi(S_i' \cap M_1) \leq -3 - 2g(M_1)$, and $\chi(S_i' \cap M_2) \leq -3 - 2g(M_2)$. Hence $2g(S) \geq 4 - \chi(S_i') > 2g(M_1) + 2g(M_2) + 4$, a contradiction.

This completes the proof of Claim 6.

We now divide the proof into the following three cases.

Case 1. No component of $\mathcal{F}$ is ∂-parallel in M_1 or M_2, and $A \cap \mathcal{F} = \emptyset$.

In this case, by Claim 6 and the assumption, we may assume that any component of $\mathscr{F}$ is contained in M_1. Let H be an outermost component of $\mathscr{F}$ in M_1. By the hypothesis of Case 1, H is essential in M_1.

Suppose $A \subset N_j = V_j' \cup_{S_j'} W_j'$. A is essential in M, and hence in N_j. Since H is essential in M_1, by Claim 3 in the proof of Lemma 3.1, we have $\chi(S_j' \cap M_1) \leq -3 - 2g(M_1)$. Now $N_j \cap M_2 = M_2$, and by Claim 4 in the proof of Lemma 3.2, we have that $\chi(S_j' \cap M_2) \leq -2g(M_2)$. Then $2g(S) \geq 4 - \chi(S_j') \geq 2g(M_1) + 2g(M_2) + 7$, a contradiction.

Thus, Case 1 cannot happen.

Case 2. No component of $\mathscr{F}$ is ∂-parallel in M_1 or M_2, and $A \cap \mathscr{F} \neq \emptyset$.

In this case, we may assume that any component of $\mathscr{F} \cap A$ is essential in both A and $\mathscr{F}$, and $|\mathscr{F} \cap A|$ is minimal. There are three subcases.

Subcase 2.1. The outermost component H of $\mathscr{F}$ with $H \cap A \neq \emptyset$ is essential in M_1 but ∂-parallel in M_2.

By Claim 6, we may assume that each component of $\mathscr{F} \cap M_1$ with boundary is essential in M_1 and each component of $\mathscr{F} \cap M_2$ with boundary is ∂-parallel in M_2. Among the components of $\mathscr{F} \cap M_2$, let B be the innermost one, that is, B cuts M_2 into two pieces M_2' and M_2'', where $M_2' \cong M_2$ and $M_2'' \cong B \times I$, and the interior of M_2' contains no component of $\mathscr{F} \cap M_2$. Let H_r be the component of $\mathscr{F}$ containing B. Then $H_r \cap M_1$ is essential in M_1 and $H_r \cap M_2$ is ∂-parallel in M_2, see Figure 1.

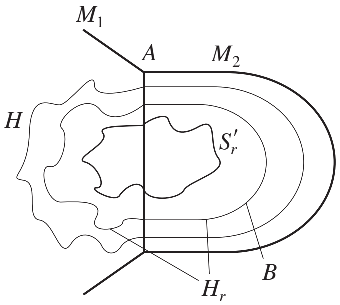

FIGURE 1.

We may assume that M_2' is contained in the submanifold $N_r = V_r' \cup_{S_r'} W_r'$ of the untelescoping. Since B is innermost, N_r is not a product. $V_r' \cup_{S_r'} W_r'$ is a strongly irreducible Heegaard splitting of N_r. Since any component of $H_r \cap M_1$ is essential in M_1, by Claim 3 in the proof of Lemma 3.1, $\chi(S_r' \cap M_1) \leq -3 - 2g(M_1)$. Note that $N_r \cap M_2 \cong M_2$, and by Claim 4 in the proof of Lemma 3.2, we have $\chi(S_r' \cap M_2) \leq -2g(M_2)$. Then $2g(S) \geq 4 - \chi(S_r') \geq 2g(M_1) + 2g(M_2) + 7$, a contradiction.

Subcase 2.2. The outermost component H of $\mathscr{F}$ with $H \cap A \neq \emptyset$ is essential in M_2 but ∂-parallel in M_1.

There are two sub-subcases.

Sub-subcase 2.2.1. Each component of $H \cap M_1$ is parallel to the same one of F_1^1 or F_1^2, say F_1^1, in M_1.

We denote the Heegaard splitting in the untelescoping between F_3 and H by $N_j = V_j' \cup_{S_j'} W_j'$. See Figure 2. Note that by Clai 6, $N_j \cap M_1 \cong M_1$, and by Claim 1 in the proof of Lemma 3.1, we have that $\chi(S_j' \cap M_1) \leq -2g(M_1)$. Since $H \cap M_2$ is essential in M_2, by Claim 3 in the proof of Lemma 3.1, $\chi(S_j' \cap M_2) \leq -3-2g(M_2)$. Then $2g(S) \geq 4-\chi(S_j') \geq 2g(M_1)+2g(M_2)+7$, a contradiction.

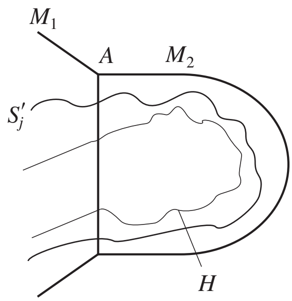

FIGURE 2.

Sub-subcase 2.2.2. At least one component of $H \cap M_1$ is parallel to F_1^1 and at least one component of $H \cap M_1$ is parallel to F_1^2 in M_1.

By Claim 6, we may assume that each component of $\mathscr{F} \cap M_1$ with boundary is ∂-parallel in M_1. Among the components of $\mathscr{F} \cap M_1$ which are parallel to F_1^1, let B_1 be the innermost one, and among the components of $\mathscr{F} \cap M_1$ which are parallel to F_1^2, let B_2 be the innermost one, that is, B_1 and B_2 cut M_1 into three pieces M_1', M_1'' and M_1''' with $M_1' \cong B_1 \times I$, $M_1'' \cong M_1$ and $M_1''' \cong B_2 \times I$, and the interior of M_1'' contains no component of $\mathscr{F} \cap M_1$. By swapping the labels of B_1 and B_2 if necessary, we may suppose that the number of components of $\mathscr{F} \cap M_1$ in M_1' is greater than the number in M_1'''. Let H_j be the component of $\mathscr{F}$ containing B_2. Then by Claim 6, we have that $H_j \cap M_1$ is ∂-parallel in M_1 and $H_j \cap M_2$ is essential in M_2. See Figure 3.

We may assume that M_1'' is contained in the submanifold $N_j = V_j' \cup_{S_j'} W_j'$ of the untelescoping. Since $H_j \cap M_2$ is essential in M_2, by Claim 3 in the proof of Lemma 3.1, $\chi(S_j' \cap M_2) \leq -3 - 2g(M_2)$. Note that $N_j \cap M_1 \cong M_1$, and by Claim 1 in the proof of Lemma 3.1, we have that $\chi(S_j' \cap M_1) \leq -2g(M_1)$. Then $2g(S) \geq 4 - \chi(S_j') \geq 2g(M_1) + 2g(M_2) + 7$, a contradiction.

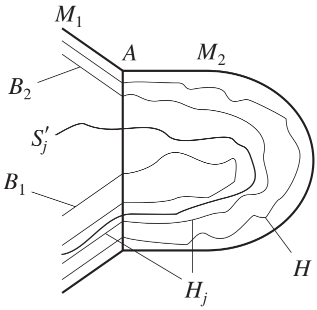

FIGURE 3.

Subcase 2.3. The outermost component H of $\mathscr{F}$ with $H \cap A \neq \emptyset$ is isotopic to F_1^* or F_2^*, say F_1^*.

We denote the Heegaard splitting in the untelescoping between F_3 and F_1^* by $N_j = V_j' \cup_{S_j'} W_j'$. Let $S_j^1 = S_j' \cap M_1$ and $S_j^2 = S_j' \cap M_2$. Now if N_j has some other boundary component H^*, then by assumption $H^* \cap A = \emptyset$, i.e., H^* is a closed essential surface in M_1 or M_2. Now $N_j \cap M_2 \cong (F_2 - int A) \times I$, hence $H^* \subset M_1$. Since H^* is an essential surface in M_1, by Claim 3 and Claim 2 in the proof of Lemma 3.1, we have that $\chi(S_j^1) \leq -3 - 2g(M_1)$, $\chi(S_j^2) \leq \chi(F_2)$. If N_j has no other boundary component, then $N_j \cap M_1 \cong M_1$. By Claim 1 and Claim 2 in the proof of Lemma 3.1, we have $\chi(S_j^1) \leq -2g(M_1)$, $\chi(S_j^2) \leq \chi(F_2)$. Hence whether N_j has some other boundary component or not, we have $\chi(S_j') = \chi(S_j^1) + \chi(S_j^2) \leq 2 - 2g(M_1) - 2g(F_2)$.

We denote the Heegaard splitting in the untelescoping on the other side of F_1^* which has F_1^* as a boundary component by $N_r = V_r' \cup_{S_r'} W_r'$. Let $S_r^i = S_r' \cap M_i$, $i = 1, 2$.

There are three sub-subcases.

Sub-subcase 2.3.1. N_r has another boundary component H' of $\mathscr{F}$ with $H' \cap M_1$ essential in M_1.

In this case, if $H' \cap M_2 = \emptyset$, then $H' \subset (F_1^1 \times I)$, which means that a compression body contains a closed essential surface, a contradiction. Hence $H' \cap M_2 \neq \emptyset$, and all components of $H' \cap M_2$ are ∂-parallel in M_2, and furthermore, by Claim 6, we may assume that each component of $(\mathscr{F} - F_1^*) \cap M_1$ with boundary is essential in M_1 and each component of $\mathscr{F} \cap M_2$ with boundary is ∂-parallel in M_2.

The following arguments are in some sense similar to those in Subcase 2.1. Take the innermost component B of $\mathscr{F} \cap M_2$, that is, B cuts M_2 into two pieces M_2' and M_2'', where $M_2' \cong M_2$ and $M_2'' \cong B \times I$, and the interior of M_2' contains no component of $\mathscr{F} \cap M_2$. Let H_i be the component of $\mathscr{F}$ containing

B. Then $H_i \cap M_1$ is essential in M_1 and $H_i \cap M_2$ is ∂-parallel in M_2. We may assume that M_2' is contained in the submanifold $N_i = V_i' \cup_{S_i'} W_i'$ of the untelescoping. $V_i' \cup_{S_i'} W_i'$ is a strongly irreducible Heegaard splitting of N_i. Since $H_i \cap M_1$ is essential in M_1, by Claim 3 in the proof of Lemma 3.1, $\chi(S_i' \cap M_1) \leq -3 - 2g(M_1)$. Note that $N_i \cap M_2 \cong M_2$, and by Claim 4 in the proof of Lemma 3.2, $\chi(S_i' \cap M_2) \leq -2g(M_2)$. Then $2g(S) \geq 4 - \chi(S_i') \geq 2g(M_1) + 2g(M_2) + 7$, a contradiction.

Sub-subcase 2.3.2. N_r has another boundary component H' of $\mathscr{F}$ with $H' \cap M_2$ essential in M_2.

In this case, $H' \cap M_2$ essential in M_2, and by Claim 3 in the proof of Lemma 3.1, we have that $\chi(S_r^2) \leq -3 - 2g(M_2)$. Whether $H' \cap M_1 = \emptyset$ or not, see Figure 4 (a), (b), by Claim 5 in the proof of Lemma 3.2, we have that $\chi(S_r^1) \leq 2\chi(F_1^1)$. Hence $2g(S) \geq 2 - \chi(S_r') - \chi(S_j') + \chi(F_1^*) \geq 2g(M_1) + 2g(M_2) + 5$, a contradiction.

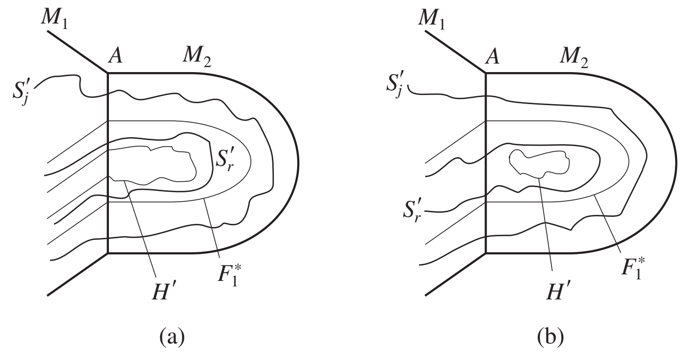

FIGURE 4.

Sub-subcase 2.3.3. N_r has no other boundary component.

In this case, $N_r \cap M_2 \cong M_2$, see Figure 5. By Claim 4 and Claim 5 in the proof of Lemma 3.2, we have that $\chi(S_r^1) \leq 2\chi(F_1^1)$, $\chi(S_r^2) \leq -2g(M_2)$. Hence $2g(S) = 2 - \chi(S_r') - \chi(S_j') + \chi(F_1^*) \geq 2g(M_1) + 2g(M_2) + 2$, a contradiction.

Therefore, Case 2 cannot happen. Thus, we have only one possibility.

Case 3. There exists one component of $\mathscr{F}$ which is ∂-parallel in M_1 or M_2. By applying Lemma 3.1 and Lemma 3.2 to this case, we have the stated conclusion.

This finishes the proof of Theorem 1.1.

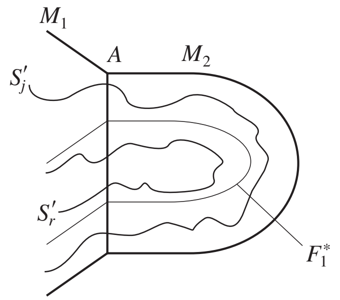

FIGURE 5.

PROPOSITION 3.3. *Any minimal Heegaard splitting of M^0 is strongly irreducible.*

PROOF. Let $V_0 \cup_{S_0} W_0$ be any minimal Heegaard splitting of M^0. Since $M^0 = F_1 \times I \cup_A F_2 \times I$, from the result of Schultens, we have $g(S_0) \leq g(F_1) + g(F_2)$. Since A is separating on F_1 and non-separating on F_2, M^0 contains only three boundary components F^1, F^2 and $F_3 = F_1^1 \cup (F_2 \backslash A_2) \cup F_1^2$. Note that $g(F_3) = g(F_1) + g(F_2) - 1$, and then $g(S_0) = g(M^0) \geq min\{g(F^1) + g(F^2), g(F^1) + g(F_3), g(F^2) + g(F_3)\} \geq g(F_1) + g(F_2)$. Hence we have $g(S_0) = g(F_1) + g(F_2)$.

Now if $V_0 \cup_{S_0} W_0$ is weakly reducible, then it has an untelescoping as $(V_1' \cup_{S_1'} W_1') \cup_{H_1} (V_2' \cup_{S_2'} W_2')$, where H_1 is isotopic to either F_1^* or F_2^*. We may assume that $F_3 \subset \partial_- V_1'$. Since F_3 is incompressible in M^0 and the length of the untelescoping is at least 2, we have $g(S_0) \geq g(S_1') + 1 \geq g(F_3) + 2 \geq g(F_1) + g(F_2) + 1$, a contradiction. Hence $V_0 \cup_{S_0} W_0$ is strongly irreducible.

Now we come to the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Since $d(S_i) \geq 2g(M_i) + 5$, Lemma 2.7 implies that S_i is the unique minimal Heegaard splitting of M_i for $i = 1, 2$. By Theorem 1.1, any minimal Heegaard splitting of M is the amalgamation of minimal Heegaard splittings of M^1, M^0, and M^2 along F^1, F^2, and $g(M) = g(M_1) + g(M_2)$. To prove Theorem 1.2, we only need to consider the minimal Heegaard splitting $V_0 \cup_{S_0} W_0$ of M^0. By Proposition 3.3, $V_0 \cup_{S_0} W_0$ is strongly irreducible. M^0 contains three boundary components F^1, F^2 and F_3, and the type of $V_0 \cup_{S_0} W_0$ is only determined by the partition of its boundary components. We may assume that $F_3 \subset \partial_- V_0$. Note that since A is separating on F_1, $g(F_1) \geq 2$.

(1) When $g(F_2) \geq 2$, since $g(F_3) = g(F_1) + g(F_2) - 1 > g(F_1), g(F_2)$, the only possibility is $F_3 = \partial_- V_0$, $F^1 \cup F^2 = \partial_- W_0$. Hence the minimal Heegaard splitting of M is unique.

(2) When $g(F_2) = 1$, $g(F_3) = g(F_1) + g(F_2) - 1 = g(F_1)$, then there are two possibilities: $F_3 = \partial_- V_0$, $F^1 \cup F^2 = \partial_- W_0$ or $F^1 = \partial_- V_0$, $F^2 \cup F_3 = \partial_- W_0$. Hence there are at most two minimal Heegaard splittings of M.

This completes the proof of Theorem 1.2.

ACKNOWLEDGEMENTS. We wish to thank the referee most warmly for numerous suggestions that have improved the exposition of this paper.

REFERENCES

1. Casson, A. J., and Gordon, C. M., *Reducing Heegaard Splittings*, Topology Appl. 27 (1987), 275–283.
2. Du, K., Ma, J., Qiu, R., and Zhang, M., *Heegaard genera of annular 3-manifolds*, J. Knot Theory Ramifications 20 (2011), 547–566.
3. Hartshorn, K., *Heegaard splittings of Haken manifolds have bounded distance*, Pacific J. Math. 204 (2002), 61–75.
4. Hempel, J., *3-manifolds as viewed from the curve complex*, Topology 40 (2001), 631–657.
5. Jaco, W., *Lectures on three manifold topology*, CBMS Conference Ser. Math. 43, Amer. Math. Soc., Providence 1980.
6. Kobayashi, T., and Qiu, R., *The amalgamation of high distance Heegaard splittings is always efficient*, Math. Ann. 341 (2008), 707–715.
7. Lei, F., and Yang, G., *A lower bound of genus of amalgamations of Heegaard splittings*, Math. Proc. Camb. Phil. Soc. 146 (2009), 615–623.
8. Li, F., Yang, G., and Lei, F., *Heegaard genera of high distance are additive under annulus sum*, Topology Appl. 157 (2010), 1188–1194.
9. Morimoto, K., *Tunnel number, connected sum and meridional essential surface*, Topology 39 (2000), 469–485.
10. Scharlemann, M., *Local detection of strongly irreducible Heegaard splittings*, Topology Appl. 90 (1998), 135–147.
11. Scharlemann, M., *Proximity in the curve complex: boundary reduction and bicompressible surfaces*, Pacific J. Math. 228 (2006), 325–348.
12. Scharlemann, M., and Schultens, J., *The tunnel number of the sum of n knots is at least n*, Topology 38 (1999), 265–270.
13. Scharlemann, M., and Thompson, A., *Thin position for 3-manifolds*, pp. 231–238 in: Geometric topology, Contemp. Math. 164, Amer. Math. Soc., Providence 1994.
14. Scharlemann, M., and Tomova, M., *Alternate Heegaard genus bounds distance*, Geom. Topol. 10 (2006), 593–617.
15. Schultens, J., *Additivity of tunnel number for small knots*, Comment. Math. Helv. 75 (2000), 353–363.
16. Schultens, J., *The classification of Heegaard splittings for (compact orientable surfaces)* $\times S^1$, Proc. London Math. Soc. 67 (1993), 425–448.
17. Yang, G., and Lei, F., *Some sufficient conditions for tunnel numbers of connected sum of two knots not to go down*, Acta Math. Sin. (Engl. Ser.) 27 (2011), 2229–2244.

SCHOOL OF MATHEMATICAL SCIENCES
DALIAN UNIVERSITY OF TECHNOLOGY
DALIAN 116024
CHINA
E-mail: dutlfl@163.com, fclei@dlut.edu.cn

DEPARTMENT OF MATHEMATICS
HARBIN INSTITUTE OF TECHNOLOGY
HARBIN 150001
CHINA
E-mail: gqyang@hit.edu.cn