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CONTINUOUS FIELDS WITH FIBRES $\mathcal{O}_\infty$

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Abstract

We study in this article a class of unital continuous C^*-bundles the fibres of which are all isomorphic to the Cuntz C^*-algebra $\mathcal{O}_\infty$. This enables us to give several equivalent reformulations for the triviality of all these C^*-bundles.

1. Introduction

A programme of classification for separable nuclear C^*-algebras through K-theoretical invariants has been launched by Elliott ([19]). Quite a number of results have already been obtained for simple C^*-algebras (see e.g. [20], [10] for an account of them).

One of the central C^*-algebras for this programme is the simple unital nuclear Cuntz C^*-algebra $\mathcal{O}_\infty$ generated by a countable family of isometries $\{s_k;\, k \in \mathsf{N}\}$ with pairwise orthogonal ranges ([11]). This C^*-algebra belongs to the category of unital strongly self-absorbing C^*-algebras systematically studied by Toms and Winter:

DEFINITION 1.1 ([34]). Let A, B and D_0 be separable unital C^*-algebras distinct from C.

a) Two unital completely positive (u.c.p.) maps θ_1 and θ_2 from A to B are said to be *approximately unitarily equivalent* (written $\theta_1 \approx_{\text{a.u.}} \theta_2$) if there is a sequence $\{v_m\}_m$ of unitaries in B such that

$$\|\theta_2(a) - v_m\theta_1(a)v_m^*\| \underset{m\to\infty}{\longrightarrow} 0 \qquad \text{for all} \quad a \in A.$$

b) The C^*-algebra D_0 is said to be *strongly self-absorbing* if there is an isomorphism $\pi : D_0 \to D_0 \otimes D_0$ such that $\pi \approx_{\text{a.u.}} \imath_{D_0} \otimes 1_{D_0}$.

REMARK 1.2. All separable unital strongly self-absorbing C^*-algebras are simple and nuclear ([34]). Besides, these C^*-algebras are K_1-injective (see e.g. Definition 2.9 for a definition and [35] for a proof of it).

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This property of strong self-absorption is pretty invariant under continuous deformation. Indeed, any separable unital continuous $C(X)$-algebra D the fibres of which are all isomorphic to a given strongly self-absorbing C^*-algebra D_0 satisfies an isomorphism of $C(X)$-algebra

$$D \cong D_0 \otimes C(X)$$

provided the second countable compact metric space X is of finite topological dimension ([7], [22], [18], [35]).

This is not anymore always the case when the compact Hausdorff space X has infinite topological dimension. Indeed, Hirshberg, Rørdam and Winter constructed non-trivial unital continuous fields of algebras over the infinite dimensional compact space $Y = \Pi_{n=1}^{\infty} S^2$ with constant strongly self-absorbing fibre D_0 in case that C^*-algebra D_0 is a UHF algebra of infinite type or the Jiang-Su algebra ([22, Examples 4.7–4.8]). Then, Dădărlat has constructed in [16] unital continuous fields of C^*-algebras A over the contractible Hilbert cube $\mathfrak{X}$ of infinite topological dimension such that $K_*(A)$ is non-trivial and each fibre A_x is isomorphic to the Cuntz C^*-algebra $\mathcal{O}_2$, so that

$$K_0(A) \oplus K_1(A) \neq K_*(C(\mathfrak{X}; \mathcal{O}_2)) = 0 \oplus 0.$$

We analyse in this present paper the case $D_0 = \mathcal{O}_\infty$ through explicit examples of Cuntz-Pimsner $C(\mathfrak{X})$-algebra associated with Hilbert $C(\mathfrak{X})$-module with infinite dimensional fibres, so that all the fibres of these Cuntz-Pimsner $C(\mathfrak{X})$-algebras are isomorphic to the C^*-algebra $\mathcal{O}_\infty$. We were not able to prove here whether these continuous $C(\mathfrak{X})$-algebras are always trivial. But we describe a number of equivalent formulations of this problem (see e.g. Proposition 5.1) and we especially look at the case of the Cuntz-Pimsner $C(\mathfrak{X})$-algebra for the non-trivial continuous field of Hilbert spaces $\mathcal{E}$ defined by Dixmier and Douady in [13].

More precisely, we first recall in section §2 the construction of the Cuntz-Pimsner $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ for any continuous field E of infinite dimensional Hilbert spaces over a compact Hausdorff space X ([13]). We show in section §3 that this $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is always locally purely infinite. We observe that basic K-theoretical arguments are not enough to decide whether such a continuous $C(X)$-algebra with fibres isomorphic to $\mathcal{O}_\infty$ is or is not trivial unlike the case where all the fibres are isomorphic to the Cuntz C^*-algebra $\mathcal{O}_2$ ([16]). We study in the following section whether this $C(X)$-algebra is at least purely infinite, i.e. whether $\mathcal{T}_{C(X)}(E) \cong \mathcal{T}_{C(X)}(E) \otimes \mathcal{O}_\infty$, before we analyse in §6 whether $\mathcal{T}_{C(X)}(E)$ is always properly infinite, i.e. whether there is always a unital embedding of $C(X)$-algebra $\mathcal{O}_\infty \otimes C(X) \hookrightarrow \mathcal{T}_{C(X)}(E)$.

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2. Preliminaries

We fix in this section a few notations which will be used all along this paper.

We shall denote by $\mathsf{N} := \{0, 1, 2, \ldots\}$ the set of *positive* integers and by $\mathsf{N}^* := \mathsf{N} \setminus \{0\}$ the subset of *strictly positive* integer. We can then define the two separable Hilbert spaces

$$\ell^2(\mathsf{N}^*) := \Big\{x = (x_m) \in \mathsf{C}^{\mathsf{N}^*} \ ; \ \sum_{m\geq 1} |x_m|^2 < +\infty\Big\}$$
(2.1)
$$\ell^2(\mathsf{N}) := \mathsf{C} \oplus \ell^2(\mathsf{N}^*).$$

Let now X be a non-zero compact Hausdorff space and denote by $C(X)$ the C^*-algebra of continuous functions on X with values in the complex field C. For all points x in the compact space X, one denotes by $C_0(X \setminus \{x\}) \subset C(X)$ the closed two-sided ideal of continuous functions on X which are zero at x.

DEFINITION 2.1 ([13]). Assume that X is a compact Hausdorff space.

a) A Banach $C(X)$-module is a Banach space E endowed with a structure of $C(X)$-module such that $1_{C(X)} \cdot e = e$ and $\|f \cdot e\| \leq \|f\| \cdot \|e\|$ for all e in E and f in $C(X)$.

b) A Hilbert $C(X)$-module is a Banach $C(X)$-module E with a $C(X)$-valued sesquilinear inner product $\langle \cdot, \cdot \rangle : E \times E \to C(X)$ such that

- $f \cdot e = e \cdot f \in E$ for all pairs (e, f) in $E \times C(X)$,
- $\langle e_1, e_2 \cdot f \rangle = \langle e_1, e_2 \rangle \cdot f$, $\langle e_2, e_1 \rangle = \langle e_1, e_2 \rangle^*$, $\langle e_1, e_1 \rangle \geq 0$ and $\langle e_1, e_1 \rangle = 0$ if and only if $e_1 = 0$ for all triples (e_1, e_2, f) in $E \times E \times C(X)$,
- the Banach space E is complete for the norm $\|e\| := \|\langle e, e \rangle\|^{1/2}$.

For all Banach $C(X)$-modules E and all points $x \in X$, the set $C_0(X \setminus \{x\}) \cdot E$ is closed in E (by Cohen's factorization theorem). The quotient $E_x := E/C_0(X \setminus \{x\}) \cdot E$ is a Hilbert space which is called the *fibre* at x of the $C(X)$-module E. One will denote in the sequel by e_x the image of a *section* $e \in E$ in the *fibre* E_x.

The map $x \mapsto \|e_x\| = \inf\{\|[1-f+f(x)]\cdot e\| \ ; \ f \in C(X)\}$ is always upper semi-continuous. If that map is actually continuous for all sections $e \in E$, then the $C(X)$-module E is said to be *continuous*.

REMARK 2.2. The continuous fields of Hilbert spaces over a compact Hausdorff space X studied by Dixmier and Douady in [13] are now called *continuous Hilbert $C(X)$-modules*.

EXAMPLES 2.3. a) If H is a Hilbert space, then the tensor product $E := H \otimes C(X)$ is a *trivial* Hilbert $C(X)$-module the fibres of which are all isomorphic to H.

b) If E is a Hilbert $C(X)$-module, one defines inductively for all $m \in \mathbb{N}$ the Hilbert $C(X)$-modules $E^{(\otimes_{C(X)})m}$ by $E^{(\otimes_{C(X)})0} = C(X)$ and $E^{(\otimes_{C(X)})m+1} = E^{(\otimes_{C(X)})m} \otimes_{C(X)} E$. Then, the *full Fock Hilbert $C(X)$-module $\mathscr{F}(E)$* of E is the sum $\mathscr{F}(E) := \bigoplus_{m\in\mathbb{N}} E^{(\otimes_{C(X)})m}$.

Note that many of these full Fock Hilbert $C(X)$-modules are trivial Hilbert $C(X)$-modules, as the referee noticed through the following lemma.

LEMMA 2.4. *Assume that X is a compact Hausdorff space and E is countably generated Hilbert $C(X)$-module.*

Then the two Hilbert $C(X)$-modules $\mathscr{F}(E)$ and $\ell^2(\mathbb{N}) \otimes C(X)$ are isomorphic if and only if the Hilbert $C(X)$-module E is full, i.e. all the fibres of E are non-zero.

PROOF. One has $\mathscr{F}(E)_x \cong \mathscr{F}(E_x) = \mathbb{C} \oplus E_x \oplus (E_x \otimes E_x) \oplus \ldots$ for all $x \in X$. And so, the existence of an isomorphism $\mathscr{F}(E) \cong \ell^2(\mathbb{N}) \otimes C(X)$ implies that each fibre E_x is non-zero, i.e. E is full.

Conversely, if one assumes that the Hilbert $C(X)$-module E is full, then for all points x in X, there exists a section ζ in E with

$$\langle \zeta, \zeta \rangle(x) = \langle\, \zeta(x), \zeta(x)\, \rangle = 1.$$

Moreover, this implies by continuity that there are a closed neighbourhood $F(x) \subset X$ of that point x and a norm 1 section ζ' in $C(X) \cdot \xi \subset E$ such that

$$\langle \zeta', \zeta' \rangle(y) = 1 \qquad \text{for all} \quad y \in F(x).$$

The fullness of the $C(X)$-module E, the continuity of the map $x \in X \mapsto \langle \zeta, \zeta \rangle(x)$ for any section $\zeta \in E$ and the compactness of the space X imply that there are:

- a finite covering $X = \mathring{F}_1 \cup \ldots \cup \mathring{F}_n$ by the interiors of closed subsets $F_1, \ldots, F_n$ in X,
- norm 1 sections $\zeta_1, \zeta_2, \ldots, \zeta_n$ in E such that:

$$\langle \zeta_k, \zeta_k \rangle(y) = 1 \qquad \text{for all} \quad k \in \{1, \ldots, n\} \ \text{ and } \ y \in F_k. \tag{2.2}$$

Fix a partition of unity $1_{C(X)} = \sum_{1\le k\le n} \phi_k$, where each positive contraction $\phi_k \in C(X)$ has support in the closed subset F_k $(1 \le k \le n)$. Denote by $\xi^{\otimes j}$

the tensor product $\xi \otimes \ldots \otimes \xi$ in $(E_x)^{\otimes j}$ for all pairs (x, j) in $X \times \mathsf{N}^*$ and all ξ in the Hilbert space E_x. For all $m \in \mathsf{N}$, let $E(m)$ be the Hilbert $C(X)$-module $E(m) := \bigoplus_{k=1}^{n} E^{\otimes_{C(X)}(mn+k)}$ and let $\xi_m \in E(m)$ be the section given by the formula

$$x \in X \mapsto \xi_m(x) := \sum_{k=1}^{n} \phi_k(x)^{1/2} \cdot \zeta_k(x)^{\otimes(mn+k)}. \tag{2.3}$$

It satisfies $\langle \xi_m, \xi_m \rangle(x) = 1$ for all $x \in X$, so that we have an Hilbert $C(X)$-module isomorphism $E(m) \cong C(X) \oplus F(m)$, where $F(m)$ is the Hilbert $C(X)$-module

$$F(m) := \{\zeta \in E(m)\ ;\ \langle \xi_m, \zeta \rangle = 0\}.$$

Thus, the Dixmier-Douady stabilization theorem for separable Hilbert $C(X)$-modules ([13, Theorem 4]) implies that

$$\begin{aligned}
\mathcal{F}(E) = C(X) \oplus \bigoplus_{m \in \mathsf{N}^*} E^{\otimes_{C(X)} m} &= C(X) \oplus \bigoplus_{m \in \mathsf{N}} E(m) \\
&\cong [\ell^2(\mathsf{N}) \otimes C(X)] \oplus C(X) \oplus \bigoplus_{m \in \mathsf{N}} F(m) \\
&\cong \ell^2(\mathsf{N}) \otimes C(X).
\end{aligned}$$

c) A non-trivial full Hilbert $C(X)$-module with infinite dimensional fibres is the following construction by Dixmier and Douady:

DEFINITION 2.5 ([13]). Let $\mathfrak{X} := \big\{x = (x_m) \in \ell^2(\mathsf{N}^*)\ ;\ \|x\|^2 = \sum_{m \geq 1} |x_m|^2 \leq 1\big\}$ be the unit ball of the Hilbert space $\ell^2(\mathsf{N}^*)$: It is called the compact *Hilbert cube* when it is endowed with the distance $d(x, y) = \big(\sum_{m \geq 1} 2^{-m} |x_m - y_m|^2\big)^{1/2}$.

Let also $\eta \in \ell^\infty(\mathfrak{X}; \ell^2(\mathsf{N})) = \ell^\infty(\mathfrak{X}; \mathsf{C} \oplus \ell^2(\mathsf{N}^*))$ be the normalized section $x \mapsto \big(\sqrt{1 - \|x\|^2}, x\big)$ and let $\theta_{\eta,\eta}$ be the projection $\zeta \mapsto \eta \cdot \langle \eta, \zeta \rangle$.

Then the *norm closure* $\mathcal{E} := \overline{(1 - \theta_{\eta,\eta}) C(\mathfrak{X}; 0 \oplus \ell^2(\mathsf{N}^*))} \subset \ell^\infty(\mathfrak{X}; \ell^2(\mathsf{N})) \cap \eta^\perp$ is a Hilbert $C(\mathfrak{X})$-module with infinite dimensional fibres (called the *Dixmier-Douady Hilbert* $C(\mathfrak{X})$*-module*), where for all sections $\zeta \in C(\mathfrak{X}; \ell^2(\mathsf{N}^*))$ and all points $x \in \mathfrak{X}$:

$$(1 - \theta_{\eta,\eta})\zeta(x) = \big(-\langle x, \zeta(x) \rangle \cdot \sqrt{1 - \|x\|^2}, \zeta(x) - \langle x, \zeta(x) \rangle \cdot x\big) \in \ell^2(\mathsf{N}).$$

REMARKS 2.6.

a) There is a canonical isomorphism of Hilbert $C(\mathfrak{X})$-module:

$$\begin{aligned}
C(\mathfrak{X}) \oplus \mathcal{E} &\cong C(\mathfrak{X}) \cdot \eta + \mathcal{E} \subset \ell^\infty(\mathfrak{X}; \ell^2(\mathsf{N})) \\
(f, \xi) &\mapsto f \cdot \eta + \xi
\end{aligned} \tag{2.4}$$

b) The constant n given in formula (2.2) is greater than 2 for the Dixmier-Douady Hilbert $C(\mathfrak{X})$-module $\mathscr{E}$ since any section ζ in that Hilbert $C(\mathfrak{X})$-module satisfies $\zeta(x) = 0$ for at least one point $x \in \mathfrak{X}$ (see [13], or [7, Proposition 3.6]).

DEFINITION 2.7. If X is a non-empty compact Hausdorff space, a $C(X)$-*algebra* is a C^*-algebra A endowed with a unital $*$-homomorphism from $C(X)$ to the centre of the multiplier C^*-algebra $\mathscr{M}(A)$ of A.

For all closed subsets $F \subset X$, the two-sided ideal $C_0(X \setminus F) \cdot A$ is closed in A (by Cohen's factorization theorem). One denotes by $A_{|F}$ the quotient $A/C_0(X \setminus F) \cdot A$.

If F is reduced to a single point x, the quotient $A_{|\{x\}}$ is also called the fibre A_x at x of the $C(X)$-algebra A and one denotes by a_x the image of a section $a \in A$ in A_x.

The $C(X)$-algebra A is said to be a *continuous* $C(X)$-algebra if the map $x \mapsto \|a_x\|$ is continuous for all $a \in A$.

We shall mainly study here the following Cuntz-Pimsner $C(X)$-algebras.

DEFINITION 2.8 ([13]). If E is a full separable Hilbert $C(X)$-module and $\hat{1}_{C(X)}$ is a unit vector generating the first direct summand of the full Fock Hilbert $C(X)$-module $\mathscr{F}(E)$, then one defines for all $\zeta \in E$ the *creation* operator $\ell(\zeta) \in \mathscr{L}_{C(X)}(\mathscr{F}(E))$ through the formulae:

(2.5)
$$\begin{aligned} \ell(\zeta)(f \cdot \hat{1}_{C(X)}) &= f \cdot \zeta = \zeta \cdot f && \text{for } f \in C(X) \quad \text{and} \\ \ell(\zeta)(\zeta_1 \otimes \ldots \otimes \zeta_k) &= \zeta \otimes \zeta_1 \otimes \ldots \otimes \zeta_k && \text{for } \zeta_1, \ldots, \zeta_k \in E \text{ if } k \geq 1. \end{aligned}$$

The Cuntz-Pimsner $C(X)$-algebra $\mathscr{T}_{C(X)}(E)$ of the Hilbert $C(X)$-module E is the $C(X)$-algebra generated in $\mathscr{L}_{C(X)}(\mathscr{F}(E))$ by all the creation operators $\ell(\zeta)$, $\zeta \in E$.

Let us recall the definition of a K_1-injective C^*-algebra.

DEFINITION 2.9. Let $\mathscr{U}(A)$ be the group of unitaries in a unital C^*-algebra A and let $\mathscr{U}^0(A)$ be the normal connected component of the unit 1_A in $\mathscr{U}(A)$.

Then the tensor product $M_m(A) := M_m(\mathsf{C}) \otimes A$ is a unital C^*-algebra for all $m \in \mathsf{N}^*$ and the group $\mathscr{U}(M_m(A))$ embeds in $\mathscr{U}(M_{m+1}(A))$ by $u \mapsto u \oplus 1$.

The C^*-algebra A is said to be K_1-injective if the canonical map $\mathscr{U}(A)/\mathscr{U}^0(A) \to K_1(A) = \lim_{m\to\infty} \mathscr{U}(M_m(A))/\mathscr{U}^0(M_m(A))$ is injective.

3. A question of local purely infiniteness

We show in this section that the Cuntz-Pimsner $C(X)$-algebra $\mathscr{T}_{C(X)}(E)$ of any Hilbert $C(X)$-module E with infinite-dimensional fibres is locally purely infinite ([8, Definition 1.3]).

PROPOSITION 3.1. *Let E be a separable Hilbert $C(X)$-module the fibres of which are all infinite dimensional Hilbert spaces. Then, the Cuntz-Pimsner $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is a locally purely infinite continuous $C(X)$-algebra the fibres of which are all isomorphic to the Cuntz C^*-algebra $\mathcal{O}_\infty$.*

PROOF. All the fibres of the $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ are isomorphic to the C^*-algebra $\mathcal{O}_\infty$ by universality ([12]).

As the C^*-algebra $\mathcal{O}_\infty$ is simple (and non-zero), the separable unital $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is continuous ([15, Lemma 2.3] or [5, Lemma 4.5]) and the C*-representation of $\mathcal{T}_{C(X)}(E)$ on the Hilbert $C(X)$-module $\mathcal{F}(E)$ is a continuous field of faithful representations ([4]).

Besides, this $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is locally purely infinite (l.p.i.) since all its fibres are simple and purely infinite (p.i.) ([8, Proposition 5.1]).

4. A question of triviality

We observe in this section that basic K-theory arguments are not enough to decide whether the Cuntz-Pimsner $C(\mathfrak{X})$-algebra $\mathcal{T}_{C(\mathfrak{X})}(E)$ associated to a Hilbert $C(\mathfrak{X})$-module E with infinite-dimensional fibres is always a trivial continuous $C(\mathfrak{X})$-algebra, i.e. , whether $\mathcal{T}_{C(\mathfrak{X})}(E) \cong C(\mathfrak{X};\mathcal{O}_\infty)$.

Let $\{\mathfrak{X}_k\}_{k\in\mathbb{N}^*}$ be an increasing sequence of closed subspaces with finite (covering) dimension in the Hilbert cube $\mathfrak{X}$ (Definition 2.5) given by

$$\mathfrak{X}_k := \{(x_m)_m \in \mathfrak{X}\ ;\ x_m = 0 \text{ for all } m > k\}.$$

Then, Dădărlat has constructed in [16] non K_*-trivial unital continuous $C(\mathfrak{X})$-algebras the restriction of which to each compact subset $\mathfrak{X}_k$ is isomorphic to the K-trivial $C(\mathfrak{X}_k)$-algebra $\mathcal{O}_2 \otimes C(\mathfrak{X}_k)$ ([18, Theorem 1.1]).

Now, this K-theory proof fails to decide whether the Cuntz-Pimsner $C(\mathfrak{X})$-algebra associated to a Hilbert $C(\mathfrak{X})$-module E with infinite dimensional fibres is trivial or not. Indeed, we only know that the restriction $\mathcal{T}_{C(\mathfrak{X})}(E)_{|\mathfrak{X}_k}$ is a unital $C(\mathfrak{X}_k)$-algebra the fibres of which are all isomorphic to the K_1-injective ([31]) strongly self-absorbing ([35]) C^*-algebra $\mathcal{O}_\infty$, so that there is an isomorphism of $C(\mathfrak{X}_k)$-algebra $\mathcal{T}_{C(\mathfrak{X})}(E)_{|\mathfrak{X}_k} \cong C(\mathfrak{X}_k;\mathcal{O}_\infty)$ for each $k \geq 1$ ([18, Theorem 1.1]).

REMARKS 4.1. a) For all $k \in \mathbb{N}$, let $\jmath_k : C(\mathfrak{X}_k;\mathcal{O}_\infty) \hookrightarrow C(\mathfrak{X}_{k+1};\mathcal{O}_\infty)$ be the embedding

$$\forall (f,x) \in C(\mathfrak{X}_k;\mathcal{O}_\infty) \times \mathfrak{X}_{k+1},\ \jmath_k(f)(x_0,\dots,x_k,x_{k+1},0,\dots) = f(x_0,\dots,x_k,0,\dots) \tag{4.1}$$

Then the inductive limit satisfies $\varinjlim_{k\in\mathsf{N}^*} (C(\mathfrak{X}_k;\mathcal{O}_\infty), \jmath_k) = C(\mathfrak{X};\mathcal{O}_\infty)$. But the diagramme

$$\begin{array}{ccc} C(\mathfrak{X}_k;\mathcal{O}_\infty)G & \xrightarrow{\ \sim\ } & \mathcal{T}_{C(\mathfrak{X})}(E)_{|\mathfrak{X}_k} \\ {\scriptstyle \jmath_k}\downarrow & & \downarrow \\ C(\mathfrak{X}_{k+1};\mathcal{O}_\infty) & \xrightarrow{\ \sim\ } & \mathcal{T}_{C(\mathfrak{X})}(E)_{|\mathfrak{X}_{k+1}} \end{array}$$

is not asked to be commutative for every $k \in \mathsf{N}$.

If this is the case, then $\mathcal{T}_{C(\mathfrak{X})}(E) \cong \mathcal{T}_{C(\mathfrak{X})}(E) \otimes \mathcal{O}_\infty$ because $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty$.

b) If X is a second countable compact Hausdorff space, a more general question would be to know when two separable Hilbert $C(X)$-modules E_1 and E_2 have isomorphic Cuntz-Pimsner $C(X)$-algebras.

Dădărlat has characterized in [17] when $\mathcal{T}_{C(X)}(E_1)$ and $\mathcal{T}_{C(X)}(E_1)$ have isomorphic quotient Cuntz $C(X)$-algebras (with simple fibres) if all the fibres of E_1 and E_2 have the same finite dimension.

5. A question of pure infiniteness

We show in this section that the $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ associated to any full separable Hilbert $C(X)$-module E is purely infinite (in the sense of [26, definition 4.1]) if and only if it tensorially absorbs $\mathcal{O}_\infty$ ([34]), a property which is local.

Proposition 5.1. *Given a second countable compact Hausdorff space X and a full separable Hilbert $C(X)$-module E, the following assertions are equivalent:*

a) *The $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is purely infinite (abbreviated p.i.),*

b) *The $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is strongly purely infinite,*

c) *One has $\mathcal{T}_{C(X)}(E) \cong \mathcal{T}_{C(X)}(E) \otimes \mathcal{O}_\infty$,*

d) *There exists a unital $*$-homomorphism*

$$\rho : \mathcal{O}_\infty \to \big(c_b(\mathsf{N};\mathcal{T}_{C(X)}(E))\big/c_0(\mathsf{N};\mathcal{T}_{C(X)}(E))\big) \cap \imath(\mathcal{T}_{C(X)}(E))',$$

e) *There are unital $*$-homomorphisms $\rho_m : \mathcal{O}_\infty \to \mathcal{T}_{C(X)}(E)$ $(m \in \mathsf{N}^*)$ such that:*

$$\|[\rho_m(s_k), b]\| \underset{m\to\infty}{\longrightarrow} 0 \quad \textit{for all pairs } (k,b) \textit{ in } \mathsf{N}\times\mathcal{T}_{C(X)}(E),$$

f) *Any point $x \in X$ has a closed neighbourhood $F(x) \subset X$ such that $\mathcal{T}_{C(X)}(E)_{|F(x)} \cong \mathcal{T}_{C(X)}(E)_{|F(x)} \otimes \mathcal{O}_\infty$,*

g) *Any point $x \in X$ has a closed neighbourhood $F(x) \subset X$ such that the quotient $\mathcal{T}_{C(X)}(E)_{|F(x)}$ is p.i.*

Let us first prove the following technical lemma:

LEMMA 5.2. *Let A_1, A_2, B, D be separable unital C^*-algebras such that:*

- *D is strongly self-absorbing* ([34, Definition 1.3]), *hence nuclear,*
- *A_1 and A_2 are D-stable, i.e. $A_i \cong A_i \otimes D$ for $i = 1, 2$,*
- *there are $*$-epimorphisms $\pi_1 : A_1 \to B$ and $\pi_2 : A_2 \to B$.*

Then the amalgamated sum $A_1 \oplus_B A_2 = \{(a_1, a_2) \in A_1 \oplus A_2 \ ; \ \pi_1(a_1) = \pi_2(a_2)\}$ is also D-stable.

PROOF. The closed two sided ideal $I := \ker \pi_2 \lhd A_2$ absorbs D tensorially ([34, Corollary 3.3]) and the sequence $0 \to I \to A_1 \oplus_B A_2 \to A_1 \to 0$ is exact. Hence, $A_1 \oplus_B A_2$ is D-stable ([34, Corollary 4.3]).

REMARKS 5.3.

a) Corollary 4.3 in [34] answers Question 9.8 in [27], i.e. if $0 \to I \to A \to B \to 0$ is an exact sequence of separable C^*-algebras, then the C^*-algebra A is $\mathcal{O}_\infty$-stable if and only if the ideal I and the quotient B are $\mathcal{O}_\infty$-stable.

b) Winter noticed that Lemma 5.2 does not hold if the C^*-morphisms π_1 and π_2 are not surjective. Indeed, if $A_1 = A_2 = D$ is a strongly self-absorbing C^*-algebra, $B = D \otimes D$ and π_1, π_2 are the first and second factor canonical unital embeddings, then the amalgamated sum $A_1 \oplus_B A_2$ is isomorphic to $\mathbb{C}$, and so it is not D-stable.

PROOF OF PROPOSITION 5.1. The equivalences a) $\Leftrightarrow$ b) $\Leftrightarrow$ c) $\Leftrightarrow$ d) $\Leftrightarrow$ e) are proved respectively in [8, Theorem 5.8], [27, Theorem 8.5], [34, Theorem 2.2], and [18, Proposition 3.7].

The implication c) $\Rightarrow$ f) is contained in [34, Corollary 3.3] while the converse implication f) $\Rightarrow$ c) follows from Lemma 5.2.

At last, the proof of f)$\Leftrightarrow$g) is the same as the one of a)$\Leftrightarrow$c).

REMARKS 5.4.

a) If the $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is purely infinite, then $\mathcal{T}_{C(X)}(E)_{|F} \cong \mathcal{T}_{C(X)}(E)_{|F} \otimes \mathcal{O}_\infty$ for all closed subsets $F \subset X$ (Proposition 5.1) and so all the quotient C^*-algebras $\mathcal{T}_{C(X)}(E)_{|F}$ are K_1-injective ([29], [9, Proposition 6.1]).

b) Given a separable Hilbert $C(X)$-module with infinite dimensional fibres, the locally purely infinite $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is isomorphic to $C(X; \mathcal{O}_\infty)$ if and only if it is at the same time purely infinite and $KK_{C(X)}$-equivalent to $C(X) \otimes \mathcal{O}_\infty$. Indeed,

- the pure infiniteness implies that $\mathcal{T}_{C(X)}(E) \cong \mathcal{T}_{C(X)}(E) \otimes \mathcal{O}_\infty$ (Proposition 5.1),
- the $KK_{C(X)}$-equivalence to $C(X) \otimes \mathcal{O}_\infty$ implies that $\mathcal{K}(\ell^2(\mathsf{N}^*)) \otimes \mathcal{T}_{C(X)}(E) \cong \mathcal{K}(\ell^2(\mathsf{N}^*)) \otimes \mathcal{O}_\infty \otimes C(X)$ ([25], [16]),
- the proper infiniteness of the two full K_0-equivalent projections $1_{\mathcal{T}_{C(X)}(E)}$ and $1_{C(X;\mathcal{O}_\infty)}$ then implies that $\mathcal{T}_{C(X)}(E) \cong C(X; \mathcal{O}_\infty)$ ([9, Lemma 2.3]).

6. A question of proper Infiniteness

Given a compact metric space X and a full separable Hilbert $C(X)$-module E, we study in this section whether the Cuntz-Pimsner $C(X)$-algebra $\mathcal{T}_{C(X)}(E)$ is always properly infinite, i.e. , if there exists a unital $*$-homomorphism $\mathcal{O}_\infty \to \mathcal{T}_{C(X)}(E)$. We also make the link between that question and several problems raised in [9].

1) Let us first state the following general lemma.

LEMMA 6.1. *Let E be a full separable Hilbert $C(X)$-module. Then the Cuntz-Pimsner $C(X)$-algebra $\mathcal{T}_{C(X)}(C(X) \oplus E)$ is a unital properly infinite $C(X)$-algebra.*

PROOF. The separable Hilbert $C(X)$-module E is full by assumption. Thus, there exist by Lemma 2.4

- a finite integer $n \in \mathsf{N}^*$,
- positive functions $\phi_1, \dots, \phi_n$ in $C(X)$ such that $1_{C(X)} = \phi_1 + \dots + \phi_n$,
- contractive sections $\zeta_1, \dots, \zeta_n$ in E such that $\|\zeta_k(x)\| = 1$ for all (k, x) in $\{1, \dots, n\} \times X$ with $\phi_k(x) > 0$.

For all integers k, k' in $\{1, \dots, n\}$ and all integers m, m' in N, a direct computation gives the equality

$$\ell(1_{C(X)} \oplus 0)^* \cdot (\ell(0 \oplus \zeta_k)^{mn+k})^* \cdot \ell(0 \oplus \zeta_{k'})^{m'n+k'} \cdot \ell(1_{C(X)} \oplus 0) = \delta_{k=k'} \cdot \delta_{m=m'} \cdot \langle \zeta_k, \zeta_k \rangle^{mn+k}.$$

Hence, the sequence $\big\{\big(\sum_{k=1}^n (\phi_k)^{1/2} \cdot \ell(0 \oplus \zeta_k)^{mn+k}\big) \cdot \ell(1_{C(X)} \oplus 0)\big\}_{m\in\mathsf{N}}$ is by linearity a countable family of isometries in $\mathcal{T}_{C(X)}(C(X) \oplus E)$ with pairwise orthogonal ranges. In other words, these isometries define by universality (see [30, Theorem 3.4]) a unital morphism of $C(X)$-algebra

$$\mathcal{O}_\infty \otimes C(X) \to \mathcal{T}_{C(X)}\big(C(X) \oplus E\big). \tag{6.1}$$

2) On the other hand, if $\mathcal{E}$ is the Dixmier-Douady Hilbert $C(\mathfrak{X})$-module (Definition 2.5), $\widetilde{\mathcal{E}}$ is the Hilbert $C(\mathfrak{X})$-module $\widetilde{\mathcal{E}} := C(\mathfrak{X}) \oplus \mathcal{E}$ and α :

$\mathcal{T}_{C(\mathfrak{X})}(\widetilde{\mathcal{E}}) \to \mathcal{T}_{C(\mathfrak{X})}(\widetilde{\mathcal{E}}) \otimes C(\mathsf{T})$ is the *only* coaction (by [30]) of the circle group $\mathsf{T} = \{z \in \mathsf{C};\, |z| = 1\}$ on the Cuntz-Pimsner $C(\mathfrak{X})$-algebra $\mathcal{T}_{C(\mathfrak{X})}(\widetilde{\mathcal{E}})$ such that

$$\alpha(\ell(\zeta)) = \ell(\zeta) \otimes z \qquad \text{for all} \quad \zeta \in \widetilde{\mathcal{E}}, \tag{6.2}$$

then the fixed point $C(\mathfrak{X})$-subalgebra $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha$ of elements $a \in \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})$ with $\alpha(a) = a \otimes 1$ is the closed linear span of $C(\mathfrak{X}) \cdot 1$ and the words of the form $\ell(\zeta_1) \dots \ell(\zeta_k)\ell(\zeta_k)^* \dots \ell(\zeta_1)^*$ for some integer $k \geq 1$ and some sections $\zeta_1, \dots, \zeta_k$ in $0 \oplus \mathcal{E}$. This unital C^*-subalgebra is not properly infinite since:

- There is a unital epimorphism of $C(\mathfrak{X})$-algebra

$$\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha \twoheadrightarrow C(\mathfrak{X}) \cdot 1_{\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})} + \mathcal{K}_{C(\mathfrak{X})}(\mathcal{E}).$$

- The C^*-algebra $\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})$ is not properly infinite ([7, Corollary 3.7]).

Thus, any unital $*$-homomorphism $\mathcal{O}_\infty \to \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha$ would give by composition a unital $*$-homomorphism from $\mathcal{O}_\infty$ to the unital C^*-subalgebra $C(\mathfrak{X}) \cdot 1_{\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})} + \mathcal{K}_{C(\mathfrak{X})}(\mathcal{E})$ of $\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})$, something which cannot be.

QUESTION 6.2. Let $D := [\ell(0 \oplus \mathcal{E}) \cdot \ell(0 \oplus \mathcal{E})^*] \subset \mathcal{L}_{C(\mathfrak{X})}(\mathcal{F}(\widetilde{\mathcal{E}})) = \mathcal{L}_{C(\mathfrak{X})}(\mathcal{F}(C(\mathfrak{X}) \oplus \mathcal{E}))$. Then the C^*-algebra $D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha \cdot D = [D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \cdot D]^\alpha$ is a non-stable C^*-algebra since:

- $C(\mathfrak{X})$ is a quotient of the fixed point $C(\mathfrak{X})$-algebra $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha$. And so D is a quotient $C(\mathfrak{X})$-algebra of $[D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \cdot D]^\alpha$.
- Any non-trivial quotient C^*-algebra of a stable C^*-algebra is stable ([33]). But the C^*-algebra D is not stable.

Is the C^*-algebra $D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \cdot D$ also not stable?

Kirchberg noticed that this would imply that the locally purely infinite C^*-algebra $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})$ is not purely infinite (see Question 3.8(iii) in [7]).

3) There is (by [14, Claim 3.4] or [5, Lemma 4.5]) a sequence of unital inclusions of $C(\mathfrak{X})$-algebras:

$$\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha \subset \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \subset \mathcal{T}_{C(\mathfrak{X})}(C(\mathfrak{X}) \oplus \mathcal{E}) \cong \mathcal{T}_{C(\mathfrak{X})}(C(\mathfrak{X}) \oplus \mathcal{E})^\alpha \rtimes \mathsf{N}.$$

Now, we can reformulate the question of whether the intermediate $C(\mathfrak{X})$-algebra $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})$ is properly infinite or not through the following result of gluing.

PROPOSITION 6.3. *Let $\mathfrak{X}$ be the compact Hilbert cube and let $\mathcal{T}_{C(\mathfrak{X})}(E)$ be the Cuntz-Pimsner $C(\mathfrak{X})$-algebra of a separable Hilbert $C(\mathfrak{X})$-module E with infinite dimensional fibres. Then there exist:*

- *a finite covering* $\mathfrak{X} = \mathring{F}_1 \cup \ldots \cup \mathring{F}_n$ *by the interiors of closed contractible subsets*
- *unital* $*$*-homomorphisms* $\sigma_k : \mathcal{O}_\infty \to \mathcal{T}_{C(\mathfrak{X})}(E)_{|F_k}$ $(1 \leq k \leq n)$
- *unitaries* $u_{i,j} \in \mathcal{U}(\mathcal{T}_{C(\mathfrak{X})}(E)_{|F_i \cap F_j})$ $(1 \leq i, j \leq n)$ *such that for all triples* $1 \leq i, j, k \leq n$ *with* $F_i \cap F_j \cap F_k \neq \emptyset$:
  (1) $\pi_{F_i \cap F_j}(\sigma_i(s_m)) = u_{i,j} \cdot \pi_{F_j \cap F_i}(\sigma_j(s_m))$ *for all* $m \in \mathsf{N}$
  (2) $u_{i,j} \oplus 1 \sim_h 1 \oplus 1$ *in* $\mathcal{U}(M_2(\mathsf{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(E)_{|F_i \cap F_j})$
  (3) $\pi_{F_i \cap F_j \cap F_k}(u_{i,j}) \cdot \pi_{F_i \cap F_j \cap F_k}(u_{j,k}) \cdot \pi_{F_i \cap F_j \cap F_k}(u_{k,i}) = 1$ *in* $\mathcal{U}(\mathcal{T}_{C(\mathfrak{X})}(E)_{|F_i \cap F_j \cap F_k})$.

PROOF. For all points $x \in \mathfrak{X}$, the fibre $[\mathcal{T}_{C(\mathfrak{X})}(E)]_x \cong \mathcal{T}_{\mathsf{C}}(E_x)$ of the continuous $C(\mathfrak{X})$-algebra $\mathcal{T}_{C(\mathfrak{X})}(E)$ is isomorphic to the semiprojective C^*-algebra $\mathcal{O}_\infty$ ([1, Theorem 3.2]). And so, there exists a closed ball $F(x) \subset \mathfrak{X}$ of strictly positive radius around the point x and a unital $*$-monomorphism from $\mathcal{O}_\infty$ to the quotient $\mathcal{T}_{C(\mathfrak{X})}(E)_{|F(x)} := \mathcal{T}_{C(\mathfrak{X})}(E)/C_0(\mathfrak{X} \setminus F(x)) \cdot \mathcal{T}_{C(\mathfrak{X})}(E)$ lifting the unital $*$-isomorphism $\mathcal{O}_\infty \xrightarrow{\sim} \mathcal{T}_{C(\mathfrak{X})}(E)_x$.

The compactness of the convex metric space $\mathfrak{X}$ implies that there are

- a finite covering $\mathfrak{X} = \mathring{F}_1 \cup \ldots \cup \mathring{F}_n$ by the interiors of closed contractible subsets $F_1, \ldots, F_n$ of $\mathfrak{X}$ and
- unital $*$-homomorphisms $\sigma_k : \mathcal{O}_\infty \to \mathcal{T}_{C(\mathfrak{X})}(E)_{|F_k}$ $(1 \leq k \leq n)$.

If two closed subsets $F_i, F_j \subset \mathfrak{X}$ of this finite covering have a non-zero intersection, the partial isometry

$$(6.3)\quad u_{i,j} := \sum_{m \in \mathsf{N}} \pi_{F_i \cap F_j}(\sigma_i(s_m)) \cdot \pi_{F_i \cap F_j}(\sigma_j(s_m)^*) \in \mathcal{L}_{C(F_i \cap F_j)}(\mathcal{F}(E)_{|F_i \cap F_j})$$

satisfies

$$(6.4)\quad u_{i,j}^* u_{i,j} = \sum_{m \in \mathsf{N}} \pi_{F_i \cap F_j}(\sigma_i(s_m s_m^*)) = \pi_{F_i \cap F_j}(\sigma_i(1_{\mathcal{O}_\infty})) = 1_{\mathcal{T}_{C(\mathfrak{X})}(E)_{|F_i \cap F_j}}$$

since the only projection in $\mathcal{O}_\infty$ which dominates all the pairwise orthogonal projections $s_m s_m^*$ $(m \in \mathsf{N})$ is $1_{\mathcal{O}_\infty}$. Similarly, $u_{i,j} u_{i,j}^* = 1_{\mathcal{T}_{C(\mathfrak{X})}(E)_{|F_i \cap F_j}}$ and so $u_{i,j}$ is a unitary in $\mathcal{L}_{C(F_i \cap F_j)}(\mathcal{F}(E)_{|F_i \cap F_j})$ which satisfies the relation (1).

As the $C(\mathfrak{X})$-linear $*$-representation $\pi : \mathcal{T}_{C(\mathfrak{X})}(E) \to \mathcal{L}_{C(\mathfrak{X})}(\mathcal{F}(E))$ is a continuous field of faithful representations and each $\pi_x(u_{i,j})$ belongs to $\mathcal{U}(\mathcal{T}(E_x)) = \mathcal{U}^0(\mathcal{T}(E_x))$ (for all $x \in F_i \cap F_j$), the unitary $u_{i,j}$ actually belongs to the unital C^*-subalgebra $\mathcal{T}_{C(\mathfrak{X})}(E)_{|F_i \cap F_j} \subset \mathcal{L}_{C(F_i \cap F_j)}(\mathcal{F}(E)_{|F_i \cap F_j})$.

This unitary $u_{i,j}$ also satisfies the relation (2) by [28, Exercise 8.11].

At last, if $\pi_{i,j,k}$ denotes the quotient map $\pi_{F_i\cap F_j\cap F_k}$ onto $\mathcal{L}_{C(F_i\cap F_j\cap F_k)}(\mathcal{F}(E)_{|F_i\cap F_j\cap F_k})$, then

$$\begin{aligned}\pi_{i,j,k}(u_{i,j})\cdot\pi_{i,j,k}(u_{j,k})\cdot\pi_{i,j,k}(u_{k,i}) &= \sum_{m\in\mathbb{N}}\pi_{i,j,k}(\sigma_i(s_m s_m^*))\\ &= 1_{C(F_i\cap F_j\cap F_k)}\end{aligned}\tag{6.5}$$

REMARKS 6.4.

a) One has $u_{i,i} = 1_{C(F_i)}$ and $u_{i,j} = (u_{j,i})^{-1}$ for all $1 \le i, j \le n$.

b) The contractibility of two closed subsets F_1, F_2 of a metric space (X, d) does not imply the contractibility of their union $F_1 \cup F_2$ or their intersection $F_1 \cap F_2$ (take e.g. $X = \mathbb{T}$, $F_1 = \{z \in \mathbb{T}\ ;\ z + z^* \ge 0\}$ and $F_2 = \{z \in \mathbb{T}\ ;\ z + z^* \le 0\}$).

c) For all indices $1 \le i, j \le n$ and all points $x \in F_i \cap F_j$, the unitary $\pi_x(u_{i,j})$ belongs to the *simple* connected component $\mathcal{U}^0(\mathcal{T}(E_x)) \cong \mathcal{U}(\mathcal{O}_\infty)$. Nonetheless this does not necessarily imply that the unitary $u_{i,j}$ belongs to the connected component $\mathcal{U}^0(\mathcal{T}_{C(\mathfrak{X})}(E)_{|F_i\cap F_j})$ as is shown in the following:

LEMMA 6.5. *Let B be a unital C^*-algebra which is the pull-back of two unital C^*-algebras B_1 and B_2 along the $*$-epimorphisms $\pi_1 : B_1 \to B_{12}$ and $\pi_2 : B_2 \to B_{12}$:*

$$\begin{array}{ccccc} & & B & & \\ & \swarrow^{\sigma_1} & & \searrow^{\sigma_2} & \\ B_1 & & & & B_2 \\ & \searrow_{\pi_1} & & \swarrow_{\pi_2} & \\ & & B_{12} & & \end{array}$$

If $\mathcal{U}(B) := \{u \in \mathcal{U}(B)\ ;\ \sigma_1(u) \in \mathcal{U}^0(B_1)$ and $\sigma_2(u) \in \mathcal{U}^0(B_2)\}$, one has the sequence of inclusions of normal subgroups: $\mathcal{U}^0(B) \triangleleft \mathcal{U}(B) \triangleleft \mathcal{U}(B)$.

Yet the subgroup $\mathcal{U}^0(B)$ is distinct from $\mathcal{U}(B)$ in general, even if the unital C^*-algebra B is K_1-injective.

PROOF. One has $\mathcal{U}(B) = \{g \in \mathcal{U}(B)\ ;\ \sigma_1(g) \in \mathcal{U}^0(B_1)\} \cap \{g \in \mathcal{U}(B)\ ;\ \sigma_2(g) \in \mathcal{U}^0(B_2)\}$, whence the expected sequence of inclusions of normal subgroups.

Now, let $B_1 = B_2 = C([0,1];\mathcal{O}_\infty)$, $B_{12} = \mathbb{C}^2 \otimes \mathcal{O}_\infty$. Set:

$$\pi_1(f) = (f(0), f(1)) \qquad \text{and} \qquad \pi_2(f) = (f(1), f(0))$$

for all $f \in C([0,1];\mathcal{O}_\infty)$.

Then, there is a C^*-isomorphism $\alpha : B = B_1 \oplus_{B_{12}} B_2 \cong C(\mathsf{T}, \mathcal{O}_\infty)$ given by:

$$\alpha(f_1, f_2)(e^{\imath\pi t}) = \begin{cases} f_1(t) & \text{if } 0 \le t \le 1 \\ f_2(1-t) & \text{if } 1 \le t \le 2 \end{cases} \qquad \text{for all} \quad (f_1, f_2) \in A.$$

Thus, the C^*-algebra B satisfies $B \cong B \otimes \mathcal{O}_\infty$ and so is K_1-injective ([31]). If $v_1(t) = e^{2\imath\pi t} \cdot 1_{\mathcal{O}_\infty}$ and $v_2(t) = 1_{\mathcal{O}_\infty}$ for $t \in [0, 1]$, then the pair (v_1, v_2) belongs to $\mathcal{U}(B) \setminus \mathcal{U}^0(B)$ since $\alpha(v_1, v_2) \sim_h (z \mapsto z \cdot 1_{\mathcal{O}_\infty})$ in $\mathcal{U}(B)$ and $[z] \neq [1]$ in $K_1(B) \cong \mathsf{Z}$.

4) If E denotes the trivial Hilbert $C(\mathfrak{X})$-module $\mathsf{E} := \ell^2(\mathsf{N}) \otimes C(\mathfrak{X})$, then Theorem 2.11 of [9] and Equation (2) of the above Proposition 6.3 imply by finite induction the sequence of unital inclusions of $C(\mathfrak{X})$-algebras:

$$\mathcal{O}_\infty \otimes C(\mathfrak{X}) = \mathcal{T}_{C(\mathfrak{X})}(\mathsf{E}) \hookrightarrow M_{2^{n-1}}(\mathsf{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(E) \subset M_{2^{n-1}}(\mathsf{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(\mathsf{E}).$$

In particular, the tensor product $M_{2^{n-1}}(\mathsf{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(E)$ is properly infinite (thus answering Question 3.8(ii) in [7]). Note that this does not *a priori* imply that the C^*-algebra $\mathcal{T}_{C(\mathfrak{X})}(E)$ itself is properly infinite (see e.g. [32, Theorem 5.3]).

Let us eventually recall the following link between the proper infiniteness of the $C(\mathfrak{X})$-algebra $\mathcal{T}_{C(\mathfrak{X})}(E)$ and the K_1-injectivity of its properly infinite quotient C^*-algebras.

LEMMA 6.6 ([9]). *Suppose that:*

- $X = \mathring{F}_1 \cup \ldots \cup \mathring{F}_n$ *is a finite covering of a compact metric space X by the interiors of closed subsets $F_k \subset X$,*
- *B is a unital continuous $C(X)$-algebra such that all the quotients C^*-algebras $B_{|F_1}, \ldots, B_{|F_n}$ are properly infinite.*

Then the C^*-algebra B is a properly infinite C^*-algebra as soon as all the properly infinite quotients $B_{|F_k \cap (F_1 \cup \ldots \cup F_{k-1})}$ are K_1-injective ($2 \le k \le n$).

This result easily derives from Proposition 2.7 in [9]. Nevertheless we sketch a self-contained proof for the completeness of this paper.

PROOF. We can construct inductively for all $k \in \{1, \ldots n\}$ a unitary $d_k \in \mathcal{U}^0(B_{|F_k})$ and a unital $*$-homomorphism σ'_k from the C^*-subalgebra $\mathcal{T}_{n+2-k} := C^*(s_1, \ldots, s_{n+2-k}) \subset \mathcal{O}_\infty$ to the quotient $B_{|F_1 \cup \ldots \cup F_k}$ such that:

$$\pi_{F_j}\big(\sigma'_k(s_m)\big) = d_j \cdot \sigma_j(s_m) \in B_{|F_k} \tag{6.6}$$

for all $j \in \{1, \ldots, k\}$ and all $m \in \{1, \ldots, n+2-k\}$ in the following way.

Set first $d_1 := 1_{|F_1}$ and let $\sigma'_1 : \mathcal{T}_{n+1} \to B_{|F_1}$ be the only unital $*$-homomorphism such that $\sigma'_1(s_m) := \sigma_1(s_m)$ for all integers $1 \le m \le n+1$.

Take now an integer k in the finite set $\{2, \ldots, n\}$ and suppose already constructed the first $k-1$ unitaries $d_1, \ldots, d_{k-1}$ and a unital homomorphism of C^*-algebra $\sigma'_{k-1} : \mathcal{T}_{n+3-k} \to B_{|F_1\cup\ldots\cup F_{k-1}}$ so that:

- $\pi_{F_j}(\sigma'_{k-1}(s_m)) = d_j \cdot \sigma_j(s_m)$ for all $1 \le j \le k-1$ and $1 \le m \le n+3-k$,
- $u_{i,j} = \pi_{F_i\cap F_j}(d_i)^* \cdot \pi_{F_i\cap F_j}(d_j)$ for all $1 \le i, j \le k-1$.

As the projection $1 - \sum_{j=1}^{n+2-k} s_j s_j^*$ is properly infinite and full in the C^*-algebra $\mathcal{T}_{n+3-k}$, Lemma 2.4 of [9] implies that there exists a unitary $z_k \in \mathcal{U}(B_{|F_k\cap(F_1\cup\ldots\cup F_{k-1})})$ such that:

- $[z_k] = [1]$ in $K_1(B_{|F_k\cap(F_1\cup\ldots\cup F_{k-1})})$,
- $\pi_{F_k\cap(F_1\cup\ldots F_{k-1})}(\sigma'_{k-1}(s_m)) = z_k \cdot \pi_{F_k\cap(F_1\cup\ldots F_{k-1})}(\sigma_k(s_m))$ for all $1 \le m \le n+2-k$.

The assumed K_1-injectivity of the quotient C^*-algebra $B_{|F_k\cap(F_1\cup\ldots\cup F_{k-1})}$ implies that the K_1-trivial unitary z_k belongs to the connected component $\mathcal{U}^0(B_{|F_k\cap(F_1\cup\ldots\cup F_{k-1})})$. Hence, it admits a lifting d_k in $\mathcal{U}^0(B_{|F_k})$ ([9, Proposition 2.7]) and there exists one and only one unital $*$-homomorphism σ'_k : $\mathcal{T}_{n+2-k} \to B_{|F_1\cup\ldots\cup F_k}$ such that for all $m \in \{1, \ldots, n+2-k\}$:

- $\pi_{F_1\cup\ldots\cup F_{k-1}} \circ \sigma'_k(s_m) = \sigma'_{k-1}(s_m)$
- $\pi_{F_k} \circ \sigma'_k(s_m) = d_k \cdot \sigma_k(s_m)$.

The composition of a unital $*$-homomorphism $\mathcal{O}_\infty \hookrightarrow \mathcal{T}_2$ with the above constructed $*$-homomorphism σ'_n gives a convenient unital $*$-homomorphism $\mathcal{O}_\infty \to B$.

REMARKS 6.7.

a) If E is a separable Hilbert $C(\mathfrak{X})$-module with infinite dimensional fibres, then the non proper infiniteness of the C^*-algebra $\mathcal{T}_{C(\mathfrak{X})}(E)$ implies by Proposition 6.3 and Lemma 6.6 that one of the quotients $\mathcal{T}_{C(\mathfrak{X})}(E)_{|F_k\cap(F_1\cup\ldots\cup F_{k-1})}$ is a unital properly infinite C^*-algebra which is not K_1-injective (see [9, Theorem 5.5]).

b) Is the C^*-algebra $\mathcal{T}_{C(\mathfrak{X})}(E)$ weakly purely infinite for any separable full Hilbert $C(\mathfrak{X})$-module E? (see [7])

c) Is a quotient of a unital K_1-injective properly infinite C^*-algebra always K_1-injective? (see [9, Theorem 5.5])

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