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DERIVATIONS OF MURRAY-VON NEUMANN ALGEBRAS

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(Dedicated to Professor Shôichirô Sakai on the occasion of his 85th birthday*)

Abstract

A Murray-von Neumann algebra is the algebra of operators affiliated with a finite von Neumann algebra. In this article, we study derivations of Murray-von Neumann algebras and their properties. We show that the "extended derivations" of a Murray-von Neumann algebra, those that map the associated finite von Neumann algebra into itself, are inner. In particular, we prove that the only derivation that maps a Murray-von Neumann algebra associated with a von Neumann algebra of type II_1 into that von Neumann algebra is 0. This result is an extension, in two ways, of Singer's seminal result answering a question of Kaplansky, as applied to von Neumann algebras: the algebra may be non-commutative and contain unbounded elements. In another sense, as we indicate in the introduction, all the derivation results including ours extend what Singer's result says, that the derivation is the 0-mapping, numerically in our main theorem and cohomologically in our theorem on extended derivations. The cohomology in this case is the Hochschild cohomology for associative algebras.

1. Introduction

At a conference held in 1953, Kaplansky asked Singer if he had an idea of what the derivations of $C(X)$, the algebra of continuous functions on a compact Hausdorff space X, might be. A day later, Singer gave Kaplansky a short, clever argument that such derivations are the 0-mapping (that is, must map all of $C(X)$ to 0). (See [10] for an account of this.) As noted in [10], Kaplansky's paper [12] and the strong interest in derivations of operator algebras grew out of Singer's result. Kaplansky showed that each derivation of a type I von Neumann algebra (for example, $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$) into itself is "inner" (that is, has the form $\mathrm{Ad}(B)$, where $\mathrm{Ad}(B)(A) = AB - BA$). In the course of his argument, Kaplansky proves that each such derivation is (norm-)continuous and conjectures that that "automatic" continuity is true for all C*-algebras. This conjecture was proved a few years later by Sakai [18] –

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an ingenious argument – and extended, later, by Ringrose to derivations of a C*-algebra into a Banach bimodule [16] – with another ingenious argument. These were among the earliest "automatic continuity" results. In [9] and [17] (see, also, [7] and [5]), it was proved that each derivation of a C*-algebra acting on a Hilbert space $\mathcal{H}$ extends to a derivation of the strong-operator closure of that algebra, a von Neumann algebra, and that each derivation of a von Neumann algebra is inner. The proof of this last result is not simple. Surprisingly enough, this theorem is an extension of Singer's result. Of course, the von Neumann algebra is a C*-algebra. If it is abelian, it is isomorphic to a $C(X)$, and each inner derivation, $\mathrm{Ad}(B)$, is the 0-mapping. One may object that not all abelian C*-algebras are von Neumann algebras; but this can be easily remedied by adducing the possibility of extending a derivation of a C*-algebra to its strong-operator closure. It is not, however, in this primitive sense that we see the von Neumann algebra derivation theorem as an extension of Singer's derivation theorem; but, rather, in the sense that it tells us that each such derivation is 0 as an element of the 1-cohomology group of the von Neumann algebra [8].

We recall that a derivation of an associative algebra $\mathcal{A}$ is a linear mapping δ of $\mathcal{A}$ into itself satisfying the (Leibnitz-differentiation) property, $\delta(AB) = A\delta(B) + \delta(A)B$ for all A and B in $\mathcal{A}$. More generally, if $\mathcal{M}$ is an $\mathcal{A}$ bimodule, and δ is a linear mapping of $\mathcal{A}$ into $\mathcal{M}$ satisfying the Leibnitz rule (precisely as just described – in that case, $\mathcal{A}$ is a bimodule over itself) δ is said to be a derivation of $\mathcal{A}$ into $\mathcal{M}$. In Hochschild's cohomology of associative algebras [3] and [4], an n-multilinear mapping φ of $\mathcal{A}$ into $\mathcal{M}$ (an "n-cochain") is transformed by a precisely defined process, the (n-coboundary) operator Δ_n, into an $n+1$-cochain $\Delta_n(\varphi)$. If $\Delta_n(\varphi) = 0$, φ is said to be an "n-cocycle." In any event, $\Delta_n(\varphi)$ is said to be an "$n+1$-coboundary" and is an $n+1$-cocycle (as $\Delta_{n+1}\Delta_n = 0$, the main property of coboundary operations). The coboundary operators are linear, from which, the n-cocycles form a linear subspace of the linear space of n-cochains ("on $\mathcal{A}$ with coefficients in $\mathcal{M}$") and the n-coboundaries form a linear subspace of the n-cocycles whose quotient (as additive groups) is the "n th cohomology group" of $\mathcal{A}$ with coefficients in $\mathcal{M}$. As it relates to our derivations, the 1-coboundaries are the mappings $\mathrm{Ad}(B)$ with B in $\mathcal{A}$, and the Leibnitz rule for derivations "embodies" the coboundary operator

$$(\Delta_1(\varphi))(A, B) = A\varphi(B) - \varphi(AB) + \varphi(A)B,$$

which is 0 for all A and B in $\mathcal{A}$ precisely when φ is a derivation. The theorem of [9] and [17], is the statement that the first cohomology group of a von Neumann algebra (with coefficients in itself) is 0 (that is, that each cocycle is a coboundary – that each derivation is $\mathrm{Ad}(B)$ for some B in $\mathcal{A}$). Singer's theorem tells us that insisting that a derivation apply to (that is, "differentiates") all functions in $C(X)$ (that is, in a commutative C*-algebra) to yield functions, once more, forces the derivation to be the 0-mapping ("numerically") on $C(X)$. This same insistence for a derivation of a non-commutative C*-algebra (or its extension to a von Neumann closure of that algebra), again, forces the derivation to be "0" ("cohomologically"). In this non-commutative case, there is enough left of the derivation, as $\mathrm{Ad}(H)$, to be of significance in modeling quantum physics (H will become the *Hamiltonian*). Without trying to be too precise, if we follow (the thrust of) Dirac's program in the first chapters of [1], we associate the bounded observables of some quantum mechanical system with the self-adjoint operators in a von Neumann algebra $\mathcal{R}$. Loosely speaking, the symmetries of the system (and the associated conservation laws) are modeled by the corresponding symmetry groups as groups of automorphisms of $\mathcal{R}$. The time-evolution of the system, with a given dynamics, corresponds to a one-parameter group of automorphisms, $t \to \alpha_t$ of $\mathcal{R}$. Again, very loosely, α_t will be $\exp(it\delta)$ for some linear mapping δ (of the "algebra" of observables). Thus

$$\begin{aligned}\frac{d(\alpha_t(A))}{dt}\bigg|_{t=0} &= \frac{d}{dt}e^{-itH}Ae^{itH}\bigg|_{t=0}\\ &= -iHe^{-itH}Ae^{itH} + e^{-itH}Ae^{itH}(iH)\big|_{t=0}\\ &= -iHA + iAH = i[A, H],\end{aligned}$$

while

$$\frac{d(\alpha_t(A))}{dt}\bigg|_{t=0} = \frac{d}{dt}e^{it\delta(A)}\bigg|_{t=0} = i\delta(A)e^{it\delta(A)}\big|_{t=0} = i\delta(A).$$

Thus $\delta(A) = [A, H]$. In the case of Hamiltonian mechanics, time-differentiation of the dynamical variable is Poisson bracketing with the Hamiltonian (the total energy). In quantum mechanics, differentiation of the "evolving observable" is Lie bracketing with the (quantum) Hamiltonian (modeled from the kinetic and potential energies for the corresponding "classical analogue" of the quantum system, when there is one). Of course, this bracketing, δ, is a derivation of the system as the other generators of the one-parameter automorphism groups of the "operator algebras" that describe our physical system and its symmetries are likely to be – hence, our interest in studying those derivations.

Now, the (physical) Hamiltonian will, in general, correspond to an *unbounded* operator on our Hilbert space $\mathcal{H}$ as will likely be the case for the other operators K such that $\mathrm{Ad}(K)$ generates a group of symmetries of the quantum system. Of course, these unbounded operators will not lie in a von Neumann algebra, but they may be "affiliated" with the von Neumann algebra corresponding to our quantum system, in a sense that we shall soon make explicit (roughly, each "bounded function" of such an affiliated generator lies in the von Neumann algebra). This makes it very desirable to study derivations of algebras that include such unbounded operators. Regrettably, the tendency of unbounded operators not to combine effectively under the operations of addition and multiplication severely limits the possibility of forming *algebras* that include these affiliated operators, and along with that, we cannot speak of "their derivations." There is, however, one intriguing exception discovered by Murray and von Neumann, the "finite" von Neumann algebras and their families of affiliated operators [13]. These algebras are the main focus of this article.

In [20], von Neumann defines a class of algebras of bounded operators on a Hilbert space that have acquired the name "von Neumann algebras." [2] (Von Neumann refers to them as "rings of operators.") Such algebras are *self-adjoint*, *strong-operator closed*, and contain the identity operator. We say that a *closed densely defined* operator T on a Hilbert space $\mathcal{H}$ is *affiliated* with a von Neumann algebra $\mathcal{R}$ when $U'T = TU'$ for each *unitary* operator U' in $\mathcal{R}'$, the commutant of $\mathcal{R}$. Murray and von Neumann show, at the end of [13], that the family of operators affiliated with a factor of type II_1 (or, more generally, affiliated with a finite von Neumann algebra, those in which the identity operator is finite) admits surprising operations of addition and multiplication that suit the formal algebraic manipulations used by the founders of quantum mechanics in their mathematical model. This is the case because of very special domain properties that are valid for finite families of operators affiliated with a finite von Neumann algebra. (Unbounded operators, even those that are closed and densely defined, can often neither be added nor multiplied usefully. They may not have *common* dense domains.) In [22], it is proved that the family of operators affiliated with a finite von Neumann algebra is a *-algebra (with unit I, the identity operator) under the operations of addition $\hat{+}$ and multiplication $\hat{\cdot}$ (in the "Murray-von Neumann" sense). We refer to such algebras as *Murray-von Neumann algebras*.

Returning to the lessons Singer's description of derivations of $C(X)$ and the von Neumann algebra derivation theorem have taught us, we have seen that "over-differentiation," in two senses, "too many" functions have derivatives and the range of derivations are too restricted, requires "payment" in some form of "self-nullification." In the first case, that of $C(X)$, the derivation must be "numerically" 0, in the second case, that of a von Neumann algebra, the derivation must be "cohomologically" 0, $\mathrm{Ad}(B)$, for some B in the algebra. To assure ourselves that "over-differentiation" is a principal cause of this "nullification," we have only to note that the (norm-dense) subalgebra of polynomials on [0, 1] has classical (one-variable) differentiation as a non-zero derivation of that function algebra into itself. Of course, all this is taking place in the commutative setting of function algebras – a restriction on the range of the derivation. In the non-commutative C*-algebra case (where the domain restriction of a full von Neumann algebra and the commutative range restriction are removed), the example that follows displays a derivation of a C*-algebra into itself that is not inner (that is, a derivation that gives rise to a non-zero element of the first cohomology group of the C*-algebra with coefficients in itself).

EXAMPLE 0. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{K}$ be the (norm-closed) two-sided ideal (in $\mathcal{B}(\mathcal{H})$) of compact operators on $\mathcal{H}$. (To recall, $\mathcal{K}$ is the unique, proper, norm-closed, two-sided ideal in $\mathcal{B}(\mathcal{H})$ and is the norm closure of the family of operators in $\mathcal{B}(\mathcal{H})$ with finite-dimensional range – another two-sided ideal in $\mathcal{B}(\mathcal{H})$, contained in every other non-zero two sided ideal in $\mathcal{B}(\mathcal{H})$.) The family $\{aI + \mathcal{K} : K \in \mathcal{K}\}$ is a C*-algebra $\mathcal{A}$ (with unit I). If $B \in \mathcal{B}(\mathcal{H})$, then $\mathrm{Ad}(B)$, restricted to $\mathcal{A}$, is a derivation δ of $\mathcal{A}$ (into itself, as $\mathcal{K}$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$). If δ is inner, there is an A in $\mathcal{K}$ and a scalar a such that $\mathrm{Ad}(aI + A)(= \mathrm{Ad}(A))$ restricted to $\mathcal{A}$ is δ. In this case,

$$0 = (\mathrm{Ad}(aI + A) - \mathrm{Ad}(B))(K) = \mathrm{Ad}(aI + A - B)(K),$$

for each K in $\mathcal{K}$. Thus $A - B$ commutes with each element of $\mathcal{A}$. As $\mathcal{A}$ acts irreducibly on $\mathcal{H}$ and is a self-adjoint family of operators, $A - B = bI$ for some scalar b (from Schur's Lemma, rather, von Neumann's Double Commutant Theorem – see [6], I, II). Thus, when δ is inner, B is $A - bI$, which lies in $\mathcal{A}$. If we choose B not in $\mathcal{A}$, δ is not inner.

The view of the basic derivation theory of operator algebras from the vantage point of Singer's seminal answer to Kaplansky's question and the corresponding result for non-commutative von Neumann algebras raises a number of highly provocative, related questions. For example, is there a restriction on the range of a derivation δ of a von Neumann algebra, say, the restriction that is present in the case of Singer's theorem, that the range be abelian, that allows us to recapture Singer's "numerical" 0-nullification? This question has an affirmative answer. We shall prove this and other, broader results related to Singer's theorem elsewhere. For the present article, we concentrate on the questions referring to derivations of the algebras of unbounded operators, where such *algebras* are present. Loosely speaking, the central questions in this connection are as follows. Are there cohomological and numerical 0-nullification results for those algebras? There are, and these are the two main results of this paper.

If $\mathcal{R}$ is a finite von Neumann algebra, we denote by '$\mathscr{A}_{\mathrm{f}}(\mathcal{R})$' its associated Murray-von Neumann algebra. The complete cohomological result would say that each derivation of $\mathscr{A}_f(\mathcal{R})$ is inner (that is, is $\mathrm{Ad}(T)$ for some T in $\mathscr{A}_f(\mathcal{R})$). The authors feel *strongly* that this is true; but it is still open. (It is a work in progress for us.) We characterize the derivations of $\mathscr{A}_f(\mathcal{R})$ that have the form $\mathrm{Ad}(B)$ (restricted to $\mathscr{A}_f(\mathcal{R})$) with B in $\mathcal{R}$ as those that map $\mathcal{R}$ into $\mathcal{R}$. (See Theorem 4.3.) We call such mappings on $\mathscr{A}_f(\mathcal{R})$ *extended derivations* of $\mathscr{A}_f(\mathcal{R})$ (because they extend a derivation of $\mathcal{R}$ into itself). It's proof makes (crucial) use of the von Neumann algebra derivation theorem of [9] and [17], which, as noted, is not easy, as well as some spectral-theoretic techniques fashioned for finite von Neumann algebras (and based on the deep results of Murray and von Neumann [13]). Given what is known, it is not a difficult argument. We use it at once to prove our other main result, the assertion that each derivation of $\mathscr{A}_f(\mathcal{R})$ with $\mathcal{R}$ a von Neumann algebra of type II_1 that maps $\mathscr{A}_f(\mathcal{R})$ into $\mathcal{R}$ is 0. In other words, the restriction that the range of the derivation is in $\mathcal{R}$, the "bounded" part of $\mathscr{A}_f(\mathcal{R})$, allows us to recapture Singer's numerical 0 nullification in the (non-commutative, unbounded) case of $\mathscr{A}_f(\mathcal{R})$. This result is difficult. The matrix techniques developed in Section 3 are indispensable for our proof. For anyone following that proof in detail, it is important to realize that all the caution of the calculations performed in Section 3 is needed to produce a valid argument for our main theorem.

2. Murray-von Neumann Algebras

DEFINITION 2.1. We say that a closed densely defined operator T is *affiliated* with a von Neumann algebra $\mathcal{R}$ and write $T\eta\mathcal{R}$ when $U^*TU = T$ for each unitary operator U commuting with $\mathcal{R}$.

Note that the equality, $U^*TU = T$, of the preceding definition is to be understood in the strict sense that U^*TU and T have the same domain and (formal) equality holds for the transforms of vectors in that domain. As far as the domains are concerned, the effect is that U transforms $\mathscr{D}(T)$ onto itself.

As mentioned in the introduction, we are interested in operators affiliated with finite von Neumann algebras. We say that a von Neumann algebra is *finite* when the identity operator I is *finite*. Murray and von Neumann define finiteness of projections as follows. Let $\mathcal{H}$ be a Hilbert space. Two projections E and F acting on $\mathcal{H}$ are said to be *orthogonal* if $EF = 0$. If the range of F is contained in the range of E (equivalently, $EF = F$), we say that F is a subprojection of E and write $F \leqslant E$. Let $\mathcal{R}$ be a von Neumann algebra acting on $\mathcal{H}$. Suppose that E and F are nonzero projections in $\mathcal{R}$. We say E is a *minimal projection* in $\mathcal{R}$ if $F \leqslant E$ implies $F = E$. Murray and von Neumann conceived the idea of comparing the "sizes" of projections in a von Neumann algebra in the following way: E and F are said to be *equivalent* (*modulo* or *relative to* $\mathcal{R}$), written $E \sim F$, when $V^*V = E$ and $VV^* = F$ for some V

in $\mathcal{R}$. (Such an operator V is called a *partial isometry* with *initial* projection E and *final* projection F.) We write $E \precsim F$ when $E \sim F_0$ and $F_0 \leqslant F$ and $E \prec F$ when E is, in addition, not equivalent to F. It is apparent that $\sim$ is an equivalence relation on the projections in $\mathcal{R}$. In addition, $\precsim$ is a *partial ordering* of the equivalence classes of projections in $\mathcal{R}$, and it is a non-trivial and crucially important fact that this partial ordering is a *total ordering* when $\mathcal{R}$ is a factor. (Factors are von Neumann algebras whose centers consist of scalar multiples of the identity operator.) Murray and von Neumann also define infinite and finite projections in this framework modeled on the set-theoretic approach. The projection E in $\mathcal{R}$ is *infinite* (relative to $\mathcal{R}$) when $E \sim F < E$, for some projection F in $\mathcal{R}$, and *finite* otherwise.

Throughout the rest of this section, $\mathcal{R}$ denotes a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and $\mathscr{A}(\mathcal{R})$ denotes the family of operators affiliated with $\mathcal{R}$.

In [22], the following are proved.

PROPOSITION 2.2. *Suppose that operators S and T are affiliated with $\mathcal{R}$, then:*

(i) *$S + T$ is densely defined, preclosed and its closure, denoted by $S \hat{+} T$, is affiliated with $\mathcal{R}$;*

(ii) *ST is densely defined, preclosed and its closure, denoted by $S \hat{\cdot} T$, is affiliated with $\mathcal{R}$.*

PROPOSITION 2.3. *Suppose that operators A, B and C are affiliated with $\mathcal{R}$, then*

$$(A \hat{\cdot} B) \hat{\cdot} C = A \hat{\cdot} (B \hat{\cdot} C),$$

that is, the associative law holds for the multiplication $\hat{\cdot}$.

PROPOSITION 2.4. *Suppose that operators A, B and C are affiliated with $\mathcal{R}$, then*

$$(A \hat{+} B) \hat{\cdot} C = (A \hat{\cdot} C) \hat{+} (B \hat{\cdot} C)$$

and

$$C \hat{\cdot} (A \hat{+} B) = (C \hat{\cdot} A) \hat{+} (C \hat{\cdot} B),$$

that is, the distributive laws hold for the multiplication $\hat{\cdot}$ relative to the addition $\hat{+}$.

PROPOSITION 2.5. *Suppose that operators A and B are affiliated with $\mathcal{R}$, then*

$$(aA \hat{+} bB)^* = \bar{a}A^* \hat{+} \bar{b}B^* \quad \textit{and} \quad (A \hat{\cdot} B)^* = B^* \hat{\cdot} A^*, \qquad (a, b \in \mathbb{C})$$

where $*$ is the usual adjoint operation on operators (possibly unbounded).

Therefore, $\mathscr{A}(\mathcal{R})$, provided with the operations $\hat{+}$ (addition) and $\hat{\cdot}$ (multiplication), is a *-algebra (with unit I). Recall, $\mathcal{R}$ is finite (and *must be*) as a von Neumann algebra for this to be valid.

DEFINITION 2.6. We use '$\mathscr{A}_{\mathrm{f}}(\mathcal{R})$' to denote the *-algebra $(\mathscr{A}(\mathcal{R}), \hat{+}, \hat{\cdot})$. We call $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ *the Murray-von Neumann algebra* associated with $\mathcal{R}$.

3. Matrix Representation of Murray-von Neumann Algebras

Let $\mathcal{R}$ be a ring with unit I, and involution $A \to A^*$ $(A \in \mathcal{R})$.

DEFINITION 3.1. We call a set $\{E_{ab}\}_{a,b\in\mathsf{A}}$ a *matrix-unit system* in $\mathcal{R}$ when each $E_{ab} \neq 0$, $E_{ab}E_{cd}$ is 0 if $b \neq c$ and $E_{ab}E_{bd} = E_{ad}$, for all a, b, c, and d in A. If, in addition, $E^*_{ab} = E_{ba}$, we say that $\{E_{ab}\}$ is a *self-adjoint matrix-unit system*. If $\{F_{cd}\}_{c,d\in\mathsf{B}}$ is a matrix-unit system in $\mathcal{R}$ such that $\mathsf{A} \subseteq \mathsf{B}$ and $\{E_{ab}\}_{a,b\in\mathsf{A}} \subseteq \{F_{cd}\}_{c,d\in\mathsf{B}}$, we say that $\{F_{cd}\}$ is a *larger* matrix-unit system than $\{E_{ab}\}$. If $\{E_{ab}\}$ is maximal relative to this partial ordering of matrix-unit systems in $\mathcal{R}$, we call $\{E_{ab}\}_{a,b\in\mathsf{A}}$ a *complete matrix-unit system for* $\mathcal{R}$. Each E_{ab} in a matrix-unit system is said to be a *matrix unit* (in the system). The matrix units E_{aa}, $a \in \mathsf{A}$, are said to be *principal* (or *diagonal*) matrix units in the system $\{E_{ab}\}_{a,b\in\mathsf{A}}$.

REMARK 3.2. The principal matrix units E_{aa} are idempotents in $\mathcal{R}$, since $E_{aa}E_{aa} = E_{aa}$. If A is finite, say, $\mathsf{A} = \{1, \dots, n\}$ and the sum of the principal matrix units, $E_{11} + \cdots + E_{nn}$, is I, then $\{E_{jk}\}_{j,k\in\{1,\dots,n\}}$ is complete in $\mathcal{R}$, for if there is a larger matrix-unit system, it has a principal matrix unit E_{aa} distinct from each of $E_{11}, \dots, E_{nn}$. In this case, $E_{aa} = E_{aa}I = E_{aa}(E_{11}+\cdots+E_{nn}) = 0$, contrary to our assumption that matrix units are non-zero.

The classic example of a system of matrix units is that of the set of $n \times n$ matrices each of which has a single non-zero entry 1. If that entry is in the jth row and kth column, the resulting matrix is E_{jk} of our matrix-unit system for $\mathcal{M}_n(\mathsf{C})$, the algebra of $n \times n$ matrices with complex entries (in which it is complete). The examples that are most relevant for our present purposes are the finite, complete, self-adjoint matrix-unit systems for factors of type II_1. If $\mathcal{M}$ is such a factor, the principal matrix units $E_{11}, \dots, E_{nn}$ are equivalent projections (self-adjoint idempotents) and each E_{jk} is a partial isometry with initial projection E_{kk} (since $E^*_{jk}E_{jk} = E_{kj}E_{jk} = E_{kk}$) and final projection E_{jj} (since $E_{jk}E^*_{jk} = E_{jj}$). The key result that allows us to begin the process of constructing matrix-unit systems is in [6] II, Section 6.5. Lemma 6.5.6 asserts that each projection in a von Neumann algebra $\mathcal{R}$ with no central portion of type I (equivalently, with no non-zero abelian projections), in particular, in a factor of type II_1, is the sum of n equivalent (orthogonal) projections in $\mathcal{R}$, where n is any preassigned positive integer. In [11], Corollary 3.15, it is proved, among other such results, that each maximal abelian, self-adjoint subalgebra of a von Neumann algebra of type II_1 has n orthogonal equivalent projections with sum I. This possibility for choosing the principal matrix units for special purposes directed by spectral analysis is a technique that will be vital to our proof of Theorem 4.10. Given the n equivalent projections $E_{11}, \ldots, E_{nn}$ with sum I in the von Neumann algebra $\mathcal{R}$, to construct a finite, complete, self-adjoint matrix-unit system with these principal matrix units, we choose a partial isometry E_{j1} with initial projection E_{11} and final projection E_{jj} (say, by use of the polar decomposition of $E_{jj}E_{11}$). When $j = 1$, it is best to use E_{11} as E_{j1}. With these choices, we define E_{jk} to be $E_{j1}E_{k1}^*(= E_{j1}E_{1k})$.

With the ring $\mathcal{R}$ and a finite, self-adjoint matrix-unit system $\{E_{jk}\}_{j,k\in\{1,\ldots,n\}}$, such that $\sum_{j=1}^n E_{jj} = I$, there is a procedure for associating a ring of matrices whose entries lie in the subring $\mathcal{T}$ of $\mathcal{R}$ consisting of the elements of $\mathcal{R}$ that commute with all the matrix units of our system. This procedure is described in [6] II, Lemma 6.6.3. That lemma directs us to assign to T in $\mathcal{R}$ the $n \times n$ matrix whose (j,k) entry T_{jk} is $\sum_{r=1}^n E_{rj}TE_{kr}$. That this element lies in $\mathcal{T}$ follows from

$$\begin{aligned} E_{st}T_{jk} &= E_{st}\left(\sum_{r=1}^n E_{rj}TE_{kr}\right) = E_{st}E_{tj}TE_{kt} = E_{sj}TE_{kt} \\ &= E_{sj}TE_{ks}E_{st} = \left(\sum_{r=1}^n E_{rj}TE_{kr}\right)E_{st} = T_{jk}E_{st}, \quad j,k \in \{1,\ldots,n\}. \end{aligned}$$

If we denote by φ the mapping that assigns to T the matrix $[T_{jk}]$ in the $n \times n$ matrix ring $n\otimes\mathcal{T}$ over $\mathcal{T}$, then $\varphi(E_{jk})$ is the matrix (in $n\otimes\mathcal{T}$) with I at the (j,k) entry and 0 at all other entries, as the following calculation shows. The (s,t) entry for $\varphi(E_{jk})$ is $\sum_{r=1}^n E_{rs}E_{jk}E_{tr} = 0$ unless $s = j$ and $k = t$, in which case that entry is $\sum_{r=1}^n E_{rj}E_{jk}E_{kr} = \sum_{r=1}^n E_{rr}$, which is I, by assumption. With the present notation:

THEOREM 3.3. *The mapping φ is a *-isomorphism of $\mathcal{R}$ onto $n \otimes \mathcal{T}$.*

PROOF. If S and T are in $\mathcal{R}$ (an arbitrary ring with unit I and involution $A \to A^*$, $A \in \mathcal{R}$),

$$\begin{aligned} (S+T)_{jk} &= \sum_{r=1}^n E_{rj}(S+T)E_{kr} = \sum_{r=1}^n E_{rj}SE_{kr} + \sum_{r=1}^n E_{rj}TE_{kr} \\ &= S_{jk} + T_{jk}. \end{aligned}$$

Thus $\varphi(S+T) = [(S+T)_{jk}] = [S_{jk} + T_{jk}] = [S_{jk}] + [T_{jk}] = \varphi(S) + \varphi(T)$.

At the same time,

$$\begin{aligned}(ST)_{jk} &= \sum_{r=1}^{n} E_{rj}STE_{kr} = \sum_{r=1}^{n} E_{rj}S\Big(\sum_{t=1}^{n} E_{tt}\Big)TE_{kr} \\ &= \sum_{r=1}^{n}\sum_{t=1}^{n} E_{rj}SE_{tt}TE_{kr} = \sum_{t=1}^{n}\sum_{r=1}^{n} E_{rj}SE_{tt}TE_{kr} \\ &= \sum_{t=1}^{n}\Big(\sum_{r=1}^{n} E_{rj}SE_{tr}\Big)\Big(\sum_{r=1}^{n} E_{rt}TE_{kr}\Big) \\ &= \sum_{t=1}^{n} S_{jt}T_{tk} = (\varphi(S)\varphi(T))_{jk}.\end{aligned}$$

Thus $\varphi(ST) = \varphi(S)\varphi(T)$.

In addition, since our matrix-unit system is self-adjoint,

$$(T^*)_{jk} = \sum_{r=1}^{n} E_{rj}T^*E_{kr} = \Big(\sum_{r=1}^{n} E_{rk}TE_{jr}\Big)^* = (T_{kj})^*.$$

Thus $\varphi(T^*) = \varphi(T)^*$ for each T in $\mathcal{R}$.

Finally, we must show that φ is one-to-one and onto $n \otimes \mathcal{T}$. Suppose $\varphi(T)(= [T_{jk}])$ is 0 for some T in $\mathcal{R}$. Then

$$\begin{aligned}0 = T_{jk} &= \sum_{r=1}^{n} E_{rj}TE_{kr} = E_{jj}\Big(\sum_{r=1}^{n} E_{rj}TE_{kr}\Big)E_{jk} \\ &= E_{jj}E_{jj}TE_{kj}E_{jk} = E_{jj}TE_{kk},\end{aligned}$$

for all j and k in $\{1, \dots, n\}$. Thus

$$T = (E_{11} + \cdots + E_{nn})T(E_{11} + \cdots + E_{nn}) = \sum_{j,k=1}^{n} E_{jj}TE_{kk} = 0.$$

It follows that φ is a one-to-one mapping ("injective"). To show that φ maps onto $n \otimes \mathcal{T}$ (that is, is "surjective"), it will suffice to show that, for each given T in $\mathcal{T}$ and each choice of j and k in $\{1, \dots, n\}$, there is an element of $\mathcal{R}$ that φ maps to the matrix with T as the (j, k) entry and 0 at all other entries. We show that TE_{jk} is that element in $\mathcal{R}$. For this, we use the fact we have proved that $\varphi(TE_{jk}) = \varphi(T)\varphi(E_{jk})$. Since T is in $\mathcal{T}$, T commutes with all the matrix units. It follows that $\sum_{r=1}^{n} E_{rs}TE_{tr} = \sum_{r=1}^{n} TE_{rs}E_{tr}$, so that T_{st} is 0 when $s \neq t$. If $s = t$, $\sum_{r=1}^{n} TE_{rs}E_{tr} = T\sum_{r=1}^{n} E_{rr} = T$. Thus $\varphi(T)$ is the $n \times n$ diagonal matrix with T at each diagonal entry. As we have noted, $\varphi(E_{jk})$ is the matrix whose only non-zero entry is I in the (j, k) position. Hence $\varphi(T)\varphi(E_{jk})(= \varphi(TE_{jk}))$ is the matrix whose only non-zero entry is T in the (j, k) position. It follows that φ is surjective.

REMARK 3.4. The proof that φ is surjective can also be effected by computing the entries of the matrix for TE_{jk} directly. Note for this, that $(TE_{jk})_{st} = \sum_{r=1}^{n} E_{rs}TE_{jk}E_{tr} = T\sum_{r=1}^{n} E_{rs}E_{jk}E_{tr} = 0$ unless $s = j$ and $k = t$, in which case, $\sum_{r=1}^{n} E_{rs}E_{jk}E_{tr} = \sum_{r=1}^{n} E_{rr} = I$. Thus $(TE_{jk})_{st} = 0$ unless $s = j$ and $k = t$, in which case $(TE_{jk})_{jk} = T$.

4. Derivations of Murray-von Neumann Algebras

4.1. Definitions and Basic Results

Throughout this subsection, $\mathcal{R}$ denotes a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and note, from Section 2, that $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ denotes the Murray-von Neumann algebra associated with $\mathcal{R}$.

DEFINITION 4.1. We say that δ, a derivation of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, is an *extended derivation* of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ if δ maps $\mathcal{R}$ into $\mathcal{R}$.

We shall show that every extended derivation δ of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ is inner; that is $\delta = \mathrm{Ad}(B)$ for some B in $\mathcal{R}$.

LEMMA 4.2. *Let T be an operator affiliated with $\mathcal{R}$. Suppose that there is a sequence $\{F_n\}$ of operators in $\mathcal{R}$ with strong-operator limit I, the identity operator, such that $TF_nx = 0$ for all x in $\mathscr{D}(TF_n)$, the domain of TF_n, and for each n. Then $Tx = 0$ for all x in $\mathcal{H}$.*

PROOF. By definition of affiliated operators, T is densely defined and closed. Since each F_n is a bounded operator, from Proposition 3.7 and Lemma 4.10 in [22], the operator TF_n is densely defined and closed.

Since $TF_n = 0$ on its dense domain for each n, and TF_n is closed, $TF_n = 0$ on the whole Hilbert space $\mathcal{H}$ and $T\big(\bigcup_{n=1}^{\infty} F_n(\mathcal{H})\big) = 0$. At the same time, the sequence $\{F_n\}$ is strong-operator convergent to the identity operator I, and so, $\bigcup_{n=1}^{\infty} F_n(\mathcal{H})$ is dense in $\mathcal{H}$. Again, $T = 0$ on a dense subset of $\mathcal{H}$, hence, $T = 0$ on $\mathcal{H}$ (since T is closed).

THEOREM 4.3. *Suppose that δ is an extended derivation of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$. Then there is an operator B in $\mathcal{R}$ such that, for each operator A in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, $\delta(A) = \mathrm{Ad}(B)(A) = A \hat{\cdot} B \hat{-} B \hat{\cdot} A$.*

PROOF. By definition of extended derivations of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, the restriction of δ on $\mathcal{R}$ is a derivation of $\mathcal{R}$. Since every derivation of a von Neumann algebra is inner ([9] and [17]), there is an operator B in $\mathcal{R}$ such that

$$\delta(A) = AB - BA \qquad \text{for all } A \text{ in } \mathcal{R}.$$

Define $\mathrm{Ad}(B) : \mathscr{A}_{\mathrm{f}}(\mathcal{R}) \to \mathscr{A}_{\mathrm{f}}(\mathcal{R})$ by

$$\mathrm{Ad}(B)(A) = A \hat{\cdot} B \mathbin{\hat{-}} B \hat{\cdot} A \qquad (A \in \mathscr{A}_{\mathrm{f}}(\mathcal{R})).$$

Note that for every A in $\mathcal{R}$,

$$\mathrm{Ad}(B)(A) = A \hat{\cdot} B \mathbin{\hat{-}} B \hat{\cdot} A = AB - BA = \delta(A).$$

Let $\delta_0 = \delta - \mathrm{Ad}(B)$. Then δ_0 is a derivation of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ and $\delta_0(\mathcal{R}) = 0$. We shall show that $\delta_0(\mathscr{A}_{\mathrm{f}}(\mathcal{R})) = 0$, which will complete the proof.

For any operator A in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, let VH be the polar decomposition of A and let E_n be the spectral projection for H corresponding to the interval $[-n, n]$ for each positive integer n. Then, the sequence $\{E_n\}$ is strong-operator convergent to I, and for each n, AE_n is a bounded everywhere-defined operator in $\mathcal{R}$. Moreover,

$$0 = \delta_0(AE_n) = A\delta_0(E_n) + \delta_0(A)E_n = \delta_0(A)E_n.$$

From the preceding lemma, $\delta_0(A) = 0$ $(A \in \mathscr{A}_{\mathrm{f}}(\mathcal{R}))$.

4.2. Main Theorem

We shall prove that the only derivation of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ that maps $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ into $\mathcal{R}$ is 0, where $\mathcal{R}$ is a von Neumann algebra of type II_1 (that is, $\mathcal{R}$ is a finite von Neumann algebra such that E is 0 when $E\mathcal{R}E$ is abelian for some projection E in $\mathcal{R}$).

The following results will be useful to us in the proof of Theorem 4.10.

DEFINITION 4.4. We say that a von Neumann algebra $\mathscr{R}$ is *diffuse* if it has no projection minimal in $\mathscr{R}$.

LEMMA 4.5. *Each von Neumann algebra $\mathscr{R}$ with no central portion of type I, in particular, a von Neumann algebra of type II_1, is diffuse.*

PROOF. Suppose E is a minimal projection in $\mathscr{R}$, then C_E, the central support of E, is a minimal projection of the center $\mathscr{C}$ of $\mathscr{R}$ and $\mathscr{R}C_E$ is a factor [6] II, Proposition 6.4.3. By assumption, $\mathscr{R}C_E$, a central portion of $\mathscr{R}$, is not of type I. Since E is in $\mathscr{R}C_E$ and E is minimal in $\mathscr{R}$, it is minimal in $\mathscr{R}C_E$, contradicting the fact that $\mathscr{R}C_E$ is a factor not of type I. Thus $\mathscr{R}$ has no such minimal projection E, and $\mathscr{R}$ is diffuse.

Despite the exclusion of "type I" as required in the preceding lemma, there are certainly diffuse type I von Neumann algebras – as is clear from the proposition that follows.

PROPOSITION 4.6. *Every maximal abelian self-adjoint subalgebra* (*masa*) $\mathscr{A}$ *in a diffuse von Neumann algebra* $\mathscr{R}$ *is diffuse.*

PROOF. We show that if a projection E is minimal in $\mathscr{A}$, it is minimal in $\mathscr{R}$. Then, since $\mathscr{R}$ is diffuse, $\mathscr{A}$ must be diffuse.

We make use of the fact that a projection F is a minimal projection in a von Neumann algebra $\mathscr{R}$ if and only if $F\mathscr{R}F$ consists of scalar multiples of F (see [6] II, Proposition 6.4.3).

Now, suppose that E is a minimal projection in $\mathscr{A}$. For each B in $\mathscr{A}$, $EBE = \lambda E$ for some scalar λ. If A is in $\mathscr{R}$, then

$$\begin{aligned} EAEB &= EAEEB = EAEBE = EA(\lambda E) = \lambda EAE, \\ BEAE &= BEEAE = EBEAE = (\lambda E)AE = \lambda EAE. \end{aligned}$$

Since B is an arbitrary element in $\mathscr{A}$ and by maximality of $\mathscr{A}$, EAE is in $\mathscr{A}$. Again, since E is minimal in $\mathscr{A}$, $E(EAE)E = \beta E$ for some scalar β. But $E(EAE)E = EAE$ so that $EAE = \beta E$ for some scalar β (depending on A) for each A is $\mathscr{R}$. That is, $E\mathscr{R}E$ consists of scalar multiples of E and hence E is minimal in $\mathscr{R}$.

LEMMA 4.7. *Suppose that B is an operator in $\mathcal{R}$, a finite von Neumann algebra, and that B is not in the center of $\mathcal{R}$. Then, if there is an operator T in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ such that* $\mathrm{Ad}(B)(T) \notin \mathcal{R}$, *there is a self-adjoint operator S in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ such that* $\mathrm{Ad}(B)(S) \notin \mathcal{R}$.

PROOF. Suppose that there is an operator T (necessarily, unbounded) in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ such that

$$\mathrm{Ad}(B)(T) = T \,\hat{\cdot}\, B \,\hat{-}\, B \,\hat{\cdot}\, T \notin \mathcal{R};$$

that is, $\mathrm{Ad}(B)(T)$ is an unbounded operator affiliated with $\mathcal{R}$. Note that T can be decomposed as $T_1 \,\hat{+}\, iT_2$ with T_1 and T_2 self-adjoint operators affiliated with $\mathcal{R}$. From the algebraic properties of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$,

$$\begin{aligned} \mathrm{Ad}(B)(T) &= \mathrm{Ad}(B)(T_1 \,\hat{+}\, iT_2) \\ &= (T_1 \,\hat{+}\, iT_2) \,\hat{\cdot}\, B \,\hat{-}\, B \,\hat{\cdot}\, (T_1 \,\hat{+}\, iT_2) \\ &= (T_1 \,\hat{\cdot}\, B \,\hat{-}\, B \,\hat{\cdot}\, T_1) \,\hat{+}\, i(T_2 \,\hat{\cdot}\, B \,\hat{-}\, B \,\hat{\cdot}\, T_2). \end{aligned}$$

At least one of $T_1 \,\hat{\cdot}\, B \,\hat{-}\, B \,\hat{\cdot}\, T_1 (= \mathrm{Ad}(B)(T_1))$ and $T_2 \,\hat{\cdot}\, B \,\hat{-}\, B \,\hat{\cdot}\, T_2 (= \mathrm{Ad}(B)(T_2))$ is unbounded (affiliated with $\mathcal{R}$) since $\mathrm{Ad}(B)(T)$ is unbounded.

LEMMA 4.8. *Suppose that B is an operator in $\mathcal{R}$, a finite von Neumann algebra, and that B is not in the center of $\mathcal{R}$. If* $\mathrm{Ad}(B)(T)$ *is in $\mathcal{R}$ for every self-adjoint operator T in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, then there is a self-adjoint operator S in $\mathcal{R}$, not in the center of the von Neumann algebra $\mathcal{R}$, such that* $\mathrm{Ad}(S)(T)$ *is in $\mathcal{R}$ for every self-adjoint operator T in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$.*

PROOF. If, for every self-adjoint operator T in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$,

$$\mathrm{Ad}(B)(T) = T \hat{\cdot} B \hat{-} B \hat{\cdot} T \in \mathcal{R},$$

then

$$-(\mathrm{Ad}(B)(T))^* = T \hat{\cdot} B^* \hat{-} B^* \hat{\cdot} T \in \mathcal{R}.$$

It follows that

$$\begin{aligned}\mathrm{Ad}(B)(T) - (\mathrm{Ad}(B)(T))^* &= (T \hat{\cdot} B \hat{-} B \hat{\cdot} T) \hat{+} (T \hat{\cdot} B^* \hat{-} B^* \hat{\cdot} T)\\ &= T \hat{\cdot} (B \hat{+} B^*) \hat{-} (B \hat{+} B^*) \hat{\cdot} T\\ &= \mathrm{Ad}(B \hat{+} B^*)(T) = \mathrm{Ad}(B + B^*)(T) \in \mathcal{R}.\end{aligned}$$

Similarly, $\mathrm{Ad}(B)(T) + (\mathrm{Ad}(B)(T))^* = \mathrm{Ad}(B - B^*)(T) \in \mathcal{R}$, and $\mathrm{Ad}(i(B - B^*)(T)) \in \mathcal{R}$. Now, both $B + B^*$ and $i(B - B^*)$ are self-adjoint operators in $\mathcal{R}$. If both were in the center of $\mathcal{R}$, $B + B^*$ and $B - B^*$, hence, B, would be in that center, contrary to assumption.

PROPOSITION 4.9. *Let $\mathscr{A}$ be an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Suppose $\{F_a\}_{a\in\mathsf{A}}$ is a family of mutually orthogonal, non-zero projections in $\mathscr{A}$ with sum F, and $\{H_a\}_{a\in\mathsf{A}}$ is a family of self-adjoint operators affiliated with $\mathscr{A}$ such that $H_aF_a = H_a$ for each a in A. Let $\mathscr{D}_a$ be $\mathscr{D}(H_a) \cap F_a(\mathcal{H})$ and $\mathscr{D}_{\mathsf{A}}$ be the linear span of $\{\{\mathscr{D}_a\}_{a\in\mathsf{A}}, (I - F)(\mathcal{H})\}$. If H_0 is the linear operator with domain $\mathscr{D}_{\mathsf{A}}$ that maps x_a in $\mathscr{D}_a$ to H_ax_a and x' in $(I - F)(\mathcal{H})$ to 0, then H_0 is closable with closure a self-adjoint operator affiliated with $\mathscr{A}$.*

PROOF. Since $H_aF_a = H_a$, F_a maps $\mathscr{D}(H_a)$ into $\mathscr{D}(H_a)$. So, $F_a(\mathscr{D}(H_a)) \subseteq \mathscr{D}(H_a) \cap F_a(\mathcal{H})$. As F_a is continuous and $\mathscr{D}(H_a)$ is dense in $\mathcal{H}$, $F_a(\mathscr{D}(H_a))$ is dense in $F_a(\mathcal{H})$. Thus $\mathscr{D}(H_a) \cap F_a(\mathcal{H})$ is dense in $F_a(\mathcal{H})$, and $\mathscr{D}_{\mathsf{A}}$ is dense in $\mathcal{H}$. We show, next, that H_0, with its dense domain $\mathscr{D}_{\mathsf{A}}$, is symmetric; that is, $\langle H_0x, y\rangle = \langle x, H_0y\rangle$, for each x and y in $\mathscr{D}_{\mathsf{A}}$. It follows, then, that H_0 is closable. After we note that this closure, H, is affiliated with $\mathscr{A}$, a finite von Neumann algebra, we conclude, from [6] IV, Exercise 6.9.53, that H is self-adjoint.

Let x be $x_{a(1)} + \cdots + x_{a(m)} + x'$ and y be $y_{b(1)} + \cdots + y_{b(n)} + y'$, where x' and y' are in $(I - F)(\mathcal{H})$, each of $\{a(1), \ldots, a(m)\}$ and $\{b(1), \ldots, b(n)\}$ is a subset of A of distinct elements, all $x_{a(j)}$ and $y_{b(j)}$ are non-zero, each $x_{a(j)}$ is in $\mathscr{D}_{a(j)}$ and each $y_{b(j)}$ is in $\mathscr{D}_{b(j)}$. Let $\{c(1), \dots, c(r)\}$ be $\{a(1), \dots, a(m)\} \cap \{b(1), \dots, b(n)\}$. Then

$$\begin{aligned}
\langle H_0 x, y\rangle &= \langle H_0 x_{a(1)} + \cdots + H_0 x_{a(m)} + H_0 x', y_{b(1)} + \cdots + y_{b(n)} + y'\rangle \\
&= \langle H_{a(1)} x_{a(1)} + \cdots + H_{a(m)} x_{a(m)}, y_{b(1)} + \cdots + y_{b(n)} + y'\rangle \\
&= \langle H_{c(1)} x_{c(1)}, y_{c(1)}\rangle + \cdots + \langle H_{c(r)} x_{c(r)}, y_{c(r)}\rangle \\
&= \langle x_{c(1)}, H_{c(1)} y_{c(1)}\rangle + \cdots + \langle x_{c(r)}, H_{c(r)} y_{c(r)}\rangle \\
&= \langle x_{a(1)} + \cdots + x_{a(m)} + x', H_{b(1)} y_{b(1)} + \cdots + H_{b(n)} y_{b(n)}\rangle \\
&= \langle x_{a(1)} + \cdots + x_{a(m)} + x', H_0 y_{b(1)} + \cdots + H_0 y_{b(n)} + H_0 y'\rangle \\
&= \langle x, H_0 y\rangle.
\end{aligned}$$

We show, now, that H, the closure of H_0, is affiliated with $\mathscr{A}$. Let U' be a unitary operator in $\mathscr{A}'$. We want to show that $HU' = U'H$, in the strict sense of identical domains and equality on these domains. From Remark 5.6.3 of [6] I, since $\mathscr{D}_{\mathsf{A}}$ is a core for H, it will suffice to show that $HU'x = U'Hx$ for each x in $\mathscr{D}_{\mathsf{A}}$. First we show that $HU'x_a = U'Hx_a$ for each x_a in $\mathscr{D}_a$. Since $H_a \eta \mathscr{A}$, $H_a U' = U' H_a$, for all a in A, and $HU'x_a = H_0 U' x_a = H_a U' x_a = U' H_a x_a = U' H_0 x_a = U' H x_a$. Now, if $x \in (I - F)(\mathcal{H})$, since each $F_a \in \mathscr{A}$, F and $I - F$ are in $\mathscr{A}$, $U'x \in (I - F)(\mathcal{H})$. Hence $HU'x = H_0 U' x = 0 = U' H_0 x = U' H x$.

THEOREM 4.10. *If $\mathcal{R}$ is a von Neumann algebra of type II_1 and B is an operator in $\mathcal{R}$ not in the center $\mathscr{C}$ of $\mathcal{R}$, then there is an operator H in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ such that* $\mathrm{Ad}(B)(H) \notin \mathcal{R}$.

PROOF. Of course, if $\mathrm{Ad}(B)(H) \notin \mathcal{R}$, with B in $\mathcal{R}$ and H in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, then $H \notin \mathcal{R}$. From Lemmas 4.7 and 4.8, it suffices to consider the case in which B is a self-adjoint operator in $\mathcal{R}$; even a stronger result should be true, *viz.*, we can find a self-adjoint H such that $\mathrm{Ad}(B)(H) \notin \mathcal{R}$. Let $\mathscr{A}$ be a maximal abelian, self-adjoint algebra (masa) in $\mathcal{R}$ containing B. Then $\mathscr{C} \subseteq \mathscr{A}$. From [6] I, Theorem 5.2.1, $\mathscr{A} \cong C(X)$ with X an extremely disconnected compact Hausdorff space. Let $A \to \widetilde{A}$ be the isomorphism and $\widetilde{\mathscr{C}}$ be the image of $\mathscr{C}$ in $C(X)$. Then, from the properties of the isomorphism of $\mathscr{A}$ with $C(X)$, $\widetilde{\mathscr{C}}$ is a ("sup") norm-closed subalgebra of $C(X)$ closed under complex conjugation, and containing the constant functions. M. H. Stone [19], Theorem 5 identifies the norm-closed, self-adjoint subalgebras of $C(X)$ containing the scalar multiples of I, such as $\widetilde{\mathscr{C}}$, as those corresponding to an equivalence relation $\sim$ on the points of X defined by: $x \sim x'$ when $f(x) = f(x')$ for each f in $\widetilde{\mathscr{C}}$. In the usual way, $\sim$ is associated with a partition of X into (closed) subsets of

X (each of which is the equivalence class of every point in it), and $\widetilde{\mathscr{C}}$ consists of all functions g (the image of an operator in $\mathscr{C}$) in $C(X)$ such that g takes constant value on each of the partition sets. As $B \notin \mathscr{C}$, there are two points x and x' in X, necessarily distinct, such that $\widetilde{B}(x) \neq \widetilde{B}(x')$ and $x \sim x'$. We may suppose, without loss of generality, that $\widetilde{B}(x') < \widetilde{B}(x)$. If P is a projection in $\mathscr{C}$ such that $1 = \widetilde{P}(x)(= \widetilde{P}(x'))$, then $\widetilde{BP}(x) = \widetilde{B}(x)\widetilde{P}(x) = \widetilde{B}(x)$ and $\widetilde{BP}(x') = \widetilde{B}(x')$. It will suffice to find H in $\mathscr{A}_{\mathrm{f}}(\mathcal{R}P)$ for BP such that $\mathrm{Ad}(BP)(H) \notin \mathcal{R}P$ (since $\mathrm{Ad}(BP)(H) = (\mathrm{Ad}(B)(H))P$). In particular, C_B, the central carrier of B serves as such a P. To see this, note that $C_B B = B$. Thus $\widetilde{C_B B}(x) = \widetilde{C_B}(x)\widetilde{B}(x) = \widetilde{B}(x)$ and $\widetilde{C_B B}(x') = \widetilde{C_B}(x')\widetilde{B}(x') = \widetilde{B}(x')$. It follows that $\widetilde{C_B}(x')\widetilde{B}(x') \neq \widetilde{C_B}(x)\widetilde{B}(x)$. Now, $x \sim x'$, whence $\widetilde{C_B}(x) = \widetilde{C_B}(x')$ ($C_B \in \mathscr{C}$). Hence $\widetilde{C_B}(x)$ and $\widetilde{C_B}(x')$ are non-zero. However, $\widetilde{C_B}$ is an idempotent function. Thus $\widetilde{C_B}(x) = \widetilde{C_B}(x') = 1$.

If we replace B by $B - \frac{1}{2}[\widetilde{B}(x) + \widetilde{B}(x')]C_B$, then $\mathrm{Ad}(B)$ is unchanged, for all scalar multiples of C_B are in $\mathscr{C}$. At the same time, $\widetilde{B}(x)$ is replaced by $\frac{1}{2}[\widetilde{B}(x) - \widetilde{B}(x')]$, and $\widetilde{B}(x')$ is replaced by $-\frac{1}{2}[\widetilde{B}(x) - \widetilde{B}(x')]$. If we find H as desired, for this new B, we are through. At the same time, $\mathrm{Ad}(aB) = a\,\mathrm{Ad}(B)$ for each scalar a. If $a \neq 0$, finding H for $\mathrm{Ad}(aB)$ will complete our argument (using $a^{-1}H$ of course). Replacing the new B by aB where a is $2[\widetilde{B}(x) - \widetilde{B}(x')]^{-1}$, we may now assume that $\widetilde{B}(x) = 1$ and $\widetilde{B}(x') = -1$.

Let S_0 be the closure of the open set in X where $\widetilde{B}$ takes values greater than $\frac{7}{8}$ and less than $\frac{9}{8}$, and let S_0' be the closure of the open set on which $\widetilde{B}$ takes values less than $-\frac{7}{8}$ and greater than $-\frac{9}{8}$. These sets, S_0 and S_0', are non-null since $x \in S_0$ and $x' \in S_0'$. Let E_0 and E_0' be the projections in $\mathscr{A}$ corresponding to the characteristic functions of E_0 and E_0', respectively. From the function representation in $C(X)$,

$$\tfrac{9}{8}E_0 \geqslant BE_0 \geqslant \tfrac{7}{8}E_0 \qquad \text{and} \qquad -\tfrac{7}{8}E_0' \geqslant BE_0' \geqslant -\tfrac{9}{8}E_0.$$

It follows, from the definition of central carrier, that $C_{E_0} = C_{BE_0}$ and $C_{E_0'} = C_{BE_0'}$. By choice of E_0 and E_0', $\widetilde{E_0}(x) = 1$ and $\widetilde{E_0'}(x') = 1$. Now, $E_0 = C_{E_0}E_0$ and $E_0' = C_{E_0'}E_0'$, whence $1 = \widetilde{E_0}(x) = \widetilde{C_{E_0}}(x)\widetilde{E_0}(x) = \widetilde{C_{E_0}}(x)$ and $1 = \widetilde{E_0'}(x') = \widetilde{C_{E_0'}}(x')\widetilde{E_0'}(x') = \widetilde{C_{E_0'}}(x')$. As $x \sim x'$ and C_{E_0} and $C_{E_0'}$ are in $\mathscr{C}$, $\widetilde{C_{E_0}}(x') = \widetilde{C_{E_0}}(x) = 1 = \widetilde{C_{E_0'}}(x') = \widetilde{C_{E_0'}}(x)$. Thus $\big(\widetilde{C_{E_0}}\widetilde{C_{E_0'}}\big)(x) = 1$ and $C_{E_0}C_{E_0'} \neq 0$. Let P be $C_{E_0}C_{E_0'}$. Then $C_{PE_0} = P = C_{PE_0'}$. We restrict our considerations to $\mathcal{R}P$ acting on $P(\mathcal{H})$ and relabel BP, $P(\mathcal{H})$, and $\mathcal{R}P$ as B, $\mathcal{H}$, and $\mathcal{R}$. This replacement carries over to the function representation, so that X is now the clopen subset of the original X whose characteristic function is $\widetilde{P}$. We retain the designation $\sim$ for the isomorphism of (the relabeled) $\mathcal{R}$ (that is, $\mathcal{R}P$) with (the relabeled) $C(X)$ (that is, $C(X')$, where X' is $\{x : x \in X,\ \widetilde{P}(x) =$

$1\}$). In this new notation, $P = I$, $C_{E_0} = C_{E'_0} = I$. From the definition of central carrier, $QE_0 > 0$ and $QE'_0 > 0$ for each non-zero central projection Q in $\mathcal{R}$. Thus, with τ the center-valued trace on $\mathcal{R}$, $Q\tau(E_0) = \tau(QE_0) > 0$ and $Q\tau(E'_0) = \tau(QE'_0) > 0$ for each such Q. It follows, now, that the (closed) subset of X_0, where $C(X_0)$ is the function representation of $\mathscr{C}$, on which $\widetilde{\tau(E_0)}$ is zero is nowhere dense (for, if its interior were non-null, the closure of that interior would be a non-null clopen set corresponding to a non-zero central projection Q such that $Q\tau(E_0)$ is not zero as noted before and, yet, zero since $\widetilde{\tau(E_0)}$ is zero on that clopen set.) Similarly, the subset of X_0 where $\widetilde{\tau(E'_0)}$ is zero is nowhere dense. From continuity of $\widetilde{\tau(E_0)}$, there is an open set in X_0 on which $\widetilde{\tau(E_0)}$ is bounded below by some positive constant a; and the same is true on the closure O of that open set. As the zero set of $\widetilde{\tau(E'_0)}$ is nowhere dense, $\widetilde{\tau(E'_0)}$ takes a positive value at some point of O. Again, from continuity of $\widetilde{\tau(E'_0)}$, $\widetilde{\tau(E'_0)}$ is bounded below on a clopen subset O' of O by $\frac{1}{n}$ for some (possibly, large) positive integer n. Of course, we may choose n large enough so that $\frac{1}{n} \leqslant a$. Thus, for the central projection Q corresponding to O' and some (possibly, large) positive integer n, $\tau(E_0) \geqslant \frac{1}{n}Q$, $\tau(E'_0) \geqslant \frac{1}{n}Q$. Hence $\tau(QE_0) = Q\tau(E_0) \geqslant \frac{1}{n}Q$ and $\tau(QE'_0) \geqslant \frac{1}{n}Q$. We now restrict attention to $\mathcal{R}Q$, BQ, on $Q(\mathcal{H})$ and relabel these as $\mathcal{R}$, B, and H. In this notation, we have $C_{E_0} = C_{E'_0} = I = Q$, $\tau(E_0) \geqslant \frac{1}{n}I$, and $\tau(E'_0) \geqslant \frac{1}{n}I$.

Applying Corollary 3.14 of [11], there are subprojections E and E' in $\mathscr{A}$ of E_0 and E'_0, respectively, such that $\tau(E) = \tau(E') = \frac{1}{n}I$. Let E be E_1 and E' be E_n. From Corollary 3.15 of [11], there are $n-2$ orthogonal equivalent projections in $\mathscr{A}$, each with trace $\frac{1}{n}I$, $E_2, E_3, \ldots, E_{n-1}$, with sum $I - E_1 - E_n$. Let F be $I - E_1 - E_n$. According to the cited result, there are $n-2$ orthogonal equivalent projections in $\mathscr{A}F$ with sum F, the identity of $\mathscr{A}F$.

Let V_j be the partial isometry with initial projection E_1 and final projection E_j. Then $V_j^*V_j = E_1$ and $V_jV_j^* = E_j$. Let

$$E_{jk} = V_jV_k^*;$$

that is, E_{jk} is the partial isometry with initial projection E_k and final projection E_j. Then

$$E_{jj} = V_jV_j^* = E_j \quad (j = 1, 2, \ldots, n), \qquad \sum_{j=1}^{n} E_{jj} = \sum_{j=1}^{n} E_j = I;$$

$$E_{jk}E_{kl} = V_jV_k^*V_kV_l^* = V_jE_1V_l^* = V_jV_l^* = E_{jl};$$

$$E_{jk}E_{lm} = V_jV_k^*V_lV_m^* = 0 \qquad \text{if} \quad k \neq l;$$

$$E_{jk}^* = E_{kj}.$$

Hence, $\{E_{jk}\}_{j,k=1,\ldots,n}$ is a self-adjoint system of $n \times n$ matrix units for $\mathcal{R}$ (and for $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ as well).

Consider the matrix unit E_{1n} in $\mathcal{R}$. When we compute its matrix in $n \otimes \mathcal{T}$, the $n \times n$ matrices over $\mathcal{T}$ (the subalgebra $\mathcal{T}$ of $\mathcal{R}$ consisting of elements in $\mathcal{R}$ commuting with all matrix units in $\mathcal{R}$ in the chosen self-adjoint matrix unit system), the result is the $n \times n$ matrix with I at the $(1, n)$ position and 0 at all other positions. The mapping from $\mathcal{R}$ to $n \otimes \mathcal{T}$ described in [6] II, Section 6.6 is a *-isomorphism of $\mathcal{R}$ onto $n \otimes \mathcal{T}$. That mapping is effected by assigning to T in $\mathcal{R}$ the $n \times n$ matrix whose entry in the (j, k) position is

$$\sum_{r=1}^{n} = E_{rj} T E_{kr}.$$

Of course, we must show that this entry is an element of $\mathcal{T}$, that is, that the entry commutes with each of the matrix units in the chosen self-adjoint matrix unit system. Note, for this, that, for each h and l,

$$E_{hl}\left(\sum_{r=1}^{n} E_{rj} T E_{kr}\right) = E_{hj} T E_{kl} = \left(\sum_{r=1}^{n} E_{rj} T E_{kr}\right) E_{hl}.$$

Next, suppose B is, as chosen earlier, a self-adjoint element in the masa $\mathscr{A}$, from our construction, $\mathscr{A}$ contains the principal matrix units $E_{11}, \ldots, E_{nn}$ of our matrix unit system $\{E_{jk}\}_{j,k=1,\ldots,n}$, and $BE_{11} \geqslant \frac{7}{8} E_{11}$, $BE_{nn} \leqslant -\frac{7}{8} E_{nn}$. Suppose, also, that we have chosen H, a self-adjoint operator in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ as well as in the algebra of operators affiliated with $\mathscr{A}$. Without specifying H precisely, at this point, we assume that $HE_{11} \geqslant E_{11}$ and $H \,\hat{\cdot}\, B (= HB) \notin \mathcal{R}$. Our goal, now, is to show that $HE_{1n} (= H \,\hat{\cdot}\, E_{1n})$ and B form a commutator $(\mathrm{Ad}(B)(HE_{1n}))$ that is not in $\mathcal{R}$ (hence, is in $\mathscr{A}_{\mathrm{f}}(\mathcal{R}) \setminus \mathcal{R}$).

The final step is a precise construction of the operator H. For this step, we make use of the fact that each masa in a von Neumann algebra of type II_1 is diffuse (see Proposition 4.6). Using this, we construct a sequence of non-zero mutually orthogonal subprojections $F_1, F_2, \ldots$ of E_{11} in $\mathscr{A}$. We note, from Proposition 4.9, that $2F_1 + 3F_2 + 4F_3 + \cdots$ is an operator with closure H affiliated with $\mathscr{A}$ (here, $\mathsf{A} = \{1, 2, \ldots\}$, $H_j = (j+1)F_j$, $\mathscr{D}(H_j) = \mathcal{H}$, $\mathscr{D}_j = F_j(\mathcal{H})$), and that $HE_{11} = H$. Moreover, $E_{11}F_j = F_j$, and $F_j \,\hat{\cdot}\, H = HF_j = (j+1)F_j$, since $F_j F_k = 0$ when $j \neq k$. Recall that, if T is a closed operator and B is a bounded operator on the Hilbert space $\mathcal{H}$, then the operator TB is closed. So, we write HF_j instead of $H \,\hat{\cdot}\, F_j$. Now, F_j and B are in $\mathscr{A}$. Thus

$$\begin{aligned} F_j \,\hat{\cdot}\, HB &= (j+1)F_j B = (j+1)BF_j = (j+1)BE_{11}F_j \\ &\geqslant (j+1)\tfrac{7}{8} E_{11} F_j = \tfrac{7}{8}(j+1) F_j \end{aligned}$$

for each j. As F_j is a non-zero projection, $\|F_j \hat{\cdot} HB\| \geqslant \frac{7}{8}(j+1)\|F_j\| = \frac{7}{8}(j+1)$ for each j. Thus HB is unbounded and affiliated with $\mathscr{A}$. At the same time,

$$\|F_j \hat{\cdot} HBE_{1n}\| \geqslant \left\|\tfrac{7}{8}(j+1)F_j E_{1n}\right\| = \tfrac{7}{8}(j+1),$$

since E_{1n} is a partial isometry with final space $E_{11}(\mathcal{H})$, containing $F_j(\mathcal{H})$. It follows that HBE_{1n} is an unbounded operator in $\mathscr{A}_f(\mathcal{R})$. We shall use this construction to provide us with the desired commutator $\mathrm{Ad}(B)(HE_{1n})$ in $\mathscr{A}_f(\mathcal{R}) \setminus \mathcal{R}$.

The operator $B \hat{\cdot} HE_{1n}$ corresponds to the $n \times n$ matrix over $\mathcal{T}$ with

$$\sum_{r=1}^{n} E_{rj} B \hat{\cdot} HE_{1n} E_{kr}$$

at the (j, k) entry. Since B is in $\mathscr{A}$ and H is affiliated with $\mathscr{A}$, they commute with all the principal matrix units E_{kk} ($k = 1, \ldots, n$), this (j, k) entry is

$$\sum_{r=1}^{n} E_{rj} B \hat{\cdot} HE_{jj} E_{1n} E_{kk} E_{kr},$$

which is 0 unless $j = 1$ and $k = n$. It follows that the (j, k) entry for the $n \times n$ matrix corresponding to $B \hat{\cdot} HE_{1n}$ is 0 at all entries except, possibly, the $(1, n)$ entry, which is

$$\sum_{r=1}^{n} E_{r1} B \hat{\cdot} HE_{1r}.$$

At the same time, B, H, and $B \hat{\cdot} H$ have diagonal matrices in $n \otimes \mathcal{T}$ corresponding to them. To see this, note that the (j, k) entry of the matrix corresponding to B is B_{jk}, where

$$B_{jk} = \sum_{r=1}^{n} E_{rj} BE_{kr},$$

which is

$$\sum_{r=1}^{n} E_{rj} E_{jj} BE_{kk} E_{kr} = \sum_{r=1}^{n} E_{rj} BE_{jj} E_{kk} E_{kr}.$$

Since $E_{jj} E_{kk}$ is 0 unless $j = k$, in which case $E_{jj} E_{kk} = E_{jj}$, the (j, k) entry of the matrix corresponding to B is 0 unless $j = k$, in which case, the (j, j) entry is B_{jj}, where

$$B_{jj} = \sum_{r=1}^{n} E_{rj} BE_{jr}$$

for each j. Thus B corresponds to the diagonal matrix with B_{jj} at the diagonal position (j, j) $(j = 1, \dots, n)$, and 0 at every off-diagonal position. If we compute $\mathrm{Ad}(B)(HE_{1n})(= (HE_{1n}) \mathbin{\hat{\cdot}} B \mathbin{\hat{-}} B \mathbin{\hat{\cdot}} (HE_{1n}))$ in terms of the $n \times n$ matrices corresponding to it, we have that $\mathrm{Ad}(B)(HE_{1n})$ corresponds to the $n \times n$ matrix with (j, k) entry,

$$\begin{aligned}
&\sum_{r=1}^{n} E_{rj} \mathbin{\hat{\cdot}} HE_{1n}BE_{kr} \mathbin{\hat{-}} \sum_{r=1}^{n} E_{rj}B \mathbin{\hat{\cdot}} HE_{1n}E_{kr} \\
&\qquad\qquad = \sum_{r=1}^{n} E_{rj} \mathbin{\hat{\cdot}} HE_{jj}E_{1n}E_{kk}BE_{kr} \mathbin{\hat{-}} \sum_{r=1}^{n} E_{rj}B \mathbin{\hat{\cdot}} HE_{jj}E_{1n}E_{kk}E_{kr},
\end{aligned}$$

which is 0 unless $j = 1$ and $k = n$, in which case it is the $(1, n)$ entry,

$$\begin{aligned}
&\sum_{r=1}^{n} E_{r1} \mathbin{\hat{\cdot}} HE_{1n}BE_{nr} \mathbin{\hat{-}} \sum_{r=1}^{n} E_{r1}B \mathbin{\hat{\cdot}} HE_{1r} \\
&\quad = \left(\sum_{r=1}^{n} E_{r1} \mathbin{\hat{\cdot}} HE_{1r}\right)\left(\sum_{s=1}^{n} E_{sn}BE_{ns}\right) \mathbin{\hat{-}} \left(\sum_{r=1}^{n} E_{r1}BE_{1r}\right) \mathbin{\hat{\cdot}} \left(\sum_{s=1}^{n} E_{s1} \mathbin{\hat{\cdot}} HE_{1s}\right) \\
&\quad = H_{11}B_{nn} \mathbin{\hat{-}} B_{11} \mathbin{\hat{\cdot}} H_{11}.
\end{aligned}$$

We want to show that this entry is not in $\mathscr{R}$ (and is, hence, unbounded). If this $(1, n)$ entry is in $\mathscr{R}$, then multiplying it on the left by $-E_{11}$ and on the right by E_{11} results in

$$\begin{aligned}
&-E_{11}\big(H_{11}B_{nn} \mathbin{\hat{-}} B_{11} \mathbin{\hat{\cdot}} H_{11}\big) \mathbin{\hat{\cdot}} E_{11} \\
&\qquad = -E_{11}\left(\sum_{r=1}^{n} E_{r1} \mathbin{\hat{\cdot}} HE_{1n}BE_{nr} \mathbin{\hat{-}} \sum_{r=1}^{n} E_{r1}B \mathbin{\hat{\cdot}} HE_{1r}\right)E_{11} \\
&\qquad = B \mathbin{\hat{\cdot}} HE_{11} \mathbin{\hat{-}} HE_{1n}BE_{n1},
\end{aligned}$$

which is also in $\mathscr{R}$. We argue, by contradiction, to show that this is not the case.

In the construction of H, we defined non-zero projections F_j in $\mathscr{A}$ such that $F_j \mathbin{\hat{\cdot}} H = (j + 1)F_j$. Thus,

$$\begin{aligned}
\|B \mathbin{\hat{\cdot}} HE_{11} \mathbin{\hat{-}} HE_{1n}BE_{n1}\| &= \|F_j\|\,\|B \mathbin{\hat{\cdot}} HE_{11} \mathbin{\hat{-}} HE_{1n}BE_{n1}\|\,\|F_j\| \\
&\geqslant (j + 1)\|BF_jE_{11}F_j - F_jE_{1n}BE_{n1}F_j\| \\
&\geqslant (j + 1)\|BF_j - F_jE_{1n}BE_{n1}F_j\|.
\end{aligned}$$

Now, by choice of E_{11},

$$BF_j = BE_{11}F_j \geqslant \big(\tfrac{7}{8}E_{11}\big)F_j = \tfrac{7}{8}F_j,$$

while

$$\begin{aligned}-F_j E_{1n} B E_{n1} F_j &= -F_j E_{1n} B E_{nn} E_{n1} F_j \\ &\geqslant F_j E_{1n}\left(\tfrac{7}{8} E_{nn}\right) E_{n1} F_j = \tfrac{7}{8} F_j E_{11} F_j = \tfrac{7}{8} F_j.\end{aligned}$$

Hence,

$$BF_j - F_j E_{1n} B E_{n1} F_j \geqslant \tfrac{14}{8} F_j, \qquad \|BF_j - F_j E_{1n} B E_{n1} F_j\| \geqslant \tfrac{14}{8},$$

and

$$\|B \hat{\cdot} HE_{11} \hat{-} HE_{1n} B E_{n1}\| \geqslant \tfrac{14}{8}(j+1) > j,$$

for each positive integer j. It follows that $B \hat{\cdot} HE_{11} \hat{-} HE_{1n} B E_{n1}$ is not bounded, not in $\mathcal{R}$, and that $\mathrm{Ad}(B)(HE_{1n}) \in \mathscr{A}_{\mathrm{f}}(\mathcal{R}) \setminus \mathcal{R}$.

COROLLARY 4.11. *Suppose that δ is a derivation of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ that maps $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ into $\mathcal{R}$, where $\mathcal{R}$ is a von Neumann algebra of type II_1. Then $\delta(A) = 0$ for every A in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$.*

PROOF. Since δ maps $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ into $\mathcal{R}$, δ maps $\mathcal{R}$ into $\mathcal{R}$. So, δ is an extended derivation of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$. From Theorem 4.3, δ is inner, that is, there is an operator B in $\mathcal{R}$ such that, for each operator A in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, $\delta(A) = \mathrm{Ad}(B)(A) = A \hat{\cdot} B \hat{-} B \hat{\cdot} A$. If the operator B is in the center of $\mathcal{R}$, then B is in the center of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ (see Proposition 30 of [23]) and hence for each operator A in $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$, $\mathrm{Ad}(B)(A) = A \hat{\cdot} B \hat{-} B \hat{\cdot} A = 0$. If B is not in the center of $\mathcal{R}$, from Theorem 4.10, there is an operator H in $\mathscr{A}_{\mathrm{f}}(\mathcal{R}) \setminus \mathcal{R}$ such that $\mathrm{Ad}(B)(H) \notin \mathcal{R}$. Contrary to the assumption that δ maps $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ into $\mathcal{R}$. Thus the only derivation of $\mathscr{A}_{\mathrm{f}}(\mathcal{R})$ into $\mathcal{R}$ is 0.

REMARK 4.12. At first, we proved Theorem 4.10 and Corollary 4.11 for factors of type II_1 (those whose centers consist of scalar multiples of I) in order to simplify a complicated argument, to a certain extent. As is often the case with von Neumann algebras, much of the essence of the result being proved is present in the case of a factor. For a von Neumann algebra of type II_1, quite a bit of difficulty resides in the nature of the center. This should not be surprising; we are dealing with derivations and (Lie) bracketing and the crucial hypothesis in Theorem 4.10 is that the operator B, about which the assertion is made, does not lie in the center. Before we can succeed in constructing what we need in the case where the von Neumann algebra has a robust center, we must transform the condition of "non-centrality" into detailed spectral information about B. This transformation is the substance of the first part of the proof of Theorem 4.10. The use of the masa $\mathscr{A}$ containing B and the $C(X)$ to which it is isomorphic, with X extremely disconnected, and the representation of operators as functions, is the powerful form of spectral theory that we use.

The "struggle" that is apparent in our manipulation of central carriers to find a non-zero central projection over which B has distinct spectrum (bounded apart) is entirely unnecessary in the factor case; the center is isomorphic to "functions" on a one-point space. Stone's characterization of norm-closed, self-adjoint subalgebras of $C(X)$, in particular of $\mathscr{C}$ in $\mathscr{A}$, is not needed in that instance. As noted, the proof simplifies considerably in the factor case, though it remains complicated. In the end, we felt that it was worthwhile to include the general case, given the context of Lie bracketing.

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REFERENCES

1. Dirac, P. A. M., *The Principles of Quantum Mechanics*, 3 ed., Clarendon Press, Oxford 1947.
2. Dixmier, J., *Les Algèbres d'Opérateurs dans l'Espace Hilbertien*, 2 ed., Gauthier-Villars, Paris 1969.
3. Hochschild, G., *On the cohomology groups of an associative algebra*, Ann. of Math. 46 (1945), 58–67.
4. Hochschild, G., *On the cohomology theory for associative algebras*, Ann. of Math. 47 (1946), 568–579.
5. Johnson, B. E., and Ringrose, J. R., *Derivations of operator algebras and discrete group algebras*, Bull. London Math. Soc. 1 (1969), 70–74.
6. Kadison, R. V., and Ringrose, J. R., *Fundamentals of the Theory of Operator Algebras*, vols. I, II, III and IV, Academic Press, New York 1983, 1986 and 1991.
7. Kadison, R. V., and Ringrose, J. R., *Derivations of operator group algebras*, Amer. J. Math. 88 (1966), 562–576.
8. Kadison, R. V., and Ringrose, J. R., *Cohomology of operator algebras. I. Type I von Neumann algebras*, Acta Math. 126 (1971), 227–243.
9. Kadison, R. V., *Derivations of operator algebras*, Ann. of Math. 83 (1966), 280–293.
10. Kadison, R. V., *Which Singer is that?*, pp. 347–373 in: Surveys in differential geometry, Surv. Differ. Geom., VII, Int. Press, Somerville 2000.
11. Kadison, R. V., *Diagonalizing matrices*, Amer. J. Math. 106 (1984), 1451–1468.
12. Kaplansky, I., *Modules over operator algebras*, Amer. J. Math. 75 (1953), 839–858.
13. Murray, F. J., and von Neumann, J., *On rings of operators*, Ann. of Math. 37 (1936), 116–229.
14. Murray, F. J., and von Neumann, J., *On rings of operators*, II, Trans. Amer. Math. Soc. 41 (1937), 208–248.
15. Murray, F. J., and von Neumann, J., *On rings of operators*, IV, Ann. of Math. 44 (1943), 716–808.
16. Ringrose, J. R., *Automatic continuity of derivations of operator algebras*, J. London Math. Soc. (2) 5 (1972), 432–438.
17. Sakai, S., *Derivations of W^*-algebras*, Ann. of Math. 83 (1966), 273–279.

18. Sakai, S., *On a conjecture of Kaplansky*, Tôhoku Math. J. 12 (1960), 31–33.
19. Stone, M. H., *The Generalized Weierstrass Approximation Theorem*, Math. Mag. 21 (1948), 167–184.
20. von Neumann, J., *Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren*, Math. Ann. 102 (1929–30), 370–427.
21. von Neumann, J., *On rings of operators*, III, Ann. of Math. 41 (1940), 94–161.
22. Zhe Liu, *On some mathematical aspects of the Heisenberg relation*, Sci. China Math. 54 (2011), 2427–2452.
23. Zhe Liu, *A double commutant theorem for Murray-von Neumann algebras*, Proc. Nat. Acad. Sci. U.S.A. 109 (2012), 7676–7681.

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