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APPLICATION OF LOCALIZATION TO THE MULTIVARIATE MOMENT PROBLEM

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Abstract

It is explained how the localization technique introduced by the author in [19] leads to a useful reformulation of the multivariate moment problem in terms of extension of positive semidefinite linear functionals to positive semidefinite linear functionals on the localization of $\mathsf{R}[\underline{x}]$ at $p = \prod_{i=1}^{n}(1+x_i^2)$ or $p' = \prod_{i=1}^{n-1}(1+x_i^2)$. It is explained how this reformulation can be exploited to prove new results concerning existence and uniqueness of the measure μ and density of $\mathsf{C}[\underline{x}]$ in $\mathscr{L}^s(\mu)$ and, at the same time, to give new proofs of old results of Fuglede [11], Nussbaum [21], Petersen [22] and Schmüdgen [27], results which were proved previously using the theory of strongly commuting self-adjoint operators on Hilbert space.

1. Introduction

For $n \geq 1$, we denote the polynomial ring $\mathsf{R}[x_1, \dots, x_n]$ by $\mathsf{R}[\underline{x}]$ for short. For a linear map $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$, we consider the set of positive Borel measures μ on R^n such that $L(f) = \int f\, d\mu\ \forall\, f \in \mathsf{R}[\underline{x}]$. The *multivariate moment problem* is to understand this set of measures, for a given linear map $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$. In particular, one wants to know:

(i) When is this set non-empty?

(ii) In case it is non-empty, when is it a singleton set?

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathsf{N}^n$, we denote the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ by $\underline{x}^\alpha$ for short. The positive Borel measures μ that we are interested in have *finite moments*, i.e., $\int \underline{x}^\alpha\, d\mu$ is a finite real number $\forall\, \alpha \in \mathsf{N}^n$. If μ is any positive Borel measure on R^n having finite moments then $L_\mu : \mathsf{R}[\underline{x}] \to \mathsf{R}$ defined by $L_\mu(f) = \int f\, d\mu\ \forall\, f \in \mathsf{R}[\underline{x}]$ is a well-defined linear map. This is clear.

For positive Borel measures μ, ν on R^n, each having finite moments, we write $\mu \sim \nu$ to indicate that μ and ν have the same moments, i.e., $L_\mu = L_\nu$. We say μ is *determinate* if $\mu \sim \nu \Rightarrow \mu = \nu$ and *indeterminate* if this is not the case.

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A linear map $L : A \to \mathsf{R}$, where A is an R-algebra, is said to be PSD (positive semidefinite) if $L(f^2) \geq 0\ \forall\ f \in A$. $\sum A^2$ denotes the set of all (finite) sums of squares of elements of A. For a linear map $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$, a necessary condition for the set in (i) to be non-empty is that L is PSD.

The multivariate moment problem has also been considered in the more general context of semigroup algebras [2], [6]. There is a one-to-one correspondence between functions $s : \mathsf{N}^n \to \mathsf{R}$ and linear maps $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ given by $L(\underline{x}^\alpha) = s(\alpha)$ for all $\alpha \in \mathsf{N}^n$, and L is PSD if and only if s is positive definite in the sense of [2], [6].

In the 1-dimensional case the literature on the moment problem is extensive; see [1] and [28]. In particular, one has the following result:

THEOREM 1.1. *For a linear map $L : \mathsf{R}[x] \to \mathsf{R}$:*

(1) *There exists a positive Borel measure μ on R such that $L = L_\mu$ iff L is* PSD.

(2) *The measure μ in* (1) *is determinate iff there exists a sequence Q_k of polynomials in $\mathsf{C}[x]$ such that $Q_k(i) = 1$ and $L(|Q_k|^2) \to 0$ as $k \to \infty$.*

PROOF. See [1, Theorem 2.1.1] and [1, Theorem 2.5.1].

For a positive Borel measure μ on a locally compact Hausdorff space X and a Borel measurable function $f : X \to \mathsf{C}$, define $\|f\|_{s,\mu} := [\int |f|^s d\mu]^{1/s}$, and define

$$\mathscr{L}^s(\mu) := \{f : X \to \mathsf{C} \mid f \text{ is Borel measurable and } \|f\|_{s,\mu} < \infty\}.$$

The condition that there exists a sequence Q_k of polynomials in $\mathsf{C}[x]$ such that $Q_k(i) = 1$ and $\int |Q_k|^2 d\mu \to 0$ as $k \to \infty$ is equivalent to the assertion that $\mathsf{C}[x]$ is dense in $\mathscr{L}^2((1+x^2)\mu)$.[1] It implies, in particular, that $\mathsf{C}[x]$ is dense in $\mathscr{L}^2(\mu)$. This is well-known. See Corollary 3.4 for a more general result.

For a PSD linear map $L : \mathsf{R}[x] \to \mathsf{R}$, the *Carleman condition*

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt[2i]{L(x^{2i})}} = \infty$$

is a well-known sufficient condition for the measure μ satisfying $L = L_\mu$ (which exists by Theorem 1.1(1)) to be unique. In fact the following holds:

THEOREM 1.2. *If the Carleman condition holds then $\mathsf{C}[x]$ is dense in $\mathscr{L}^s(\mu)$ for all real $s \geq 1$.*

PROOF. See [4, Théorème 3].

[1] $g\mu$ denotes the measure ν satisfying $\nu(E) := \int_E g\, d\mu$.

The Carleman condition holds if μ drops off sufficiently rapidly as $x \to \pm\infty$, e.g., this holds if μ has compact support or, more generally, if $\int e^{a|x|}\, d\mu < \infty$ for some real $a > 0$ [10, p. 80].

For a subset K of R^n, denote by $\operatorname{Pos}(K)$ the set of polynomials $f \in \mathsf{R}[\underline{x}]$ such that $f \geq 0$ on K (i.e., $f(a) \geq 0$ for all $a \in K$). The following general result is known; see [12] and [13].

THEOREM 1.3 (Haviland). *For a linear map $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ and a closed subset K in R^n, there exists a positive Borel measure μ on K such that $L = L_\mu$ iff $L(f) \geq 0$ holds for all $f \in \operatorname{Pos}(K)$.*

For $n = 1$, $\sum \mathsf{R}[\underline{x}]^2 = \operatorname{Pos}(\mathsf{R}^n)$. For $n \geq 2$, $\sum \mathsf{R}[\underline{x}]^2$ is a proper subset of $\operatorname{Pos}(\mathsf{R}^n)$ [14] and the condition that $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ is PSD is no longer sufficient for the set in (i) to be non-empty, see [5], [26].

For $n \geq 2$ the theory is not very well developed. See [27, Section 3] for open problems. A variety of partial results are known; see [23] for a survey. Some of these results are about the uniqueness of the measure, e.g., the results of Fuglede [11], Petersen [22] and Putinar and Vasilescu [25]. There are results about the density of $\mathsf{C}[\underline{x}]$ in $\mathscr{L}^s(\mu)$, $1 \leq s < \infty$, both in the case $n = 1$ and in the case $n \geq 2$ in [3], [4], [11] and [22]. There are also results about the existence of the measure, by Devinatz [8], Eskin [9], Nussbaum [21], Putinar and Schmüdgen [23], and Schmüdgen [27]. All these results, with the exception of [25] and the 1-dimensional results, are proved in the framework of unbounded operators on Hilbert space.

In [25] a different approach is taken which is based on the localization method developed in [24], but the localization method developed in [24] is still essentially a functional-analytic one, since, in the end, it is based on the theory of strongly commuting self-adjoint operators.

In [19] (also see [17] and [18]) the localization method is developed in a purely algebraic setting. First and foremost a Positivstellensatz is developed (see Theorem 2.1 below) which is based on Jacobi's representation theorem [15]. There is also a refined version of this Positivstellensatz (see Theorem 4.1 below) which is based on a result for cylinders with compact cross-section, established in [17] and [19], which is itself a corollary of Jacobi's representation theorem. There is very little functional analysis in the approach taken in [19], the one exception being a certain extension of Haviland's theorem [19, Theorem 3.1] which seems to be useful.

In preparing the present paper, the immediate goal was to exploit the localization method in [19] to give new algebraic proofs of the various partial results referred to above. The proofs were to be simpler than the existing ones. It was also hoped that this new way of looking at things would allow one to prove new results which were stronger than those that were previously known.

We leave it to the reader to decide how well these various goals have been accomplished.

We refer the reader to [20] for a more comprehensive treatment of positive polynomials, sums of squares and the moment problem. See [20, Theorem 5.4.4] for a simple proof of Jacobi's representation theorem. The paper [16] of Krivine, only recently rediscovered, is one of the earliest to bridge the gap between the moment problem and semialgebraic geometry. See [16, Théorème 12] for an early version of Jacobi's result.

In Section 2, we recall two results from [19] (see Theorems 2.1 and 2.3) and use Theorem 2.3 to give a new formulation of the multivariate moment problem in terms of localizations (see Corollary 2.5). We use Corollary 2.5 to prove a uniqueness result (Corollary 2.7) which extends results of Fuglede [11, Theorem, Section 7] and Petersen [22, Theorem 3]. In Section 3, we prove two results concerning density of $\mathsf{C}[\underline{x}]$ and $\mathsf{C}[\underline{x}]_p$ in $\mathscr{L}^s(\mu)$ (see Theorem 3.1 and Corollary 3.2), results which may be well-known but don't seem to be explicitly mentioned anywhere. We apply these results to obtain several corollaries, including a new proof of [4, Théorème 1] (see Corollary 3.5) and a strengthened version of [22, Proposition] (see Corollary 3.6). In Section 4, we apply the cylinder results from [19, Section 5] to obtain a new strengthened version of Haviland's Theorem (see Theorem 4.5). We use Theorem 4.5 to derive some non-trivial corollaries including a new proof of Nussbaum's multivariate Carleman result [21, Theorem 10] (see Theorem 4.10) and a new proof of a generalization of the Nussbaum result due to Schmüdgen [27, Proposition 1] (see Theorem 4.11). An interesting question that remains open is whether it is possible to prove the related results of Devinatz [8] and Eskin [9] by the method introduced in Section 4. The author was not able to do this, but, of course, this does not mean that it cannot be done.

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2. Reformulation of the problem

For A a commutative ring with 1 and $p \in A$, we denote by A_p the localization of A at p, i.e.,

$$A_p := \left\{ \frac{a}{p^k} \;\middle|\; a \in A, k \geq 0 \right\}.$$

The ring operations on A_p are defined in the standard way. If A is an R-algebra then so is A_p. We are interested here in the case $A = \mathsf{R}[\underline{x}]$. The results in [19] which we use are valid for various choices of p and various choices of a quadratic module. We restrict our attention here to the quadratic module

$\sum \mathsf{R}[\underline{x}]^2$ of $\mathsf{R}[\underline{x}]$ and its extension $\sum \mathsf{R}[\underline{x}]_p^2$ to $\mathsf{R}[\underline{x}]_p$, and we always take

$$p := \prod_{i=1}^{n} (1 + x_i^2).$$

We recall the Positivstellensatz from [19].

THEOREM 2.1. *Suppose* $f \in \mathsf{R}[\underline{x}]_p$. *The following are equivalent:*

(1) $f \geq 0$ *on* R^n.

(2) $\exists\, k \geq 0$ *such that* $\forall$ *real* $\epsilon > 0$ $f + \epsilon p^k \in \sum \mathsf{R}[\underline{x}]_p^2$.

PROOF. See [19, Corollary 4.3].

REMARK 2.2. (1) In the proof of Theorem 2.1 given in [19] one considers the subalgebra B of $\mathsf{R}[\underline{x}]_p$ consisting of algebraically bounded elements, i.e.,

$$B := \left\{ f \in \mathsf{R}[\underline{x}]_p \;\middle|\; \exists k \in \mathsf{N} \text{ such that } k \pm f \in \sum \mathsf{R}[\underline{x}]_p^2 \right\},$$

and the preordering $M := B \cap \sum \mathsf{R}[\underline{x}]_p^2$ of B. M is an archimedean preordering of B. Let

$$X_M := \left\{ \alpha : B \to \mathsf{R} \mid \alpha \text{ is a (unitary) ring homomorphism}, \alpha(M) \subseteq \mathsf{R}_{\geq 0} \right\},$$

define $\hat{f}$, for $f \in B$, by $\hat{f}(\alpha) = \alpha(f)$, and give X_M the weakest topology such that each $\hat{f}$, $f \in B$, is continuous. Since M is archimedean, X_M is compact. R^n is naturally embedded in X_M via $a \mapsto \alpha_a$ where $\alpha_a(f) := f(a)$. $X_M \backslash \mathsf{R}^n$ consists of those $\alpha \in X_M$ such that $\alpha(\frac{1}{p}) = 0$. In particular, $X_M \backslash \mathsf{R}^n$ is closed in X_M (so R^n is open in X_M). All this is explained in detail in [19].

(2) Fix $f \in \mathsf{R}[\underline{x}]_p$. Write f in the form $f = \frac{g}{p^m}$, $g \in \mathsf{R}[\underline{x}]$, $m \geq 0$. Say $g = \sum g_\alpha \underline{x}^\alpha$, $g_\alpha \in \mathsf{R}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathsf{N}^n$. We *claim* that the following are equivalent:

(i) $f \in B$.

(ii) f is geometrically bounded, i.e., $\exists\, k \in \mathsf{N}$ such that $|f(a)| \leq k\ \forall\, a \in \mathsf{R}^n$.

(iii) $\forall \alpha \in \mathsf{N}^n$, $g_\alpha \neq 0 \Rightarrow \alpha_j \leq 2m$, $j = 1, \dots, n$.

(iv) $f \in \mathsf{R}\left[\frac{1}{1+x_j^2}, \frac{x_j}{1+x_j^2} \;\middle|\; j = 1, \dots, n\right]$

PROOF. Since $k \pm f \in \sum \mathsf{R}[\underline{x}]_p^2 \Rightarrow |f(\underline{x})| \leq k\ \forall\ \underline{x} \in \mathsf{R}^n$, we see that (i)$\Rightarrow$(ii). Suppose now that (iii) fails, i.e., $\exists\ \alpha$ such that $g_\alpha \neq 0$ but $\alpha_j > 2m$ for some j. Reindexing, we can assume $j = 1$. Then $g = \sum_{i=0}^{k} a_i x_1^i$, with $k > 2m$, $a_i \in \mathsf{R}[x_2, \dots, x_n]$, $a_k \neq 0$. Fixing $x_2, \dots, x_n \in \mathsf{R}$ so that

$a_k(x_2, \dots, x_n) \neq 0$ and letting $x_1 \to \infty$, we obtain $|f(\underline{x})| \to \infty$, so (ii) fails. This proves (ii) ⇒ (iii). Suppose now that (iii) holds. Thus

$$f = \sum_\alpha h_\alpha \prod_{j=1}^n \frac{x_j^{\alpha_j}}{(1+x_j^2)^m}$$

with $\alpha_j \leq 2m$ for each j. If $\alpha_j \leq m$ write

$$\frac{x_j^{\alpha_j}}{(1+x_j^2)^m} = \left[\frac{x_j}{1+x_j^2}\right]^{\alpha_j}\left[\frac{1}{1+x_j^2}\right]^{m-\alpha_j}.$$

If $m < \alpha_j \leq 2m$ write

$$\frac{x_j^{\alpha_j}}{(1+x_j^2)^m} = \left[\frac{x_j}{1+x_j^2}\right]^{t_j}\left[1-\frac{1}{1+x_j^2}\right]^{u_j},$$

where $t_j + 2u_j = \alpha_j$, $t_j + u_j = m$. This proves that (iii) ⇒ (iv). Finally, since $1 \pm \frac{1}{1+x_j^2}$ and $1 \pm \frac{x_j}{1+x_j^2}$ are sum of squares in $\mathsf{R}[\underline{x}]_p$, $\frac{1}{1+x_j^2}$ and $\frac{x_j}{1+x_j^2}$ belong to B, so (iv) ⇒ (i).

(3) One can also check that $M = \sum B^2$.

Proof. Let $f \in M$, say $f = \sum_{k=1}^{\ell}[\frac{g_k}{p^m}]^2$, $g_k \in \mathsf{R}[\underline{x}]$, $k = 1, \dots, \ell$. The degree of $\sum_{k=1}^{\ell} g_k^2$ in the variable x_j is equal to the maximum of the degrees of the g_k^2 in the variable x_j, $k = 1, \dots, \ell$. Since $f \in B$, the implication (i) ⇒ (iii) in (2) shows the degree of $\sum_{k=1}^{\ell} g_k^2$ in x_j is $\leq 4m$. It follows that $\deg_{x_j}(g_k) \leq 2m$, $j = 1, \dots, n$, $k = 1, \dots, \ell$ so, by the implication (iii) ⇒ (i) in (2), $\frac{g_k}{p^m} \in B$.

(4) It is a consequence of (2) and (3) that X_M is identified with the real variety consisting of all points $(y_1, z_1, \dots, y_n, z_n) \in \mathsf{R}^{2n}$ satisfying

$$\left(y_j - \frac{1}{2}\right)^2 + z_j^2 = \frac{1}{4}, \qquad j = 1, \dots, n,$$

(an n-torus), B is identified with the coordinate ring of this variety, and the embedding $\mathsf{R}^n \hookrightarrow X_M$ is identified with the n-fold stereographic projection

$$(x_1, \dots, x_n) \mapsto \left(\frac{1}{1+x_1^2}, \frac{x_1}{1+x_1^2}, \dots, \frac{1}{1+x_n^2}, \frac{x_n}{1+x_n^2}\right).$$

(5) Analogs of (2), (3) and (4) for the localization of $\mathsf{R}[\underline{x}]$ at $p = 1+\sum_{i=1}^n x_i^2$ are established in [19, Example 8.1]. The real variety in this case is the Veronese variety, see [19, Example 8.1] and [24, Section 3].

(6) The non-trivial implication in the proof of Theorem 2.1 is (1) $\Rightarrow$ (2). k is chosen so that $\frac{f}{p^k} \in B$. From (4) one sees that R^n is dense in X_M, so $\frac{f}{p^k}$ is non-negative on all of X_M (not just on R^n). Jacobi's representation theorem [15] implies that for any real $\epsilon > 0$, $\frac{f}{p^k} + \epsilon \in M$. Multiplying by p^k yields (2).

THEOREM 2.3. *If $L : \mathsf{R}[\underline{x}]_p \to \mathsf{R}$ is a* PSD *linear map there exists a unique positive Borel measure μ on R^n such that $L(f) = \int f d\mu$ for all $f \in \mathsf{R}[\underline{x}]_p$.*

PROOF. See [19, Corollary 4.4]. Let $f \in \mathsf{R}[\underline{x}]_p$, $f \geq 0$ on R^n. By Theorem 2.1 $\exists$ $k \geq 0$ such that $\forall$ real $\epsilon > 0$ $f + \epsilon p^k \in \sum \mathsf{R}[\underline{x}]_p^2$. Thus $L(f + \epsilon p^k) \geq 0$. Letting $\epsilon \to 0$, we see that $L(f) \geq 0$. By the extension of Haviland's Theorem proved in [19, Theorem 3.1], there exists a positive Borel measure μ on R^n such that $L(f) = \int f d\mu$ for all $f \in \mathsf{R}[\underline{x}]_p$.[2] To prove uniqueness of μ, let $\phi : \mathsf{R}^n \to \mathsf{R}$ be any continuous function with compact support. We use the notation of Remark 2.2(1). Extend ϕ to X_M by setting $\phi = 0$ on $X_M \backslash \mathsf{R}^n$. By the Stone-Weierstrass approximation theorem $\exists$ a sequence $f_k \in B$ such that $|\hat{f_k} - \phi| \leq \frac{1}{k}$ pointwise on X_M. This implies, in particular, that $|\int (f_k - \phi)\, d\mu| \leq \frac{1}{k}\mu(\mathsf{R}^n)$, so $\int \phi\, d\mu = \lim_{k\to\infty} L(f_k)$. Uniqueness of μ follows now, by the Riesz representation theorem.

REMARK 2.4. The measure μ in Theorem 2.3 has finite moments. Conversely, if μ is any positive Borel measure on R^n having finite moments, then $L : \mathsf{R}[\underline{x}]_p \to \mathsf{R}$ defined by $L(f) = \int f d\mu$ $\forall$ $f \in \mathsf{R}[\underline{x}]_p$ is a well-defined map which is linear and PSD. This is clear.

COROLLARY 2.5. *For any linear map $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$, the set of positive Borel measures μ on R^n such that $L = L_\mu$ is in natural one-to-one correspondence with the set of* PSD *linear maps $L' : \mathsf{R}[\underline{x}]_p \to \mathsf{R}$ extending L.*

PROOF. If μ is a positive Borel measure on R^n such that $L = L_\mu$, the corresponding extension of L to a PSD linear map $L' : \mathsf{R}[\underline{x}]_p \to \mathsf{R}$ is defined by $L'(f) = \int f d\mu$. The correspondence $\mu \mapsto L'$ has the desired properties by Theorem 2.3.

REMARK 2.6. Corollary 2.5 allows one to reformulate the multivariate moment problem as follows: The *multivariate moment problem* is to understand the set of extensions of L to a PSD linear map $L' : \mathsf{R}[\underline{x}]_p \to \mathsf{R}$, for a given linear map $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$. In particular, one wants to know:

(i) When is this set non-empty?

(ii) In case it is non-empty, when is it a singleton set?

[2] Alternatively, existence of μ can be deduced by applying Haviland's theorem to $\overline{L} : \mathsf{R}[\underline{x}, y] \to \mathsf{R}$ defined by $\overline{L}(f(\underline{x}, y)) = L\big(f\big(\underline{x}, \frac{1}{p(\underline{x})}\big)\big)$ and $K \subseteq \mathsf{R}^{n+1}$ defined by $K = \{\big(a, \frac{1}{p(a)}\big) \mid a \in \mathsf{R}^n\}$.

Our next result explains how one half of Theorem 1.1(2) is valid for arbitrary n.

COROLLARY 2.7. *Suppose $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ is linear and, for each $j \in \{1, \dots, n\}$, there exists a sequence $p_{jk} \in \mathsf{C}[\underline{x}]$ such that $L(|1-(x_j-i)p_{jk}|^2) \to 0$ as $k \to \infty$. Then there is at most one positive Borel measure μ on R^n such that $L = L_\mu$.*

PROOF. Suppose μ and ν are positive Borel measures on R^n such that $L = L_\mu = L_\nu$. In view of Theorem 2.3 it suffices to show that $\int f d\mu = \int f d\nu$ $\forall f \in \mathsf{C}[\underline{x}]_p$, where $p := \prod_{j=1}^n (1+x_j^2)$. Observe that $1 + x_j^2 = (x_j - i)(x_j + i)$ so $\frac{1}{x_j - i}$, $\frac{1}{x_j + i}$ are elements of $\mathsf{C}[\underline{x}]_p$. The proof is by induction on the number of factors of the form $x_j \pm i$ appearing in the denominator of f. Suppose $x_j - i$ appears in the denominator of f. By assumption $\exists$ $p_{jk} \in \mathsf{C}[\underline{x}]$ so that $L(|Q_{jk}|^2) \to 0$ as $k \to \infty$ where $Q_{jk} := 1 - (x_j - i)p_{jk}$. By induction,

$$\int (x_j - i)p_{jk} f\, d\mu = \int (x_j - i)p_{jk} f\, d\nu.$$

Applying the Cauchy-Schwartz inequality,

$$\left|\int Q_{jk} f\, d\mu\right| \le L(|Q_{jk}|^2)^{1/2} \left[\int |f|^2 d\mu\right]^{1/2} \to 0 \qquad \text{as} \quad k \to \infty,$$

so

$$\int (x_j - i)p_{jk} f\, d\mu \to \int f\, d\mu \qquad \text{as} \quad k \to \infty.$$

Similarly,

$$\left|\int Q_{jk} f\, d\nu\right| \le L(|Q_{jk}|^2)^{1/2} \left[\int |f|^2 d\nu\right]^{1/2} \to 0 \qquad \text{as} \quad k \to \infty,$$

so

$$\int (x_j - i)p_{jk} f d\nu \to \int f\, d\nu \qquad \text{as} \quad k \to \infty.$$

It follows that $\int f d\mu = \int f d\nu$. The case where $x_j + i$ appears in the denominator of f is dealt with similarly, replacing Q_{jk} by $\overline{Q_{jk}}$.

REMARK 2.8. (1) In [22, Theorem 3] Petersen proves that a positive Borel measure μ on R^n with finite moments is determinate if each of the projection measures $\pi_j(\mu)$, $j = 1, \dots, n$ is determinate. Since $L_{\pi_j(\mu)} = L_\mu|_{\mathsf{R}[x_j]}$, Theorem 1.1(2) implies that $\pi_j(\mu)$ is determinate iff $\exists$ a sequence p_{jk} in $\mathsf{C}[x_j]$ such that $\int |1 - (x_j - i)p_{jk}|^2 d\mu \to 0$ as $k \to \infty$. In this way [22, Theorem 3] can be viewed as a special case of Corollary 2.7.

(2) In [11, Section 7] Fuglede proves that a positive Borel measure μ on R^n with finite moments is determinate if $\mathsf{C}[\underline{x}]$ is dense in $\mathscr{L}^2((1+x_j^2)\mu)$ for each $j = 1, \dots, n$. Since $\mathsf{C}[\underline{x}]$ dense in $\mathscr{L}^2((1+x_j^2)\mu) \Rightarrow \frac{1}{x_j - i}$ belongs to the closure of $\mathsf{C}[\underline{x}]$ in $\mathscr{L}^2((1+x_j^2)\mu) \Leftrightarrow \exists$ a sequence $p_{jk} \in \mathsf{C}[\underline{x}]$ such that $\int |1-(x_j-i)p_{jk})|^2 d\mu \to 0$ as $k \to \infty$, Fuglede's result is a special case of Corollary 2.7.

3. Density results

We fix a positive Borel measure μ on R^n having finite moments.

THEOREM 3.1. *For any $1 \le s < \infty$, $\mathsf{C}[\underline{x}]_p$ is dense in $\mathscr{L}^s(\mu)$, equivalently, $\mathsf{R}[\underline{x}]_p$ is dense in the real part of $\mathscr{L}^s(\mu)$.*

PROOF. It suffices to show that the step functions $\sum_{j=1}^m a_j \chi_{A_j}$, $a_j \in \mathsf{C}$, $A_j \subseteq \mathsf{R}^n$ a Borel set, belong to the closure of $\mathsf{C}[\underline{x}]_p$. Using the triangle inequality we are reduced further to the case $m = 1$, $a_1 = 1$. Let $A \subseteq \mathsf{R}^n$ be a Borel set. Choose K compact, U open such that $K \subseteq A \subseteq U$, $\mu(U \backslash K) < \epsilon$. We make use of the terminology introduced in Remark 2.2(1). By Urysohn's lemma there exists a continuous function $\phi : X_M \to \mathsf{R}$ such that $0 \le \phi \le 1$ on X_M, $\phi = 1$ on K, $\phi = 0$ on $X_M \backslash U$. Extend μ to a positive Borel measure μ' on X_M defined by $\mu'(E) := \mu(E \cap \mathsf{R}^n)$. Then $\|\chi_A - \phi\|_{s,\mu'} \le \epsilon^{1/s}$. Use the Stone-Weierstrass approximation theorem to get $f \in B$ such that $\|\phi - \hat{f}\|_\infty < \epsilon$, where $\|\cdot\|_\infty$ denotes the sup-norm. Then $\|\phi - \hat{f}\|_{s,\mu'} \le \epsilon\mu(\mathsf{R}^n)^{1/s}$. Putting these things together yields $\|\chi_A - f\|_{s,\mu} = \|\chi_A - \hat{f}\|_{s,\mu'} \le \|\chi_A - \phi\|_{s,\mu'} + \|\phi - \hat{f}\|_{s,\mu'} \le \epsilon^{1/s} + \epsilon\mu(\mathsf{R}^n)^{1/s}$.

COROLLARY 3.2. *For $1 \le s < \infty$, the following are equivalent:*

(1) *$\mathsf{C}[\underline{x}]$ is dense in $\mathscr{L}^s(\mu)$.*

(2) *$\mathsf{C}[\underline{x}]$ is dense in $\mathsf{C}[\underline{x}]_p$ in the topology induced by the norm $\|\cdot\|_{s,\mu}$.*

Suppose now that $n = 1$, so μ is a positive Borel measure on R having finite moments, $\mathsf{C}[\underline{x}] = \mathsf{C}[x]$ and $p = 1 + x^2$. Observe that $1 + x^2 = (x-i)(x+i)$ so $\frac{1}{x-i}, \frac{1}{x+i}$ are elements of $\mathsf{C}[x]_{1+x^2}$.

COROLLARY 3.3. *For $1 \le s < \infty$, the following are equivalent:*

(1) *$\mathsf{C}[x]$ is dense in $\mathscr{L}^s(\mu)$.*

(2) *$\exists$ a sequence $q_k \in \mathsf{C}[x]$ such that $\left\| q_k - \frac{1}{x-i} \right\|_{s,\mu} \to 0$ as $k \to \infty$.*

(3) *$\exists$ a sequence $Q_k \in \mathsf{C}[x]$ such that $Q_k(i) = 1$ and $\left\| \frac{Q_k}{x-i} \right\|_{s,\mu} \to 0$ as $k \to \infty$.*

PROOF. Clearly (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3), so it remains to show (2) $\Rightarrow$ (1). In view of Corollary 3.2 it suffices to show $\mathsf{C}[x]$ is dense in $\mathsf{C}[x]_{1+x^2}$. Denote by $\overline{\mathsf{C}[x]}$ the closure of $\mathsf{C}[x]$ in $\mathsf{C}[x]_{1+x^2}$. By (2), $\frac{1}{x-i} \in \overline{\mathsf{C}[x]}$. Conjugating, $\frac{1}{x+i} \in \overline{\mathsf{C}[x]}$. Using the identities

$$\frac{1}{1+x^2} = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right] \quad \text{and} \quad \frac{x}{1+x^2} = \frac{1}{2}\left[\frac{1}{x-i} + \frac{1}{x+i}\right],$$

and the division algorithm, we see that $\frac{f(x)}{1+x^2} \in \overline{\mathsf{C}[x]}$, for each $f(x) \in \mathsf{C}[x]$. Fix $f(x) \in \mathsf{C}[x]$ and choose $g_k(x) \in \mathsf{C}[x]$ so that $\|g_k(x) - \frac{f(x)}{1+x^2}\|_{s,\mu} \to 0$ as $k \to \infty$. Using the fact that $1 + x^2 \geq 1$ on R, we see that for each $\ell \geq 1$, $\left\| \frac{g_k(x)}{(1+x^2)^\ell} - \frac{f(x)}{(1+x^2)^{\ell+1}} \right\|_{s,\mu} \to 0$ as $k \to \infty$. If follows by induction on ℓ that $\frac{f(x)}{(1+x^2)^\ell} \in \overline{\mathsf{C}[x]}$ for all $\ell \geq 1$.

Corollary 3.4. *For $1 \leq s < \infty$, consider the conditions:*

(1) *$\exists$ a sequence Q_k in $\mathsf{C}[x]$ such that $Q_k(i) = 1$ and $\|Q_k\|_{s,\mu} \to 0$ as $k \to \infty$.*

(2) *$\mathsf{C}[x]$ is dense in $\mathscr{L}^s((1+x^2)^{s/2}\mu)$.*

(3) *$\mathsf{C}[x]$ is dense in $\mathscr{L}^s(\mu)$.*

(4) *$\exists$ a sequence Q_k in $\mathsf{C}[x]$ such that $Q_k(i) = 1$ and, $\forall\ 1 \leq s' < s$, $\|Q_k\|_{s',\mu} \to 0$ as $k \to \infty$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Leftrightarrow$ (2): Apply Corollary 3.3 to the measure $(1+x^2)^{s/2}\mu$. (2) $\Rightarrow$ (3): Since $1 + x^2 \geq 1$ this is clear. (3) $\Rightarrow$ (4): By Corollary 3.3 $\exists\ Q_k \in \mathsf{C}[x]$ such that $Q_k(i) = 1$ and $\left\| \frac{Q_k}{x-i} \right\|_{s,\mu} \to 0$ as $k \to \infty$. For $1 \leq s' < s$ an easy application of the Hölder inequality yields:

$$\begin{aligned}\|Q_k\|_{s',\mu} = \left[\int |Q_k|^{s'} d\mu\right]^{1/s'} &= \left[\int \left|\frac{Q_k}{x-i}\right|^{s'} |x-i|^{s'} d\mu\right]^{1/s'} \\ &\leq \left\|\frac{Q_k}{x-i}\right\|_{s,\mu} \cdot \|x-i\|_{\frac{ss'}{s-s'},\mu} \to 0 \qquad \text{as} \quad k \to \infty.\end{aligned}$$

Corollary 3.5.

$$\begin{aligned}&\sup\{s \mid \mathsf{C}[x] \textit{ is dense in } \mathscr{L}^s(\mu)\} \\ &\quad = \sup\{s \mid \exists Q_k \in \mathsf{C}[x] \textit{ such that } Q_k(i) = 1 \textit{ and } \lim_{k\to\infty} \|Q_k\|_{s,\mu} = 0\}.\end{aligned}$$

Proof. Immediate from Corollary 3.4. See [4, Théorème 1] for another proof.

We remark that a certain weak variant of Corollary 3.3 holds for $n \geq 2$. The following result extends [22, Proposition].

COROLLARY 3.6. *Suppose* $1 < s < \infty$. *Suppose for each* $j = 1, \dots, n$ $\exists$ $q_{jk} \in \mathsf{C}[\underline{x}]$ *such that* $\left\| q_{jk} - \frac{1}{x_j - i} \right\|_{s,\mu} \to 0$ *as* $k \to \infty$. *Then* $\mathsf{C}[\underline{x}]$ *is dense in* $\mathscr{L}^{s'}(\mu)$ *for each* $1 \leq s' < s$.

PROOF. Arguing as in the proof of Corollary 3.3, we see that $\mathsf{C}[x_j]_{1+x_j^2}$ is contained in the closure of $\mathsf{C}[\underline{x}]$ with respect to the norm $\|\cdot\|_{s,\mu}$, for $j = 1, \dots, n$. Every element of $\mathsf{C}[\underline{x}]_p$ is expressible as a sum of products of the form $f_1 \cdots f_n$, $f_j \in \mathsf{C}[x_j]_{1+x_j^2}$, $j = 1, \dots, n$. Choosing $g_{jk} \in \mathsf{C}[\underline{x}]$ so that $\|f_j - g_{jk}\|_{s,\mu} \to 0$ as $k \to \infty$, writing

$$\begin{aligned} f_1 \cdots f_n - g_{1k} \cdots g_{nk} = (f_1 - g_{1k}) f_2 \cdots f_n + (f_2 - g_{2k}) g_{1k} f_3 \cdots f_n \\ + \cdots + (f_n - g_{nk}) g_{1k} \cdots g_{n-1k}, \end{aligned}$$

and applying Hölder's inequality to each term, we see that

$$\|f_1 \cdots f_n - g_{1k} \cdots g_{nk}\|_{s',\mu} \to 0 \qquad \text{as} \quad k \to \infty,$$

for each $1 \leq s' < s$.

We also recall the following result of Fuglede; see [11, Sections 7, 8 and 10]:

COROLLARY 3.7. *Consider the following conditions:*

(1) $\mathsf{C}[\underline{x}]$ *is dense in* $\mathscr{L}^s(\mu)$ *for some* $2 < s < \infty$.

(2) $\mathsf{C}[\underline{x}]$ *is dense in* $\mathscr{L}^2((1 + x_1^2 + \ldots + x_n^2)\mu)$.

(3) $\mathsf{C}[\underline{x}]$ *is dense in* $\mathscr{L}^2((1 + x_j^2)\mu)$ *for* $j = 1, \dots, n$.

(4) μ *is determinate.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

PROOF. (1) $\Rightarrow$ (2). Let $f \in \mathsf{C}[\underline{x}]_p$ and choose $g_k \in \mathsf{C}[\underline{x}]$ so that $\|f - g_k\|_{s,\mu} \to 0$ as $k \to \infty$. By the Hölder inequality,

$$\begin{aligned} \|f - g_k\|_{2,(1+\sum x_t^2)\mu} &= \left[\int |f - g_k|^2 \left(1 + \sum x_t^2\right) d\mu \right]^{1/2} \\ &\leq \|f - g_k\|_{s,\mu} \cdot \left\| \sqrt{1 + \sum x_t^2} \right\|_{\frac{2s}{s-2},\mu} \to 0 \end{aligned}$$

as $k \to \infty$.

(2) $\Rightarrow$ (3). Follows from the fact that $1 + x_j^2 \leq 1 + x_1^2 + \cdots + x_n^2$.

(3) $\Rightarrow$ (4). As explained already in Remark 2.8(2), this follows from Corollary 2.7.

REMARK 3.8. Fuglede defines a positive Borel measure μ on R^n to be *ultradeterminate* if condition (2) of Corollary 3.7 holds and *strongly determinate* if condition (3) of Corollary 3.7 holds. Examples of Schmüdgen in [27, Section 1] show that conditions (2), (3) and (4) of Corollary 3.7 are not equivalent if $n \geq 2$. Examples of Berg and Thill in [7] show that μ determinate does not imply $\mathsf{C}[\underline{x}]$ is dense in $\mathscr{L}^2(\mu)$ if $n \geq 2$.

4. Extendibility results

In this section we apply the result on cylinders from [19, Section 5]. Let

$$p' := \prod_{i=1}^{n-1}(1+x_i^2).$$

Note: If $n = 1$ then $p' = 1$. Observe that $\mathsf{R}[\underline{x}]_{p'} = \mathsf{R}[\underline{x}']_{p'}[x_n]$ (the polynomial ring in the single variable x_n with coefficients in $\mathsf{R}[\underline{x}']_{p'}$), where $\underline{x}' := (x_1, \dots, x_{n-1})$.

THEOREM 4.1. *Suppose $f \in \mathsf{R}[\underline{x}]_{p'}$. The following are equivalent:*

(1) $f \geq 0$ *on* R^n.

(2) $\exists\, k, \ell \geq 0$ *such that* $\forall$ *real* $\epsilon > 0$ $f + \epsilon p'^k(1+x_n^2)^\ell \in \sum \mathsf{R}[\underline{x}]^2_{p'}$.

PROOF. See [19, Corollary 5.3].

REMARK 4.2. (1) The difference between Theorem 4.1 and Theorem 2.1 is that in Theorem 4.1 we do not need to invert as much: $\mathsf{R}[\underline{x}]_{p'}$ is a proper subalgebra of $\mathsf{R}[\underline{x}]_p$.

(2) In the proof of Theorem 4.1 given in [19] one considers the subalgebra B' of $\mathsf{R}[\underline{x}']_{p'}$ consisting of algebraically bounded elements, i.e.,

$$B' := \left\{ f \in \mathsf{R}[\underline{x}']_{p'} \;\middle|\; \exists k \in \mathsf{N} \text{ such that } k \pm f \in \sum \mathsf{R}[\underline{x}']^2_{p'} \right\},$$

and the preordering $N := B'[x_n] \cap \sum \mathsf{R}[\underline{x}]^2_{p'}$ of $B'[x_n]$. Let

$$X_N := \{\alpha : B'[x_n] \to \mathsf{R} \mid \alpha \text{ is a ring homomorphism}, \alpha(N) \subseteq \mathsf{R}_{\geq 0}\},$$

define $\hat{f}$, for $f \in B'[x_n]$, by $\hat{f}(\alpha) = \alpha(f)$, and give X_N the weakest topology such that each $\hat{f}$, $f \in B'[x_n]$, is continuous. Since $M' := N \cap B'$ is an archimedean preordering of B', $X_N = X_{M'} \times \mathsf{R}$ is a cylinder with compact cross-section. R^n is naturally embedded in X_N via $a \mapsto \alpha_a$ where $\alpha_a(f) := f(a)$. All this is explained in detail in [19].

(3) Concrete descriptions of B' and M' are provided by (2) and (3) of Remark 2.2. Using these descriptions, we see that X_N is identified with the real variety consisting of all points $(y_1, z_1, \dots, y_{n-1}, z_{n-1}, x_n) \in \mathsf{R}^{2n-1}$ satisfying

$$\left(y_j - \frac{1}{2}\right)^2 + z_j^2 = \frac{1}{4}, \qquad j = 1, \dots, n-1,$$

$B'[x_n]$ is identified with the coordinate ring of this variety and the embedding $\mathsf{R}^n \hookrightarrow X_N$ is identified with the map

$$(x_1, \dots, x_n) \mapsto \left(\frac{1}{1+x_1^2}, \frac{x_1}{1+x_1^2}, \dots, \frac{1}{1+x_{n-1}^2}, \frac{x_{n-1}}{1+x_{n-1}^2}, x_n\right).$$

(4) The non-trivial implication in the proof of Theorem 4.1 is (1) ⇒ (2). k is chosen so that $\frac{f}{p'^k} \in B'[x_n]$. From (3) one sees that R^n is dense in X_N, so $\frac{f}{p'^k}$ is non-negative on all of X_N (not just on R^n). By [19, Theorem 5.1] there exists an integer $\ell \geq 0$ such that for any real $\epsilon > 0$, $\frac{f}{p'^k} + \epsilon(1 + x_n^2)^\ell \in N$. Multiplying by p'^k yields (2).

THEOREM 4.3. *If $L : \mathsf{R}[\underline{x}]_{p'} \to \mathsf{R}$ is a* PSD *linear map there exists a positive Borel measure μ on R^n such that $L(f) = \int f d\mu$ for all $f \in \mathsf{R}[\underline{x}]_{p'}$.*

PROOF. Argue as in the proof of Theorem 2.3 but use Theorem 4.1 now instead of Theorem 2.1.

REMARK 4.4. (1) There is no claim in Theorem 4.3 that the measure μ (equivalently, the extension of L to a PSD linear map from $\mathsf{R}[\underline{x}]_p$ to R) is unique. In fact, it is not unique in general. (2) A sufficient condition for the measure μ to be unique is that there exists a sequence q_k in $\mathsf{R}[\underline{x}]_{p'}$ such that $L(|1 - (x_n - i)q_k|^2) \to 0$ as $k \to \infty$. The proof of this fact is similar to the proof of Corollary 2.7. (3) If $n = 1$ this sufficient condition is also necessary, by Theorem 1.1(2).

THEOREM 4.5. *For a linear map $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$, the following are equivalent:*

(1) *There exists a positive Borel measure μ on R^n such that $L = L_\mu$.*
(2) *L extends to a* PSD *linear map $L : \mathsf{R}[\underline{x}]_{p'} \to \mathsf{R}$.*
(3) *$L \geq 0$ on $\sum \mathsf{R}[\underline{x}]_{p'}^2 \cap \mathsf{R}[\underline{x}]$.*
(4) *For all $m \geq 0$, $p'^m f \in \sum \mathsf{R}[\underline{x}]^2 \Rightarrow L(f) \geq 0$.*

PROOF. (1) ⇒ (2). Extend L to $\mathsf{R}[\underline{x}]_{p'}$ in the obvious way, i.e., $L(f) = \int f d\mu$ for all $f \in \mathsf{R}[\underline{x}]_{p'}$.

(2) $\Rightarrow$ (3). $L \geq 0$ on $\sum \mathsf{R}[\underline{x}]^2_{p'}$ so $L \geq 0$ on $\sum \mathsf{R}[\underline{x}]^2_{p'} \cap \mathsf{R}[\underline{x}]$.

(3) $\Rightarrow$ (4). Suppose that $f \in \mathsf{R}[\underline{x}]$, $p'^m f \in \sum \mathsf{R}[\underline{x}]^2$. Then $f = \frac{p'^m f}{p'^m} = (\frac{1}{p'})^{2m}(p'^m)(p'^m f) \in \sum \mathsf{R}[\underline{x}]^2_{p'}$, so $L(f) \geq 0$.

(4) $\Rightarrow$ (1). Suppose $f \in \mathsf{R}[\underline{x}]$, $f \geq 0$ on R^n. By Theorem 4.1, there exist integers $k, \ell \geq 0$ such that, for all $\epsilon > 0$, $f + \epsilon p'^k(1 + x_n^2)^\ell \in \sum \mathsf{R}[\underline{x}]^2_{p'}$, so $p'^{2m}(f + \epsilon p'^k(1 + x_n^2)^\ell) \in \sum \mathsf{R}[\underline{x}]^2$, for some $m \geq 0$. By (4) this implies $L(f + \epsilon p'^k(1 + x_n^2)^\ell) \geq 0$. Since this is valid for any $\epsilon > 0$, this implies $L(f) \geq 0$. Thus (1) follows, by Haviland's Theorem 1.3.

REMARK 4.6. (1) Theorem 4.5 strengthens Haviland's Theorem. Instead of having to check $f \geq 0$ on $\mathsf{R}^n \Rightarrow L(f) \geq 0$, one only has to check that $p'^m f \in \sum \mathsf{R}[\underline{x}]^2 \Rightarrow L(f) \geq 0$. (2) Observe that if $n = 1$ then $p' = 1$, so Theorem 4.5 coincides with Theorem 1.1(1) in this case. (3) There is also a weak version of Theorem 4.5, obtained by replacing p' by p. The proof is the same except that Theorem 4.1 is replaced now by Theorem 2.1.

We turn our attention to applications of the implication (4) $\Rightarrow$ (1) of Theorem 4.5.

COROLLARY 4.7. *If $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ is a linear map which is* PSD *and, for each $f \in \mathsf{R}[\underline{x}]$ and each $j \in \{1, \dots, n-1\}$,*

(4.1) $$L_1(q) := L(q(1 + x_j^2)f) \text{ is PSD} \Rightarrow L_2(q) := L(qf) \text{ is PSD},$$

then $L = L_\mu$ for some positive Borel measure μ on R^n.

PROOF. We show condition (4) of Theorem 4.5 holds. Suppose $p'^m f \in \sum \mathsf{R}[\underline{x}]^2$. Then $L'(q) := L(qp'^m f)$ is PSD. Applying (4.1) repeatedly, we deduce that $L''(q) := L(qf)$ is PSD. In particular, $L(f) = L''(1) \geq 0$.

COROLLARY 4.8. *If $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ is a linear map which is* PSD *and, for each $g \in \mathsf{R}[\underline{x}]$ and each $j \in \{1, \dots, n-1\}$,*

(4.2) $$\exists\, p_k = p_{gjk} \in \mathsf{C}[\underline{x}] \text{ such that } L(g - (1 + x_j^2)p_k\overline{p_k}g) \to 0$$

as $k \to \infty$, then $L = L_\mu$ for some positive Borel measure μ on R^n.

PROOF. Apply (4.2) with $g = h\overline{h}f$ to deduce that the hypothesis of Corollary 4.7 holds.

THEOREM 4.9. *Suppose $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ is linear and* PSD *and, for each $j = 1, \dots, n-1$,*

(4.3) $$\exists\, p_k = p_{jk} \in \mathsf{C}[\underline{x}] \text{ such that } L(|1 - (x_j - i)p_k|^4) \to 0$$

as $k \to \infty$. Then there exists a positive Borel measure μ on R^n such that $L = L_\mu$. If condition (4.3) *holds also for $j = n$ then the measure is determinate.*

PROOF. Fix $g \in \mathsf{R}[\underline{x}]$, $j \in \{1, \dots, n-1\}$. Set $Q_k = 1 - (x_j - i)p_k$, so

$$g - (1 + x_j^2)p_k\overline{p_k}g = g - (1 - Q_k)(1 - \overline{Q_k})g = Q_k g + \overline{Q_k} g - |Q_k|^2 g.$$

Extending L to $\mathsf{C}[\underline{x}]$ in the obvious way, and applying the Cauchy-Schwartz inequality to the inner product on $\mathsf{C}[\underline{x}]$ defined by $\langle f, g\rangle := L(f\overline{g})$, we see that

$$\begin{aligned}
&|L(g - (1 + x_j^2)p_k\overline{p_k}g)| \\
&\qquad \le |L(Q_k g)| + |L(\overline{Q_k} g)| + |L(|Q_k|^2 g)| \\
&\qquad \le 2[L(|Q_k|^2)]^{1/2}[L(g^2)]^{1/2} + |L(|Q_k|^2 g)| \\
&\qquad \le 2[L(|Q_k|^4)]^{1/4}[L(1)]^{1/4}[L(g^2)]^{1/2} + [L(|Q_k|^4)]^{1/2}[L(g^2)]^{1/2} \to 0
\end{aligned}$$

as $k \to \infty$.

The first assertion follows from this, by Corollary 4.8. Since

$$L(|1 - (x_j - i)p_k|^2) \le [L(|1 - (x_j - i)p_k|^4)]^{1/2}[L(1)]^{1/2} \to 0$$

as $k \to \infty$, the second assertion is immediate, by Corollary 2.7.

Combining Theorem 4.9 with Theorem 1.2 yields the following result of Nussbaum [21, Theorem 10]:

THEOREM 4.10. *Suppose $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ is linear and* PSD *and the Carleman condition*

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt[2i]{L(x_j^{2i})}} = \infty \tag{4.4}$$

holds for $j = 1, \dots, n-1$. Then there exists a positive Borel measure μ on R^n such that $L = L_\mu$. If condition (4.4) *holds also for $j = n$ then the measure is determinate.*

PROOF. Let μ_j be a positive Borel measure on R such that $L_{\mu_j} = L|_{\mathsf{R}[x_j]}$. According to Theorem 1.2, condition (4.4) implies that $\mathsf{C}[x_j]$ is dense in $\mathscr{L}^s(\mu_j)$ for $1 \le s < \infty$. In particular, $\mathsf{C}[x_j]$ is dense in $\mathscr{L}^{4+\epsilon}(\mu_j)$ for $\epsilon > 0$, which implies, by Corollary 3.4, that $\exists\ p_k = p_{jk} \in \mathsf{C}[x_j]$ such that $L(|1 - (x_j - i)p_k|^4) \to 0$ as $k \to \infty$. Now apply Theorem 4.9.

We conclude by mentioning another result, similar to Theorem 4.9, which, like Theorem 4.9, is of sufficient strength to imply Theorem 4.10. See Schmüdgen [27, Proposition 1] for a different proof of this result.[3]

THEOREM 4.11. *Suppose* $L : \mathsf{R}[\underline{x}] \to \mathsf{R}$ *is linear and* PSD. *Fix a positive Borel measure* μ_j *on* R *such that* $L|_{\mathsf{R}[x_j]} = L_{\mu_j}$ *and suppose, for each* $j = 1, \dots, n-1$, $\mathsf{C}[x_j]$ *is dense in* $\mathscr{L}^4(\mu_j)$, *i.e.,*

$$\text{(4.5)} \qquad \exists\, Q_k = Q_{k,j} \in \mathsf{C}[x_j] \text{ such that } Q_k(i) = 1 \text{ and } \left\| \frac{Q_k}{x_j - i} \right\|_{4,u_j} \to 0$$

as $k \to \infty$. *Then there exists a positive Borel measure* μ *on* R^n *such that* $L = L_\mu$. *If condition* (4.5) *holds also for* $j = n$ *then the measure is determinate.*

PROOF. By the proof of Theorem 4.9 it suffices to show, for each $g \in \mathsf{R}[\underline{x}]$ and for each j, that condition (4.5) implies $L(Q_k\overline{Q_k}g) \to 0$ as $k \to \infty$. Let $x := x_j$, $\mu := \mu_j$, and define measures μ' and μ'' on R by

$$\mu' = \frac{\mu}{(1+x^2)^2}, \qquad \mu'' = (1+x^2)^2\mu.$$

Claim 1. For each $q \in \mathsf{C}[x]$ and each $\ell \in \{0, 1\}$,

$$|L(x-i)^{2\ell}qg)| \le C \cdot [L(q\overline{q})]^{1/2}$$

where

$$C = \max\left\{[L(g^2)]^{1/2}, [L((x-i)^2(x+i)^2g^2)]^{1/2}\right\}.$$

This is an immediate consequence of the Cauchy-Schwartz inequality.

Claim 2. The measure μ'' is determinate. This follows from

$$\int Q_k\overline{Q_k}\,d\mu'' = \int Q_k\overline{Q_k}(1+x^2)^2 d\mu = \int \frac{Q_k\overline{Q_k}}{1+x^2}(1+x^2)^3 d\mu$$
$$\le \left[\int \left[\frac{Q_k\overline{Q_k}}{1+x^2}\right]^2 d\mu\right]^{1/2} \left[\int (1+x^2)^6 d\mu\right]^{1/2} \to 0$$

as $k \to \infty$.

Claim 3. $|L(pg)| \le C \cdot \left[\int |p|^2 d\mu'\right]^{1/2}$ for each $p \in \mathsf{C}[x]$. From Claim 2 and Corollary 3.4 it follows that $\mathsf{C}[x]$ is dense in $\mathscr{L}^2(\mu'')$ so $\exists$ a sequence

[3] According to Fuglede [11, p. 62], Theorem 4.11 is an unpublished result of J. P. R. Christensen, 1981.

q_k in $\mathsf{C}[x]$ such that $q_k \to \frac{p}{(x-i)^2}$ in $\mathscr{L}^2(\mu'')$. Applying Claim 1 with $\ell = 0$, $q = (x-i)^2 q_k - p$ and noting that

$$[L(q\overline{q})]^{1/2} = \left[\int |q|^2 d\mu\right]^{1/2} = \left[\int \left|q_k - \frac{p}{(x-i)^2}\right|^2 d\mu''\right]^{1/2} \to 0$$

as $k \to \infty$, we see that $L((x-i)^2 q_k g) \to L(pg)$ as $k \to \infty$. Because

$$\int \left|q_k - \frac{p}{(x-i)^2}\right|^2 d\mu \leq \int \left|q_k - \frac{p}{(x-i)^2}\right|^2 d\mu''$$

(using the fact that $1 + x^2 \geq 1$ on R), we see that $\int \left|q_k - \frac{p}{(x-i)^2}\right|^2 d\mu \to 0$ as $k \to \infty$. Then, applying Claim 1 again, with $\ell = 1$, $q = q_k$, we see that

$$\begin{aligned} |L((x-i)^2 q_k g)| &\leq C \cdot \left[L(q_k \overline{q_k})\right]^{1/2} = C \cdot \left[\int |q_k|^2 d\mu\right]^{1/2} \\ &\to C \cdot \left[\int \left|\frac{p}{(x-i)^2}\right|^2 d\mu\right]^{1/2} = C \cdot \left[\int |p|^2 d\mu'\right]^{1/2} \quad \text{as} \quad k \to \infty. \end{aligned}$$

Putting these things together, we see that $|L(pg)| \leq C \cdot \left[\int |p|^2 d\mu'\right]^{1/2}$.

Applying Claim 3 with $p = Q_k \overline{Q_k}$, we see that $|L(Q_k \overline{Q_k} g)| \leq C \cdot \left[\int |Q_k \overline{Q_k}|^2 d\mu'\right]^{1/2}$, so $|L(Q_k \overline{Q_k} g)| \to 0$ as $k \to \infty$.

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