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MAXIMAL OPERATOR IN VARIABLE EXPONENT LEBESGUE SPACES ON UNBOUNDED QUASIMETRIC MEASURE SPACES

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Abstract

We study the Hardy-Littlewood maximal operator M on $L^{p(\cdot)}(X)$ when X is an unbounded (quasi)metric measure space, and p may be unbounded. We consider both the doubling and general measure case, and use two versions of the log-Hölder condition. As a special case we obtain the criterion for a boundedness of M on $L^{p(\cdot)}(\mathsf{R}^n, \mu)$ for arbitrary, possibly non-doubling, Radon measures.

1. Introduction

A major breakthrough in the investigation of variable exponent spaces occurred in the beginning of the millennium when Lars Diening [5] proved the boundedness of the maximal operator on the space $L^{p(\cdot)}(\mathsf{R}^n)$. He assumed that the exponent satisfies a local continuity condition and that it is constant outside some ball. The latter condition is quite unnatural, and it was subsequently improved to a metric decay condition in [4] and a more general integral decay condition in [18].

These results were quickly generalized from R^n to the (quasi)metric measure space setting [11], [13], [14]. However, strangely these papers considered either the bounded space case or used the same unnatural ball condition as Diening. Later several papers (e.g., [7], [8], [9]) have appeared dealing with other operators on $L^{p(\cdot)}(X, d, \mu)$, but to the best of our knowledge they all had the same restrictions on the behavior at infinity.

The purpose of this paper is to investigate the Hardy-Littlewood maximal operator M on $L^{p(\cdot)}(X)$ when X is an unbounded (quasi)metric measure space. Further, we consider the case of unbounded exponents, also apparently new outside R^n. Both of these improvements follow from the machinery in [6]. Perhaps the largest novelty is in our delineation and study of the relationship

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between different conditions on the exponent, which we call the metric and the measure log-Hölder conditions, see Theorem 1.4.

Thus, we are able to prove the boundedness of the maximal operator in Theorem 1.7 without any doubling condition on the measure. This allows us to prove the boundedness on R^n for all Radon measures (Corollary 1.9), which was not previously known.

Let us move on to the details. We say that (X, d) is a *quasimetric* space if $d : X \times X \to [0, \infty)$ satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$ for all x, y,
(3) $d(x, y) \leqslant A(d(x, z) + d(y, z))$ for all x, y, z and some constant $A \geqslant 1$.

The third condition is called the *quasitriangle inequality*. A metric is quasimetric with $A = 1$. A typical example of a quasimetric which is not a metric is $d(x, y) = |x - y| + |x - y|^\alpha$ for $\alpha > 1$ and $x, y \in \mathsf{R}^n$. In the above definition we follow [12, Chapter 14], however some authors discussed quasimetrics with the inequality $d(x, y) \leqslant Ld(y, x)$ in place of symmetry in the second condition (see e.g. [15]). Balls in the topology given by d need not be open, but there always exists an equivalent quasimetric with this property, see [16]. In fact, for a suitable choice of $\varepsilon(A)$ one can show that d^ε is bilipschitz equivalent to a metric on X (see [12, Proposition 14.5]).

Let X be a quasimetric space with a distance function d and a measure μ, such that $0 < \mu(B) < \infty$ for any ball $B \subset X$. We say that the measure μ is *doubling* if there exists a *doubling constant* $C_\mu \geqslant 1$ such that for all balls $B \subset X$

$$\mu(2B) \leqslant C_\mu \mu(B). \tag{1.1}$$

Quasimetric spaces equipped with a doubling measure are often called *spaces of homogeneous type*, see e.g. [2, Chapter 6] or [3, Chapter 3]. Among examples of such spaces let us mention smooth Riemannian manifolds, graphs of Lipschitz functions, some Cantor type sets, some connected Lie groups with a left-invariant Riemannian metric and Carnot-Carathéodory spaces. We refer to [2, Chapter 6] for further examples of homogeneous spaces and to [10, Chapter 11] for a comprehensive introduction to Carnot-Carathéodory spaces.

We next define conditions on the exponent, using the metric and the measure.

DEFINITION 1.2. Let $\Omega \subset X$. We say that $\alpha \colon \Omega \to \mathsf{R}$ is *locally* log-*Hölder continuous* in Ω if there exists $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leqslant \frac{c_1}{\log(e + 1/d(x, y))}$$

for all $x, y \in \Omega$. We say that α satisfies the log-*Hölder decay condition* with basepoint $x_0 \in X$ if there exist $\alpha_\infty \in \mathsf{R}$ and a constant $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{c_2}{\log(e + d(x, x_0))}$$

for all $x \in \Omega$. We say that α is log-*Hölder continuous* in Ω if both conditions are satisfied. The maximum $\max\{c_1, c_2\}$ is called *the* log-*Hölder constant* of α.

We define a class of exponents p whose reciprocal is log-Hölder continuous:

$$\mathscr{P}_d^{\log}(\Omega) := \big\{p : \Omega \to [1, \infty] \;\big|\; \tfrac{1}{p} \text{ is log-Hölder continuous}\big\}.$$

By $c_{\log}(p)$ or $c_{\log}$ we denote the log-Hölder constant of $\frac{1}{p}$.

As usual we use the convention $\frac{1}{\infty} := 0$. A priori the decay condition depends on the basepoint x_0, but we show in Lemma 2.1 that the choice of basepoint only affects the value of the constant c_2.

Next we have the condition related to the measure. For a function α we denote by $\alpha_A^\pm$ the essential supremum and infimum of over a set A.

DEFINITION 1.3. We say that a function $p : \Omega \to [1, \infty]$ belongs to $\mathscr{P}_\mu^{\log}(\Omega)$ if there exists $c > 0$ such that

$$\mu(B)^{\frac{1}{p_B^+} - \frac{1}{p_B^-}} \leqslant c$$

for every ball $B \subset \Omega$ and there exists $p_\infty \in [1, \infty]$ such that

$$1 \in L^{s(\cdot)}(\Omega), \qquad \text{where} \quad \tfrac{1}{s(x)} := \big|\tfrac{1}{p(x)} - \tfrac{1}{p_\infty}\big|.$$

Note that if $p \in \mathscr{P}_\mu^{\log}(\Omega)$ and $c > 1/p^-$, then $cp \in \mathscr{P}_\mu^{\log}(\Omega)$. The next result relates exponents in $\mathscr{P}_d^{\log}(X)$ and in $\mathscr{P}_\mu^{\log}(X)$.

THEOREM 1.4. *If $p \in \mathscr{P}_d^{\log}(X)$ and μ is doubling, then $p \in \mathscr{P}_\mu^{\log}(X)$.*

The proof of Theorem 1.4 follows directly from Lemmas 2.2 and 3.1, and Corollary 3.5. The converse is not true, even in the Euclidean setting since the integral condition in $\mathscr{P}_\mu^{\log}(X)$ does not imply the existence of a limit at infinity.

We will show that $\mathscr{P}_d^{\log}(X)$ and $\mathscr{P}_\mu^{\log}(X)$ are sufficient for the boundedness of the maximal operator with suitable auxiliary conditions. We start by recalling the definition.

DEFINITION 1.5. For a measurable function f we define the (Hardy-Littlewood) *maximal function* Mf by

$$Mf(x) := \sup_{B \ni x} \fint_B |f(y)| \, d\mu(y)$$

for all $x \in X$.

Our discussion is valid also for the centered maximal operator since it is dominated by the non-centered one. The centered maximal operator is defined as above except that the supremum is taken over balls centered at x, not all balls containing x.

Let us recall some classical results for the maximal operator M, see for example Stein [19] for the Euclidean case, Coifman-Weiss [3] for homogeneous spaces and Heinonen [12] for metric measure spaces. For $f \in L^1_{\mathrm{loc}}(X)$ the function $Mf\colon X \to [0, \infty]$ is lower semicontinuous and satisfies $|f| \leqslant Mf$ almost everywhere in X. For any $q \in [1, \infty]$ and $f \in L^q(X)$ the function Mf is almost everywhere finite. Moreover, for $1 < q \leqslant \infty$ the maximal operator is bounded in the sense that

$$\|Mf\|_q \leqslant \frac{cq}{q-1} \|f\|_q. \tag{1.6}$$

On the other hand, M is not bounded from $L^1(X)$ to $L^1(X)$. Actually, $Mf \notin L^1(X)$ for every non-zero $f \in L^1(X)$. We generalize the boundedness (1.6) to $L^{p(\cdot)}(X)$ as follows.

THEOREM 1.7. *Let X be a quasimetric space and $p \in \mathcal{P}^{\log}_\mu(X)$ with $p^- > 1$. Assume that $M : L^{p^-}(X) \to L^{p^-}(X)$ is bounded for the constant exponent p^-. Then there exists $c > 0$ depending on p such that*

$$\|Mf\|_{p(\cdot)} \leqslant c \, \|f\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}(X)$.

The previous theorem features general conditions, which are not so simple to check in particular cases. So we provide some useful special cases. If μ is a doubling measure, then M is bounded on $L^q(X)$ with boundedness constant depending only on $Q = \log_2 C_\mu$ and q (see Remark 2.5 in [12]). Thus Theorems 1.4 and 1.7 give the following corollary.

By standard arguments and Proposition 2.4 we can derive from this the boundedness of the maximal operator on $L^{p(\cdot)}(Y)$ for $Y \subset X$, under the same assumption on p.

COROLLARY 1.8. *Let X be a metric space with doubling measure μ. Let $p \in \mathcal{P}_d^{\log}(X)$ with $p^- > 1$. Then*

$$\|Mf\|_{L^{p(\cdot)}(X)} \leqslant \frac{cp^-}{p^- - 1} \|f\|_{L^{p(\cdot)}(X)}$$

for all $f \in L^{p(\cdot)}(X)$, where c depends only on $\mu(B(x_0, 1))$, C_μ and $c_{\log}(p)$.

Note that Theorem 1.7 is not limited to the case of doubling measures. For instance, Theorem 2.19 in [17] states that the (centered) maximal operator is bounded for $p \in (1, \infty]$ and every Radon measure μ on R^n. Thus we obtain also the following corollary.

COROLLARY 1.9. *Let μ be a Radon measure on R^n and let $p \in \mathcal{P}_d^{\log}(\mathsf{R}^n)$ with $p^- > 1$. Then there exists constant c depending on μ and $c_{\log}$ such that, for the centered maximal operator,*

$$\|Mf\|_{L^{p(\cdot)}(\mathsf{R}^n,\mu)} \leqslant \frac{cp^-}{p^- - 1} \|f\|_{L^{p(\cdot)}(\mathsf{R}^n,\mu)}$$

for all $f \in L^{p(\cdot)}(\mathsf{R}^n, \mu)$.

For the history of the proof of Theorem 1.7 in the Euclidean setting we refer to discussion after Theorem 4.3.8 in Chapter 4 of [6].

Notation and background

For $t \geqslant 0$ and $1 \leqslant p < \infty$ we define $\varphi_p(t) := t^p$ and for $p = \infty$ we set

$$\varphi_\infty(t) := \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant 1, \\ \infty & \text{for } 1 < t < \infty. \end{cases} \tag{1.10}$$

We will use t^p as an abbreviation for $\varphi_p(t)$, also in the case $p = \infty$. Similarly, $t^{1/p}$ will denote the inverse function $\varphi_p^{-1}(t)$; note that in case $p = \infty$ we have $t^{1/\infty} = \varphi_\infty^{-1}(t) = \chi_{(0,\infty)}(t)$.

For a measurable function $f \colon X \to \mathsf{R}$ we define the modular

$$\varrho_{p(\cdot)}(f) = \int_X |f(y)|^{p(y)}\, d\mu(y)$$

and the norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 \colon \varrho_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)} := L^{p(\cdot)}(X, d, \mu)$ consists of those measurable functions $f \colon X \to \mathsf{R}$ for which $\|f\|_{p(\cdot)} < \infty$. It is a Banach space. By the *unit ball property*, we mean that $\|f\|_{p(\cdot)} \leqslant 1$ if and only if $\varrho_{p(\cdot)}(f) \leqslant 1$. For more information see [6, Lemmas 3.2.4 and 3.4.2].

Let X and Y be normed subspaces of some vector space. Then the intersection $X \cap Y$ equipped with the norm $\|z\|_{X \cap Y} = \max\{\|z\|_X, \|z\|_Y\}$ and the sum $X + Y := \{x + y \,:\, x \in X, y \in Y\}$ equipped with the norm

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y \,:\, x \in X, y \in Y, z = x + y\}$$

are normed spaces. If X and Y are Banach spaces, then so are $X \cap Y$ and $X + Y$.

By c we denote a generic constant whose value may change between appearances.

2. Auxiliary results

We first establish the independence of the decay condition of the chosen basepoint.

LEMMA 2.1. *Let $x_0, y_0 \in X$. Assume, that $\alpha : X \to \mathsf{R}$ satisfies the* log-*Hölder decay condition with basepoint x_0. Then it satisfies the* log-*Hölder decay condition with basepoint y_0 possibly with a different constant.*

PROOF. Recall that by A we denote the constant from the quasitriangle inequality and let $x \in X$. We first observe that

$$\frac{e + d(x, y_0)}{e + d(x, x_0)} \leqslant \frac{e + A(d(x, x_0) + d(x_0, y_0))}{e + d(x, x_0)} \leqslant A + A\, d(x_0, y_0).$$

Set $m := 1 + \log(A + A\, d(x_0, y_0))$ and take logarithms in the above inequality. Then we obtain $\log(e + d(x, y_0)) \leqslant m - 1 + \log(e + d(x, x_0)) \leqslant m \log(e + d(x, x_0))$. Hence

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{c(x_0)}{\log(e + d(x, x_0))} \leqslant \frac{m c(x_0)}{\log(e + d(x, y_0))}.$$

The next lemma is an generalization of [11, Lemma 3.6] in that we do not require the continuity of the exponent function that was used in the cited reference.

LEMMA 2.2. *Let $\alpha \in L^\infty(X)$. The following conditions are equivalent:*

(1) *For all balls $B \subset X$ we have $\mu(B)^{\alpha_B^- - \alpha_B^+} \leqslant c$.*
(2) *For all balls $B \subset X$ and all $x \in B$ we have $\mu(B)^{\alpha_B^- - \alpha(x)} \leqslant c$.*
(3) *For all balls $B \subset X$ and all $x \in B$ we have $\mu(B)^{\alpha(x) - \alpha_B^+} \leqslant c$.*
(4) *For all balls $B \subset X$ and all $x, y \in B$ we have $\mu(B)^{-|\alpha(x) - \alpha(y)|} \leqslant c$.*

PROOF. In all cases the inequality holds (with $c = 1$) when $\mu(B) > 1$. So we consider only $\mu(B) \leqslant 1$. Assume that (2) holds. Choose points $x_i \in B$ such that $\lim \alpha(x_i) = \alpha_B^+$. Then

$$\mu(B)^{\alpha_B^- - \alpha_B^+} = \mu(B)^{\alpha_B^- - \lim \alpha(x_i)} = \lim \mu(B)^{\alpha_B^- - \alpha(x_i)} \leqslant c,$$

which is to say that (1) holds. For the opposite implication we note that $\mu(B)^{\alpha_B^- - \alpha(x)} \leqslant \mu(B)^{\alpha_B^- - \alpha_B^+}$ since $\mu(B) \leqslant 1$. The equivalence of (1) and (3), and (1) and (4) are proved similarly.

Many results below are stated for variable exponents p which are defined on the whole space X. However, sometimes initially the variable exponent is only given on a subset $Y \subset X$, i.e. $q \in \mathcal{P}_d^{\log}(Y)$. The following result ensures that such a variable exponent q can always be extended to X without changing its fundamental properties. The proof is the same as in the Euclidean case, cf. [6, Proposition 4.1.7, p. 102].

LEMMA 2.3. *Let $Y \subset X$ be metric spaces. Then $p \in \mathcal{P}_d^{\log}(Y)$ has an extension $q \in \mathcal{P}_d^{\log}(X)$ with $c_{\log}(q) = c_{\log}(p)$, $q^- = p^-$, and $q^+ = p^+$. If Y is unbounded, then additionally $q_\infty = p_\infty$.*

We can extend this to the quasimetric space case with some additional arguments:

PROPOSITION 2.4. *Let $Y \subset X$ be quasimetric spaces. Then $p \in \mathcal{P}_d^{\log}(Y)$ has an extension $q \in \mathcal{P}_d^{\log}(X)$ with $c_{\log}(q) \leqslant c_A c_{\log}(p)$, $q^- = p^-$, and $q^+ = p^+$. If Y is unbounded, then additionally $q_\infty = p_\infty$.*

PROOF. As mentioned in the introduction, there exists ε such that d^ε is bilipschitz equivalent to a metric d'. From the inequalities

$$\log(e + t) \approx \log(e + Ct) \approx \log(e + t^\varepsilon),$$

with $C > 0$ a fixed constant, we conclude that $\mathcal{P}_d^{\log}(Y) = \mathcal{P}_{d'}^{\log}(Y)$. By Lemma 2.3 we get an extension $q \in \mathcal{P}_{d'}^{\log}(X)$. Finally, by the same reasoning $\mathcal{P}_d^{\log}(X) = \mathcal{P}_{d'}^{\log}(X)$, which concludes the proof.

In the definition of $\mathcal{P}_\mu^{\log}$ we did not require a priori the exponent to be continuous. In fact, in certain cases this is implied by the made assumptions, as is clarified in the next statement.

LEMMA 2.5. *Assume that for all balls $B \subset X$ we have $\mu(B)^{\alpha_B^- - \alpha_B^+} \leqslant c$. If $\mu(\{x\}) = 0$, then α is essentially continuous at x.*

PROOF. Let us assume that α is not essentially continuous at x, i.e. that

$$\limsup_{r\to 0}(\alpha^+_{B(x,r)} - \alpha^-_{B(x,r)}) > 0.$$

Choose a sequence (r_i) such that $r_i \to 0$ and $\alpha^+_{B(x,r_i)} - \alpha^-_{B(x,r_i)} \to m > 0$. Then

$$\begin{aligned} c &\geqslant \limsup_{r\to 0} \mu(B(x,r))^{\alpha^-_{B(x,r)} - \alpha^+_{B(x,r)}} \\ &\geqslant \lim_{i\to\infty} \mu(B(x,r_i))^{\alpha^-_{B(x,r_i)} - \alpha^+_{B(x,r_i)}} \\ &= \mu(\{x\})^{-m}. \end{aligned}$$

Hence $\mu(\{x\}) > 0$, from which the claim follows by contraposition.

In order to clarify the relationship between the local conditions in the classes $\mathscr{P}^{\log}_d$ and $\mathscr{P}^{\log}_\mu$, we need some more rigid regularity of the measure.

DEFINITION 2.6. Let $Q \in (0,\infty)$. We say that a measure μ is *locally lower Ahlfors Q-regular* if there is a positive constant c such that

$$\mu(B(x,r)) \geqslant cr^Q$$

for all $x \in X$ and all $r \in (0,1)$. A measure μ is *locally upper Ahlfors Q-regular* if there is a positive constant c such that

$$\mu(B(x,r)) \leqslant cr^Q$$

for all $x \in X$ and all $r \in (0,1)$. We call a measure *locally Ahlfors Q-regular* if it is lower and upper Ahlfors Q-regular.

In each case we omit "*locally*" if the inequality holds for $r \in (0,\infty)$.

It is not difficult to show that an Ahlfors Q-regular measure is doubling; in this case we may take $Q = \log_2 C_\mu$. The opposite does not hold. A simple example of a space with doubling measure which is not Ahlfors regular is weighted R^n with $d\mu = |x|^\alpha\,dx$, $\alpha > -n$, cf. Example 3.5 in [1] for details.

The following lemma provides a characterization of local log-Hölder continuity. In particular, it relates the local conditions in $\mathscr{P}^{\log}_d(X)$ and $\mathscr{P}^{\log}_\mu(X)$.

LEMMA 2.7 (Lemma 3.6, [11]). *Let $\alpha \in L^\infty(X)$ and define two conditions:*

(1) *α is locally* log-*Hölder continuous.*

(2) *For all balls $B \subset X$ we have $\mu(B)^{\alpha^-_B - \alpha^+_B} \leqslant c$.*

If μ is locally lower Ahlfors regular, then (1) *implies* (2). *If μ is locally upper Ahlfors regular, then* (2) *implies* (1).

3. Logarithmic Hölder continuity variants

For the rest of the paper we fix a basepoint $x_0 \in X$ and denote $d(x) := d(x, x_0)$.

From the doubling condition we can derive the following *lower mass bound*:

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \geqslant \frac{1}{C_\mu^2}\Big(\frac{r}{R}\Big)^Q,$$

where $0 < r < R$ and $Q = \log_2 C_\mu$.

We now prove Lemma 3.1 and Corollary 3.5 which together with Lemma 2.2 directly give Theorem 1.4.

LEMMA 3.1. *Let $p \in \mathcal{P}_d^{\log}(X)$ and μ be doubling. For all balls $B \subset X$ and all $x, y \in B$,*

$$\mu(B)^{-|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant c.$$

Here c depends only on A, C_μ, $\mu(B(x_0,1))$ and $c_{\log}(p)$.

PROOF. Let $k := 2\lceil \log_2 4A\rceil + 2$, where A is from the quasitriangle inequality, and fix $B_0 := B(x_0, 1)$, $C_1 := \mu(B_0)/C_\mu^k$, and $B := B(x, r)$ and $y \in B$. Let $\mu(B) \geqslant C_1$ and note that $\sup\big|\frac{1}{p(x)} - \frac{1}{p(y)}\big| \leqslant \frac{1}{p^-} \leqslant 1$. Then

$$\left(\frac{1}{\mu(B)}\right)^{|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant \max\left\{1, \frac{1}{C_1}\right\}^{\sup|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant \max\left\{1, \frac{1}{C_1}\right\} =: C(\mu),$$

which is the assertion in this case. So we next assume that $\mu(B) < C_1$.

Assume first that the radius r of B is at least 1. If $d(x,y) \leqslant 1$, then by the quasi-triangle inequality we have $d(x_0, y) \leqslant A\,(d(x_0,x) + d(x,y)) \leqslant A\,(1 + d(x))$. Hence $\frac{1}{r}B \subset A\,(1 + d(x))B_0$. Using the lower mass bound estimate we find that

$$\begin{aligned}\mu(B_0) &\leqslant \mu\big(A(1+d(x))B_0\big) \leqslant cA^Q(1+d(x))^Q\mu\big(\tfrac{1}{r}B\big)\\ &\leqslant cA^Q(1+d(x))^Q\mu(B).\end{aligned}$$

Absorbing $\mu(B_0)$ and A into the constant, we obtain

$$\mu(B)^{-|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant c\,(1+d(x))^{Q|\frac{1}{p(x)}-\frac{1}{p(y)}|}. \tag{3.2}$$

We next want to estimate the right hand side of the previous inequality. For this we need some geometric reasoning. Suppose that $B_0 \cap (2A)B \neq \emptyset$. Let $z \in B_0 \cap (2A)B$ and let $w \in B_0$. Then $d(x,w) \leqslant A(d(x,z) + d(z,w)) \leqslant 2A(1 + Ar) \leqslant 4A^2r$, since $A, r \geqslant 1$. This means that $B_0 \subset (4A^2)B$. By the lower mass bound, $\mu(B) \geqslant \mu((4A^2)B)/C_\mu^k \geqslant \mu(B_0)/C_\mu^k = C_1$, which is a contradiction. Hence $B_0 \cap (2A)B = \emptyset$ and in particular $x_0 \notin (2A)B$. It follows that $d(x) > 2Ar$. By the quasitriangle inequality

$$d(x) \leqslant A(d(x,y) + d(y)) \leqslant A(r + d(y)).$$

Since $d(x) > 2Ar$, we conclude that $d(y) \geqslant r$. Therefore $d(x) \leqslant 2A\,d(y)$, and so

$$(1+d(x))^{|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant (1+d(x))^{|\frac{1}{p(x)}-\frac{1}{p_\infty}|}(1+2A\,d(y))^{|\frac{1}{p(y)}-\frac{1}{p_\infty}|} \leqslant c$$

by the decay condition and thus (3.2) gives the claim in this case.

It remains to consider the case $r < 1$. Using again the lower mass bound estimate we have that

$$\mu(B)^{-|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant c r^{-Q|\frac{1}{p(x)}-\frac{1}{p(y)}|}\mu\big(\tfrac{1}{r}B\big)^{-|\frac{1}{p(x)}-\frac{1}{p(y)}|}.$$

We can argue as in (3.2) for the ball $\frac{1}{r}B$ to show that the last term is bounded by a constant; and by local log-Hölder continuity $r^{-|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant c$. We have proved that

$$\mu(B)^{-|\frac{1}{p(x)}-\frac{1}{p(y)}|} \leqslant c,$$

where x in the center of B and $y \in B$.

We consider then the case $y, z \in B$, not necessarily at the center. If $\mu(B) \geqslant 1$, then the claim holds with $c = 1$. So let us assume that $\mu(B) < 1$, x is the center of B. We obtain

$$\begin{aligned}\mu(B)^{-|\frac{1}{p(z)}-\frac{1}{p(y)}|} &= \mu(B)^{-|\frac{1}{p(z)}-\frac{1}{p(x)}+\frac{1}{p(x)}-\frac{1}{p(y)}|}\\ &\leqslant \mu(B)^{-\left(|\frac{1}{p(z)}-\frac{1}{p(x)}|+|\frac{1}{p(x)}-\frac{1}{p(y)}|\right)}\\ &\leqslant \mu(B)^{-|\frac{1}{p(z)}-\frac{1}{p(x)}|}\mu(B)^{-|\frac{1}{p(y)}-\frac{1}{p(x)}|},\end{aligned}$$

which is bounded by two applications of the first part of the proof.

We now deal with the decay condition. The following is a part of [6, Proposition 4.1.8]. For completeness we provide a proof.

LEMMA 3.3. *Let $p \in \mathcal{P}_d^{\log}(X)$. If $s : X \to [1,\infty]$ is given by $\frac{1}{s} := \big|\frac{1}{p} - \frac{1}{p_\infty}\big|$, then for every $m > 0$ there exists $\gamma \in (0,1)$ only depending on $c_{\log}(p)$ such that*

$$\gamma^{s(x)} \leqslant (e + d(x))^{-m}$$

for all $x \in X$.

PROOF. If $s(y) = \infty$, then $\gamma^\infty = 0$. So we assume that $s(y) < \infty$. Since $p \in \mathcal{P}_d^{\log}(X)$,

$$\frac{1}{s(y)} = \left|\frac{1}{p(y)} - \frac{1}{p_\infty}\right| \leqslant \frac{c_{\log}(p)}{\log(e + d(y))}$$

for all $y \in X$ and thus $s(y) \geqslant \log(e+d(y))/c_{\log}(p)$. If $\gamma := \exp(-mc_{\log}(p))$, then

$$\gamma^{s(y)} \leqslant \gamma^{\log(e+d(y))/c_{\log}(p)} = \exp\bigl(-m\log(e + d(y))\bigr) = (e + d(y))^{-m}.$$

LEMMA 3.4. *Let X be a quasimetric measure space with doubling measure μ. If $m > \log_2 C_\mu$, then $(e + d(\cdot))^{-m} \in L^1(X)$.*

PROOF. Denote $B_k := B(x_0, 2^k)$. We split X into a disjoint union of dyadic annular regions and use the doubling property of μ to estimate

$$\begin{aligned}
&\int_X (e + d(x))^{-m}\, d\mu(x)\\
&\quad= \sum_{k=3}^{\infty} \int_{B_k\setminus B_{k-1}} (e + d(x))^{-m}\, d\mu(x) + \int_{B_2} (e + d(x))^{-m}\, d\mu(x)\\
&\quad\leqslant \sum_{k=3}^{\infty} \int_{B_k\setminus B_{k-1}} (e + 2^{k-1})^{-m}\, d\mu(x) + \frac{1}{e^m} C_\mu^2\, \mu(B_0)\\
&\quad\leqslant \sum_{k=3}^{\infty} C_\mu^k \frac{\mu(B_0)}{(2^{k-1})^m} + C_\mu^2\, \mu(B_0)\\
&\quad= \mu(B_0)\biggl(C_\mu \sum_{k=2}^{\infty}\Bigl(\frac{C_\mu}{2^m}\Bigr)^k + C_\mu^2\biggr).
\end{aligned}$$

The last sum is finite provided that $m > \log_2 C_\mu$.

COROLLARY 3.5. *Let $p \in \mathcal{P}_d^{\log}(X)$ and let μ be doubling. Then $1 \in L^{s(\cdot)}(X)$, where s is as in Lemma 3.3.*

PROOF. By Lemma 3.3 for every $m > 0$ there exists $\gamma \in (0, 1)$ such that $\gamma^{s(x)} \leqslant (e+d(x))^{-m}$, and by Lemma 3.4 the upper bound is integrable, hence $\gamma^{s(\cdot)} \in L^1(X)$, i.e. $\varrho_{s(\cdot)}(\gamma) < \infty$, which by definition means that $1 \in L^{s(\cdot)}(X)$.

4. The boundedness of the maximal operator

To prove the boundedness of the maximal operator we use the following pointwise result. The proof of the proposition is mutatis mutandis the same as in the Euclidean case.

PROPOSITION 4.1. *Let $p \in \mathcal{P}_\mu^{\log}(X)$. Then for any $\gamma > 0$ there exists $\beta \in (0,1)$ depending on C_μ and $c_{\log}(p)$ such that*

$$\big(\beta Mf(x)\big)^{p(x)} \leqslant M\big(|f|^{p(\cdot)}\big)(x) + h(x),$$

for all $f \in L^{p(\cdot)}(X) + L^\infty(X)$ with $\|f\|_{L^{p(\cdot)}(X)+L^\infty(X)} \leqslant 1$ and all $x \in \mathsf{R}^n$, where $h(x) := 2M\big(\gamma^{s(\cdot)}\big)(x)$ and $\frac{1}{s(x)} := \big|\frac{1}{p(x)} - \frac{1}{p_\infty}\big|$.

PROOF. Let $\gamma > 0$; from Theorem A.5 it follows that there exists $\beta > 0$ such that

$$\left(\beta \fint_B |f(y)|\, d\mu(y)\right)^{p(x)} \leqslant \fint_B |f(y)|^{p(y)}\, d\mu(y) + \gamma^{s(x)} + \fint_B \gamma^{s(y)}\, d\mu(y)$$

for $f \in L^{p(\cdot)}(X) + L^\infty(X)$ with $\|f\|_{L^{p(\cdot)}(X)+L^\infty(X)} \leqslant 1$ and all $x \in B$. We take the supremum over all balls $B \subset X$ with $x \in B$ and use that $|g| \leqslant Mg$:

$$\begin{aligned}\big(\beta Mf(x)\big)^{p(x)} &\leqslant M\big(|f|^{p(\cdot)}\big)(x) + \gamma^{s(x)} + M\big(\gamma^{s(\cdot)}\big)(x)\\ &\leqslant M\big(|f|^{p(\cdot)}\big)(x) + h(x).\end{aligned}$$

We need to introduce some further standard concepts. The *weak Lebesgue space* w-L^q with $q \in [1,\infty]$ is defined by the quasinorm

$$\|f\|_{\text{w-}L^q} := \sup_{\lambda>0} \|\lambda \chi_{\{|f|>\lambda\}}\|_q.$$

The quasinorm satisfies the triangle inequality $\|f+g\|_{\text{w-}L^q} \leqslant 2\big(\|f\|_{\text{w-}L^q} + \|g\|_{\text{w-}L^q}\big)$, while the other norm properties remain true. It is known that M is of weak type $(1,1)$ [12, Theorem 2.1, p. 10], i.e. M maps $L^1(X)$ to w-$L^1(X)$. Also, we note that the embedding

$$\text{w-}L^1(X) \cap L^\infty(X) \hookrightarrow L^q(X), \tag{4.2}$$

for $q \in (1,\infty]$ follows from the estimate

$$\|f\|_{L^q(X)}^q = q\int_0^{\|f\|_\infty} t^{q-1}\big|\{|f|>t\}\big|\, dt \leqslant q\|f\|_{\text{w-}L^q(X)} \int_0^{\|f\|_\infty} t^{q-2}\, dt < \infty.$$

We are now ready to prove the main theorem of this section, the boundedness of M when $p \in \mathcal{P}_d^{\log}$.

PROOF OF THEOREM 1.7. Let $q := \frac{p}{p^-}$, so that $q \in \mathcal{P}_\mu^{\log}(X)$ with $q^- = 1$. Let $f \in L^{p(\cdot)}(X)$ with $\|f\|_{p(\cdot)} \leqslant \frac14$, and note that $\|f\|_{L^{q(\cdot)}(X)+L^\infty(X)} \leqslant 1$ by [6, Theorem 3.3.11]. Hence Proposition 4.1 implies that

$$\big(\beta Mf(x)\big)^{q(x)} \leqslant M\big(|f|^{q(\cdot)}\big)(x) + h(x)$$

with $h(x) := 2M\big(\gamma^{s(\cdot)}\big)(x)$. Here, β and γ are as in Proposition 4.1. Thus

$$\begin{aligned}\big(\beta Mf(x)\big)^{p(x)} = \Big(\big(\beta Mf(x)\big)^{q(x)}\Big)^{p^-} &\leqslant \Big(M\big(|f|^{q(\cdot)}\big)(x) + h(x)\Big)^{p^-}\\ &\leqslant c\Big(M\big(|f|^{q(\cdot)}\big)(x)^{p^-} + h(x)^{p^-}\Big).\end{aligned}$$

Integration over X yields

$$\varrho_{p(\cdot)}(\beta Mf) \leqslant c\big(\|M(|f|^{q(\cdot)})\|_{p^-}^{p^-} + \|h\|_{p^-}^{p^-}\big).$$

Since $\gamma^{s(\cdot)} \in L^1(X)$ by assumption, and since M is of weak type (1, 1), we conclude that $h \in$ w-$L^1(X)$. Since $h \in L^\infty(X)$, it follows from (4.2) that $h \in L^{p^-}$. Moreover, $\|f\|_{p(\cdot)} \leqslant 1$ implies that $\big\||f|^{q(\cdot)}\big\|_{p^-} \leqslant 1$. Then we use the boundedness of M on $L^{p^-}(X)$ (cf. (1.6)):

$$\varrho_{p(\cdot)}(\beta Mf) \leqslant \frac{cp^-}{p^- - 1}\Big(\big\||f|^{q(\cdot)}\big\|_{p^-}^{p^-} + \|\gamma^{s(\cdot)}\|_{p^-}^{p^-}\Big) \leqslant \frac{cp^-}{p^- - 1}.$$

If the modular is bounded, then so is the norm, by [6, Lemma 3.2.5], and thus we obtain that $\|Mf\|_{p(\cdot)} \leqslant \frac{c\,p^-}{p^- - 1}$ for $\|f\|_{p(\cdot)} \leqslant \frac{1}{4}$.

For general non-zero f we set $\tilde{f} := f/(4\|f\|_{p(\cdot)})$. Then by the above conclusion, $\|M\tilde{f}\|_{p(\cdot)} \leqslant \frac{cp^-}{p^- - 1}$, from which it follows by the linearity of the norm and the homogeneity of M that $\|Mf\|_{p(\cdot)} \leqslant \frac{4cp^-}{p^- - 1}\|f\|_{p(\cdot)}$.

Appendix A. Point-wise estimates

For completeness we provide the proof of the following results, even though the proofs are identical to those presented in [6, Section 4.2]. Note, however, that we simplified the notation comparing to that used in [6].

LEMMA A.1. *Let $p \in \mathscr{P}_\mu^{\log}(X)$. Then there exists $\beta \in (0, 1)$, which depends only on $c_{\log}(p)$, such that*

$$\left(\beta\left(\frac{\lambda}{\mu(B)}\right)^{1/p_B^-}\right)^{p(x)} \leqslant \frac{\lambda}{\mu(B)},$$

for all $\lambda \in [0, 1]$, any ball $B \subset X$ and any $x \in B$.

PROOF. If $\lambda = 0$, then the claim follows since $0^{1/p_B^-} = 0$ and $0^{p(x)} = 0$. So let us assume in the following that $\lambda > 0$. If $p_B^- = \infty$, then, by continuity of $\frac{1}{p}$, $p(x) = \infty$ for all $x \in X$ and $\varphi_\infty\big(\frac{1}{2}\varphi_\infty^{-1}(\lambda\mu(B)^{-1})\big) = \varphi_\infty\big(\frac{1}{2}\big) = 0$, since $\varphi_\infty^{-1} = \chi_{(0,\infty)}$. (For the definition of φ_∞ see (1.10).) Assume now that $p_B^- < \infty$

and $p(x) < \infty$. Since $p \in \mathscr{P}^{\log}_\mu(X)$, by Lemma 2.2 there exists $\beta \in (0,1)$ such that

$$\beta\mu(B)^{\frac{1}{p(x)}-\frac{1}{p_B^-}} \leqslant 1. \tag{A.2}$$

Now, multiply this by $\mu(B)^{-\frac{1}{p(x)}}$ and raise the result to the power of $p(x)$ to prove the claim for $\lambda = 1$. For the case $0 \leqslant \lambda < 1$ we have

$$\Big(\beta\big(\lambda\mu(B)^{-1}\big)^{1/p_B^-}\Big)^{p(x)} = \lambda^{\frac{p(x)}{p_B^-}}\Big(\beta\mu(B)^{-1/p_B^-}\Big)^{p(x)} \leqslant \lambda\mu(B)^{-1}.$$

It remains to consider the case $p(x) = \infty$ and $p_B^- < \infty$. Now

$$\varphi_\infty\Big(\beta\big(\lambda\,\mu(B)^{-1}\big)^{1/p_B^-}\Big) = 0$$

if $\beta\big(\lambda\,\mu(B)^{-1}\big)^{1/p_B^-} \leqslant 1$. However, by log-Hölder continuity,

$$c \geqslant \mu(B)^{\frac{1}{p(x)}-\frac{1}{p_B^-}} = \mu(B)^{-\frac{1}{p_B^-}} \geqslant \big(\lambda\,\mu(B)^{-1}\big)^{\frac{1}{p_B^-}},$$

so the condition holds with $\beta = 1/c$.

For constant $q \in [1,\infty]$, $f \in L^q(B)$ and a ball $B \subset X$ we have by Jensen's inequality

$$\bigg(\fint_B |f(y)|\,d\mu(y)\bigg)^q \leqslant \fint_B |f(y)|^q\,d\mu(y).$$

We now generalize this to the variable exponent context, which gives an additional error term.

LEMMA A.3. *Let $p \in \mathscr{P}^{\log}_\mu(X)$. Define $q \in \mathscr{P}^{\log}_d(X\times X)$ by*

$$\frac{1}{q(x,y)} := \max\bigg\{\frac{1}{p(x)} - \frac{1}{p(y)}, 0\bigg\}.$$

Then for any $\gamma \in (0,1)$ there exists $\beta \in (0,1)$ only depending on γ and $c_{\log}(p)$ such that

$$\bigg(\beta\fint_B |f(y)|\,d\mu(y)\bigg)^{p(x)} \leqslant \fint_B |f(y)|^{p(y)}\,d\mu(y) + \fint_B \gamma^{q(x,y)}\chi_{\{0<|f(y)|\leqslant 1\}}\,d\mu(y)$$

for every ball $B \subset X$, $x \in B$, and

$$f \in L^{p(\cdot)}(X) + L^\infty(X) \qquad \textit{with} \quad \|f\|_{L^{p(\cdot)}(X)+L^\infty(X)} \leqslant 1.$$

PROOF. The proof of the lemma is similar to the proof Lemma 4.2.2 in [6].

By convexity of $t^{p(y)}$ it suffices to prove the claim separately for $\|f\|_{p(\cdot)} \leqslant 1$ and $\|f\|_\infty \leqslant 1$. Let $B \subset X$ be a ball and $x \in B$.

If $p_B^- = \infty$, then $p(y) = \infty$ for all $y \in B$ and the claim is just Jensen's inequality for the convex function φ_∞. So we assume in the following $p_B^- < \infty$.

Let $\beta \in (0, 1)$ be the constant from Lemma A.1. We can assume that $\beta \leqslant \gamma$. We split f into three parts

$$\begin{aligned}
f_1(y) &:= f(y)\chi_{\{y\in B:|f(y)|>1\}},\\
f_2(y) &:= f(y)\chi_{\{y\in B:0<|f(y)|\leqslant 1, p(y)\leqslant p(x)\}},\\
f_3(y) &:= f(y)\chi_{\{y\in B:0<|f(y)|\leqslant 1, p(y)>p(x)\}}.
\end{aligned}$$

Then $f = f_1+f_2+f_3$ and $|f_j| \leqslant |f|$, so $\varrho_{p(\cdot)}(f_j) \leqslant \varrho_{p(\cdot)}(f) \leqslant 1$, $j = 1, 2, 3$. By convexity of $t^{p(x)}$,

$$\begin{aligned}
\left(\frac{\beta}{3}\fint_B |f(y)|\,d\mu(y)\right)^{p(x)} &\leqslant \frac{1}{3}\sum_{j=1}^{3}\left(\beta\fint_B |f_j(y)|\,d\mu(y)\right)^{p(x)}\\
&=: \frac{1}{3}(I_1 + I_2 + I_3).
\end{aligned}$$

(Note that here $\frac{\beta}{3}$ corresponds to the β in the statement of the result.) So it suffices to consider the functions f_1, f_2, and f_3 separately. We start with f_1. The convexity of $t^{p_B^-}$ and Jensen's inequality imply that

$$I_1 \leqslant \left(\beta\left(\fint_B |f_1(y)|^{p_B^-}\,d\mu(y)\right)^{1/p_B^-}\right)^{p(x)},$$

where we have used that $t^{p(x)}$ and t^{1/p_B^-} are non-decreasing Since $|f_1(y)| > 1$ or $|f_1(y)| = 0$ and $p_B^- \leqslant p(y)$, we have $|f_1(y)|^{p_B^-} \leqslant |f_1(y)|^{p(y)}$ and thus

$$I_1 \leqslant \left(\beta\left(\fint_B |f_1(y)|^{p(y)}\,d\mu(y)\right)^{1/p_B^-}\right)^{p(x)}.$$

If $\|f\|_\infty \leqslant 1$, then $f_1 = 0$ and $I_1 = 0$. If on the other hand $\|f\|_{p(\cdot)} \leqslant 1$, then by the unit ball property $\varrho_{p(\cdot)}(f) \leqslant 1$ and thus $\int_B |f_1(y)|^{p(y)}\,d\mu(y) \leqslant 1$. So by Lemma A.1 it follows with $\lambda = \int_B |f(y)|^{p(y)}\,d\mu(y)$ that

$$I_1 \leqslant \fint_B |f_1(y)|^{p(y)}\,d\mu(y) \leqslant \fint_B |f(y)|^{p(y)}\,d\mu(y).$$

Jensen's inequality implies that

$$I_2 \leqslant \fint_B \beta |f_2(y)|^{p(x)}\, d\mu(y).$$

Since $\beta|f_2(y)| \leqslant |f_2(y)| \leqslant 1$ and $t^{p(x)} \leqslant t^{p(y)}$ for all $t \in [0,1]$ when $p(y) \leqslant p(x)$, we find that

$$\begin{aligned} I_2 &\leqslant \fint_B \big(\beta|f_2(y)|\big)^{p(y)}\, d\mu(y) \leqslant \fint_B \big(|f_2(y)|\big)^{p(y)}\, d\mu(y) \\ &\leqslant \fint_B |f(y)|^{p(y)}\, d\mu(y). \end{aligned}$$

Finally, for I_3 we get with Jensen's inequality

$$I_3 \leqslant \fint_B (\beta|f(y)|)^{p(x)} \chi_{\{y\in B: 0<|f(y)|\leqslant 1, p(y)>p(x)\}}\, d\mu(y).$$

Now, Young's inequality (see e.g. Lemma 3.2.15 in [6]), the definition of $q(x,y)$ and $\beta \leqslant \gamma$ give that

$$\begin{aligned} I_3 &\leqslant \fint_B \left(\left(\beta \frac{|f(y)|}{\gamma} \right)^{p(y)} + \gamma^{q(x,y)} \right) \chi_{\{y\in B: 0<|f(y)|\leqslant 1, p(y)>p(x)\}}\, d\mu(y) \\ &\leqslant \fint_B |f(y)|^{p(y)}\, d\mu(y) + \fint_B \gamma^{q(x,y)} \chi_{\{y\in B: 0<|f(y)|\leqslant 1, p(y)>p(x)\}}\, d\mu(y). \end{aligned}$$

This proves the lemma.

In the case where the limit $\frac{1}{p_\infty} = \lim_{|x|\to\infty} \frac{1}{p(x)}$ exists, it is useful to split the second integral in the previous estimate into two parts by means of the following lemma:

LEMMA A.4. *Let q be as in Lemma A.3 and define $s : X \to [1,\infty]$ by $\frac{1}{s(x)} := \big|\frac{1}{p(x)} - \frac{1}{p_\infty}\big|$. Then*

$$t^{q(x,y)} \leqslant t^{\frac{s(x)}{2}} + t^{\frac{s(y)}{2}}$$

for every $t \in [0,1]$.

PROOF. For $x, y \in X$

$$0 \leqslant \frac{1}{q(x,y)} = \max\left\{ \frac{1}{p(x)} - \frac{1}{p(y)}, 0 \right\} \leqslant \frac{1}{s(x)} + \frac{1}{s(y)}.$$

Thus $q(x,y) \geqslant \min\big\{\frac{s(x)}{2}, \frac{s(y)}{2}\big\}$ and so the claim follows since $t \leqslant 1$.

By combining Lemmas A.3 and A.4, we obtain the following result. Note that here γ corresponds to $\gamma^{1/2}$ from Lemma A.3.

THEOREM A.5. *Let $p \in \mathcal{P}^{\log}_{\mu}(X)$. Then for every $\gamma > 0$ there exists $\beta \in (0, 1)$ depending on the doubling constant and p such that*

$$\left(\beta \fint_B |f(y)|\, d\mu(y)\right)^{p(x)} \leqslant \fint_B |f(y)|^{p(y)}\, d\mu(y) + \fint_B \left(\gamma^{s(x)} + \gamma^{s(y)}\right) \chi_{\{0<|f(y)|\leqslant 1\}}\, d\mu(y)$$

for every ball $B \subset X$, all $x \in B$, and all $f \in L^{p(\cdot)}(X) + L^{\infty}(X)$ with $\|f\|_{L^{p(\cdot)}(X)+L^{\infty}(X)} \leqslant 1$.

REFERENCES

1. Björn, A., and Björn, J., *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts Math. 17, European Math. Soc., Zürich 2011.
2. Christ, M., *Lectures on singular integral operators*, CBMS Reg. Conf. Ser. Math. 77, Amer. Math. Soc., Providence 1990.
3. Coifman, R., and Weiss, G., *Analyse harmonique non-commutative sur certains espaces homogénes. Étude de certaines intégrales singuliéres*, Lect. Notes Math. 242, Springer, Berlin 1971.
4. Cruz-Uribe, D., Fiorenza, A., and Neugebauer, C. J., *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. 28 (2003), 223–238, 29 (2004), 247–249.
5. Diening, L., *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. 7 (2004), 245–253.
6. Diening, L., Harjulehto, P., Hästö, P., and Růžička, M., *Lebesgue and Sobolev spaces with variable exponents*, Lect. Notes Math. 2017, Springer, Berlin 2011.
7. Futamura, T., Mizuta, Y., and Shimomura, T., *Sobolev embeddings for variable exponent Riesz potentials on metric spaces*, Ann. Acad. Sci. Fenn. Math. 31 (2006), 495–522.
8. Gorosito, O., Pradolini, G., and Salinas, O., *Boundedness of fractional operators in weighted variable exponent spaces with non doubling measures*, Czechoslovak Math. J. 60(135) (2010), 1007–1023.
9. Hajibayov, M. G., and Samko, S., *Generalized potentials in variable exponent Lebesgue spaces on homogeneous spaces*, Math. Nachr. 284 (2011), 53–66.
10. Hajłasz, P., and Koskela, P., *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145 (2000), no. 688.
11. Harjulehto, P., Hästö, P., and Pere, M., *Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator*, Real Anal. Exchange 30 (2004/05), 87–103.
12. Heinonen, J., *Lectures on analysis on metric spaces*, Universitext, Springer, Berlin 2001.
13. Khabazi, M., *The maximal operator in spaces of homogeneous type*, Proc. A. Razmadze Math. Inst. 138 (2005), 17–25.
14. Kokilashvili, V., and Samko, S., *The maximal operator in weighted variable spaces on metric measure spaces*, Proc. A. Razmadze Math. Inst. 144 (2007), 137–144.
15. Kokilashvili, V., Meskhi, A., and Sarwar, M., *Potential operators in variable exponent Lebesgue spaces: two-weight estimates*, J. Inequal. Appl. 2010 Art. ID 329571, 27 pp.

16. Macías, R., and Segovia, C., *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. 33 (1979), 257–270.
17. Mattila, P., *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambr. Stud. Adv. Math. 44, Cambr. Univ. Press, Cambridge 1995.
18. Nekvinda, A., *Hardy-Littlewood maximal operator on* $L^{p(x)}(\mathsf{R}^n)$, Math. Inequal. Appl. 7 (2004), 255–265.
19. Stein, E., *Singular integrals and differentiability properties of functions*, Princeton Math. Ser. 30, Princeton Univ. Press, Princeton 1970.

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