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HOMEOMORPHISMS OF FINITE INNER DISTORTION: COMPOSITION OPERATORS ON ZYGMUND-SOBOLEV AND LORENTZ-SOBOLEV SPACES

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Abstract

Let $p > n-1$ and $\alpha \in \mathsf{R}$ and suppose that $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a homeomorphism in the Zygmund-Sobolev space $WL^p \log^\alpha L_{\text{loc}}(\Omega,\mathsf{R}^n)$. Define $r=\frac{p}{p-n+1}$. Assume that $u\in WL^r \log^{-\alpha(r-1)} L_{\text{loc}}(\Omega)$. Then $u \circ f^{-1} \in \text{BV}_{\text{loc}}(\Omega')$. We obtain a similar result whenever f is a homeomorphism in the Lorentz-Sobolev space $WL^{p,q}_{\text{loc}}(\Omega, \mathsf{R}^n)$ with $p, q > n-1$ and $u \in WL^{r,s}_{\text{loc}}(\Omega)$ with r as before and $s = \frac{q}{q-n+1}$. Moreover, if we further assume that f has finite inner distortion we obtain in both cases $u \circ f^{-1} \in W^{1,1}_{\text{loc}}(\Omega')$.

1. Introduction

Let Ω and Ω' be bounded open subsets of R^n, $n \geq 2$ and let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a Sobolev homeomorphism of the class $W^{1,p}_{\text{loc}}(\Omega, \mathsf{R}^n)$ with $p \geq 1$.

To each such homeomorphism we associate the composition operator T_f generated by f and defined by the rule $T_f(u) = u \circ f$, for each measurable function $u : \Omega' \to \mathsf{R}$. It is well known (see [38]) that if $f : \Omega \to \mathsf{R}^n$ is a bi-Lipschitz map, then $T_f(u) \in W^{1,p}_{\text{loc}}(\Omega)$ for any function $u \in W^{1,p}_{\text{loc}}(\Omega')$ with $p \geq 1$. Moreover, T_f maps $W^{1,n}_{\text{loc}}(\Omega')$ in $W^{1,n}_{\text{loc}}(\Omega)$ if $f : \Omega \to \Omega'$ is a quasiconformal mapping, see [5], [3]. It is worth pointing out that the Sobolev space $W^{1,n}$ is not the only one which is stable under quasiconformal changes of variables. Indeed, quasiconformal mappings (and their suitable generalizations) turn to be the class of homeomorphisms for which the composition operator acts continously between spaces of functions of bounded mean oscillation [4], [11], [35], spaces of exponentially integrable functions [9], [10], logarithmic Orlicz-Sobolev spaces [22], fractional Sobolev spaces [24], [32], spaces of functions which are absolutely continuous [21]. More than that, the study of composition operators between Sobolev spaces seems to have a connection with the problem of the regularity of the inverse of a Sobolev homemorphism considered for instance in [7], [13], [25], [33]. Actually, in the previous cases

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the role of f and its inverse f^{-1} can be interchanged, but in general this is not possible (see e.g. [23], [31]).

We recall that if $f : \Omega \to \Omega'$ is a homeomorphism of the class $W^{1,p}_{\mathrm{loc}}$ with $n-1 \le p < n$, the inverse is only differentiable in a weak sense, namely it has bounded variation, $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$ (see [7], [8]). Moreover if f has finite inner distortion, then $f^{-1} \in W^{1,1}_{\mathrm{loc}}$ and has finite outer distortion (see [13]).

The goal of this paper is to find conditions on f under which $u \circ f^{-1}$ has bounded variation or has derivative in a Zygmund space.

To this aim, we denote by $WL^p \log^\alpha L_{\mathrm{loc}}(\Omega)$ the space of functions in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with weak derivatives in the Zygmund space $L^p \log^\alpha L_{\mathrm{loc}}(\Omega)$ (see Preliminaries). Our first result is the following

THEOREM 1.1. *Let $p > n-1$ and $\alpha \in \mathsf{R}$. Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism and let $f \in WL^p \log^\alpha L_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$. Assume that $u \in WL^r \log^{-\alpha(r-1)} L_{\mathrm{loc}}(\Omega)$ where*

$$r = \frac{p}{p-n+1},$$

and we further assume that u is continuous for $r > n$ or $r = n$ and $\alpha < -1$. Then $u \circ f^{-1} \in \mathrm{BV}_{\mathrm{loc}}(\Omega')$. Moreover, for every $E' \subset\subset \Omega'$ we have

$$|\nabla(u \circ f^{-1})|(E') \le C \|Df\|^{n-1}_{L^p \log^\alpha L(f^{-1}(E'))} \|\nabla u\|_{L^r \log^{-\alpha(r-1)} L(f^{-1}(E'))}, \tag{1.1}$$

for some constant C depending only on n, p, α.

As it is well known, for $r > n$ there is a continuous representative of u. This is true also if $r = n$ and $\alpha < -1$, so that $\beta = -\alpha(r-1) > n-1$, see [29]. Indeed, this can also be deduced easily using Hölder's inequality in Zygmund spaces (2.3).

Our next result shows that $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(\Omega')$ if one assumes the additional assumption that f has finite inner distortion.

We say that the homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ has finite inner distortion if its Jacobian $J_f \in L^1_{\mathrm{loc}}(\Omega)$, $J_f \ge 0$ a.e. and

$$J_f(x) = 0 \implies |\mathrm{adj}\, Df(x)| = 0 \qquad \text{a.e.},$$

where $\mathrm{adj}\, Df$ is the adjugate of the differential matrix Df of f.

Our result reads as follows.

THEOREM 1.2. *Let $p > n-1$ and $\alpha \in \mathsf{R}$. Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism of finite inner distortion and let $f \in WL^p \log^\alpha L_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$. Assume that $u \in WL^r \log^{-\alpha(r-1)} L_{\mathrm{loc}}(\Omega)$ where*

$$r = \frac{p}{p-n+1},$$

and we further assume that u is continuous for $r > n$ or $r = n$ and $\alpha < -1$. Then $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(\Omega')$. Moreover, for every $E' \subset\subset \Omega'$ we have

$$\|\nabla(u \circ f^{-1})\|_{L^1(E')} \leq C\|Df\|^{n-1}_{L^p \log^\alpha L(f^{-1}(E'))}\|\nabla u\|_{L^r \log^{-\alpha(r-1)} L(f^{-1}(E'))}, \tag{1.2}$$

for some constant C depending only on n, p, α.

Our next result may be regarded as a limit case of Theorem 1.2 ($p = n-1$) and features the space of functions with gradient in the space $\exp_{\frac{1}{\alpha}}(\Omega)$, that is the closure in the Orlicz space $\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega)$ of functions which are bounded on Ω (see [6] and Section 2.1 below).

THEOREM 1.3. *Let $\alpha > 0$. Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism of finite inner distortion and let $f \in WL^{n-1}\log^\alpha L_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Assume that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $|\nabla u| \in \exp_{\frac{1}{\alpha}}(\Omega)$. Then $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(\Omega')$. Moreover, for every $E' \subset\subset \Omega'$ we have*

$$\|\nabla(u \circ f^{-1})\|_{L^1(E')} \leq C\|Df\|^{n-1}_{L^{n-1}\log^\alpha L(f^{-1}(E'))}\|\nabla u\|_{\mathrm{Exp}_{\frac{1}{\alpha}}(f^{-1}(E'))}, \tag{1.3}$$

for some constant C depending only on n, α.

We also prove that the analogous of Theorem 1.1 and Theorem 1.2 hold in the framework of Lorentz spaces (see Section 3 and 4).

Observe that the composition is weakly differentiable, i.e. $u \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(\Omega')$, as long as we assume that f has finite inner distortion. Our results generalize some already known facts in two directions. First, in the previous papers [18], [19] the composition results are obtained under the assumption that f has finite outer distortion, while we only require that f has finite inner distortion. On the other hand, our setting is more general and recover previous results when $\alpha = 0$ (see [15]).

The paper is organized as follows. Section 2 is devoted to the preliminary results and notation. The proof of Theorem 1.1 will be given in Section 3. The proofs of Theorem 1.2 and Theorem 1.3 will be given in Section 4.

We remark that, under all the above assumptions, one cannot expect that $u \circ f^{-1} \in W^{1,1+\delta}_{\mathrm{loc}}(\Omega)$ for some $\delta > 0$ (see Remark 4.1 and Remark 4.2 below). However, in Section 4 we also prove Theorem 4.2 and Theorem 4.4 where a better regularity for $u \circ f^{-1}$ is obtained under suitable integrability assumptions on the inner distortion function.

2. Preliminaries

2.1. Orlicz Spaces

We need to recall some basic properties of Orlicz spaces; for more details we refer to [1].

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function, that is $\Phi(0) = 0$, Φ is increasing and convex. If Ω is a open subset of R^n, we define the Orlicz space $L^\Phi(\Omega)$ generated by the Young function Φ as the set of measurable functions $u : \Omega \to \mathsf{R}$ such that

$$\int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < \infty,$$

for some $\lambda > 0$. This space is equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \le 1\right\}.$$

We define the space $WL^\Phi(\Omega)$ as the set

$$WL^\Phi(\Omega) = \{u \in W^{1,1}(\Omega) : |\nabla u| \in L^\Phi(\Omega)\}.$$

For further developments, we shall need to recall that a Young function Φ is said to satisfy the Δ'-condition if

$$\Phi(ab) \le C\Phi(a)\Phi(b) \qquad \text{for every} \quad a, b \ge 0,$$

for some constant $C > 0$.

Moreover, we say that $\Phi(t)$ satisfies the Δ_2 condition if

$$\Phi(2t) \le C\Phi(t) \qquad \text{for every} \quad t \ge t_0 \ge 0,$$

for some constant $C > 0$. The Zygmund space $L^p \log^\alpha L(\Omega)$, for $1 \le p < \infty$, $\alpha \in \mathsf{R}$ ($\alpha \ge 0$ for $p = 1$), is defined as the Orlicz space $L^\Phi(\Omega)$ when the Young function Φ is given by

$$\Phi(t) \simeq t^p \log^\alpha(e + t) \qquad \text{for every} \quad t \ge t_0 \ge 0. \tag{2.1}$$

Therefore, a measurable function u on Ω belongs to $L^p \log^\alpha L(\Omega)$ if

$$\int_\Omega |u|^p \log^\alpha(e + |u|)\, dx < \infty.$$

From the elementary inequalities

$$\log(e + ab) \le \log((e + a)(e + b)) \le 2\log(e + a)\log(e + b) \tag{2.2}$$

for $a, b \geq 0$, we see that, if $\alpha \geq 0$, then $\Phi(t) = t^p \log^\alpha(e+t)$, $t \geq 0$, is a Young function and satisfies the Δ'-condition. Indeed, for every $s, t \geq 1$ we have

$$\frac{s+t}{2} \leq st.$$

Inserting $s = \log(e+a)$ and $t = \log(e+b)$ in the equation above, we get

$$\frac{\log(e+a) + \log(e+b)}{2} \leq \log(e+a)\log(e+b).$$

This readily implies (2.2).

For $\alpha = 0$ we have the ordinary Lebesgue spaces. We will need to use the following Hölder type inequality for Zygmund spaces

$$\|u_1 \dots u_k\|_{L^p \log^\alpha L} \leq C\, \|u_1\|_{L^{p_1} \log^{\alpha_1} L} \dots \|u_k\|_{L^{p_k} \log^{\alpha_k} L}, \tag{2.3}$$

where $p_i > 1$, $\alpha_i \in \mathbb{R}$, $u_i \in L^{p_i} \log^{\alpha_i} L$ for $i = 1, \dots, k$, and

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_k}, \qquad \frac{\alpha}{p} = \frac{\alpha_1}{p_1} + \dots + \frac{\alpha_k}{p_k}.$$

The following inclusions hold

$$L^p \log^\beta L(\Omega) \subset L^p(\Omega) \subset L^p \log^\alpha L(\Omega)$$

with continuous embeddings if $\alpha < 0 < \beta$. For $\alpha > 0$, the dual Orlicz space to $L \log^\alpha L(\Omega)$ is the space $\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega)$ generated by a function $\Psi(t) \simeq \exp\big(t^{\frac{1}{\alpha}}\big) - 1$ for $t \geq t_0 \geq 0$. For more details see [27, Section 4.12], [16] and [17].

As $\Psi(t)$ does not have Δ_2-property, then $L^\infty(\Omega)$ is not dense in $\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega)$. Then we will denote by $\exp_{\frac{1}{\alpha}}(\Omega)$ the closure in $\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega)$ of the space of functions u which are bounded on Ω and have bounded support in $\bar{\Omega}$ (see [6]). We recall that $C_0^\infty(\Omega)$ is dense in $\exp_{\frac{1}{\alpha}}(\Omega)$ (see [1], Theorem 8.20).

We define the space $WL^p \log^\alpha L(\Omega)$ as the set

$$WL^p \log^\alpha L(\Omega) = \big\{u \in W^{1,1}(\Omega) : |\nabla u| \in L^p \log^\alpha L(\Omega)\big\}.$$

2.2. Lorentz Spaces

Our main source here will be [38]. Let Ω be a bounded domain in $\mathbb{R}^n$ and $g : \Omega \to \mathbb{R}$ be a measurable function.

For $1 \leq p < \infty$, $0 < q < \infty$ the Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions g defined on Ω such that

$$\|g\|^q_{L^{p,q}(\Omega)} = p \int_0^\infty \big|\{x \in \Omega : |g(x)| > t\}\big|^{\frac{q}{p}} t^{q-1}\, dt < \infty.$$

For $p > 1$ and $q \geq 1$, $\|\cdot\|_{L^{p,q}}$ is equivalent to a norm under which $L^{p,q}$ is a Banach space. For $p = q$, the space $L^{p,p}$ coincides with the usual L^p space and if $1 < q < r < \infty$, the following inclusions hold

$$L^{p,1} \subset L^{p,q} \subset L^{p,r}.$$

For $q = \infty$, the class $L^{p,\infty}$ is equivalent to the Marcinkiewicz space weak-L^p and consists of all functions g defined on Ω such that

$$\|g\|_{p,\infty}^p = \sup_{t>0} t^p \big|\{x \in \Omega : |g(x)| > t\}\big| < \infty.$$

A useful property of the Lorentz norm is given by the following identities

$$\||g|^\alpha\|_{L^{p,\infty}}^p = \|g\|_{L^{\alpha p,\infty}}^{\alpha p} \qquad \text{and} \qquad \||g|^\alpha\|_{L^{p,q}}^p = \|g\|_{L^{\alpha p,\alpha q}}^{\alpha p} \tag{2.4}$$

for $\alpha > 0$.

For $1 < r < p$, $0 < q < \infty$

$$L^{p,q} \subset L^{p,\infty} \subset L^r.$$

The Hölder-type inequality for Lorentz-space

$$\|u_1 \ldots u_k\|_{L^1} \leq \|u_1\|_{L^{p_1,q_1}} \ldots \|u_k\|_{L^{p_k,q_k}}, \tag{2.5}$$

holds if $1 < p_i < \infty$, $1 \leq q_i \leq \infty$, $u_i \in L^{p_i,q_i}$ for $i = 1, \ldots, k$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1, \qquad \frac{1}{q_1} + \cdots + \frac{1}{q_k} = 1.$$

For a proof see [14].

We define the space $WL^{p,q}(\Omega)$ as the set

$$WL^{p,q}(\Omega) = \big\{u \in W^{1,1}(\Omega) : |\nabla u| \in L^{p,q}(\Omega)\big\}.$$

2.3. *Homeomorphism of finite distortion*

A homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is said to be a *bi-Sobolev map* if f belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ and its inverse f^{-1} belongs to $W^{1,1}_{\mathrm{loc}}(\Omega', \mathsf{R}^n)$. More specifically, if $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ and $f^{-1} \in W^{1,p}_{\mathrm{loc}}(\Omega', \mathsf{R}^n)$, $1 \leq p < \infty$, then we say that f is $W^{1,p}$-bi-Sobolev.

Bi-Sobolev maps have a close connection with the homeomorphisms with finite distortion. Recall that a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ has finite outer distortion if $J_f \in L^1_{\mathrm{loc}}(\Omega)$, $J_f \geq 0$ a.e. and

$$J_f(x) = 0 \implies |Df(x)| = 0 \qquad \text{a.e.}$$

Here and in what follows, $|A|$ denotes the operator norm of the $n \times n$ matrix A defined as $|A| = \sup\{|A\xi| : \xi \in \mathsf{R}^n, |\xi| = 1\}$.

The outer distortion function is defined as

$$K_{O,f}(x) = \begin{cases} \dfrac{|Df(x)|^n}{J_f(x)} & \text{for } J_f(x) > 0 \\ 1 & \text{otherwise.} \end{cases}$$

It is immediate to check that the inner and the outer distortion functions coincide in the planar case, i.e. for $n = 2$, while for $n > 2$ they are related by the inequality

$$K_{I,f}(x) \le K_{O,f}^{n-1}(x),$$

which follows from the classical Hadamard's inequality. On the other hand, the reverse estimate

$$K_{O,f}(x) \le K_{I,f}^{n-1}(x)$$

holds if $J_f(x) > 0$.

Here, we denote with $K_{I,f}$ the inner distortion function, namely

$$K_{I,f}(x) = \begin{cases} \dfrac{|\operatorname{adj} Df(x)|^n}{J_f(x)^{n-1}} & \text{for } J_f(x) > 0 \\ 1 & \text{otherwise.} \end{cases}$$

In the special case $K_{O,f} \in L^\infty(\Omega)$ with $K_{O,f}(x) \le K$ for a.e. $x \in \Omega$, we say that f is a K-quasiconformal mapping.

The deep connection between bi-Sobolev mappings and mappings with finite distortion is given in [26]. Finally we recall the following result that will be very useful later.

THEOREM 2.1 ([13]). *Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism such that $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathsf{R}^n)$ and*

$$|\operatorname{adj} Df(x)|^n \le K(x) J_f(x)^{n-1} \qquad \textit{for a.e.} \quad x \in \Omega,$$

for some Borel function $K : \Omega \to [1, \infty)$. Then, f^{-1} is a $W^{1,1}_{\text{loc}}(\Omega', \mathsf{R}^n)$ mapping of finite distortion. Moreover,

$$|Df^{-1}(y)|^n \le K(f^{-1}(y)) J_{f^{-1}}(y) \qquad \textit{for a.e.} \quad y \in \Omega',$$

and

$$\int_{\Omega'} |Df^{-1}(y)|\, dy = \int_\Omega |\operatorname{adj} Df(x)|\, dx.$$

A homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ satisfies the Lusin (N) condition if the implication

$$|E| = 0 \implies |f(E)| = 0,$$

holds for any measurable set $E \subset \Omega$. In [36] it is proved that each homeomorphism $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ satisfies the Lusin (N) condition, while may fail if $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ for $1 \le p < n$ (see [34]).

For a homeomorphism f in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$, it is well known the following inequality

$$\int_B \eta(f(x))\,|J_f(x)|\,dx \le \int_{f(B)} \eta(y)\,dy$$

where η is a nonnegative Borel measurable function on R^n and $B \subset \Omega$ is a Borel set (see Theorem 3.1.8 in [12]). We say that the area formula holds for the homeomorphism f in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ if the equality

$$\int_B \eta(f(x))\,|J_f(x)|\,dx = \int_{f(B)} \eta(y)\,dy \tag{2.6}$$

is verified. If f is a homeomorphism that satisfies the Lusin (N) condition on B, then the area formula holds.

REMARK 2.1 (Validity of the area formula). We recall that if f is a homeomorphism in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ and $\mathscr{D}_f$ is the set of points in Ω where f is differentiable, then the area formula (2.6) holds in $\mathscr{D}_f$. Equality (2.6) is proved by covering $\mathscr{D}_f$ with a countable family of measurable sets such that the restriction of f to each member of the family is a Lipschitz map [12, Theorem 3.1.8] and by applying the usual area formula for Lipschitz maps. In particular, if $\mathscr{Z}_f$ is the subset of Ω where f is differentiable and $J_f(x) = 0$, we have that $|f(\mathscr{Z}_f)| = 0$. This can be viewed as a *weak version of the classical Sard's theorem*.

3. On the weak differentiability of the composition

A function $h \in L^1(\Omega)$ is of bounded variation, $h \in \mathrm{BV}(\Omega)$, if the distributional partial derivatives of h are measures with finite total variation in Ω, that is there exists Radon signed measures $D_1 h, \dots, D_n h$ in Ω such that for $i = 1, \dots, n$, $|D_i h|(\Omega) < \infty$ and

$$\int_\Omega h D_i \varphi\,dx = -\int_\Omega \varphi\,dD_i h(x),$$

for all $\varphi \in C_0^1(\Omega)$. The gradient of h is a vector-valued measure with finite total variation

$$|\nabla h|(\Omega) = \sup\left\{ \int_\Omega h \operatorname{div} \varphi \, dx : \varphi \in C_0^1(\Omega, \mathsf{R}^n), \, \|\varphi\|_\infty \leq 1 \right\} < \infty.$$

We say that $f \in L^1(\Omega, \mathsf{R}^n)$ is mapping of bounded variation, $f \in \mathrm{BV}(\Omega, \mathsf{R}^n)$, if the coordinate functions of f belong to the space $\mathrm{BV}(\Omega)$.

From now on we assume that Ω, Ω' are bounded domains of R^n. The aim of this section is to find conditions under which the composition $u \circ f^{-1}$ belongs to BV. In the setting of Zygmund spaces we are able to prove the following result, which was known before for $\alpha = 0$ (see [19]).

Proof of Theorem 1.1. Arguing as in the proof of Theorem 1.1 in [19], for every ball $B \subset\subset \Omega'$, we obtain

$$|\nabla(w \circ f^{-1})|(B) \leq C \int_{f^{-1}(B)} |\nabla w||Df|^{n-1} \, dx \tag{3.1}$$

for any smooth function w defined in Ω. Using Hölder inequality in Zygmund spaces (2.3), we deduce from (3.1) that

$$|\nabla(w \circ f^{-1})|(B) \leq C \, \|\nabla w\|_{L^r \log^{-\alpha(r-1)} L(f^{-1}(B))} \, \|Df\|^{n-1}_{L^p \log^\alpha L(f^{-1}(B))}. \tag{3.2}$$

Let now u be an arbitrary function in $WL^r \log L^{-\alpha(r-1)}_{\mathrm{loc}}(\Omega)$ and let $\{u_j\}$ be a sequence of smooth functions which approximate u by standard mollification. We take two indices i, j and we apply (3.2) to $w = u_i - u_j$. We see that $\{\nabla(u_j \circ f^{-1})\}$ is a Cauchy sequence in the space of Radon measures. Hence, $u_j \circ f^{-1}$ forms a Cauchy sequence in $L^1(B)$.

If $n - 1 < p < n$ and thus $r > n$, we assume that $u \in C(\Omega)$. By standard approximation u_j converges to u locally uniformly and then $u_j \circ f^{-1}$ converges to $u \circ f^{-1}$ locally uniformly. Since $u_j \circ f^{-1}$ is a Cauchy sequence in $\mathrm{BV}(B)$ then $u \circ f^{-1} \in \mathrm{BV}(B)$ (see [2]). In the same way we can argue when $p = r = n$ and $\alpha < -1$ (see [29]).

If $p \geq n$ and $\alpha \geq -1$ then f satisfies the Lusin (N) condition (see [30]) and hence $u \circ f^{-1}$ does not depend on the representative of u. Since u_j converges to u in $L^1(f^{-1}(B))$, we may assume, up to a non-relabeled subsequence, that u_j converges to u a.e. in $f^{-1}(B)$. It follows that $u_j \circ f^{-1}$ converges to $u \circ f^{-1}$ a.e. in B. Since $u_j \circ f^{-1}$ is a Cauchy sequence in $\mathrm{BV}(B)$ then $u_j \circ f^{-1}$ converges to $u \circ f^{-1}$ in $\mathrm{BV}(B)$.

We apply (3.2) to $w = u_j$ and we obtain

$$|\nabla(u_j \circ f^{-1})|(B) \leq C \, \|\nabla u_j\|_{L^r \log^{-\alpha(r-1)} L(f^{-1}(B))} \, \|Df\|^{n-1}_{L^p \log^\alpha L(f^{-1}(B))}. \tag{3.3}$$

By lower semicontinuity and passing to liminf as $j \to \infty$ in (3.3), we obtain (1.1). This ends the proof.

In the setting of Lorentz spaces we are able to prove the following.

THEOREM 3.1. *Let $n-1 < p < \infty$ and $n-1 < q < \infty$, $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism and let $f \in WL^{p,q}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$. Assume that $u \in WL^{r,s}_{\mathrm{loc}}(\Omega)$ where*

$$r = \frac{p}{p-n+1} \qquad \textit{and} \qquad s = \frac{q}{q-n+1},$$

and we further assume that u is continuous for $r > n$. Then $u \circ f^{-1} \in$ $\mathrm{BV}_{\mathrm{loc}}(\Omega')$. Moreover, for every $E' \subset\subset \Omega'$ we have

$$|\nabla(u \circ f^{-1})|(E') \leq C \|Df\|^{n-1}_{L^{p,q}(f^{-1}(E'))} \|\nabla u\|_{L^{r,s}(f^{-1}(E'))}, \tag{3.4}$$

for some constant C depending only on n, p, q.

For $r > n$ it is well known that $u \in WL^{r,s}_{\mathrm{loc}}(\Omega)$ admits a continuous representative since $\nabla u \in L^{r,s}$ implies $\nabla u \in L^{\gamma}$ for $\gamma \in (n, r)$. The choice of u continuous avoids problems in defining $u \circ f^{-1}$.

PROOF OF THEOREM 3.1. Arguing as in the proof of Theorem 1.1 in [19], for every ball $B \subset\subset \Omega'$, we obtain

$$|\nabla(w \circ f^{-1})|(B) \leq C \int_{f^{-1}(B)} |\nabla w||Df|^{n-1}\, dx, \tag{3.5}$$

for any smooth function w defined in Ω. Using Hölder inequality in Lorentz spaces (2.5), we deduce from (3.5) that

$$|\nabla(w \circ f^{-1})|(B) \leq C\, \|\nabla w\|_{L^{r,s}(f^{-1}(B))}\, \|Df\|^{n-1}_{L^{p,q}(f^{-1}(B))}. \tag{3.6}$$

Let now u be an arbitrary function in $WL^{r,s}_{\mathrm{loc}}(\Omega)$ and let $\{u_j\}$ be a sequence of smooth functions which approximate u by standard mollification. We take two indices i, j and we apply (3.6) to $w = u_i - u_j$. We see that $\{\nabla(u_j \circ f^{-1})\}$ is a Cauchy sequence in the space of Radon measures. Hence, $u_j \circ f^{-1}$ forms a Cauchy sequence in $L^1(B)$.

If $n-1 < p < n$ and thus $r > n$, we assume that $u \in C(\Omega)$. By standard approximation u_j converges to u locally uniformly and then $u_j \circ f^{-1}$ converges to $u \circ f^{-1}$ locally uniformly. Since $u_j \circ f^{-1}$ is a Cauchy sequence in $\mathrm{BV}(B)$ then $u \circ f^{-1} \in \mathrm{BV}(B)$ (see [2]).

If $p \geq n$ and thus $r \leq n$, we make use of the following inclusion (see [17])

$$L^{p,q}_{\mathrm{loc}}(\Omega) \subset L^p \log^{-1} L_{\mathrm{loc}}(\Omega),$$

and we see that $|Df| \in L^{p,q}_{\mathrm{loc}}(\Omega)$ implies $|Df| \in L^p \log^{-1} L_{\mathrm{loc}}(\Omega)$. Then, f satisfies the Lusin (N) condition (see [28]) and hence $u \circ f^{-1}$ does not depend on the representative of u. Since u_j converges to u in $L^1(f^{-1}(B))$, we may assume, up to a non-relabeled subsequence, that u_j converges to u a.e. in $f^{-1}(B)$. It follows that $u_j \circ f^{-1}$ converges to $u \circ f^{-1}$ a.e. in B. Since $u_j \circ f^{-1}$ is a Cauchy sequence in $\mathrm{BV}(B)$ then $u_j \circ f^{-1}$ converges to $u \circ f^{-1}$ in $\mathrm{BV}(B)$.

We apply (3.6) to $w = u_j$ and we obtain

$$|\nabla(u_j \circ f^{-1})|(B) \leq C \, \|\nabla u_j\|_{L^r \log^{-\alpha(r-1)} L(f^{-1}(B))} \, \|Df\|^{n-1}_{L^p \log^{\alpha} L(f^{-1}(B))}. \tag{3.7}$$

By lower semicontinuity and passing to liminf as $j \to \infty$ in (3.7), we obtain (3.4). This ends the proof.

4. On the weak differentiability and the regularity of the composition

4.1. The composition operator in Zygmund spaces

We are in a position to prove Theorem 1.3.

PROOF OF THEOREM 1.3. The homeomorphism f satisfies the assumptions of Theorem 2.1 and therefore f^{-1} belongs to $W^{1,1}_{\mathrm{loc}}(\Omega', \mathbb{R}^n)$ and has finite outer distortion. Let $w \in C^\infty(\Omega)$, we have $w \circ f^{-1} \in W^{1,1}_{\mathrm{loc}}(\Omega')$ (see Lemma 8.31 in [1]) and the chain rule formula holds

$$\nabla(w \circ f^{-1})(y) = \nabla w(f^{-1}(y)) Df^{-1}(y) \qquad \text{for a.e.} \quad y \in \Omega'.$$

We decompose the domain Ω as follows

$$\Omega = \mathscr{R}_f \cup \mathscr{Z}_f \cup \mathscr{E}_f$$

where

$$\begin{aligned} \mathscr{R}_f &= \{x \in \Omega : f \text{ is differentiable at } x \text{ and } J_f(x) \neq 0\}, \\ \mathscr{Z}_f &= \{x \in \Omega : f \text{ is differentiable at } x \text{ and } J_f(x) = 0\}, \\ \mathscr{E}_f &= \{x \in \Omega : f \text{ is not differentiable at } x\}. \end{aligned}$$

Recall [37] that f is differentiable a.e. in Ω, that is $|\mathscr{E}_f| = 0$. Using the area formula we see that

$$\int_{f(\mathscr{E}_f)} J_{f^{-1}}(y)\, dy \leq |\mathscr{E}_f| = 0.$$

Therefore, $J_{f^{-1}} = 0$ a.e. in $f(\mathscr{E}_f)$. Since f^{-1} is a mapping of finite outer distortion, we deduce that $\nabla(w \circ f^{-1})(y) = 0$ for a.e. $y \in f(\mathscr{E}_f)$. By the weak version of Sard's theorem (see Remark 2.1) $|f(\mathscr{Z}_f)| = 0$ and therefore

$\nabla(w \circ f^{-1})(y) = 0$ for a.e. $y \in \Omega' \setminus f(\mathscr{R}_f)$. Moreover, for all $x \in \mathscr{R}_f$ we have

$$Df^{-1}(f(x)) = [Df(x)]^{-1}, \qquad J_{f^{-1}}(f(x)) = \frac{1}{J_f(x)}.$$

Let $B \subset\subset \Omega'$ be an open ball and $A = B \cap f(\mathscr{R}_f)$. Using the area formula it follows that

$$\begin{aligned}
\int_B |\nabla(w \circ f^{-1})|\, dy &= \int_A |\nabla(w \circ f^{-1})|\, dy \\
&\le \int_A |\nabla w(f^{-1}(y))| \frac{|Df^{-1}(y)|}{J_{f^{-1}}(y)} J_{f^{-1}}(y)\, dy \\
&\le \int_{f^{-1}(A)} |\nabla w(x)| \frac{|Df^{-1}(f(x))|}{J_{f^{-1}}(f(x))}\, dx \qquad (4.1)\\
&= \int_{f^{-1}(A)} |\nabla w(x)||\operatorname{adj} Df(x)|\, dx \\
&\le \int_{f^{-1}(B)} |\nabla w(x)||Df(x)|^{n-1}\, dx.
\end{aligned}$$

Using the elementary inequality

$$\log(e + t^{n-1}) \le C(n) \log(e + t) \qquad \text{for} \quad t \ge 0,$$

we get

$$|Df|^{n-1} \log^{\alpha}(e + |Df|^{n-1}) \le C|Df|^{n-1} \log^{\alpha}(e + |Df|)$$

and we conclude that $|Df|^{n-1} \in L \log^{\alpha}(f^{-1}(B))$ with

$$\| |Df|^{n-1} \|_{L \log^{\alpha} L(f^{-1}(B))} \le C \|Df\|^{n-1}_{L^{n-1} \log^{\alpha} L(f^{-1}(B))}$$

and then, by a duality argument, we deduce from (4.1) that

$$\int_B |\nabla(w \circ f^{-1})|\, dx \le C\, \|\nabla w\|_{\mathrm{Exp}_{\frac{1}{\alpha}}(f^{-1}(B))}\, \|Df\|^{n-1}_{L^{n-1} \log^{\alpha} L(f^{-1}(B))}. \qquad (4.2)$$

Let now u be an arbitrary function in $W^{1,1}(f^{-1}(B))$ such that $|\nabla u| \in \exp_{\frac{1}{\alpha}}(\Omega)$. By Theorem 8.20 in [1], there exists a sequence of standard mollifiers ρ_j such that

$$\lim_{j \to \infty} \|(\nabla u) * \rho_j - \nabla u\|_{\mathrm{Exp}_{\frac{1}{\alpha}}(f^{-1}(B))} = 0. \qquad (4.3)$$

Clearly,

$$\int_{f^{-1}(B)} [(\nabla u) * \rho_j]\varphi \, dx = - \int_{f^{-1}(B)} [u * \rho_j]\nabla\varphi \, dx \tag{4.4}$$

for every test function $\varphi \in C_0^\infty(f^{-1}(B))$. From (4.3) we have

$$\lim_{j\to\infty} \|\nabla(u * \rho_j) - \nabla u\|_{\mathrm{Exp}_{\frac{1}{\alpha}}(f^{-1}(B))} = 0. \tag{4.5}$$

We set $u_j = u * \rho_j$ and we take two indices i, j. We apply (4.2) to $w = u_i - u_j$. We see that $\{\nabla(u_j \circ f^{-1})\}$ is a Cauchy sequence in $L^1(B, \mathsf{R}^n)$ and we can find $g \in L^1(B, \mathsf{R}^n)$ such that $\nabla(u_j \circ f^{-1})$ converges strongly in $L^1(B, \mathsf{R}^n)$ to g. On the other hand, by Sobolev-Poincaré inequality and by (4.2), the sequence $\{u_j \circ f^{-1}\}$ is a Cauchy sequence in $L^1(B)$ and converges to $u \circ f^{-1}$. After passing to a limit, we get

$$\int_B g(y)\psi(y) \, dy = - \int_B (u \circ f^{-1})(y)\nabla\psi(y) \, dy$$

for every test function $\psi \in C_0^\infty(B)$. It follows that $\{\nabla(u_j \circ f^{-1})\}$ converges in $L^1(B)$ to $\nabla(u \circ f^{-1})$. The estimate (1.3) follows from (4.2).

We are in a position to prove Theorem 1.2.

PROOF OF THEOREM 1.2. We start by observing that f satisfies the assumptions of Theorem 2.1 and therefore f^{-1} belongs to $W^{1,1}_{\mathrm{loc}}(\Omega', \mathsf{R}^n)$ and has finite outer distortion. If w is smooth, then arguing as in the proof of Theorem 1.3 (see (4.1)) we get

$$\int_B |\nabla(w \circ f^{-1})| \, dy \le \int_{f^{-1}(B)} |\nabla w(x)||Df(x)|^{n-1} \, dx \tag{4.6}$$

for every ball $B \subset\subset \Omega'$. Thus, using Hölder inequality in Zygmund spaces, by (4.6), we get

$$\int_B |\nabla(w \circ f^{-1})| \, dx \le C \, \|\nabla w\|_{L^r \log^{-\alpha(r-1)} L(f^{-1}(B))} \, \|Df\|^{n-1}_{L^p \log^\alpha L(f^{-1}(B))}. \tag{4.7}$$

Let now u be an arbitrary function in $WL^r \log L^{-\alpha(r-1)}_{\mathrm{loc}}(\Omega)$ and let $\{u_j\}$ be a sequence of smooth functions which approximate u by standard mollification. Given $B \subset\subset \Omega'$ a ball, if $w = u_i - u_j$ we may conclude that $\nabla(u_j \circ f^{-1})$ is a Cauchy sequence in $L^1(B)$ and that converges in $L^1(B)$ to $\nabla(u \circ f^{-1})$. The argument used at the end of Theorem 1.1 leads us to the desired result. The estimate (1.2) follows by (4.7).

Now, we prove the following result.

PROPOSITION 4.1. *Let* $f : \Omega \xrightarrow{\text{onto}} \Omega'$ *be a homeomorphism of finite inner distortion and let* $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathsf{R}^n)$. *Assume that* $u \in W^{1,\infty}_{\text{loc}}(\Omega)$. *Then* $u \circ f^{-1} \in W^{1,1}_{\text{loc}}(\Omega')$. *Moreover, for every* $E' \subset\subset \Omega'$ *we have*

$$\|\nabla(u \circ f^{-1})\|_{L^1(E')} \leq \|Df\|^{n-1}_{L^{n-1}(f^{-1}(E'))}\|\nabla u\|_{L^\infty(f^{-1}(E'))}.$$

PROOF. The proof follows by Theorem 2.1 and by Lemma 8.31 in [1] to the mapping f^{-1}.

Under suitable assumptions on $K_{I,f}$ we obtain a better regularity for $u \circ f^{-1}$, as it is shown in next result.

THEOREM 4.2. *Let* $f : \Omega \xrightarrow{\text{onto}} \Omega'$ *be a homeomorphism of finite inner distortion and let* $f \in WL^n \log^{-1} L_{\text{loc}}(\Omega, \mathsf{R}^n)$. *Assume that* $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ *and that*

$$K_{I,f} \in L^{1+\alpha}_{\text{loc}}(\Omega), \tag{4.8}$$

with $\alpha \geq 0$. *Then* $u \circ f^{-1} \in WL^n \log^\alpha L_{\text{loc}}(\Omega')$.

PROOF. From Theorem 2.1 we see that f^{-1} is a homeomorphism with finite outer distortion which belongs to $W^{1,1}_{\text{loc}}(\Omega', \mathsf{R}^n)$. By the assumption (4.8) and appealing to Lemma 4.2 in [18] we conclude that $f^{-1} \in WL^n \log^\alpha L_{\text{loc}}(\Omega', \mathsf{R}^n)$. We set for simplicity $\Phi(t) = t^n \log^\alpha(e + t)$. We start by proving that for every $E \subset\subset \Omega'$ we have

$$\int_E \Phi\left(\frac{|\nabla(u \circ f^{-1})|}{\lambda}\right) dy \leq C\Phi(1) \int_E \Phi\left(\frac{|Df^{-1}|}{\lambda}\right) dy, \tag{4.9}$$

for every $\lambda > 0$, if we further assume $\|\nabla u\|_{L^\infty(f^{-1}(E))} \leq 1$. Using chain rule, the monotonicity of Φ and the Δ'-condition for Φ, we get

$$\begin{aligned}\int_E \Phi\left(\frac{|\nabla(u \circ f^{-1})(y)|}{\lambda}\right) dy &\leq \int_E \Phi\left(\frac{|\nabla u(f^{-1}(y))||Df^{-1}(y)|}{\lambda}\right) dy \\ &\leq C \int_E \Phi(|\nabla u(f^{-1}(y))|)\Phi\left(\frac{|Df^{-1}(y)|}{\lambda}\right) dy \\ &\leq C\Phi(1) \int_E \Phi\left(\frac{|Df^{-1}|}{\lambda}\right) dy,\end{aligned}$$

hence (4.9) is established. Note that in the steps above we use the fact that both f and f^{-1} satisfy the Lusin (N) condition. We fix $\lambda > \|Df^{-1}\|_{L^n \log^\alpha L(E)}$. Therefore

$$\int_E \Phi\left(\frac{|Df^{-1}|}{\lambda}\right) dy \le 1,$$

and hence from (4.9) we deduce

$$\int_E \Phi\left(\frac{|\nabla(u \circ f^{-1})(y)|}{\lambda}\right) dy \le C\Phi(1).$$

The condition above clearly implies

$$\|\nabla(u \circ f^{-1})\|_{L^n \log^\alpha L(E)} \le C\lambda.$$

Taking the limit as $\lambda \to \|Df^{-1}\|_{L^n \log^\alpha L(E)}$ we conclude that

$$\|\nabla(u \circ f^{-1})\|_{L^n \log^\alpha L(E)} \le C\|Df^{-1}\|_{L^n \log^\alpha L(E)}.$$

This in turn proves that

$$\|\nabla(u \circ f^{-1})\|_{L^n \log^\alpha L(E)} \le C\|Df^{-1}\|_{L^n \log^\alpha L(E)}.\|\nabla u\|_{L^\infty(f^{-1}(E))}$$

for every $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

REMARK 4.1. The assumptions of both Theorem 1.2 and Theorem 4.1 lead to $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$. This type of condition can not be weakened; indeed, in view of Example 1.3 in [20] there exists a homeomorphism of finite outer distortion $f \in WL^{n-1} \log^{-\alpha} L_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ with $\alpha > 0$ such that $f^{-1} \notin W^{1,1}_{\mathrm{loc}}(\Omega', \mathsf{R}^n)$. Hence

$$T_{f^{-1}}\big(W^{1,\infty}_{\mathrm{loc}}(\Omega)\big) \not\subset W^{1,1}_{\mathrm{loc}}(\Omega').$$

The regularity assumption on f of Theorem 1.2 is stronger than the one in Theorem 4.1. The advantage of this better regularity is that $T_{f^{-1}}$ acts on the space $WL^r \log^{-\alpha(r-1)} L_{\mathrm{loc}}(\Omega)$ which properly contains $W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

REMARK 4.2. Under the assumptions of both Theorem 1.2 and Theorem 4.1, we cannot expect any inclusion of the type

$$T_{f^{-1}}\big(W^{1,\infty}_{\mathrm{loc}}(\Omega)\big) \subset W^{1,1+\delta}_{\mathrm{loc}}(\Omega') \tag{4.10}$$

for some $\delta > 0$. To this aim, let us recall that for every $\delta > 0$ and $n - 1 < p < \infty$ there exists a homeomorphism of finite distortion $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{R}^n)$ such that $f^{-1} \notin W^{1,1+\delta}_{\mathrm{loc}}(\Omega', \mathsf{R}^n)$ (see Example 6.1 in [25]). The inclusion (4.10) fails for such a homeomorphism.

4.2. The composition operator in Lorentz spaces

THEOREM 4.3. *Let* $n-1 < p < \infty$ *and* $n-1 < q < \infty$. *Let* $f : \Omega \xrightarrow{\text{onto}} \Omega'$ *be a homeomorphism of finite inner distortion and let* $f \in WL^{p,q}_{\text{loc}}(\Omega, \mathbb{R}^n)$. *Assume that* $u \in WL^{r,s}_{\text{loc}}(\Omega)$ *where*

$$r = \frac{p}{p-n+1} \qquad \textit{and} \qquad s = \frac{q}{q-n+1},$$

and we further assume that u *is continuous for* $r > n$. *Then,* $u \circ f^{-1} \in W^{1,1}_{\text{loc}}(\Omega')$. *Moreover, for every* $E' \subset\subset \Omega'$ *we have*

$$\|\nabla(u \circ f^{-1})\|_{L^1(E')} \leq \|Df\|^{n-1}_{L^{p,q}(f^{-1}(E'))}\|\nabla u\|_{L^{r,s}(f^{-1}(E'))}.$$

PROOF OF THEOREM 4.3. Arguing as in the proof of Theorem 1.2, we deduce that

$$\int_B |\nabla(w \circ f^{-1})|dy \leq \int_{f^{-1}(B)} |\nabla w(x)||Df(x)|^{n-1}\, dx, \tag{4.11}$$

for every ball $B \subset\subset \Omega'$ and for every $w \in C^\infty(\Omega)$.

Using Hölder inequality in Lorentz spaces (2.5), we deduce from (4.11) that

$$\int_B |\nabla(w \circ f^{-1})|\, dx \leq \|\nabla w\|_{L^{r,s}(f^{-1}(B))}\, \|Df\|^{n-1}_{L^{p,q}(f^{-1}(B))}. \tag{4.12}$$

Let now u be an arbitrary function in $WL^{r,s}_{\text{loc}}(\Omega)$ and let $\{u_j\}$ be a sequence of smooth functions which approximate u by standard mollification. We take two indices i, j and we apply (4.12) to $w = u_i - u_j$. We see that $\{\nabla(u_j \circ f^{-1})\}$ is a Cauchy sequence in $L^1(B, \mathbb{R}^n)$ and converges strongly in $L^1(B, \mathbb{R}^n)$. By means of Theorem 3.47 in [2], it follows that $\{u_j \circ f^{-1}\}$ is a Cauchy sequence in $L^1(B)$. The argument used at the end of Theorem 3.1 leads us to the desired result.

THEOREM 4.4. *Let* $f : \Omega \xrightarrow{\text{onto}} \Omega'$ *be a homeomorphism of finite inner distortion and let* $f \in WL^{n-1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ *with*

$$K_{I,f} \in L^{1,\infty}_{\text{loc}}(\Omega).$$

Assume that $u \in W^{1,\infty}_{\text{loc}}(\Omega)$. *Then,* $u \circ f^{-1} \in W^{1,n-1}_{\text{loc}}(\Omega')$. *Moreover, for every* $E' \subset\subset \Omega'$ *we have*

$$\|\nabla(u \circ f^{-1})\|_{L^{n-1}(E')} \leq \|Df^{-1}\|_{L^{n-1}(E')}\|\nabla u\|_{L^\infty(f^{-1}(E'))}. \tag{4.13}$$

PROOF. From $|Df| \in L^{n-1,1}_{\text{loc}}(\Omega)$, Hadamard's inequality

$$|\text{adj}\, Df| \leq |Df|^{n-1} \qquad \text{a.e in} \quad \Omega,$$

and from (2.4) we get

$$\|\text{adj}\, Df\|^{\frac{1}{n-1}}_{L^{1,\frac{1}{n-1}}(E)} \leq \||Df|^{n-1}\|^{\frac{1}{n-1}}_{L^{1,\frac{1}{n-1}}(E)} = \|Df\|_{L^{n-1,1}(E)}$$

for every $E \subset\subset \Omega$. Thus $|\text{adj}\, Df| \in L^{1,\frac{1}{n-1}}_{\text{loc}}(\Omega)$. Now, we are in position to apply Theorem 1.2 in [33] and we obtain $|Df^{-1}| \in L^{n-1}_{\text{loc}}(\Omega)$. Finally, the claimed result follows from Theorem 2.1 in [15].

It remains to prove the estimate (4.13). To this end, let $u \in W^{1,\infty}_{\text{loc}}(\Omega)$; we observe that

$$\nabla(u \circ f^{-1})(y) = \nabla u(f^{-1}(y))Df^{-1}(y) \qquad \text{for a.e} \quad y \in \Omega'.$$

Hence, for every $E' \subset\subset \Omega'$ we have

$$\begin{aligned} &\int_{E'} |\nabla(u \circ f^{-1})(y)|^{n-1}\, dy \\ &\leq \int_{E'} |\nabla(u(f^{-1}(y))|^{n-1}|Df^{-1}(y)|^{n-1}\, dy \\ &\leq \|Df^{-1}\|^{n-1}_{L^{n-1}(E')}\|\nabla u\|^{n-1}_{L^{\infty}(f^{-1}(E'))} \end{aligned} \tag{4.14}$$

Therefore (4.13) holds, and the proof is complete.

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